Web Appendix O - Derivations of the Properties of the *z* Transform

O.1 Linearity

Let $\mathrm{z}[n] = \alpha\,\mathrm{x}[n] + \beta\,\mathrm{y}[n]$ where α and β are constants. Then

$$\mathrm{Z}(z) = \sum_{n=0}^{\infty}\left(\alpha\,\mathrm{x}[n] + \beta\,\mathrm{y}[n]\right)z^{-n} = \alpha\sum_{n=0}^{\infty}\mathrm{x}[n]z^{-n} + \beta\sum_{n=0}^{\infty}\mathrm{y}[n]z^{-n} = \alpha\,\mathrm{X}(z) + \beta\,\mathrm{Y}(z)$$

and the linearity property is

$$\alpha\,\mathrm{x}[n] + \beta\,\mathrm{y}[n] \xleftrightarrow{\mathcal{Z}} \alpha\,\mathrm{X}(z) + \beta\,\mathrm{Y}(z)\ .$$

O.2 Time Shifting

There are two different cases to consider, negative and positive shifts in discrete time (Figure O-1).

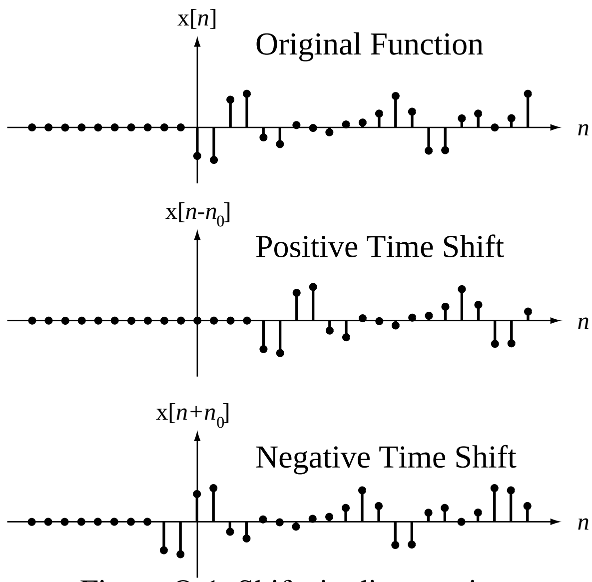

Figure O-1 Shifts in discrete time

Case 1. Positive shifts in discrete time

The signal is causal. Therefore positive shifts in discrete time simply shift in leading zeros.

$$g[n-n_0] \xleftrightarrow{\mathcal{Z}} \sum_{n=0}^{\infty} g[n-n_0] z^{-n} = \sum_{n=n_0}^{\infty} g[n-n_0] z^{-n} \quad , \quad n_0 \geq 0$$

Let $m = n - n_0$. Then

$$g[n-n_0] \xleftrightarrow{\mathcal{Z}} \sum_{m=0}^{\infty} g[m] z^{-(m+n_0)} = z^{-n_0} \sum_{m=0}^{\infty} g[m] z^{-m} = z^{-n_0} G(z)$$

$$g[n-n_0] \xleftrightarrow{\mathcal{Z}} z^{-n_0} G(z), \; n_0 \geq 0 \tag{O.1}$$

This property applies only to causal signals. Otherwise a positive shift could shift in new non-zero signal values and the relationship between the transforms of the original and shifted signals would not be unique.

Case 2. Negative shifts in discrete time

In this case, using the unilateral z transform, we are in general truncating some of the non-zero part of the signal by shifting it to the left because the transform summation begins at $n = 0$. Therefore, if the property is to apply generally we must find a way to restore the missing information. Otherwise the transform of the unshifted signal and the shifted signal cannot be uniquely related.

Start with the definition of the z transform

$$g[n+1] \xleftrightarrow{\mathcal{Z}} \sum_{n=0}^{\infty} g[n+1] z^{-n} = z \sum_{n=0}^{\infty} g[n+1] z^{-(n+1)} .$$

We have a summation which does not include the effect of $g[0]$ in the original function. Let $m = n + 1$, then

$$g[n+1] \xleftrightarrow{\mathcal{Z}} z \sum_{m=1}^{\infty} g[m] z^{-m} = z\left(\sum_{m=0}^{\infty} g[m] z^{-m} - g[0] \right) = z\left(G(z) - g[0]\right)$$

By this process we now are including the effect of $g[0]$ and the transform of the original signal and the shifted signal are uniquely related. We can extend this method to greater shifts, for example, a negative shift of 2 in discrete time.

