Sampling Example

Let $\mathrm{x}(t)=\cos(14\pi t)\cos(2\pi t)$. The fundamental frequency of $\cos(14\pi t)$ is 7 Hz. The fundamental frequency of $\cos(2\pi t)$ is 1 Hz. Then

$$\mathrm{X}_f(f)=(1/2)[\delta(f-7)+\delta(f+7)]*(1/2)[\delta(f-1)+\delta(f+1)].$$

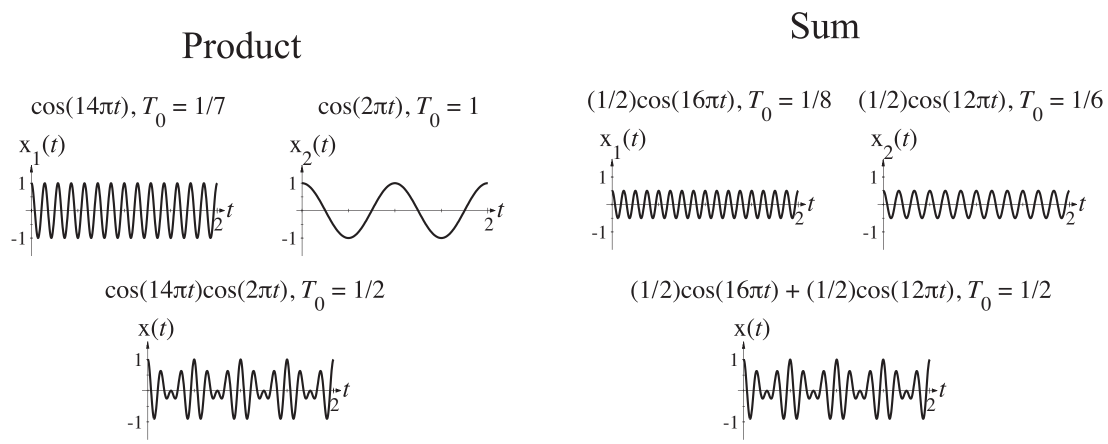
$$\mathrm{X}_f(f)=(1/4)[\delta(f-8)+\delta(f-6)+\delta(f+6)+\delta(f+8)] .$$

Taking pairs of impulses at a time, the inverse CTFT is

$$\mathrm{x}(t)=(1/2)[\cos(16\pi t)+\cos(12\pi t)].$$

The fundamental frequency of $\cos(16\pi t)$ is 8 Hz. The fundamental frequency of $\cos(12\pi t)$ is 6 Hz. Therefore the fundamental frequency of $\mathrm{x}(t)=(1/2)[\cos(16\pi t)+\cos(12\pi t)]$ is 2 Hz. Since we have now shown that

$$\cos(14\pi t)\cos(2\pi t)=(1/2)[\cos(16\pi t)+\cos(12\pi t)],$$



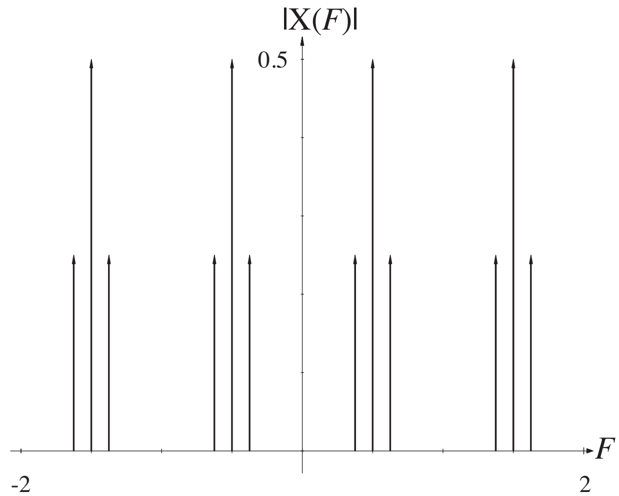
that means that the fundamental frequency of $\cos(14\pi t)\cos(2\pi t)$ is also 2 Hz and its fundamental period is 1/2 second. Also, the highest frequency in $\cos(14\pi t)\cos(2\pi t)$ is 8 Hz.

