Web Appendix L - The DFT in Relation to the Other Fourier Methods - With Examples

L.1 Approximating the CTFS Using the DFT

The harmonic function of a periodic signal with period T_F is

$$\mathrm{X}[k] = \frac{1}{T_F}\int_{T_F} \mathrm{x}(t) e^{-j2\pi k f_F t} dt .$$

Since the starting point of the integral is arbitrary, for convenience set it to $t = 0$

$$\mathrm{X}[k] = \frac{1}{T_F}\int_0^{T_F} \mathrm{x}(t) e^{-j2\pi k f_F t} dt .$$

Suppose we don't know the function $\mathrm{x}(t)$ but we have a set of N_F samples over one period starting at $t = 0$, the time between samples is $T_s = T_F / N_F$. Then we can approximate the integral by the sum of several integrals, each covering a time of length T_s

$$\mathrm{X}[k] \cong \frac{1}{T_F}\sum_{n=0}^{N_F-1}\left[\int_{nT_s}^{(n+1)T_s} \mathrm{x}(nT_s) e^{-j2\pi k f_F nT_s} dt\right] \quad \text{(L.1)}$$

(Figure L-1).

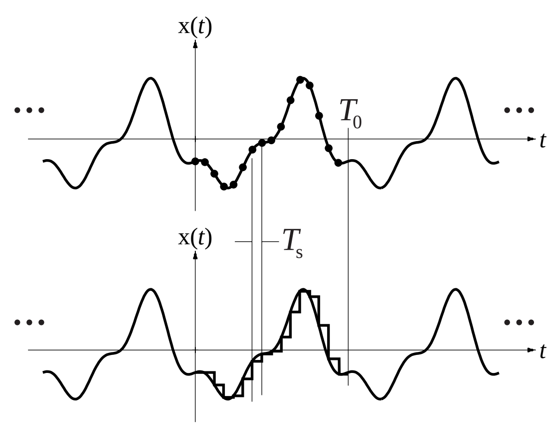

Figure L-1 Sampling an arbitrary periodic signal to estimate its CTFS harmonic function

(In Figure L-1, the samples extend over one fundamental period but they could extend over any period and the analysis would still be correct.) If the samples are close enough together $\mathrm{x}(t)$ does not change much between samples and the integral (L.1) becomes a good approximation. We can now complete the integration.

$$\mathrm{X}[k] \cong \frac{1}{T_F}\sum_{n=0}^{N_F-1}\left[\mathrm{x}(nT_s)\int_{nT_s}^{(n+1)T_s} e^{-j2\pi k f_F t}\,dt\right] = \frac{1}{T_F}\sum_{n=0}^{N_F-1}\mathrm{x}(nT_s)\left[\frac{e^{-j2\pi k f_F t}}{-j2\pi k f_F}\right]_{nT_s}^{(n+1)T_s}$$

$$\mathrm{X}[k] \cong \frac{1}{T_F}\sum_{n=0}^{N_F-1}\mathrm{x}(nT_s)\left[\frac{e^{-j2\pi k f_F nT_s} - e^{-j2\pi k f_F (n+1)T_s}}{j2\pi k f_F}\right] = \frac{1-e^{-j2\pi k f_F T_s}}{j2\pi k f_F T_F}\sum_{n=0}^{N_F-1}\mathrm{x}(nT_s)e^{-j2\pi k f_F nT_s}$$

Using $T_s = T_F / N_F$,

$$\mathrm{X}[k] \cong \frac{1-e^{-j2\pi k/N_F}}{j2\pi k}\sum_{n=0}^{N_F-1}\mathrm{x}(nT_s)e^{-j2\pi nk/N_F} = e^{-j\pi k/N_F}\,\frac{\overbrace{e^{j\pi k/N_F} - e^{-j\pi k/N_F}}^{j2\sin(\pi k/N_F)}}{j2\pi k}\sum_{n=0}^{N_F-1}\mathrm{x}(nT_s)e^{-j2\pi nk/N_F}$$

$$\mathrm{X}[k] \cong e^{-j\pi k/N_F}\,\frac{1}{N_F}\underbrace{\frac{\sin(\pi k/N_F)}{\pi k/N_F}}_{\mathrm{sinc}(k/N_F)}\sum_{n=0}^{N_F-1}\mathrm{x}(nT_s)e^{-j2\pi nk/N_F}$$

$$\cong e^{-j\pi k/N_F}\,\frac{\mathrm{sinc}(k/N_F)}{N_F}\sum_{n=0}^{N_F-1}\mathrm{x}(nT_s)e^{-j2\pi nk/N_F}$$

For harmonic numbers $|k| << N_F$ we can further approximate the harmonic function as

$$\mathrm{X}[k] \cong \frac{1}{N_F}\sum_{n=0}^{N_F-1}\mathrm{x}(nT_s)e^{-j2\pi nk/N_F}$$

or

$$\mathrm{X}[k] \cong (1/N_F)\times \mathsf{DFT}\left(\mathrm{x}(nT_s)\right).$$

This result returns a set of harmonic function values in the range $0 \le k < N_F$ which repeat periodically with period N_F. The values of $\mathrm{X}[k]$ for $|k| << N_F$ and k negative can be found from $\mathrm{X}[k] = \mathrm{X}[k+N_F]$. So, for example, to find $\mathrm{X}[-1]$, find its periodic repetition $X[N_F - 1]$ which is included in the range $0 \le k < N_F$. The set of harmonic function values in the range $-N_F << -k_{max} \le k \le k_{max} << N_F$ is the approximation of the actual harmonic function. Smaller values of T_s, implying larger values of N_F, yield better approximations.

Now assume that we know numerical values of a harmonic function $X[k]$ of a periodic continuous-time function $x(t)$, in the range $-N_F << -k_{max} \le k \le k_{max} << N_F$ and that the signal power is negligible outside that range and we want to find samples from one period of $x(t)$. The CTFS is

$$x(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi k f_F t} \cong \sum_{k=-k_{max}}^{k_{max}} X[k] e^{j2\pi k f_F t} .$$

If the time samples of $x(t)$ are taken at integer multiples of $T_s = T_F / N_F$, then

$$x(nT_s) \cong \sum_{k=-k_{max}}^{k_{max}} X[k] e^{j2\pi kn/N_F} = \sum_{k=-N_F/2}^{N_F/2-1} X_{ext}[k] e^{j2\pi kn/N_F}$$

and

$$x(nT_s) \cong \sum_{k=-N_F/2}^{-1} X_{ext}[k] e^{j2\pi kn/N_F} + \sum_{k=0}^{N_F/2-1} X_{ext}[k] e^{j2\pi kn/N_F}$$

where $X_{ext}[k] = \begin{cases} X[k] & , \quad -k_{max} \le k \le k_{max} \\ 0 & , \quad k_{max} < |k| < N_F/2 \end{cases}$ and $X_{ext}[k] = X_{ext}[k + mN_F]$ where m

is any integer. Then, taking advantage of the periodicity of $X_{ext}[k]$,

$$x(nT_s) \cong \sum_{k=N_F/2}^{N_F-1} X_{ext}[k] e^{j2\pi kn/N_F} + \sum_{k=0}^{N_F/2-1} X_{ext}[k] e^{j2\pi kn/N_F} = N_F \underbrace{(1/N_F) \sum_{k=0}^{N_F-1} X_{ext}[k] e^{j2\pi kn/N_F}}_{\mathrm{DFT}^{-1}(X_{ext}[k])}$$

or

$$x(nT_s) \cong N_F \times \mathrm{DFT}^{-1}(X_{ext}[k]).$$

L.2 Computing the DTFS Using the DFT

The forward and inverse DTFS formulas are

$$\mathrm{X}[k] = \frac{1}{N_F} \sum_{n=n_0}^{n_0+N_F-1} \mathrm{x}[n] e^{-j2\pi kn/N_F} \quad , \quad \mathrm{x}_F[n] = \sum_{k=\langle N_F \rangle} \mathrm{X}[k] e^{j2\pi kn/N_F} \qquad \text{(L.2)}$$

If the time-domain function $\mathrm{x}[n]$ is bounded on the representation time $n_0 \le n < n_0 + N_F$ the harmonic function can always be found and is itself bounded because it is a finite summation of bounded terms.

The summations in (L.2) should look familiar. In

$$\sum_{n=n_0}^{n_0+N_F-1} \mathrm{x}[n] e^{-j2\pi kn/N_F}$$

if we let n_0 be zero we get

$$\sum_{n=0}^{N_F-1} \mathrm{x}[n] e^{-j2\pi kn/N_F}$$

and this is the DFT, first encountered in approximating the CTFS numerically in Chapter 8. So the DFT and the DTFS are very similar, differing only by a scale constant N_F if the choice of the first *n* in the summation

$$\sum_{n=n_0}^{n_0+N_F-1} \mathrm{x}[n] e^{-j2\pi kn/N_F}$$

is $n_0 = 0$. As was true for the DFT, the DTFS harmonic function is periodic with period N_F. Summarizing, if a signal $\mathrm{x}[n]$ has a fundamental period N_0, and N_F is an integer multiple of N_0, its DTFS harmonic function is

$$\mathrm{X}[k] = (1/N_F)\,\mathrm{DFT}(\mathrm{x}[n]) \ , \ 0 \le k < N_F \qquad \text{(L.3)}$$

where $0 \le n < N_F$. Conversely, $\mathrm{x}[n]$ can be found from $\mathrm{X}[k]$ as

$$\mathrm{x}[n] = \mathrm{DFT}^{-1}(N_F\,\mathrm{X}[k]) \ , \ 0 \le n < N_F .$$

L.3 Approximating the CTFT Using the DFT

In cases in which the signal to be transformed is not readily describable by a mathematical function or the Fourier-transform integral cannot be done analytically, we can sometimes find an approximation to the CTFT numerically using the DFT. The CTFT of a signal $x(t)$ is

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \ .$$

If we apply this to a signal that is causal we get

$$X(f) = \int_{0}^{\infty} x(t) e^{-j2\pi ft} dt \, .$$

We can write this integral in the form

$$X(f) = \sum_{n=0}^{\infty} \int_{nT_s}^{(n+1)T_s} x(t) e^{-j2\pi ft} dt \, .$$

If T_s is small enough, the variation of $x(t)$ in the time interval $nT_s \leq t < (n+1)T_s$ is small and the CTFT can be approximated by

$$X(f) \cong \sum_{n=0}^{\infty} x(nT_s) \int_{nT_s}^{(n+1)T_s} e^{-j2\pi ft} dt \, .$$

or

$$X(f) \cong \sum_{n=0}^{\infty} x(nT_s) \frac{e^{-j2\pi fnT_s} - e^{-j2\pi f(n+1)T_s}}{j2\pi f}$$

or

$$X(f) \cong \frac{1 - e^{-j2\pi fT_s}}{j2\pi f} \sum_{n=0}^{\infty} x(nT_s) e^{-j2\pi fnT_s} = T_s e^{-j\pi fT_s} \operatorname{sinc}(T_s f) \sum_{n=0}^{\infty} x(nT_s) e^{-j2\pi fnT_s}$$

(Figure L-2).

