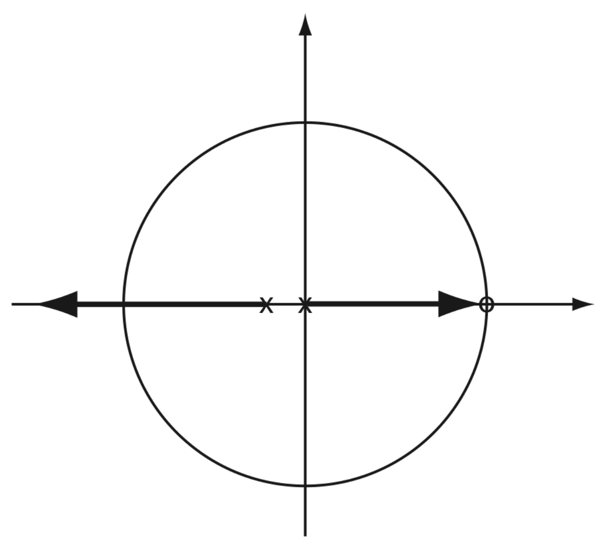
A feedback system is characterized by a forward-path transfer function $H_1(z) = \dfrac{z-1}{z+0.2}$ and a feedback-path transfer function $H_2(z) = Kz^{-1}$. At what positive value of K will the overall feedback system transition from stable to unstable? From a root locus we can see that the poles of the overall feedback system migrate from a pole at $z=0$ to a zero at $z=1$ and from a pole at $z=-0.2$ to a zero an infinite distance away on the negative real axis.



Using the feedback formulas we get

$$\mathrm{H}(z)=\frac{\dfrac{z-1}{z+0.2}}{1+\dfrac{K(z-1)}{z(z+0.2)}}=\frac{z(z-1)}{z(z+0.2)+K(z-1)}=\frac{z(z-1)}{z^2+(0.2+K)z-K}$$

This system has poles at $z^2+(0.2+K)z-K=0$

$$\text{or } z=\frac{-0.2-K\pm\sqrt{0.04+0.4K+K^2+4K}}{2}$$

$$=-0.1-K/2\pm\sqrt{0.01+1.1K+K^2/4}$$

We know one of the poles must leave the unit circle at $z=-1$. If we choose the negative sign on the square root and set $z=-1$ we get

$$-0.1-K/2-\sqrt{0.01+1.1K+K^2/4}=-1$$

Solving, $K=0.4$ and this must be the positive value of K we seek.

If we choose the positive sign on the square root and set $z=-1$ we get

$$-0.1-K/2+\sqrt{0.01+1.1K+K^2/4}=-1$$

$$\sqrt{0.01+1.1K+K^2/4}=-0.9+K/2$$

Squaring both sides we get

$$0.01+1.1K+K^2/4=0.81-0.9K+K^2/4$$

and that leads to an apparent solution $K=0.4$. But how can $K=0.4$ yield an equality for both signs in

$-0.1-K/2\pm\sqrt{0.01+1.1K+K^2/4}=-1$?