Now sample it at $f_s=16$ to get $\mathrm{x}[n]=\cos(14\pi n/16)\cos(2\pi n/16)$. This is sampling at the Nyquist rate. Therefore the sampling is not quite fast enough to avoid aliasing. The DTFT of $\mathrm{x}[n]$ is

$$\mathrm{X}_F(F)=(1/2)[\delta_1(F-7/16)+\delta_1(F+7/16)]\circledast(1/2)[\delta_1(F-1/16)+\delta_1(F+1/16)]$$

$$\mathrm{X}_F(F) = (1/4)\left[\delta_1(F - 7/16) + \delta_1(F + 7/16)\right] * \left[\delta(F - 1/16) + \delta(F + 1/16)\right]$$

$$\mathrm{X}_F(F) = (1/4)\left[\delta_1(F - 1/2) + \delta_1(F - 3/8) + \delta_1(F + 3/8) + \delta_1(F + 1/2)\right]$$



The inverse DTFT of $\mathrm{X}_F(F)$, is $\mathrm{x}[n] = \int_1 \mathrm{X}_F(F) e^{j2\pi Fn}\, dF$. The integration range of one can lie anywhere in *F*. Let the integral be

$$\mathrm{x}[n] = \int_0^1 \mathrm{X}_F(F) e^{j2\pi Fn}\, dF = (1/4)\int_0^1 \left[\delta_1(F - 8/16) + \delta_1(F - 6/16) + \delta_1(F + 6/16) + \delta_1(F + 8/16)\right] e^{j2\pi Fn}$$

$$\mathrm{x}[n] = (1/4)\int_0^1 \left[2\delta(F - 1/2) + \delta(F - 3/8) + \delta(F - 5/8)\right] e^{j2\pi Fn}\, dF$$

$$\mathrm{x}[n] = (1/4)\left[2\underbrace{e^{j\pi n}}_{e^{-j\pi n}} + e^{j3\pi n/4} + \underbrace{e^{j5\pi n/4}}_{=e^{-j3\pi n/4}}\right] = (1/4)\left[e^{j\pi n} + e^{-j\pi n} + e^{j3\pi n/4} + e^{-j3\pi n/4}\right]$$

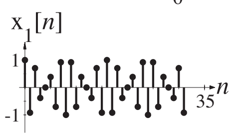
$$\mathrm{x}[n] = (1/2)\left[\cos(\pi n) + \cos(3\pi n/4)\right]$$

Therefore $\cos(14\pi n/16)\cos(2\pi n/16) = (1/2)\left[\cos(\pi n) + \cos(3\pi n/4)\right]$ which agrees with the trigonometric identity $\cos(x)\cos(y) = (1/2)\left[\cos(x - y) + \cos(x + y)\right]$.

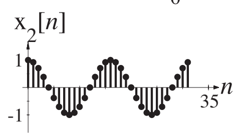
Product

Sum

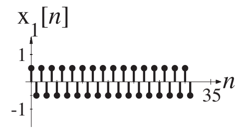
$\cos(14\pi n/16), N_0 = 16$



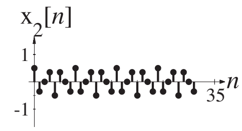
$\cos(2\pi n/16), N_0 = 16$



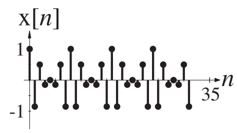
$(1/2)\cos(16\pi n/16), N_0 = 2$



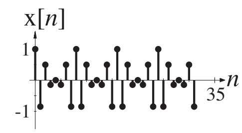
$(1/2)\cos(12\pi n/16), N_0 = 8$



$\cos(14\pi n/16)\cos(2\pi n/16), N_0 = 8$



$(1/2)\cos(16\pi n/16) + (1/2)\cos(12\pi n/16), N_0 = 8$



The relation between the CTFT and the DTFT is $X_F(F) = f_s \sum_{k=-\infty}^{\infty} X_f(f_s(F-k))$. If sampling is done according to the sampling theorem, these multiple aliases of $X_f(f)$ don't overlap and we could recover $X_f(f)$ from $X_F(F)$ by multiplying $X_F(F)$ by $\text{rect}(F)$ to cut off all the aliases, replacing F by f/f_s, and then dividing by f_s. If we do that in this case the stages of the transition are