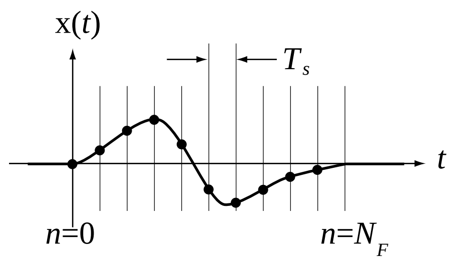

Figure L-2 A signal and multiple intervals on which the CTFT integral can be evaluated

If $x(t)$ is an energy signal then beyond some finite time its size must become negligible and we can replace the infinite range of *n* in the summation with a finite range $0 \leq n < N_F$ yielding

$$\mathrm{X}(f) \cong T_s e^{-j\pi f T_s} \operatorname{sinc}(T_s f) \sum_{n=0}^{N_F - 1} \mathrm{x}(nT_s) e^{-j2\pi f n T_s} .$$

Now if we compute the CTFT only at integer multiples of $f_s / N_F = f_F$, which is the frequency-domain resolution of this approximation to the CTFT,

$$\mathrm{X}(kf_F) \cong T_s e^{-j\pi k f_F T_s} \operatorname{sinc}(T_s k f_F) \sum_{n=0}^{N_F - 1} \mathrm{x}(nT_s) e^{-j2\pi k n f_F T_s}$$

or

$$\mathrm{X}(kf_F) \cong T_s e^{-j\pi k / N_F} \operatorname{sinc}(k / N_F) \sum_{n=0}^{N_F - 1} \mathrm{x}(nT_s) e^{-j2\pi k n / N_F} .$$

The summation in this equation is the DFT of $\mathrm{x}(nT_s)$. Therefore

$$\mathrm{X}(kf_F) \cong T_s e^{-j\pi k / N_F} \operatorname{sinc}(k / N_F) \times \mathcal{DFT}\left(\mathrm{x}(nT_s)\right)$$

where the notation $\mathcal{DFT}(\cdot)$ means "discrete Fourier transform of". For $|k| << N_F$,

$$\mathrm{X}(kf_F) \cong T_s \times \mathcal{DFT}\left(\mathrm{x}(nT_S)\right) . \tag{L.4}$$

So if the signal to be transformed is a causal energy signal and we sample it over a time containing practically all of its energy and if the samples are close enough together that the signal does not change appreciably between samples, the approximation in (L.4) becomes accurate for $|k| << N_F$.

The inverse CTFT is $\mathrm{x}(t) = \int_{-\infty}^{\infty} \mathrm{X}(f) e^{j2\pi f t} df$. If we know $\mathrm{X}(kf_F)$ in the range $-N_F << -k_{\max} \le k \le k_{\max} << N_F$ and if the magnitude of $\mathrm{X}(kf_F)$ is negligible outside that range then

$$\mathrm{x}(t) \cong \sum_{k=-k_{\max}}^{k_{\max}} \int_{kf_F}^{(k+1) f_F} \mathrm{X}(kf_F) e^{j2\pi f t} df \cong \sum_{k=-k_{\max}}^{k_{\max}} \mathrm{X}(kf_F) \int_{kf_F}^{(k+1) f_F} e^{j2\pi f t} df$$

$$\mathrm{x}(t) \cong \sum_{k=-k_{\max}}^{k_{\max}} \mathrm{X}(kf_F) \left[\frac{e^{j2\pi f t}}{j2\pi t} \right]_{kf_F}^{(k+1) f_F} = \sum_{k=-k_{\max}}^{k_{\max}} \mathrm{X}(kf_F) \left[\frac{e^{j2\pi t (k+1) f_F} - e^{j2\pi t k f_F}}{j2\pi t} \right]$$

$$\mathrm{x}(t) = \frac{e^{j2\pi t f_F} - 1}{j2\pi t} \sum_{k=-k_{\max}}^{k_{\max}} \mathrm{X}(kf_F) e^{j2\pi t k f_F}$$

If we compute $\mathrm{x}(t)$ only at a discrete set of points nT_s,

$$\mathrm{x}(nT_s) \cong \frac{e^{j2\pi n/N_F} - 1}{j2\pi nT_s} \sum_{k=-k_{\max}}^{k_{\max}} \mathrm{X}(kf_F) e^{j2\pi nk/N_F} = e^{j\pi n/N_F} \frac{\mathrm{sinc}(n/N_F)}{T_s N_F} \sum_{k=-k_{\max}}^{k_{\max}} \mathrm{X}(kf_F) e^{j2\pi nk/N_F}$$

$$\mathrm{x}(nT_s) \cong e^{j\pi n/N_F} \frac{\mathrm{sinc}(n/N_F)}{T_s N_F} \sum_{k=-N_F/2}^{N_F/2-1} \mathrm{X}_{ext}(kf_F) e^{j2\pi nk/N_F}$$

where $\mathrm{X}_{ext}(kf_F) = \begin{cases} \mathrm{X}(kf_F) & , \ -k_{\max} \le k \le k_{\max} \\ 0 & , \ k_{\max} < |k| \le N_F/2 \end{cases}$ and $\mathrm{X}_{ext}(kf_F) = \mathrm{X}_{ext}((k + mN_F) f_F)$

where *m* is an integer. Taking advantage of the periodicity of $\mathrm{X}_{ext}(kf_F)$,

$$\mathrm{x}(nT_s) \cong e^{j\pi n/N_F} \frac{\mathrm{sinc}(n/N_F)}{T_s N_F} \sum_{k=0}^{N_F-1} \mathrm{X}_{ext}(kf_F) e^{j2\pi nk/N_F} \quad .$$

Then, for $n << N_F$,

$$\mathrm{x}(nT_s) \cong \frac{1}{T_s} \underbrace{\frac{1}{N_F} \sum_{k=0}^{N_F-1} \mathrm{X}_{ext}(kf_F) e^{j2\pi nk/N_F}}_{\mathrm{DFT}^{-1}(\mathrm{X}_{ext}(kf_F))} \quad .$$

Therefore $\mathrm{x}(nT_s) \cong (1/T_s) \times \mathrm{DFT}^{-1}(\mathrm{X}_{ext}(kf_F))$.

L.4 Approximating the DTFT Using the DFT

The DFT can be used to compute the DTFT for a restricted class of signals. If the signal to be transformed $\mathrm{x}[n]$ is a causal energy signal, the general formula for the DTFT

$$\mathrm{X}(F) = \sum_{n=-\infty}^{\infty} \mathrm{x}[n] e^{-j2\pi Fn}$$

can be restricted to the form

$$\mathrm{X}(F) \cong \sum_{n=0}^{N_F-1} \mathrm{x}[n] e^{-j2\pi Fn}$$

where $n = N_F$ is the time beyond which the signal energy of $\mathrm{x}[n]$ is negligible. Then, if we compute estimates of the DTFT at discrete values $F = k/N_F$ we get

$$\mathrm{X}(k/N_F) \cong \sum_{n=0}^{N_F-1} \mathrm{x}[n] e^{-j2\pi kn/N_F} .$$

The summation is the DFT of $\mathrm{x}[n]$ in the discrete-time range $0 \le n < N_F$. So we can summarize by saying that the DTFT of $\mathrm{x}[n]$ computed at frequencies $F = k/N_F$ is

$$\mathrm{X}(k/N_F) \cong \mathcal{DFT}(\mathrm{x}[n]). \tag{L.5}$$

The inverse DTFT is

$$\mathrm{x}[n] = \int_1 \mathrm{X}(F) e^{j2\pi Fn} dF .$$

If we have the values of $\mathrm{X}(k/N_F)$(which is periodic with period 1) in the range $0 \le k < N_F - 1$ then

$$\mathrm{x}[n] \cong \sum_{k=0}^{N_F-1} \mathrm{X}(k/N_F) \int_{k/N_F}^{(k+1)/N_F} e^{j2\pi Fn} dF .$$

$$\mathrm{x}[n] \cong \sum_{k=0}^{N_F-1} \mathrm{X}(k/N_F) \left[\frac{e^{j2\pi Fn}}{j2\pi n} \right]_{k/N_F}^{(k+1)/N_F} = \frac{1}{j2\pi n} \sum_{k=0}^{N_F-1} \mathrm{X}(k/N_F) \left[e^{j2\pi(k+1)n/N_F} - e^{j2\pi kn/N_F} \right]$$

$$\mathrm{x}[n] \cong \frac{e^{j2\pi n/N_F} - 1}{j2\pi n} \sum_{k=0}^{N_F-1} \mathrm{X}(k/N_F) e^{j2\pi kn/N_F} = e^{j2\pi n/N_F} \frac{\mathrm{sinc}(n/N_F)}{N_F} \sum_{k=0}^{N_F-1} \mathrm{X}(k/N_F) e^{j2\pi kn/N_F}$$

For $n << N_F$,

$$\mathrm{x}[n] \cong \frac{1}{N_F} \sum_{k=0}^{N_F-1} \mathrm{X}(k/N_F) e^{j2\pi kn/N_F} = \mathcal{DFT}^{-1}(\mathrm{X}(k/N_F)) .$$

L.5 Approximating Continuous-Time Convolution Using the DFT

A common use of the DFT is to approximate the convolution of two CT signals using samples from them. Suppose we want to convolve two aperiodic CT energy signals $\mathrm{x}(t)$ and $\mathrm{h}(t)$ to form $\mathrm{y}(t)$. We know that

$$y(t) = x(t) * h(t) \xleftrightarrow{\mathcal{F}} X(f)H(f) = Y(f)$$

If x and y are both causal and practically time-limited to the time range $0 < t < T_F$, we can approximate $X(kf_F)$ and $H(kf_F)$ using $X(kf_F) \cong T_s \times \mathcal{DFT}(x(nT_S))$ and form $Y(kf_F) = X(kf_F)H(kf_F)$ from samples of $x(t)$ and $h(t)$. Then to get back to the time domain we need to approximate an inverse CTFT. The process of finding this approximation mirrors the derivation of the approximations in $X(kf_F) \cong T_s \times \mathcal{DFT}(x(nT_S))$. Assuming $x(t)$ and $h(t)$ are properly sampled we can write

$$y(t) = \int_{-\infty}^{\infty} Y(f) e^{j2\pi ft} df = \int_{-f_s/2}^{f_s/2} Y(f) e^{j2\pi ft} df \cong \sum_{k=-N_F/2}^{N_F/2-1} \int_{kf_F}^{(k+1)f_F} Y(kf_F) e^{j2\pi ft} df$$