The answer is, it cannot. If we substitute $K=0.4$ into

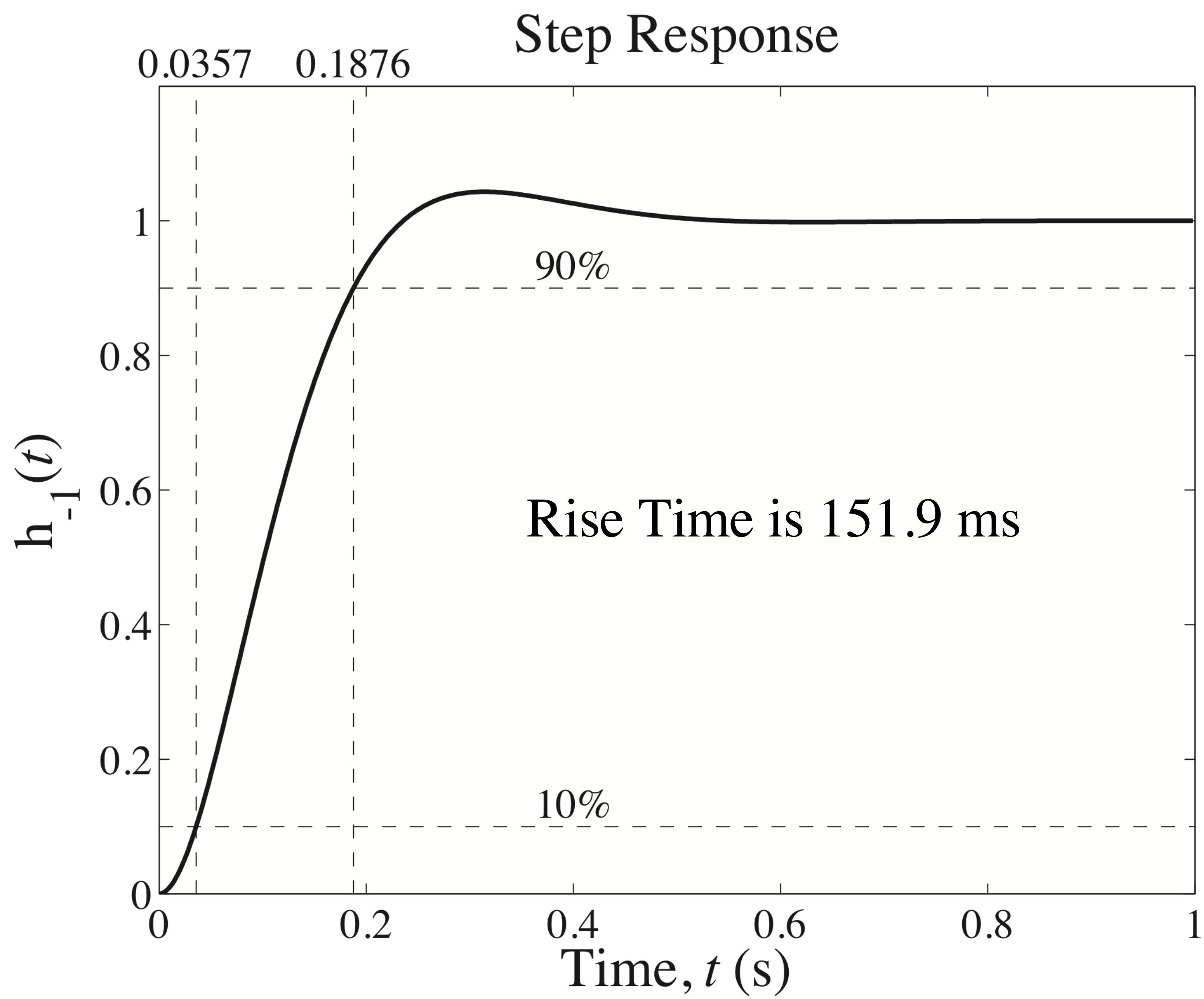
$-0.1-K/2+\sqrt{0.01+1.1K+K^2/4}$ we get

$-0.1-K/2+\sqrt{0.01+1.1K+K^2/4}=0.4$, not -1. In solving for K by squaring both sides of $\sqrt{0.01+1.1K+K^2/4}=-0.9+K/2$ we found a solution of $0.01+1.1K+K^2/4=0.81-0.9K+K^2/4$ or

$0.01+1.1K=0.81-0.9K$ but not of $-0.1-K/2+\sqrt{0.01+1.1K+K^2/4}=-1$.

Rise time is defined as the time required for a system response to a step to rise from 10% of its final value to 90% of its final value. Find the rise time of the system whose transfer function is

$\mathrm{H}(s)=\dfrac{200}{s^2+20s+200}$. The Laplace transform of the unit step response is $\mathrm{H}_{-1}(s)=\dfrac{200}{s(s^2+20s+200)}=\dfrac{1}{s}-\dfrac{s+20}{s^2+20s+200}$ and the step response is $\mathrm{h}_{-1}(t)=\left[1+1.414e^{-10t}\cos(10t+2.3562)\right]\mathrm{u}(t)$. The final value is 1. So we need to find solutions to $\mathrm{h}_{-1}(t_{10\%})=0.1$ and $\mathrm{h}_{-1}(t_{90\%})=0.9$. We can only find the solutions numerically because the equation is transcendental.