Multiply by $\text{rect}(F)$:

$$(1/4)\left[\delta_1(F-1/2)+\delta_1(F-3/8)+\delta_1(F+3/8)+\delta_1(F+1/2)\right]\text{rect}(F)$$

$$(1/4)\left[\delta(F-1/2)+\delta(F-3/8)+\delta(F+3/8)+\delta(F+1/2)\right]$$

Replace F by f/f_s:

$$(1/4)\left[\delta(f/f_s-1/2)+\delta(f/f_s-3/8)+\delta(f/f_s+3/8)+\delta(f/f_s+1/2)\right]$$

Use the scaling property of the impulse.

$$f_s(1/4)\left[\delta(f-8)+\delta(f-6)+\delta(f+6)+\delta(f+8)\right]$$

Divide by f_s:

$$(1/4)\left[\delta(f-8)+\delta(f-6)+\delta(f+6)+\delta(f+8)\right]$$

Figures showing these steps

The inverse CTFT of this last expression is

$$(1/2)\cos(16\pi t)+(1/2)\cos(12\pi t) = x(t).$$

In this case we violated the sampling theorem by sampling at the Nyquist rate but got the correct answer anyway. This occurred because the highest frequency term in

$$x(t) = (1/2)\cos(16\pi t) + (1/2)\cos(12\pi t)$$

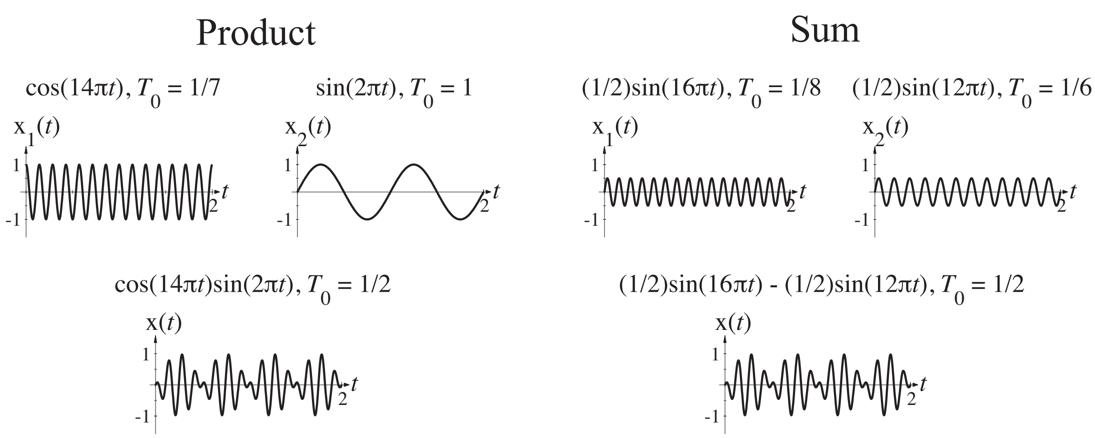
is a cosine. If it had been a sine, the reconstruction of the original continuous-time signal would have been wrong.

Let $x(t) = \cos(14\pi t)\sin(2\pi t)$. Then

$$X_f(f) = (1/2)[\delta(f-7) + \delta(f+7)] * (j/2)[\delta(f+1) - \delta(f-1)]$$

$$X_f(f) = (j/4)[\delta(f-6) - \delta(f-8) + \delta(f+8) - \delta(f+6)].$$

The inverse CTFT is $x(t) = (1/2)[\sin(16\pi t) - \sin(12\pi t)]$ which agrees with the trigonometric identity $\cos(x)\sin(y) = (1/2)[\sin(x+y) - \sin(x-y)]$. If we sample at 16 Hz, $x[n] = \cos(14\pi n/16)\sin(2\pi n/16)$ and