Using the periodicity of the DFT,

$$y(t) \cong \sum_{k=0}^{N_F-1} Y(kf_F) \int_{kf_F}^{(k+1)f_F} e^{j2\pi ft} df = \sum_{k=0}^{N_F-1} Y(kf_F) \frac{e^{j2\pi(k+1)f_F t} - e^{j2\pi kf_F t}}{j2\pi t}$$

$$y(t) \cong \frac{e^{j2\pi f_F t} - 1}{j2\pi t} \sum_{k=0}^{N_F-1} Y(kf_F) e^{j2\pi kf_F t} = e^{j\pi f_F t} \frac{e^{j\pi f_F t} - e^{-j\pi f_F t}}{j2\pi t} \sum_{k=0}^{N_F-1} Y(kf_F) e^{j2\pi kf_F t}$$

$$y(t) \cong e^{j\pi f_F t} \frac{\sin(\pi f_F t)}{\pi t} \sum_{k=0}^{N_F-1} Y(kf_F) e^{j2\pi kf_F t} .$$

Now we can find approximations to $y(t)$ at the sample times $t = nT_s$

$$y(nT_s) \cong N_F e^{j\pi n/N_F} \frac{\sin(\pi n / N_F)}{\pi n T_s} \underbrace{\frac{1}{N_F} \sum_{k=0}^{N_F-1} Y(kf_F) e^{j2\pi nk/N_F}}_{\text{inverse DFT of } Y(kf_F)}$$

$$y(nT_s) \cong e^{j\pi n/N_F} f_s \operatorname{sinc}\left(\frac{n}{N_F}\right) \underbrace{\frac{1}{N_F} \sum_{k=0}^{N_F-1} Y(kf_F) e^{j2\pi nk/N_F}}_{\text{inverse DFT of } Y(kf_F)}$$

For $|n| << N_F$,

$$y(nT_s) \cong f_s \times \mathcal{DFT}^{-1}(Y(kf_F)).$$

From $\mathrm{X}(kf_F) \cong T_s \times \mathsf{DFT}(\mathrm{x}(nT_S))$,

$$\mathrm{Y}(kf_F) = \mathrm{X}(kf_F)\mathrm{H}(kf_F) \cong T_s\,\mathsf{DFT}(\mathrm{x}(nT_s)) \times T_s\,\mathsf{DFT}(\mathrm{h}(nT_s))$$

Therefore

$$\mathrm{y}(nT_s) = \left[\mathrm{x}(t) * \mathrm{h}(t)\right]_{t \to nT_s} \cong T_s \times \mathsf{DFT}^{-1}\left(\mathsf{DFT}(\mathrm{x}(nT_s)) \times \mathsf{DFT}(\mathrm{h}(nT_s))\right) \quad \text{(L.6)}$$

L.6 Approximating Discrete-Time Convolution Using the DFT

If $\mathrm{x}[n]$ is a DT energy signal and most or all of its energy occurs in the time range $0 \le n < N_F$ then

$$\mathrm{X}(F) \cong \sum_{n=0}^{N_F-1} \mathrm{x}[n] e^{-j2\pi Fn}$$

and

$$\mathrm{X}(k/N_F) \cong \sum_{n=0}^{N_F-1} \mathrm{x}[n] e^{-j2\pi nk/N_F} = \mathsf{DFT}(\mathrm{x}[n]) \quad \text{(L.7)}$$

Convolving two DT signals through the use of the DTFT,

$$\mathrm{y}[n] = \mathrm{x}[n] * \mathrm{h}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(F)\mathrm{H}(F) = \mathrm{Y}(F)$$

$$\mathrm{y}[n] = \int_1 \mathrm{Y}(F) e^{j2\pi Fn} dF \cong \sum_{k=0}^{N_F-1} \int_{k/N_F}^{(k+1)/N_F} \mathrm{Y}(F) e^{j2\pi Fn} dF$$

Assuming $\mathrm{Y}(F)$ does not change much in any F-interval of width $1/N_F$,

$$\mathrm{y}[n] \cong \sum_{k=0}^{N_F-1} \mathrm{Y}\left(\frac{k}{N_F}\right) \int_{k/N_F}^{(k+1)/N_F} e^{j2\pi Fn} dF = \sum_{k=0}^{N_F-1} \mathrm{Y}\left(\frac{k}{N_F}\right) \frac{e^{j2\pi(k+1)n/N_F} - e^{j2\pi kn/N_F}}{j2\pi n}$$

$$\mathrm{y}[n] \cong N_F \frac{e^{j2\pi n/N_F} - 1}{j2\pi n} \underbrace{\frac{1}{N_F} \sum_{k=0}^{N_F-1} \mathrm{Y}\left(\frac{k}{N_F}\right) e^{j2\pi nk/N_F}}_{\text{Inverse DFT of } \mathrm{Y}(k/N_F)}$$

$$\mathrm{y}[n] \cong e^{j\pi n/N_F}\,\mathrm{sinc}(n/N_F) \times \mathrm{DFT}^{-1}\left(\mathrm{Y}(k/N_F)\right)$$

and for $|n| << N_F$,

$$\mathrm{y}[n] \cong \mathrm{DFT}^{-1}\left(\mathrm{Y}(k/N_F)\right).$$

Using (L.7),

$$\mathrm{y}[n] \cong \mathrm{DFT}^{-1}\left(\mathrm{X}(k/N_F)\mathrm{H}(k/N_F)\right)$$

$$\mathrm{y}[n] = \mathrm{x}[n] * \mathrm{h}[n] \cong \mathrm{DFT}^{-1}\left(\mathrm{DFT}\left(\mathrm{x}[n]\right) \times \mathrm{DFT}\left(\mathrm{h}[n]\right)\right) \qquad \text{(L.8)}$$

L.7 Approximating Continuous-Time Periodic Convolution Using the DFT

We can also approximate CT periodic convolution using the DFT. Let $\mathrm{x}(t)$ and $\mathrm{h}(t)$ be two periodic CT signals with a common period T_F and sample them over exactly that time at a rate f_s above the Nyquist rate, taking N_F samples of each signal. Let $\mathrm{y}(t)$ be the periodic convolution of $\mathrm{x}(t)$ with $\mathrm{h}(t)$. Then

$$\mathrm{y}(t) = \mathrm{x}(t) \circledast \mathrm{h}(t) \xleftrightarrow{\mathcal{FS}} T_F\,\mathrm{X}_{CTFS}[k]\mathrm{H}_{CTFS}[k] = \mathrm{Y}_{CTFS}[k].$$

Then, from $\mathrm{X}_{DFT}[k] = \mathrm{DFT}\left(\mathrm{x}(nT_S)\right) = N_F\,\mathrm{X}_{CTFS}[k] * \delta_{N_F}[k]$,

$$\mathrm{X}_{CTFS}[k] = \frac{\mathrm{DFT}\left(\mathrm{x}(nT_s)\right)}{N_F} \quad , \quad \mathrm{H}_{CTFS}[k] = \frac{\mathrm{DFT}\left(\mathrm{h}(nT_s)\right)}{N_F} \quad , \quad -\frac{N_F}{2} \le k < \frac{N_F}{2}$$

$$\mathrm{y}(t) = \sum_{k=-\infty}^{\infty} \mathrm{Y}_{CTFS}[k] e^{+j2\pi k f_F t} \cong \sum_{k=-N_F/2}^{N_F/2-1} \mathrm{Y}_{CTFS}[k] e^{+j2\pi k f_F t}$$

$$\mathrm{y}(nT_s) \cong \sum_{k=-N_F/2}^{N_F/2-1} \mathrm{Y}_{CTFS}[k] e^{+j2\pi k f_F n T_s} = \sum_{k=-N_F/2}^{N_F/2-1} \mathrm{Y}_{CTFS}[k] e^{+j2\pi k n/N_F}$$

$$\mathrm{y}(nT_s) \cong T_F \sum_{k=-N_F/2}^{N_F/2-1} \mathrm{X}_{CTFS}[k]\mathrm{H}_{CTFS}[k] e^{+j2\pi k n/N_F}$$

$$y(nT_s) \cong \frac{T_F}{N_F^2} \sum_{k=-N_F/2}^{N_F/2-1} \left[\text{DFT}\left(x(nT_s)\right) \times \text{DFT}\left(h(nT_s)\right)\right] e^{+j2\pi kn/N_F}$$

Then, using the periodicity of the DFT,

$$y(nT_s) \cong \underbrace{\frac{T_F}{N_F}}_{=T_s} \underbrace{\frac{1}{N_F} \sum_{k=0}^{N_F-1} \left[\text{DFT}\left(x(nT_s)\right) \times \text{DFT}\left(h(nT_s)\right)\right] e^{+j2\pi kn/N_F}}_{=\text{DFT}^{-1}\left[\text{DFT}\left(x(nT_s)\right)\times \text{DFT}\left(h(nT_s)\right)\right]} .$$

Therefore

$$y(nT_s) = \left[x(t) \circledast h(t)\right]_{t \to nT_s} \cong T_s \times \text{DFT}^{-1}\left(\text{DFT}\left(x(nT_s)\right) \times \text{DFT}\left(h(nT_s)\right)\right) \quad \text{(L.9)}$$

L.8 Computing DT Periodic Convolution Using the DFT

Let $x[n]$ and $h[n]$ be two periodic DT signals with a common period N_F. Let $y[n]$ be the periodic convolution of $x[n]$ with $h[n]$. Then

$$y[n] = x[n] \circledast h[n] \xleftrightarrow{\text{FS}} N_F \, X_{DTFS}[k] H_{DTFS}[k] = Y_{DTFS}[k].$$

$$y[n] = \sum_{k=\langle N_F \rangle} Y_{DTFS}[k] e^{+j2\pi kn/N_F} = \sum_{k=0}^{N_F-1} Y_{DTFS}[k] e^{+j2\pi kn/N_F}$$

$$y[n] = N_F \sum_{k=-\infty}^{\infty} X_{DTFS}[k] H_{DTFS}[k] e^{+j2\pi k f_F t} = N_F^2 \times \underbrace{\frac{1}{N_F} \sum_{k=-\infty}^{\infty} X_{DTFS}[k] H_{DTFS}[k] e^{+j2\pi k f_F t}}_{=\text{DFT}^{-1}\left(X_{DTFS}[k] H_{DTFS}[k]\right)}$$

$$X_{DTFS}[k] = \frac{\text{DFT}\left(x[n]\right)}{N_F} \quad , \quad H_{DTFS}[k] = \frac{\text{DFT}\left(h[n]\right)}{N_F}$$

$$y[n] = N_F^2 \times \text{DFT}^{-1}\left(\frac{\text{DFT}\left(x[n]\right)}{N_F} \times \frac{\text{DFT}\left(h[n]\right)}{N_F} \right)$$

$$y[n] = x[n] \circledast h[n] = \text{DFT}^{-1}\left(\text{DFT}\left(x[n]\right) \times \text{DFT}\left(h[n]\right)\right) \quad \text{(L.10)}$$

L.9 Approximating Cross Correlation Using the DFT

Cross correlation has two definitions, one for energy signals and one for power signals.