$$g[n+2] \xleftrightarrow{\mathcal{Z}} \sum_{n=0}^{\infty} g[n+2] z^{-n} = z^2 \sum_{n=0}^{\infty} g[n+2] z^{-(n+2)}$$

Let $m = n + 2$, then

$$g[n+2] \xleftrightarrow{\mathcal{Z}} z^2 \sum_{m=2}^{\infty} g[m] z^{-m} = z^2 \left(\sum_{m=0}^{\infty} g[m] z^{-m} - g[0] - z^{-1} g[1] \right)$$

$$g[n+2] \xleftrightarrow{\mathcal{Z}} z^2 \left(G(z) - g[0] - z^{-1} g[1] \right)$$

Then, by induction, for greater shifts,

$$\mathrm{g}\left[n+n_0\right] \xleftrightarrow{\mathcal{Z}} z^{n_0}\left(\mathrm{G}\left(z\right)-\sum_{m=0}^{n_0-1}\mathrm{g}\left[m\right]z^{-m}\right) \ , \ \ n_0>0 \tag{O.2}$$

O.3 Change of Scale

If we compress or expand the z-transform of a signal in the z domain, the equivalent effect in the DT domain is a multiplication by a complex exponential.

$$\alpha^n\,\mathrm{g}\left[n\right] \xleftrightarrow{\mathcal{Z}} \sum_{n=0}^{\infty}\alpha^n\,\mathrm{g}\left[n\right]z^{-n}=\sum_{n=0}^{\infty}\mathrm{g}\left[n\right]\left(z/\alpha\right)^{-n}=\mathrm{G}\left(z/\alpha\right)$$

$$\alpha^n\,\mathrm{g}\left[n\right] \xleftrightarrow{\mathcal{Z}} \mathrm{G}\left(z/\alpha\right) \tag{O.3}$$

O.4 Initial Value Theorem

The initial-value theorem is similar to its counterpart in the Laplace transform. If we take the limit as z approaches infinity of the z transform $\mathrm{G}\left(z\right)$ of any function $\mathrm{g}\left[n\right]$ all the terms except the $\mathrm{g}\left[0\right]z^0$ term approach zero leaving only the first term.

$$\lim_{z\to\infty}\mathrm{G}\left(z\right)=\lim_{z\to\infty}\sum_{n=0}^{\infty}\mathrm{g}\left[n\right]z^{-n}=\lim_{z\to\infty}\left[\mathrm{g}\left[0\right]+\frac{\mathrm{g}\left[1\right]}{z}+\frac{\mathrm{g}\left[2\right]}{z^2}+\cdots\right]=\mathrm{g}\left[0\right]$$

$$\mathrm{g}\left[0\right]=\lim_{z\to\infty}\mathrm{G}\left(z\right) \tag{O.4}$$

O.5 z-Domain Differentiation

Differentiation in the z domain is related to a multiplication by $-n$ in the DT domain.

$$\mathrm{G}\left(z\right)=\sum_{0}^{\infty}\mathrm{g}\left[n\right]z^{-n}$$

$$\frac{d}{dz}\mathrm{G}\left(z\right)=\frac{d}{dz}\sum_{n=0}^{\infty}\mathrm{g}\left[n\right]z^{-n}=\sum_{n=0}^{\infty}\mathrm{g}\left[n\right]\left(-nz^{-(n+1)}\right)=-z^{-1}\underbrace{\sum_{0}^{\infty}n\,\mathrm{g}\left[n\right]z^{-n}}_{\mathcal{Z}\left(n\,\mathrm{g}\left[n\right]\right)}$$

Therefore

$$-n\,g[n] \xleftrightarrow{Z} z\frac{d}{dz}G(z) \tag{O.5}$$

O.6 Convolution in Discrete Time

We have seen in the Fourier and Laplace transforms that there is an important relationship between convolution in one domain and multiplication in the other domain. A similar relationship exists for the z transform. Start with the definition of the convolution sum

$$g[n]*h[n] = \sum_{m=-\infty}^{\infty} g[m]h[n-m].$$

Take the z transform of both sides,

$$\mathcal{Z}\left(g[n]*h[n]\right) = \sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} g[m]h[n-m]z^{-n}$$

$$g[n]*h[n] \xleftrightarrow{Z} \sum_{m=-\infty}^{\infty} g[m]\underbrace{\sum_{n=0}^{\infty} h[n-m]z^{-n}}_{z^{-m}H(z)} = \sum_{m=-\infty}^{\infty} g[m]z^{-m}H(z)$$

Using the fact that $g[n]$ is causal,

$$g[n]*h[n] \xleftrightarrow{Z} H(z)\sum_{m=0}^{\infty} g[m]z^{-m} = H(z)G(z)$$

$$g[n]*h[n] \xleftrightarrow{Z} H(z)G(z) \tag{O.6}$$

In words, convolution of two DT functions in the DT domain corresponds to multiplication of their z transforms in the z domain, exactly as was true for the Fourier and Laplace transforms. Consideration of the z transform of the product of two DT-domain functions is beyond the scope of this text.