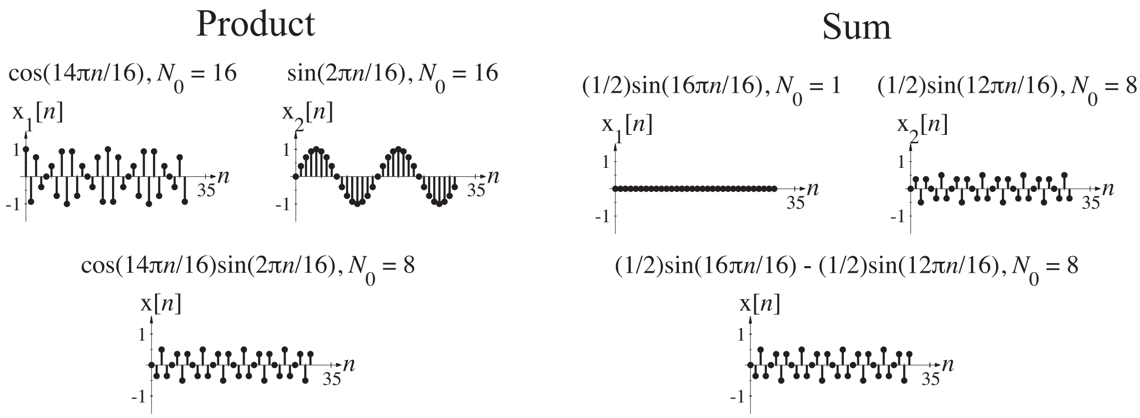


If we sample the signals at 16 Hz we get

$$x[n] = \cos(14\pi n/16)\sin(2\pi n/16)$$

and, using the trigonometric identity $\cos(x)\sin(y) = (1/2)[\sin(x+y) - \sin(x-y)]$ we get

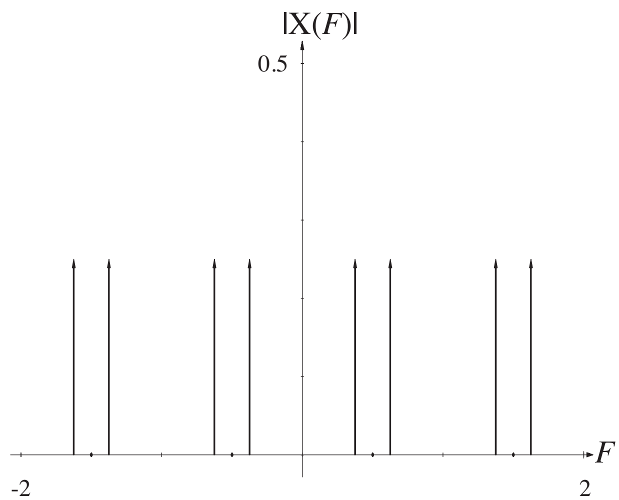
$$\cos(14\pi n/16)\sin(2\pi n/16) = (1/2)\left[\underbrace{\sin(16\pi n/16)}_{=0} - \sin(12\pi n/16)\right].$$



$$\mathrm{X}_F(F) = (j/4)\left[\delta_1(F-3/8) - \delta_1(F-1/2) + \delta_1(F+1/2) - \delta_1(F+3/8)\right]$$

The terms $-\delta_1(F-1/2) + \delta_1(F+1/2)$ add to zero everywhere and

$$\mathrm{X}_F(F) = (j/4)\left[\delta_1(F-3/8) - \delta_1(F+3/8)\right].$$



The inverse DTFT is $\mathrm{x}[n] = -(1/2)\sin(12\pi n/16)$ which does equal $\cos(14\pi n/16)\sin(2\pi n/16)$ because if we apply $\cos(x)\sin(y) = (1/2)\left[\sin(x+y) - \sin(x-y)\right]$ to $\cos(14\pi n/16)\sin(2\pi n/16)$ we get

$$\cos(14\pi n/16)\sin(2\pi n/16) = (1/2)\left[\sin(16\pi n/16) - \sin(12\pi n/16)\right].$$

But $\sin(16\pi n/16) = \sin(\pi n) = 0$ for any integer value of n. If we now try to reconstruct the original continuous-time function from the samples we get these steps

Multiply by $\mathrm{rect}(F)$:

$$(j/4)\left[\delta_1(F-3/8)-\delta_1(F+3/8)\right]\text{rect}(F)$$

$$(j/4)\left[\delta(F-3/8)-\delta(F+3/8)\right]$$

Replace F by f/f_s:

$$(j/4)\left[\delta(f/f_s-3/8)-\delta(f/f_s+3/8)\right]$$

Use the scaling property of the impulse.

$$f_s(j/4)\left[\delta(f-6)-\delta(f+6)\right]$$

Divide by f_s:

$$(j/4)\left[\delta(f-6)-\delta(f+6)\right]$$

Figures showing these steps

The inverse CTFT of this last expression is

$$-(1/2)\sin(12\pi t)\neq \mathrm{x}(t).$$

The sine at half the sampling rate is missing. This is an error due to aliasing.