Energy Signals

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\,y(t+\tau)\,dt$$

Using the definition of convolution,

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\,y(t-\tau)\,d\tau$$

it follows that

$$x(-t) * y(t) = \int_{-\infty}^{\infty} x(-\tau)\,y(t-\tau)\,d\tau$$

and, making a change of variable, $\lambda = -\tau$,

$$x(-t) * y(t) = \int_{-\infty}^{\infty} x(\lambda)\,y(\lambda+t)\,d\lambda \Rightarrow x(-\tau) * y(\tau) = \int_{-\infty}^{\infty} x(t)\,y(t+\tau)\,dt = R_{xy}(\tau)$$

Therefore $R_{xy}(\tau) = x(-\tau) * y(\tau)$.

If both signals are causal and effectively time-limited to the time range $0 < t < T_F$ we can write

$$R_{xy}(\tau) = \int_{0}^{T_F} x(t)\,y(t+\tau)\,dt$$

and, using a result from L.5,

$$y(nT_s) = \left[x(t) * h(t)\right]_{t \to nT_s} \cong T_s \times \mathrm{DFT}^{-1}\left(\mathrm{DFT}\left(x(nT_s)\right) \times \mathrm{DFT}\left(h(nT_s)\right)\right)$$

and the fact that $x(-\tau) \xleftrightarrow{\mathcal{F}} X^*(f)$,

$$R_{xy}(nT_s) \cong T_s \times \mathrm{DFT}^{-1}\left(\left[\mathrm{DFT}\left(x(nT_s)\right)\right]^* \times \mathrm{DFT}\left(y(nT_s)\right)\right)$$

[Some authors define cross correlation as $R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)\,y(t-\tau)\,dt$. With that definition

$$R_{xy}(nT_s) \cong T_s \times \mathrm{DFT}^{-1}\left(\mathrm{DFT}\left(x(nT_s)\right) \times \left[\mathrm{DFT}\left(y(nT_s)\right)\right]^*\right)$$
]

Power Signals

$$R_{xy}(\tau) = E\left(x(t)\,y(t+\tau)\right) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)\,y(t+\tau)\,dt$$

Using the definition of convolution,

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\,y(t-\tau)\,d\tau$$

it follows that

$$R_{xy}(\tau) \cong (1/T_F)\,x(-\tau) * y(\tau)$$

where T_F is a finite time over which we sample the signal to make the estimate of their cross correlation. Using a result from L.5

$$y(nT_s) = \left[x(t) * h(t)\right]_{t\to nT_s} \cong T_s \times \mathrm{DFT}^{-1}\left(\mathrm{DFT}\left(x(nT_s)\right) \times \mathrm{DFT}\left(h(nT_s)\right)\right)$$

and the fact that $x(-\tau) \xleftrightarrow{\mathcal{F}} X^*(f)$,

$$R_{xy}(nT_s) \cong (T_s/T_F) \times \mathrm{DFT}^{-1}\left(\left[\mathrm{DFT}\left(x(nT_s)\right)\right]^* \times \mathrm{DFT}\left(y(nT_s)\right)\right)$$

or

$$R_{xy}(nT_s) \cong (1/N_F) \times \mathrm{DFT}^{-1}\left(\left[\mathrm{DFT}\left(x(nT_s)\right)\right]^* \times \mathrm{DFT}\left(y(nT_s)\right)\right)$$

[Some authors define cross correlation as $R_{xy}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)\,y(t-\tau)\,dt$. With that definition

$$R_{xy}(nT_s) \cong (1/N_F) \times \mathrm{DFT}^{-1}\left(\mathrm{DFT}\left(x(nT_s)\right) \times \left[\mathrm{DFT}\left(y(nT_s)\right)\right]^*\right)$$
]

L.10 Examples of the Use of the Discrete Fourier Transform

The following examples will illustrate some of the features and limitations of the DFT as a Fourier analysis tool.

Example L-1 Comparing the DFT and CTFS of a bandlimited periodic signal sampled at the Nyquist rate over one fundamental period

The bandlimited, periodic signal $\mathrm{x}(t) = 1 + \cos(8\pi t) + \sin(4\pi t)$ is sampled at the Nyquist rate (Figure L-3).

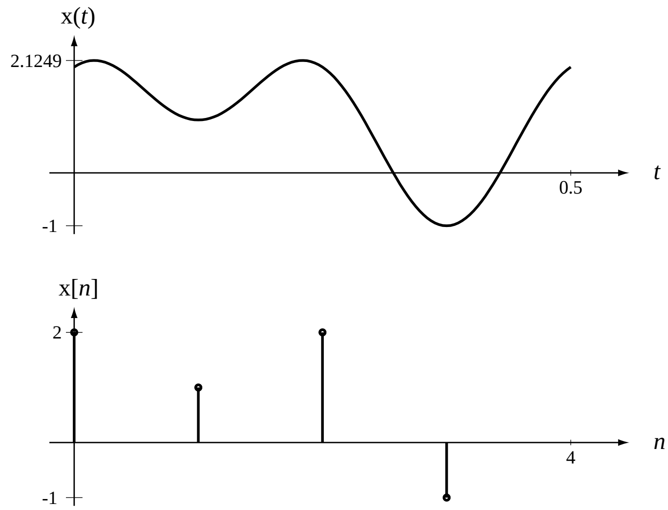

Figure L-3 A CT signal and a DT signal formed by sampling it at its Nyquist rate over one fundamental period

Find the sample values over one fundamental period and find the DFT of the sample values. Find the CTFS harmonic function of the signal.

The highest frequency present in the signal is $f_m = 4$ Hz. Therefore the samples must be taken at 8 Hz. The fundamental period of the signal is 0.5 seconds. Therefore 4 samples are required. Assuming that the first sample is taken at time $t = 0$ the samples are

$$\{\mathrm{x}[0], \mathrm{x}[1], \mathrm{x}[2], \mathrm{x}[3]\} = \{2, 1, 2, -1\}.$$

From the DFT definition,

$$\mathrm{X}_{DFT}[k] = \sum_{n=0}^{N_F - 1} \mathrm{x}[n] e^{-j2\pi nk/N_F}$$

$$\mathrm{X}_{DFT}[0] = \sum_{n=0}^{3} \mathrm{x}[n] = 4 \quad , \quad \mathrm{X}_{DFT}[1] = \sum_{n=0}^{3} \mathrm{x}[n] e^{-j\pi n/2} = 2 - j - 2 - j = -j2$$

$$\mathrm{X}_{DFT}[2] = \sum_{n=0}^{3} \mathrm{x}[n] e^{-j\pi n} = 2 - 1 + 2 + 1 = 4$$

$$\mathrm{X}_{DFT}[3] = \sum_{n=0}^{3} \mathrm{x}[n] e^{-j3\pi n/2} = 2 + j - 2 + j = j2$$

Therefore the DFT is

$$\left\{X_{DFT}[0], X_{DFT}[1], X_{DFT}[2], X_{DFT}[3]\right\} = \{4, -j2, 4, j2\}.$$

The CTFT of the original signal is

$$X(f) = \delta(f) + \frac{1}{2}\left[\delta(f-4) + \delta(f+4)\right] + \frac{j}{2}\left[\delta(f+2) - \delta(f-2)\right]$$

or, ordering the impulses with increasing frequency,

$$X(f) = \frac{1}{2}\delta(f+4) + \frac{j}{2}\delta(f+2) + \delta(f) - \frac{j}{2}\delta(f-2) + \frac{1}{2}\delta(f-4)$$

which is of the form

$$X(f) = \sum_{k=-N_0/2}^{N_0/2} X_{CTFS}[k]\delta(f - kf_0)$$

where $X_{CTFS}[k]$ is the CTFS harmonic function, $f_0 = 1/T_0$ and T_0 is the fundamental period of the signal. So the CTFS harmonic function of the bandlimited, periodic signal from which samples (over one fundamental period) were taken is

$$\left\{X_{CTFS}[-2], X_{CTFS}[-1], X_{CTFS}[0], X_{CTFS}[1], X_{CTFS}[2]\right\} = \left\{\frac{1}{2}, +\frac{j}{2}, 1, -\frac{j}{2}, \frac{1}{2}\right\}.$$

If we divide the DFT results by the number of points 4 we get

$$\frac{1}{4}\left\{X_{DFT}[0], X_{DFT}[1], X_{DFT}[2], X_{DFT}[3]\right\} = \left\{1, -\frac{j}{2}, 1, +\frac{j}{2}\right\}.$$

Using the periodicity of the DFT we see that we get the correct values for $X_{CTFS}[-1], X_{CTFS}[0]$ and $X_{CTFS}[1]$ but not for $X_{CTFS}[2]$ and $X_{CTFS}[-2]$. They are wrong by a factor of two because of aliasing. We did not sample <u>above</u> the Nyquist rate, we sampled <u>at</u> the Nyquist rate.

In the above example the signal was sampled at exactly the Nyquist rate for exactly one fundamental period. What would happen if we sampled at twice the Nyquist rate for exactly one fundamental period or at the Nyquist rate for exactly two fundamental periods?

Example L-2 Comparing the DFT and CTFS of a bandlimited periodic signal sampled at twice the Nyquist rate over one fundamental period

The bandlimited periodic signal $\mathrm{x}(t) = 1 + \cos(8\pi t) + \sin(4\pi t)$ is sampled at twice the Nyquist rate (Figure L-4).

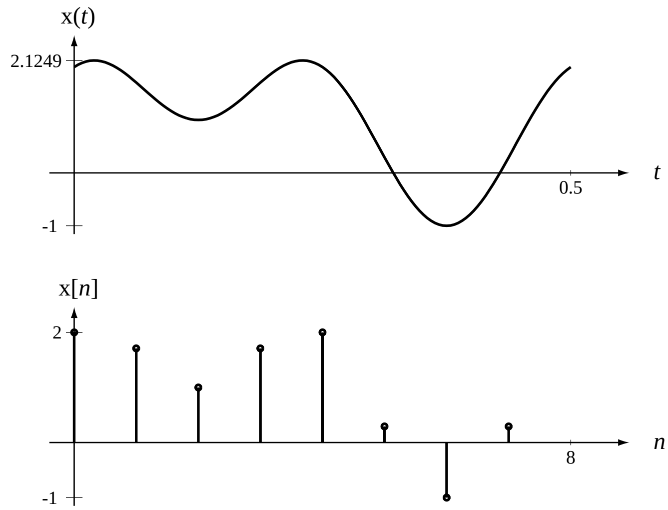

Figure L-4 A CT signal and a DT signal formed by sampling it at twice its Nyquist rate over one fundamental period

Find the sample values over one fundamental period and find the DFT of the sample values. Also find the CTFS harmonic function of the signal.