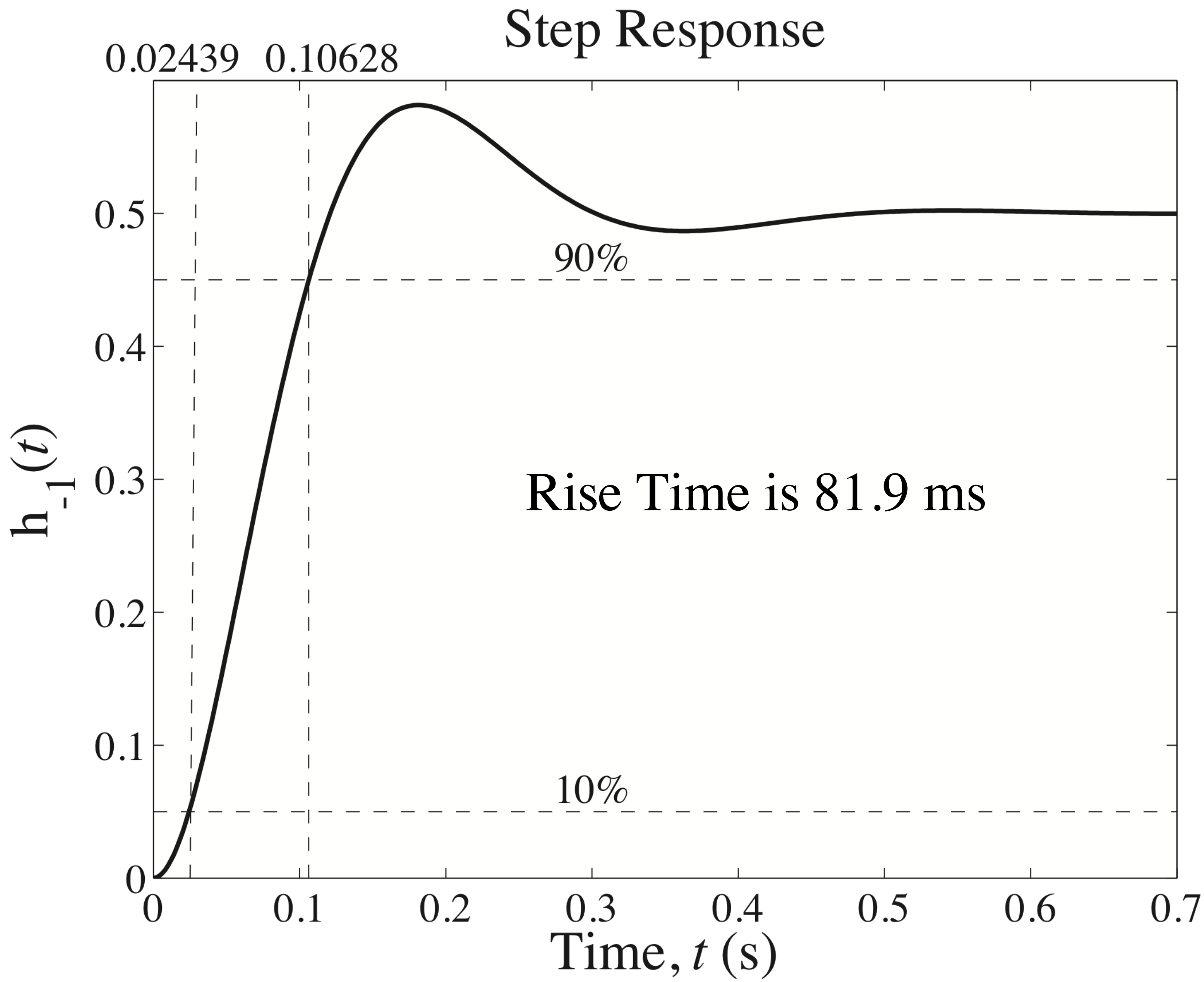
Step Response
0.0357
0.1876
90%
10%
Rise Time is 151.9 ms
$h_{-1}(t)$
Time, t (s)
0
0.2
0.4
0.6
0.8
1

Now let the system be the forward path in a unity-gain feedback system and find the rise time of the overall feedback system.

$$\mathrm{H}_1(s) = \frac{200}{s^2 + 20s + 200} \Rightarrow \mathrm{H}(s) = \frac{200}{s^2 + 20s + 400}$$

$$\mathrm{H}_{-1}(s) = \frac{200}{s\left(s^2 + 20s + 400\right)} = \frac{0.5}{s} - \frac{0.5s + 10}{s^2 + 20s + 400}$$