O.7 Differencing

Differencing is the DT operation that is analogous to CT differentiation. The first backward difference of $g[n]$ is $g[n]-g[n-1]$. Using the time-shifting property (for causal functions), the z transform of this difference forms the pair

$$g[n]-g[n-1] \xleftrightarrow{Z} G(z)-z^{-1}G(z) = \left(1-z^{-1}\right)G(z).$$

Therefore

$$\mathrm{g}[n]-\mathrm{g}[n-1] \xleftrightarrow{\mathcal{Z}} \left(1-z^{-1}\right)\mathrm{G}(z) . \qquad \text{(O.7)}$$

O.8 Accumulation

Accumulation is the DT operation which is analogous to CT integration and the proof of the property can be done in an analogous manner. First realize that accumulation is equivalent to convolution with a unit sequence

$$\mathrm{u}[n]*\mathrm{g}[n]=\sum_{m=-\infty}^{\infty}\mathrm{u}[m]\mathrm{g}[n-m]=\sum_{m=0}^{n}\mathrm{g}[m] .$$

The last summation has an upper limit of n because $\mathrm{g}[n]$ is causal.

$$\sum_{m=0}^{n}\mathrm{g}[m]=\mathrm{u}[n]*\mathrm{g}[n] \xleftrightarrow{\mathcal{Z}} \mathrm{G}(z)\mathrm{U}(z)=\frac{z}{z-1}\mathrm{G}(z)$$

Therefore

$$\sum_{m=0}^{n}\mathrm{g}[m] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-1}\mathrm{G}(z)=\frac{1}{1-z^{-1}}\mathrm{G}(z) \qquad \text{(O.8)}$$

O.9 Final Value Theorem

Begin the derivation of the final-value theorem by considering the z transform of the difference between a DT function and a shifted version of the same function (a first forward difference),

$$\mathcal{Z}\left(\mathrm{g}[n+1]-\mathrm{g}[n]\right)=\lim_{n\to\infty}\sum_{m=0}^{n}\left(\mathrm{g}[m+1]-\mathrm{g}[m]\right)z^{-m}$$

Taking the z transform and using the time-shifting property,

$$z\left(\mathrm{G}(z)-\mathrm{g}[0]\right)-\mathrm{G}(z)=\lim_{n\to\infty}\sum_{m=0}^{n}\left(\mathrm{g}[m+1]-\mathrm{g}[m]\right)z^{-m} .$$

Now take the limit as $z\to 1$ on both sides,

$$\lim_{z\to 1}\left\{(z-1)\mathrm{G}(z)-z\,\mathrm{g}[0]\right\}=\lim_{z\to 1}\left\{\lim_{n\to\infty}\sum_{m=0}^{n}\left(\mathrm{g}[m+1]-\mathrm{g}[m]\right)z^{-m}\right\} .$$

Taking the z limit first,

$$\lim_{z\to 1}\left\{(z-1)\mathrm{G}(z)\right\}-\mathrm{g}[0]=\lim_{n\to\infty}\sum_{m=0}^{n}\left(\mathrm{g}[m+1]-\mathrm{g}[m]\right).$$

$$\lim_{z\to 1}\left\{(z-1)\mathrm{G}(z)\right\}-\mathrm{g}[0]=\lim_{n\to\infty}\left(\mathrm{g}[1]-\mathrm{g}[0]+\mathrm{g}[2]-\mathrm{g}[1]+\cdots+\mathrm{g}[n+1]-\mathrm{g}[n]\right)$$

$$\lim_{z\to 1}\left\{(z-1)\mathrm{G}(z)\right\}-\mathrm{g}[0]=\lim_{n\to\infty}\left(\mathrm{g}[n+1]-\mathrm{g}[0]\right)$$

$$\lim_{n\to\infty}\mathrm{g}[n]=\lim_{z\to 1}(z-1)\mathrm{G}(z)$$