The highest frequency present in the signal is $f_m = 4$ Hz. Therefore the samples must be taken at 16 Hz. The fundamental period of the signal is 0.5 seconds. Therefore 8 samples are required. Assuming that the first sample is taken at time $t = 0$ the samples are

$$\{\mathrm{x}[0],\ldots,\mathrm{x}[7]\} = \left\{2, 1+\frac{1}{\sqrt{2}}, 1, 1+\frac{1}{\sqrt{2}}, 2, 1-\frac{1}{\sqrt{2}}, -1, 1-\frac{1}{\sqrt{2}}\right\}$$

and the DFT of those samples is

$$\{\mathrm{X}_{DFT}[0],\ldots,\mathrm{X}_{DFT}[7]\} = \{8, -j4, 4, 0, 0, 0, 4, j4\}.$$

The CTFS harmonic function of the original signal is the same as in Example L-1,

$$\{\mathrm{X}_{CTFS}[-2], \mathrm{X}_{CTFS}[-1], \mathrm{X}_{CTFS}[0], \mathrm{X}_{CTFS}[1], \mathrm{X}_{CTFS}[2]\} = \left\{\frac{1}{2}, +\frac{j}{2}, 1, -\frac{j}{2}, \frac{1}{2}\right\}$$

and dividing the DFT result by the number of points 8,

$$\frac{1}{8}\left\{X_{DFT}\left[0\right],\ldots,X_{DFT}\left[7\right]\right\} = \left\{1,-\frac{j}{2},\frac{1}{2},0,0,0,\frac{1}{2},+\frac{j}{2}\right\}.$$

Using the periodicity of the DFT we see that these results agree. In this case, we sampled twice as fast as in Example L-1. What we got for our trouble was information about higher frequencies that might have been present in the signal and no aliasing because we sampled above the Nyquist rate. Of course, since we used the same signal, there were not any higher frequencies present and the extra $X\left[k\right]$'s $\left\{X_{DFT}\left[3\right],X_{DFT}\left[4\right],X_{DFT}\left[5\right]\right\}$ were all zero.

Example L-3 Comparing the DFT and CTFS of a bandlimited periodic signal sampled at the Nyquist rate over two fundamental periods

The bandlimited periodic signal $x\left(t\right) = 1 + \cos\left(8\pi t\right) + \sin\left(4\pi t\right)$ is sampled at the Nyquist rate (Figure L-5).

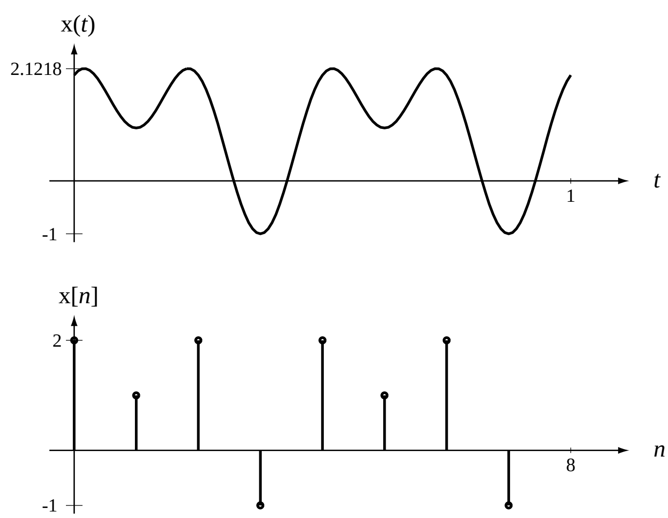

Figure L-5 A CT signal and a DT signal formed by sampling it at its Nyquist rate over two fundamental periods

Find the sample values over two fundamental periods, find the DFT of the sample values. Also find the CTFS harmonic function of the signal.

The highest frequency present in the signal is $f_m = 4$ Hz. Therefore the samples must be taken at 8 Hz. The fundamental period of the signal is 0.5 seconds. Therefore 8 samples are required. Assuming that the first sample is taken at time $t = 0$ the samples are

$$\left\{x\left[0\right],\ldots,x\left[7\right]\right\} = \{2,1,2,-1,2,1,2,-1\}$$

and the DFT of those samples is

$$\left\{X_{DFT}[0],\ldots,X_{DFT}[7]\right\} = \left\{8,0,-j4,0,8,0,j4,0\right\}.$$

The CTFS harmonic function of the original signal is still the same as in Example L-1

$$\left\{X_{CTFS}[-2],X_{CTFS}[-1],X_{CTFS}[0],X_{CTFS}[1],X_{CTFS}[2]\right\} = \left\{\frac{1}{2},+\frac{j}{2},1,-\frac{j}{2},\frac{1}{2}\right\}$$

Comparing the CTFS harmonic function and the DFT,

$$\frac{1}{8}\left\{X_{DFT}[0],\ldots,X_{DFT}[7]\right\} = \left\{1,0,-\frac{j}{2},0,1,0,+\frac{j}{2},0\right\}.$$

The fundamental of the CTFS corresponds to the second harmonic of the DFT because we sampled over two fundamental periods. Therefore the results correspond correctly, again except for the highest harmonic which is wrong because of aliasing. As in Example L-1 we sampled at the Nyquist rate instead of above it. As in Example L-2 we get extra information about the signal. By sampling twice as long, we could recognize frequencies twice as low (fundamental periods twice as long) that might have been present in the signal. That made the lowest non-zero frequency in the DFT half what it was before. Also, since $X_{DFT}[k]$ occurs at integer multiples of the lowest non-zero frequency, the whole frequency-domain graph has twice the resolution it had in Example L-1 and Example L-2. The sampling rate is the same as Example L-1, therefore the highest frequency that can be found is the same as in Example L-1 and half that in Example L-2.

Example L-4 Effects of sampling rate and number of samples on the DFT as an approximation to the CTFT of a truncated sinusoid

Sample the signal $x(t) = 5\sin(\pi t)\text{rect}((t-2)/4)$ beginning at time $t = 0$

(a) 16 times at 4 Hz, (b) 32 times at 4 Hz (c) 64 times at 4 Hz

(d) 32 times at 8 Hz, and (e) 64 times at 8 Hz.

In each case find the DFT of the samples and graph comparisons of the signal and its samples in the time domain and of the magnitude of the CTFT of the signal and the magnitude of the product of the DFT of the samples and the sampling interval T_s.

The CTFT of $x(t)$ is

$$\mathrm{X}(f) = j10\left[\mathrm{sinc}\left(4\left(f+1/2\right)\right)e^{-j4\pi\left(f+1/2\right)} - \mathrm{sinc}\left(4\left(f-1/2\right)\right)e^{-j4\pi\left(f-1/2\right)}\right]$$

(a)

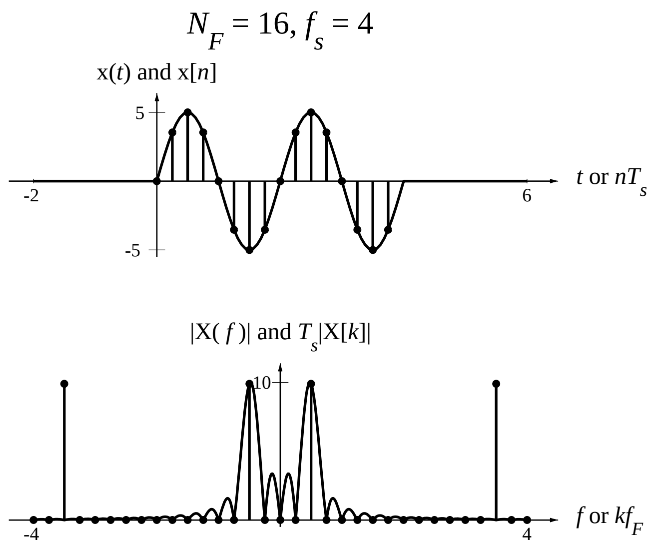

Figure L-6 Signal sampled 16 times at 4 Hz

The DFT repeats periodically with fundamental period $N_F = 16$ or, in terms of frequency, with fundamental period $f_s = N_F f_F$ Hz but in the frequency range $-f_s/2 < f < f_s/2$ the DFT (multiplied by the sampling interval T_s) seems to approximate samples of the CTFT at integer multiples of the fundamental frequency $f_F = f_s / N_F$ of the DFT. The resolution of the DFT is not very good. Since all the samples except two in the frequency range $-f_s/2 < f < f_s/2$ occur at zeros of the CTFT, if we just looked at the DFT result without knowing the CTFT, we would conclude that the CTFT had two impulses at equal positive and negative frequencies and that, therefore the original signal was a sinusoid. Remember that the DFT applies exactly to periodic signals and the set of samples used here is from exactly two fundamental periods of a sinusoid. In the absence of other information, the logical conclusion from the samples is that the sample pattern repeats periodically and that the signal is therefore a sinusoid, instead of the actual signal which is a time-limited version of a sinusoid. Taking more samples will alleviate this problem.

(b)

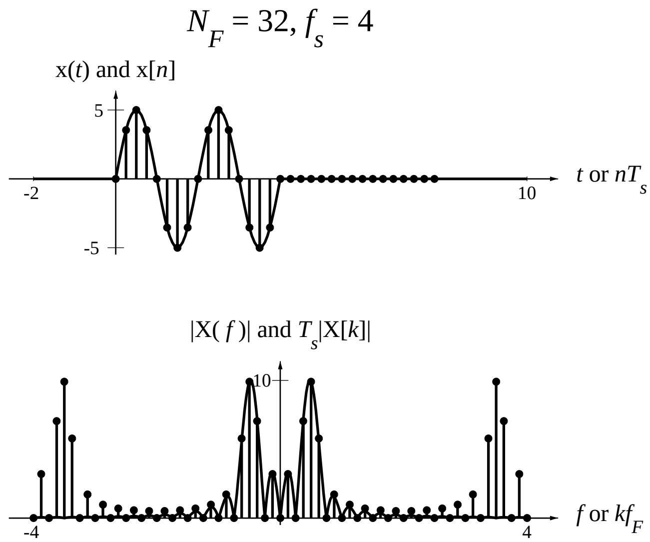

Figure L-7 Signal sampled 32 times at 4 Hz

Here twice as many samples were taken as in part (a). The extra samples were all zero. This kind of extension of the sampling of a signal with extra zeros is called *zero padding*. The inclusion of the extra zeros doubles the total sampling time and also doubles the resolution of the DFT. Now we have DFT values that fall between zero crossings of the CTFT and we can begin to see, by observing the DFT only, that the original signal is not simply a sinusoid. The agreement between the DFT and the CTFT seems very good at low frequencies, but notice that at frequencies close to the Nyquist frequency, the agreement between the DFT and CTFT is not so good. This difference is easier to see on a logarithmic magnitude graph (Figure L-8). The difference is caused by aliasing. The original signal is not bandlimited so the aliases overlap and, in this case, that causes a significant error near the Nyquist frequency.