If we use the DFT for analysis, sampling at 16 Hz

$$\mathrm{x}(t)=\cos(14\pi t)\cos(2\pi t)\Rightarrow \mathrm{x}[n]=\cos(14\pi n/16)\cos(2\pi n/16)$$

The fundamental period of each of the cosines that is multiplied is 16. But the fundamental period of the product is 8. If we view this as a product of cosines and find the DFT of each and then periodically convolve the DFT's we must use $N=16$. Using

$$\cos(2\pi qn/N)\xleftrightarrow[mN]{\mathcal{DFT}}(mN/2)\left(\delta_{mN}[k-mq]+\delta_{mN}[k+mq]\right)$$

N is the fundamental period of each cosine, 16. So in let $N=16$ and $m=1$. In the first cosine $q=7$ and

$$\cos(14\pi n/16)\xleftrightarrow[16]{\mathcal{DFT}}8\left(\delta_{16}[k-7]+\delta_{16}[k+7]\right).$$

In the second cosine $q=1$ and

$$\cos(2\pi n/16) \xleftrightarrow[16]{\mathcal{DFT}} 8\left(\delta_{16}[k-1]+\delta_{16}[k+1]\right).$$

We can now use the property of the DFT

$$\mathrm{x}[n]\mathrm{y}[n] \xleftrightarrow[N]{\mathcal{DFT}} (1/N)\mathrm{Y}[k] \circledast \mathrm{X}[k]$$

and get

$$\cos(14\pi t)\cos(2\pi t) \xleftrightarrow[16]{\mathcal{DFT}} (1/16)8\left(\delta_{16}[k-7]+\delta_{16}[k+7]\right) \circledast 8\left(\delta_{16}[k-1]+\delta_{16}[k+1]\right)$$

$$\cos(14\pi t)\cos(2\pi t) \xleftrightarrow[16]{\mathcal{DFT}} 4\left(\delta_{16}[k-7]+\delta_{16}[k+7]\right) * \left(\delta[k-1]+\delta[k+1]\right)$$

$$\cos(14\pi t)\cos(2\pi t) \xleftrightarrow[16]{\mathcal{DFT}} 4\left(\delta_{16}[k-8]+\delta_{16}[k-6]+\delta_{16}[k+6]+\delta_{16}[k+8]\right)$$

Now use the fact that

$$\cos(14\pi n/16)\cos(2\pi n/16) = (1/2)\left[\cos(\pi n)+\cos(3\pi n/4)\right]$$

we can use its fundamental period which is $N = 8$ and

$$\cos(\pi n) = \cos(2\pi n(4/8)) \xleftrightarrow[8]{\mathcal{DFT}} 4\left(\delta_8[k-4]+\delta_8[k+4]\right)$$

and

$$\cos(3\pi n/4) = \cos(2\pi n(3/8)) \xleftrightarrow[8]{\mathcal{DFT}} 4\left(\delta_8[k-3]+\delta_8[k+3]\right)$$

Then

$$(1/2)\left[\cos(\pi n)+\cos(3\pi n/4)\right] \xleftrightarrow[8]{\mathcal{DFT}} (1/2)\left[4\left(\delta_8[k-4]+\delta_8[k+4]\right)+4\left(\delta_8[k-3]+\delta_8[k+3]\right)\right]$$

$$(1/2)\left[\cos(\pi n)+\cos(3\pi n/4)\right] \xleftrightarrow[8]{\mathcal{DFT}} 2\left[\delta_8[k-4]+\delta_8[k+4]+\delta_8[k-3]+\delta_8[k+3]\right]$$

.