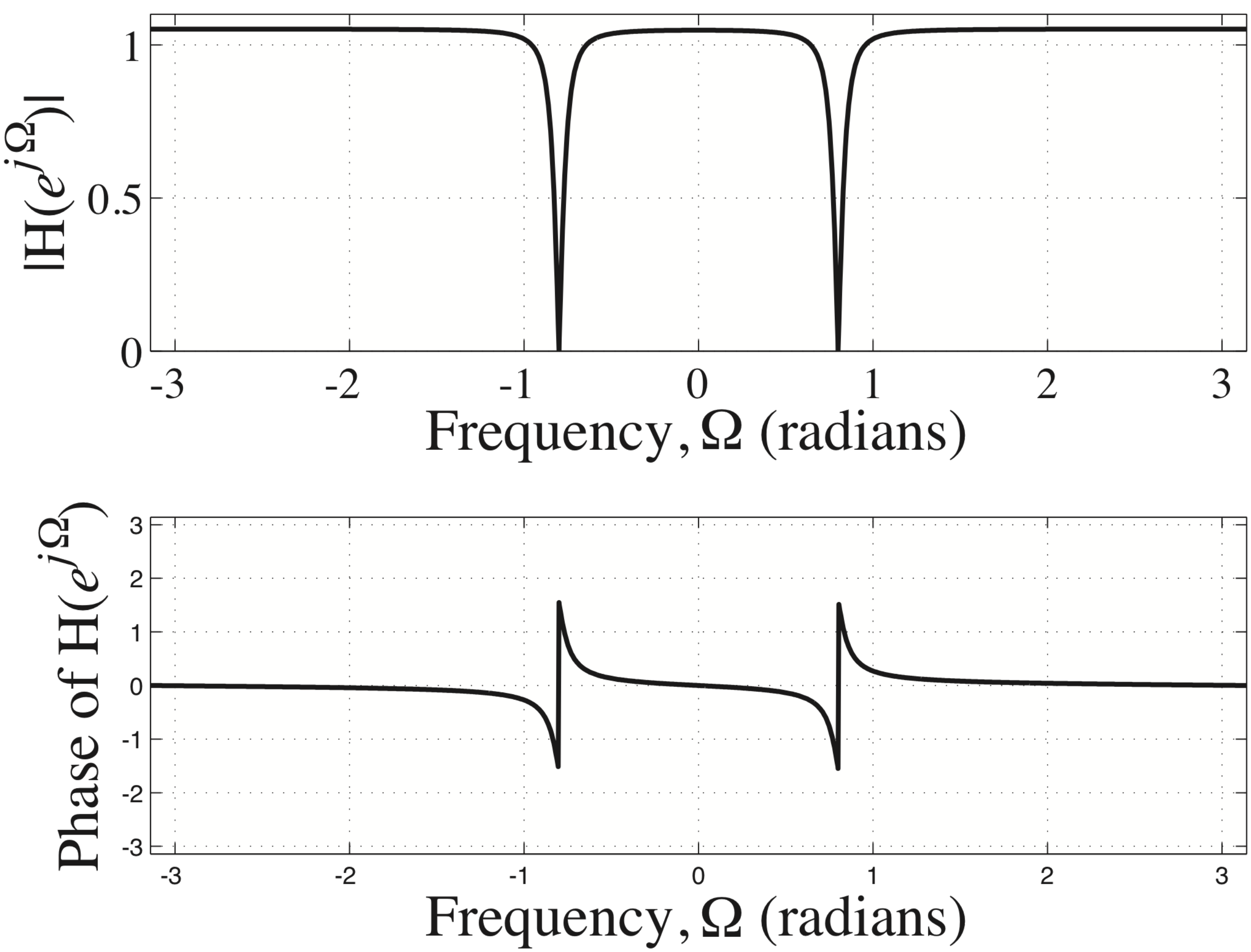
$$\mathrm{h}_{-1}(t) = \left[0.5 + 0.57735e^{-10t}\cos(17.3205t + 2.618)\right]\mathrm{u}(t)$$



Unity-gain feedback reduced the rise time by almost 50%.

Design a digital notch filter to remove a signal at a frequency $\Omega = 0.8$ by placing zeros on the unit circle at $\Omega = \pm 0.8$ and poles at the same angles but slightly inside the unit circle. Graph its frequency response.