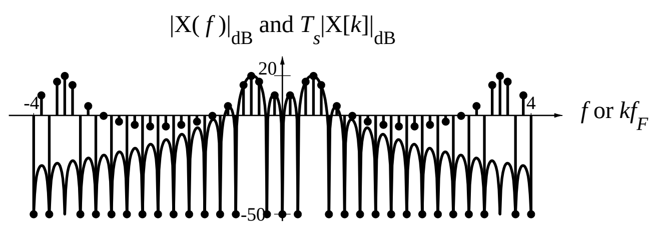

Figure L-8 Logarithmic magnitude graph, signal sampled 32 times at 4 Hz

(c)

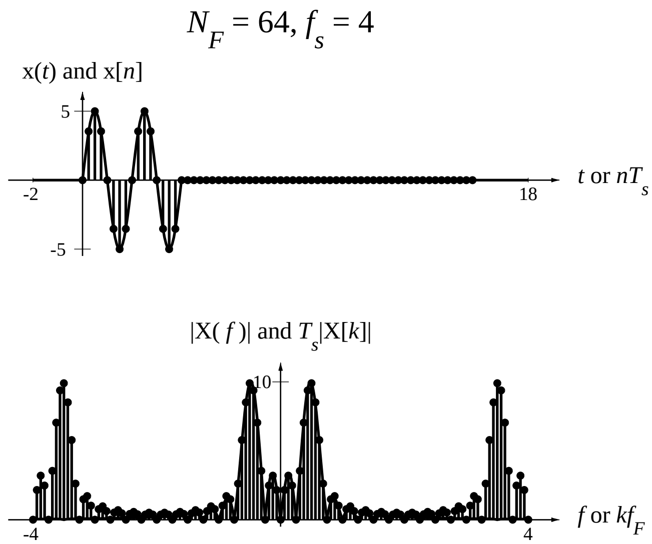

Figure L-9 Signal sampled 64 times at 4 Hz

Here the number of samples was doubled again. This again doubles the resolution of the DFT but does not really help the aliasing problem. A higher sampling rate would reduce errors due to aliasing.

(d)

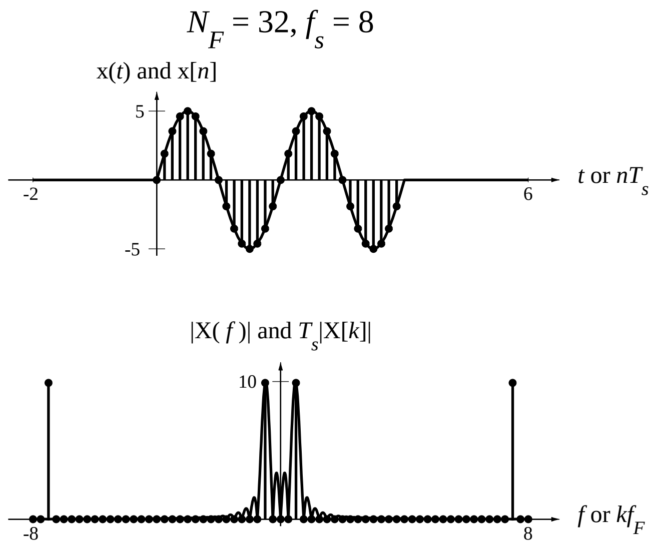

Figure L-10 Signal sampled 32 times at 8 Hz

Here the sampling rate is doubled and the number of samples is the same as in part (b). Again, as in part (a) the DFT seems to be indicating that the signal from which the samples were taken was a pure sinusoid because exactly two cycles of a sinusoid were sampled. If we now increase the number of samples at this sampling rate we will get better frequency-domain resolution and have a reduced aliasing error.

(e)

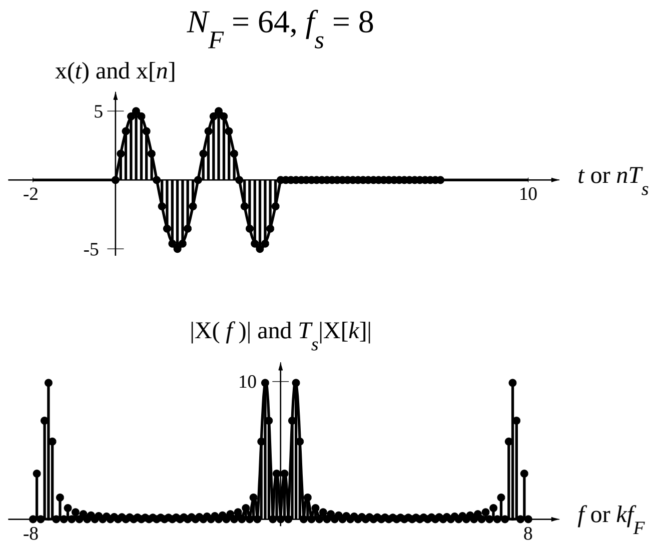

Figure L-11 Signal sampled 64 times at 8 Hz

Here we have sampled 64 times at 8 Hz. Aliasing error is reduced and the frequency-domain resolution is good enough to see that the signal is not simply a sinusoid (Figure L-12).

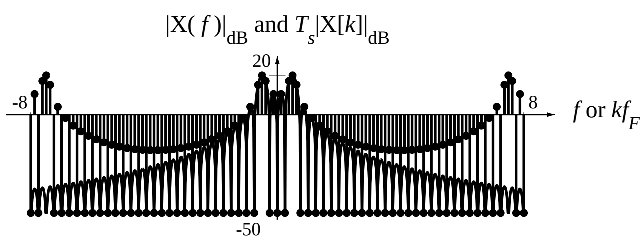

Figure L-12 Logarithmic magnitude graph, signal sampled 64 times at 8 Hz

From this example we reinforce the general principle stated earlier that sampling longer improves frequency-domain resolution and sampling at a higher rate reduces errors due to aliasing. So a good general rule in using the DFT to approximate the CTFT is to sample as fast as possible for as long as possible. In the theoretical limit in which we sample infinitely fast for an infinite time, all the information in the CTFT is preserved in the DFT. The DFT approaches the CTFT in that limit. Of course, in any practical situation there are limits imposed by real samplers. Real samplers can only sample at a finite rate and real DSP-system memories can only store a finite number of data values.

The previous examples analyzed samples from known mathematical functions to demonstrate some features of the DFT. The next example is more realistic in that the signal is not a known mathematical function.

Example L-5 DFT of samples from an unknown signal

Suppose 16 samples are taken from a signal at 1 ms intervals and that the samples are the ones graphed in Figure L-13 (with the usual assumption that the first sample occurs at time $t = 0$).

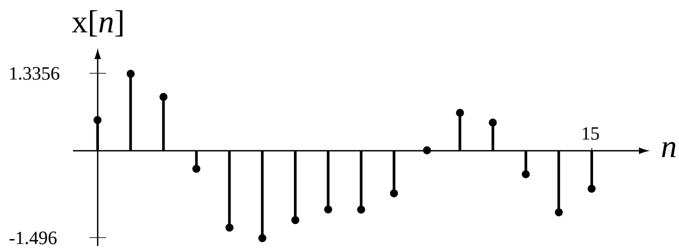

Figure L-13 A DT signal formed by sampling an unknown CT signal for a finite time

The reason for taking the samples is to gain information about the signal that was sampled. What do we know so far? We know the value of the signal at 16 points. If we are draw any more conclusions than that we must have some other information or make some assumptions.

What happened before the first sample and after the last sample? What would be reasonable to assume? We could assume that the signal varies in a similar manner outside this range of samples. That similar variation could take on many different forms. So this assumption is not mathematically precise. One possible form might be the signal in Figure L-14 (a).

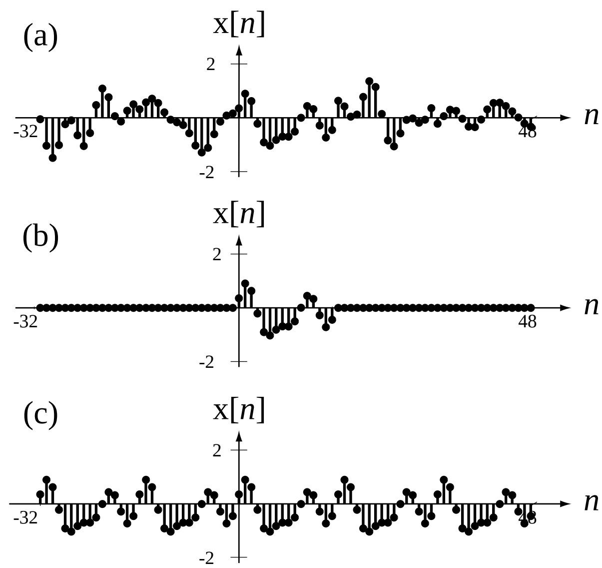

Figure L-14 Three possible extensions of the original samples

We could assume that the signal is zero outside this range of samples (Figure L-14 (b)). But, if it is, we know that we cannot sample it adequately because a signal that is time limited is not bandlimited. The usual assumption is that the set of samples we took is reasonably representative of the total signal. (If that is not true the analysis won't mean much.) That is, that the signal outside this time range is similar to the signal inside this time range. To make that assumption precise, we assume that the signal before and after the samples is as similar to the signal during the sampling as possible. We assume that if we sampled some more we would simply repeat the set of samples we got above, over and over again (Figure L-14 (c)). That is very probably not exactly true. But what would be a better assumption? If the sample set we took is typical then the assumption that the signal just keeps doing the same thing again and again is the best one we can make. Using that assumption we can say that the samples we took are from one fundamental period of a periodic signal. We assume that if we had kept sampling we would simply have repeated the samples again and again.

The next logical question is "What happened between the samples?". Again we don't really know. Below are some illustrations of what the signal that was sampled could have looked like (Figure L-15).