We can convert the last result to the same basis as the first, $N = 16$ using the Change of Period property of the DFT

$$\text{If } \mathrm{x}[n] \xleftrightarrow[N]{\mathcal{DFT}} \mathrm{X}[k] \text{ then } \mathrm{x}[n] \xleftrightarrow[mN]{\mathcal{DFT}} \begin{cases} m\mathrm{X}[k/m] & , \ k/m \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}$$

Applying it to the last result,

$$(1/2)\left[\cos(\pi n)+\cos(3\pi n/4)\right]\xleftrightarrow[16]{\mathcal{DFT}}\begin{cases}4\begin{bmatrix}\delta_8[k/2-4]+\delta_8[k/2+4]\\+\delta_8[k/2-3]+\delta_8[k/2+3]\end{bmatrix} & ,\ k/2\text{ an integer}\\ 0 & ,\ \text{otherwise}\end{cases}$$

Examine $\delta_8[k/2-4]$. It is a periodic impulse that occurs every time $k/2-4$ is an integer multiple m of 8. If $k/2-4=8m$ then $k-8=16m$. So the impulses also occur every time $k-8$ is an integer multiple of 16. Therefore $\delta_8[k/2-4]=\delta_{16}[k-8]$ and

$$(1/2)\left[\cos(\pi n)+\cos(3\pi n/4)\right]\xleftrightarrow[16]{\mathcal{DFT}}\begin{cases}4\begin{bmatrix}\delta_{16}[k-8]+\delta_{16}[k+8]\\+\delta_{16}[k-6]+\delta_{16}[k+6]\end{bmatrix} & ,\ k/2\text{ an integer}\\ 0 & ,\ \text{otherwise}\end{cases}$$

Since the impulses in $\begin{bmatrix}\delta_{16}[k-8]+\delta_{16}[k+8]\\+\delta_{16}[k-6]+\delta_{16}[k+6]\end{bmatrix}$ only occur for even values of *k*, the stipulation of "0, otherwise" is redundant and

$$(1/2)\left[\cos(\pi n)+\cos(3\pi n/4)\right]\xleftrightarrow[16]{\mathcal{DFT}}4\left[\delta_{16}[k-8]+\delta_{16}[k+8]+\delta_{16}[k-6]+\delta_{16}[k+6]\right]$$

This confirms that the two solutions using $N=8$ and $N=16$ are equivalent.

Since we are sampling a periodic signal over an integer number of periods, we can find the CTFT directly from the DFT using

$$\mathrm{X}(f)=(1/N)\sum_{k=-N/2}^{N/2-1}\mathrm{X}[k]\delta(f-kf_s/N)\ .$$

In this example,

$$\mathrm{X}(f)=(1/16)\sum_{k=-8}^{7}\left[4\left(\delta_{16}[k-8]+\delta_{16}[k-6]+\delta_{16}[k+6]+\delta_{16}[k+8]\right)\right]\delta(f-k)$$

$$\mathrm{X}(f)=(1/4)\sum_{k=-8}^{7}\left(\delta_{16}[k-8]+\delta_{16}[k-6]+\delta_{16}[k+6]+\delta_{16}[k+8]\right)\delta(f-k)$$

In the summation range of -8 to +7, the periodic impulses occur only at *k* values of 8, 6, -6 and -8. Therefore

$$\mathrm{X}(f)=(1/4)\left[\delta(f-8)+\delta(f-6)+\delta(f+6)+\delta(f+8)\right]$$

and this is the correct exact CTFT of the original continuous-time signal, $\mathrm{x}(t) = \cos(14\pi t)\cos(2\pi t)$.

If we tried to do the same kind of analysis on the signal

$$\mathrm{x}(t) = \cos(14\pi t)\sin(2\pi t)$$

using $N = 16$ and

$$\sin(2\pi q n / N) \xleftrightarrow[mN]{\mathcal{DFT}} (jmN/2)\left(\delta_{mN}[k+mq] - \delta_{mN}[k-mq]\right)$$

the DFT of the sampled signal would be

$$\cos(14\pi t)\sin(2\pi t) \xleftrightarrow[16]{\mathcal{DFT}} (1/16)8\left(\delta_{16}[k-7] + \delta_{16}[k+7]\right) \circledast j8\left(\delta_{16}[k+1] - \delta_{16}[k-1]\right)$$

which can be reduced to

$$\cos(14\pi t)\sin(2\pi t) \xleftrightarrow[16]{\mathcal{DFT}} j4\left(\delta_{16}[k-6] - \delta_{16}[k+6]\right).$$

Then trying to find the CTFT of the original continuous-time signal we get

$$\mathrm{X}(f) = (j/4)\left[\delta(f-6) - \delta(f+6)\right]$$

and this is not the correct CTFT. The sine component at half the sampling rate is missing because of aliasing.