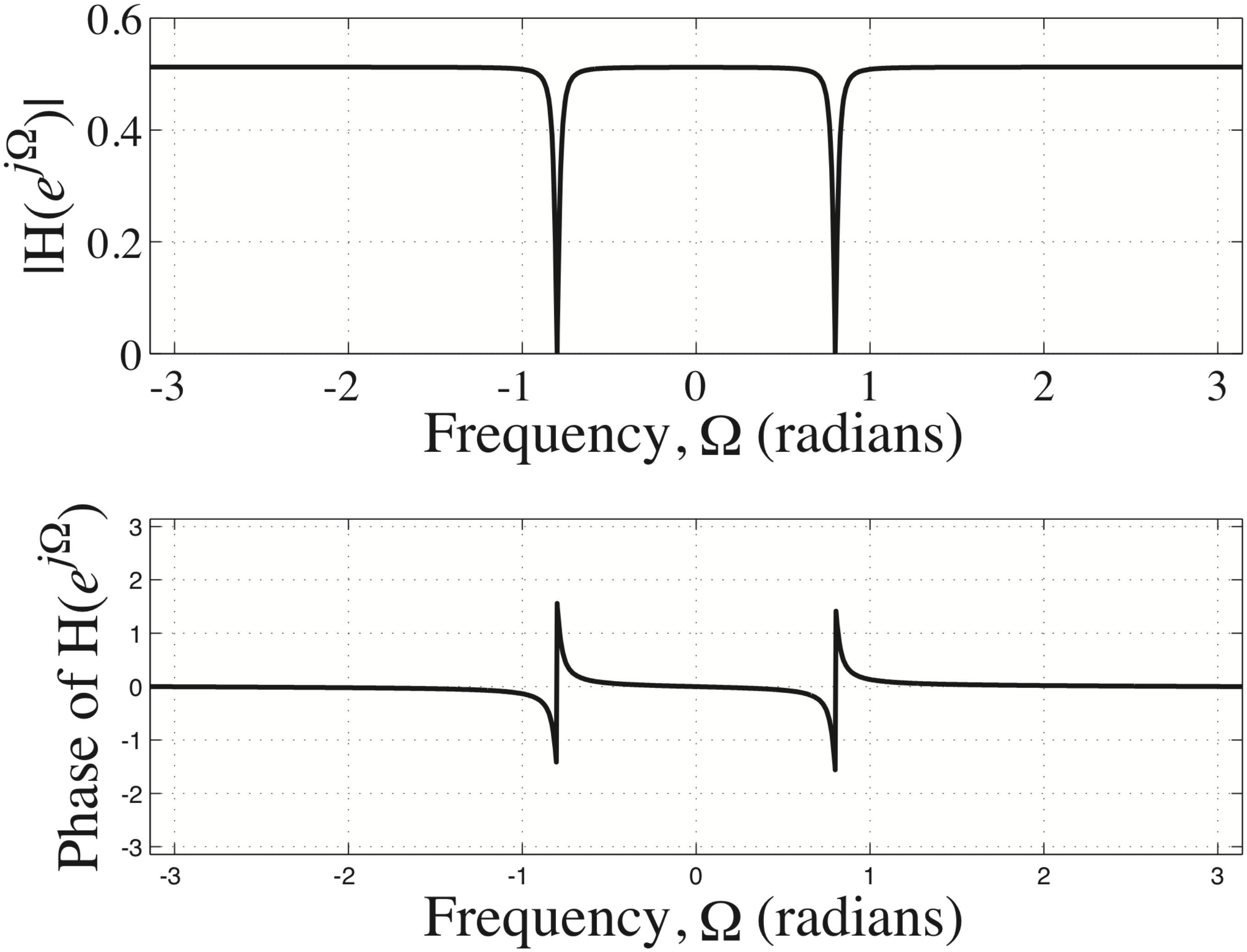
$$\mathrm{H}(z) = \frac{\left(z - e^{j0.8}\right)\left(z - e^{-j0.8}\right)}{\left(z - 0.95e^{j0.8}\right)\left(z - 0.95e^{-j0.8}\right)} = \frac{z^2 - 1.3934z + 1}{z^2 - 1.3237z + 0.9025}$$



|H($e^{j\Omega}$)|
1
0.5
0
-3
-2
-1
0
1
2
3
Frequency, Ω (radians)
Phase of H($e^{j\Omega}$)
3
2
1
0
-1
-2
-3
-3
-2
-1
0
1
2
3
Frequency, Ω (radians)

Now let the transfer function be the forward path in a unity-gain feedback system and graph the frequency response of the overall feedback system.

$$\mathrm{H}_1(z) = \frac{z^2 - 1.3934z + 1}{z^2 - 1.3237z + 0.9025} \Rightarrow \mathrm{H}(z) = 0.5\frac{z^2 - 1.3934z + 1}{z^2 - 1.359z + 0.9512}$$



The notch in the feedback system's frequency response is "sharper" than the notch in the original system's frequency response.