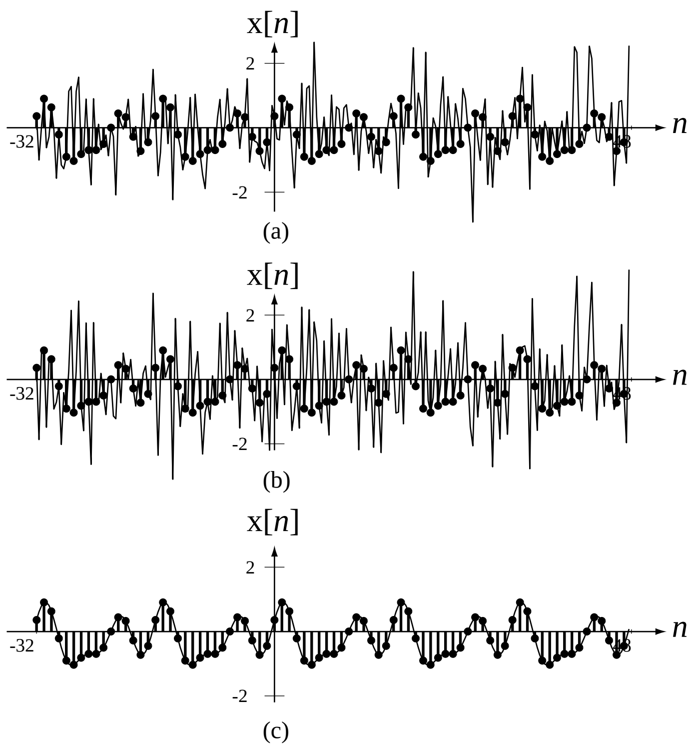

Figure L-15 Three signals, all of which have the original sample values

In each of the three signals in Figure L-15, the sample values are exactly the same but the signals are different. Unless we know something else about the signal that was sampled, any of these signals could theoretically be the actual signal sampled. But if the signal was properly sampled according to the sampling theorem (at a rate at more than twice its maximum frequency) only one of these candidate signals could be the one sampled, the last one in Figure L-15(c). So now we have narrowed down the possible signals from which the samples could have come to only one, a bandlimited periodic signal which passes through the points. We could now take the original set of samples and from it make the best estimate (based on our assumptions) of the CT signal it came from. That is exactly how the last CT signal in Figure L-15 (c) was created.

Instead of trying to reconstruct the original signal from its samples, it is more common in signal analysis to use the DFT to look at the frequency content of signals. We know how to find the CTFS harmonic function by using the DFT. What is the relation between the CTFS harmonic function and the CTFT of the original signal? It was shown previously that

$$\mathrm{X}(f) = \sum_{k=-N_0/2}^{N_0/2} \mathrm{X}_{CTFS}[k]\delta(f - kf_0)$$

That is, the CTFT for the assumed bandlimited periodic signal is a finite set of impulses spaced apart by the fundamental frequency f_0. Using the relation between the CTFS harmonic function and the DFT derived above for bandlimited periodic signals,

$$\mathrm{X}(f) = \frac{1}{N_F} \sum_{k=-N_F/2}^{N_F/2} \mathrm{X}_{DFT}[k]\delta(f - kf_0) \ , \ -N_F/2 \le k \le N_F/2 .$$

or

$$\mathrm{X}(f) = \frac{1}{N_F}\left\{\begin{array}{l} \underbrace{X_{DFT}[-N_F/2]}_{=0}\delta(f - (-N_F/2)f_0) \\ +X_{DFT}[-N_F/2+1]\delta(f - (-N_F/2+1)f_0) \\ +\cdots + X_{DFT}[0]\delta(f) + \cdots \\ +X_{DFT}[N_F/2-1]\delta(f - (N_F/2-1)f_0) \\ +\underbrace{X_{DFT}[N_F/2]}_{=0}\delta(f - (N_F/2)f_0) \end{array}\right\}.$$

(Notice that the CTFS harmonic function components at harmonic numbers, $-N_F/2$ and $N_F/2$, are always zero if the signal is properly sampled at more than twice the Nyquist frequency because then there is no signal power at the Nyquist frequency. As the sampling rate is increased more and more of the components near the Nyquist frequency will also be zero.)

This result is based on an assumption that the samples came from one period of a bandlimited periodic signal. If that assumption is correct the result is exact. If that assumption is not correct, the result is an approximation.

Example L-6 DFT of a sinusoid sampled over an integer number of fundamental periods above the Nyquist rate

Sample a sinusoidal function and find the DFT of the samples and the CTFS harmonic function of the periodic repetition.

This problem description, like many real engineering problems, is ill defined. We must make some reasonable choices for sampling rates and times so that the results will be useful. Let the CT signal be a unit-amplitude cosine and let the fundamental period be 10 ms and the total sampling time 20 ms and take 32 samples in that time. The cosine is described by $x(t) = \cos(200\pi t)$ and its CTFT is $X(f) = (1/2)[\delta(f-100) + \delta(f+100)]$. Since the cosine's frequency is 100 Hz and the sampling rate is 1.6 kHz, the signal will be oversampled and no aliasing will occur. The results are illustrated in Figure L-16.

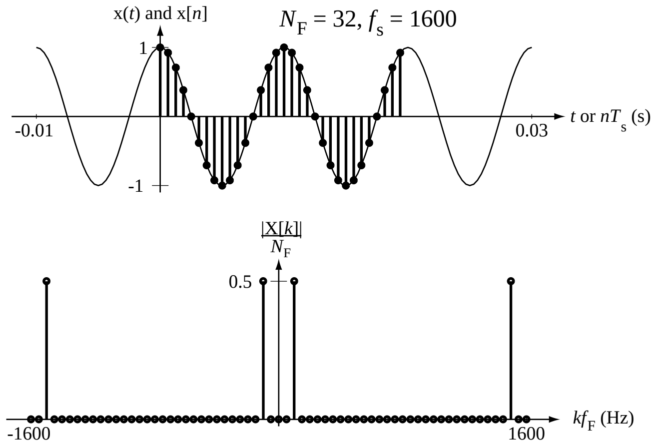

Figure L-16 A cosine sampled over two fundamental periods and the magnitude of its DFT, divided by the number of samples N_F

In this case the signal is bandlimited and periodic and the sampling is done over an integer number of fundamental periods. Therefore one should expect an exact correspondence between the CTFT of the CT signal and DFT of the samples. The CTFT of the original sinusoid has two impulses, one at $+f_0$ and the other at $-f_0$, where f_0 is the cosine's frequency. For a unit-amplitude sinusoid like this one the strengths of the impulses should each be $1/2$. The cosine's frequency is 100 Hz. The frequency domain resolution of the DFT is the sampling rate divided by the number of samples or 50 Hz. Therefore the DFT should have non-zero values only at the second harmonic of 50 Hz, which it does. When the DFT result is divided by the number of samples N_F the discrete-harmonic-number impulses in the DFT have the same strength as the continuous-frequency impulses in the CTFT of the CT sinusoid.

For the aperiodic, energy signal of Example L-4, the DFT was scaled by multiplying by the sampling interval T_s and the DFT of the samples approximated samples of the CTFT of the CT signal that was sampled. For this periodic signal, the scaling of the DFT was done by dividing it by the number of samples N_F. Why are these factors different?

First, realize that, since the CTFT of a periodic signal consists only of impulses, it cannot be sampled in any meaningful sense. So the DFT of a periodic signal must be scaled to yield the strengths of the impulses, not their amplitudes which are undefined. In the case of aperiodic energy signals, the CTFT is a continuous-frequency function with no impulses. In this case a correspondence must be made between the strengths of the DFT impulses and the samples of the CTFT. One way to see the correspondence is to realize that the CTFT is a spectral density function and therefore has the units of the signal being transformed, divided by frequency. For example, if the CT signal has units of volts (V) its CTFT has units of V/Hz. The DFT is computed by forming various linear combinations of samples of the CT function, therefore its units would be the same as the signal units, in this case, just V. To convert that to an approximation of the CTFT we must divide by some frequency to make the units right. But what frequency? If we equate the amplitude in each resolution range of the DFT to the amplitude spectral density of the CTFT the appropriate division factor is the resolution bandwidth of the DFT which is f_s / N_F. So if we take the dividing factor for periodic functions N_F and multiply it by f_s / N_F to form a new dividing factor for aperiodic energy signals f_s the effect is the same as multiplying by the sampling interval T_s because $f_s = 1/T_s$.

In the case of finding the CTFS harmonic function of a periodic signal using the DFT, the correspondence between them is different than the correspondence between the CTFT and the DFT for aperiodic energy signals. When we find a CTFS harmonic function we use the integral formula,

$$\mathrm{X}\left[k\right] = \frac{1}{T_0}\int_{T_0} \mathrm{x}\left(t\right)e^{-j2\pi\left(kf_0\right)t}dt .$$

If x has units of volts, then X also has units of volts, not V/Hz as in the CTFT. So now we simply divide by the number of points N_F (which is dimensionless) instead of the resolution bandwidth.

Example L-7 Leakage caused by sampling over a non-integer number of fundamental periods

Sample a sinusoid over a non-integer number of fundamental periods and observe the effect on the DFT.

Let the sinusoid be a cosine whose fundamental period is 66 2/3 ms and sample it 32 times in 100 ms. The results are illustrated in Figure L-17.

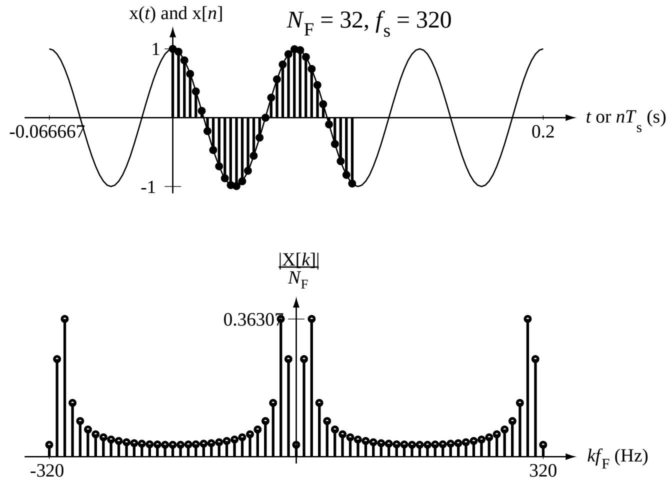

Figure L-17 A cosine sampled over one and one-half fundamental periods and the magnitude of its DFT, divided by the number of samples, N_F

The original CT cosine has a CTFT with exactly two impulses at +15 Hz and -15 Hz. But the DFT has non-zero components at every harmonic of its fundamental frequency which is 10 Hz (f_s / N_F). Since 15 Hz is not an integer multiple of 10 Hz, there is no resolved frequency component in the DFT at exactly the cosine's frequency. But the two strongest components are at 10 and 20 Hz, which bracket the actual cosine frequency of 15 Hz. Therefore one could say that the DFT is in a sense attempting to report the nature of the signal from which the samples came the best it can given the poor sampling choice. This spreading of the signal's power from the exact location into adjacent locations is an example of leakage. That is, the power at 15 Hz has leaked into components at 10 Hz, 20 Hz, 30 Hz, etc... because the original signal was not sampled over an integer number of fundamental periods. This problem could be solved by sampling over an integer number of fundamental periods. But it could also be greatly reduced by sampling for a much longer time, even if that time is not an integer multiple of the cosine's fundamental period, because with a longer sampling time, the frequency-domain resolution gets better and the bulk of the signal's power can be placed much closer to the actual frequency of 15 Hz. Figure L-18 shows the result of sampling over six and one-half fundamental periods with all other parameters unchanged.

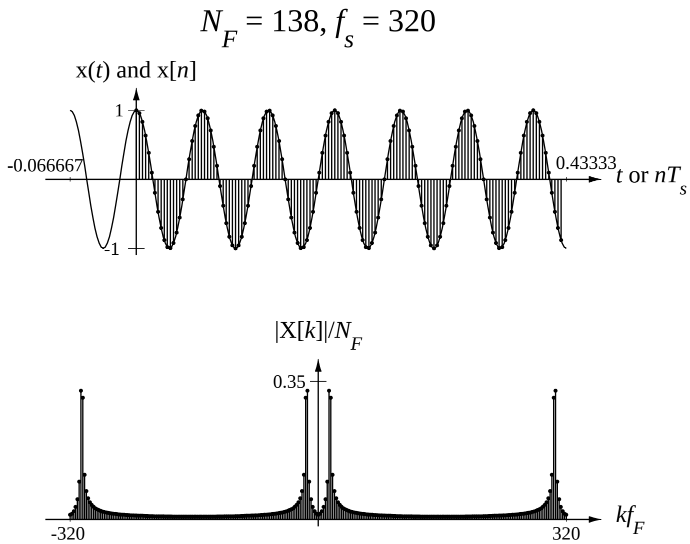

Figure L-18 A cosine sampled over six and one-half fundamental periods and the magnitude of its DFT, divided by the number of samples, N_F

Now, even though there is still not a resolved component at the frequency of the CT signal 15 Hz, because of the greater number of points and the consequent higher resolution of the DFT there are components much closer to 15 Hz than in the previous case and the leakage is spread less widely. An exact correspondence would exist if we sampled over an integer number of periods. But often in practice we are sampling signals which are either not periodic or whose period is unknown so we do not have enough information to sample over an integer number of periods.

Example L-8 Approximating a CT convolution using the DFT

Find the convolution of

$$\mathrm{x}(t) = \begin{cases} 5t & , \quad 0 < t < 2 \\ 0 & , \quad \text{otherwise} \end{cases}$$

with $\mathrm{h}(t) = 3e^{-4t}\,\mathrm{u}(t)$ using the DFT.

The CT signals and their exact CT convolution are illustrated in (Figure L-19).

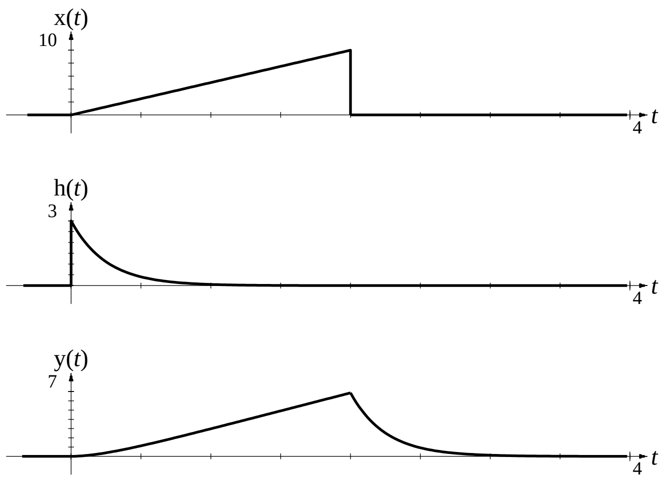

Figure L-19 Original CT signals and their exact CT convolution

Neither of these signals is bandlimited so we must find a sampling rate which will make our approximation reasonable, knowing that it can never be exact. In cases like this in which it is difficult to theoretically justify what the sampling rate should be, a strategy of iteration is often effective. We can sample at what seems intuitively to be an adequate rate, then try another rate, higher or lower, to see if it makes much difference. Once we find that sampling at a higher rate makes very little difference in the results we can be confident that the rate we have chosen is high enough. Let's start by taking 50 samples from $x(t)$ over its non-zero range. That sets a sampling rate of 25 Hz. The time constant of the exponential in $h(t)$ is 0.25 s so we will be sampling about 6 times per time constant which seems like a reasonable rate also. When we sample the signals this way and use (L.6) to compute an approximation to the convolution we get the signals in Figure L-20.

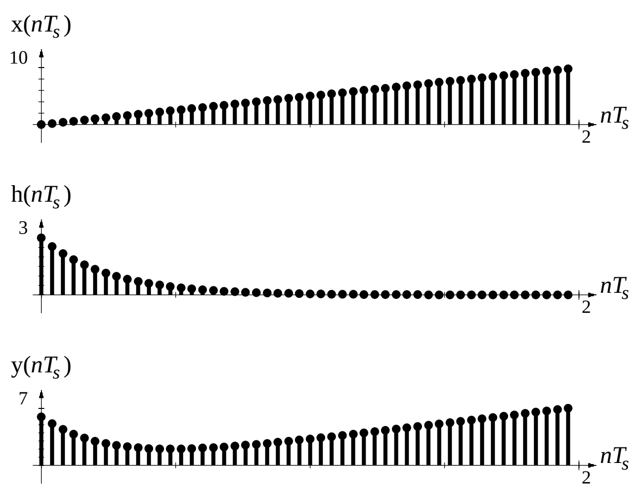

Figure L-20 Samples from two signals and a first approximation to their convolution using the DFT

Comparing this result with Figure L-19 it is obviously not an accurate approximation to the CT convolution of the original CT signals. What is wrong? Even though the signals are not bandlimited, the sampling rate seems intuitively high enough to describe these signals with reasonable accuracy and we have sampled all the time range for both of them in which they have any significant signal energy.

The problem lies in not realizing what the DFT actually does. The DFT relates a set of samples from one period of a time-domain periodic signal to a set of samples from one period of a frequency-domain periodic signal. So when we provide a set of samples to the DFT, they are presumed to have come from one period of a periodic signal (Figure L-21).

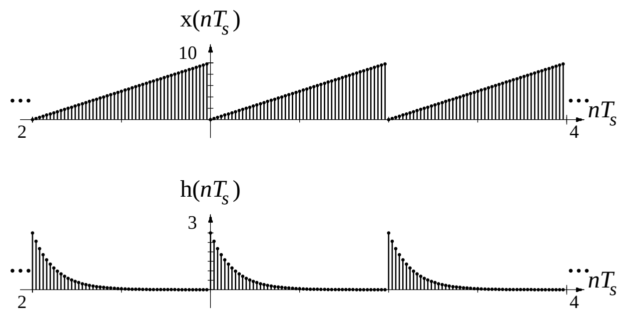

Figure L-21 Periodically repeated versions of $\mathrm{x}(nT_s)$ and $\mathrm{h}(nT_s)$

But our signals are aperiodic. When we use the DFT to perform convolution we are really doing periodic convolution. Remember that periodic convolution is the aperiodic convolution of one period of either signal with the other signal. Imagine taking only one period of $\mathrm{h}(nT_s)$, time inverting it and then shifting and looking at the sum of the product as we shift $\mathrm{h}(nT_s)$. The signal $\mathrm{h}(nT_s)$ already overlaps the periodic extension of $\mathrm{x}(nT_s)$. That is why the initial convolution value is wrong.

To avoid this problem we need to sample the signals so that we get a good approximation to the aperiodic convolution we actually want. That means that we need not only to sample the parts of the signals that have significant signal energy but also at least some of the signal where it is zero. Then when we do a periodic convolution, the convolution with the parts of the signals that are zero will also be done. We want to include enough zeros so that the non-zero portion of one signal does not get convolved with the non-zero part of the periodic repetition of the other signal. Let's try doubling the number of samples while leaving the sampling rate the same (Figure L-22).

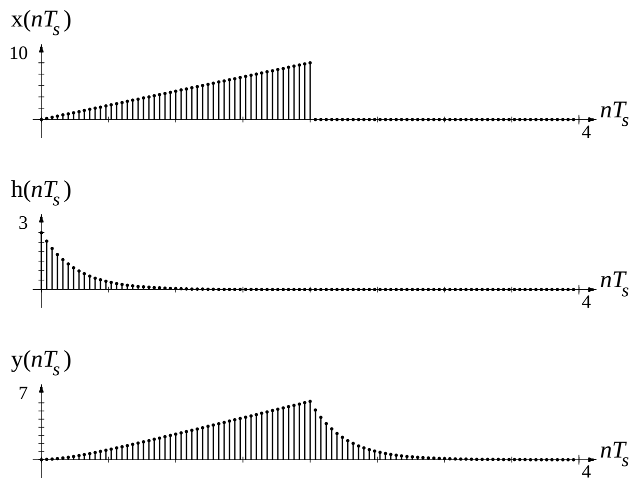

Figure L-22 Samples from two signals taken over twice the original time range and a second, and much better, approximation to their convolution using the DFT

This is another example of zero padding which is a very common technique in numerical approximations to DT operations using the DFT. By doubling the number of samples we avoided the overlap of the non-zero parts of one signal and the non-zero parts of the periodic extension of the other. This last approximation is much more accurate than the first one. If we try to improve on this approximation by now doubling the original sampling rate and quadrupling the original number of points leaving the total time of sampling the same as in Figure L-22 we get another approximation (Figure L-23). Comparing this approximation with the previous one shows that very little has been gained by doubling the sampling rate. The results are almost identical.

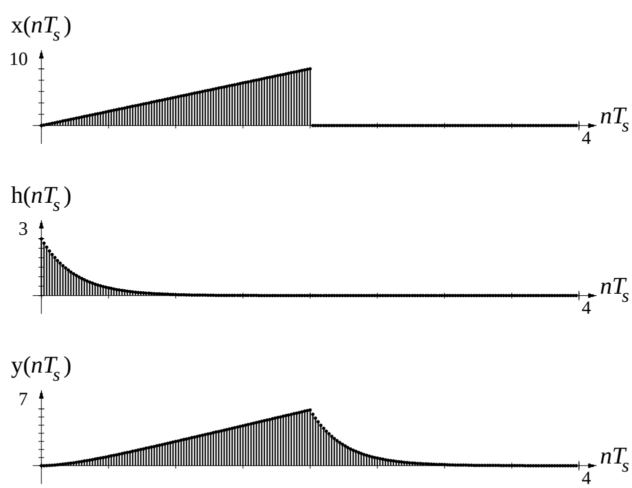

Figure L-23 A third approximation to the convolution using twice the previous sampling rate and the same total sampling time

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