Sampling and Signal Processing

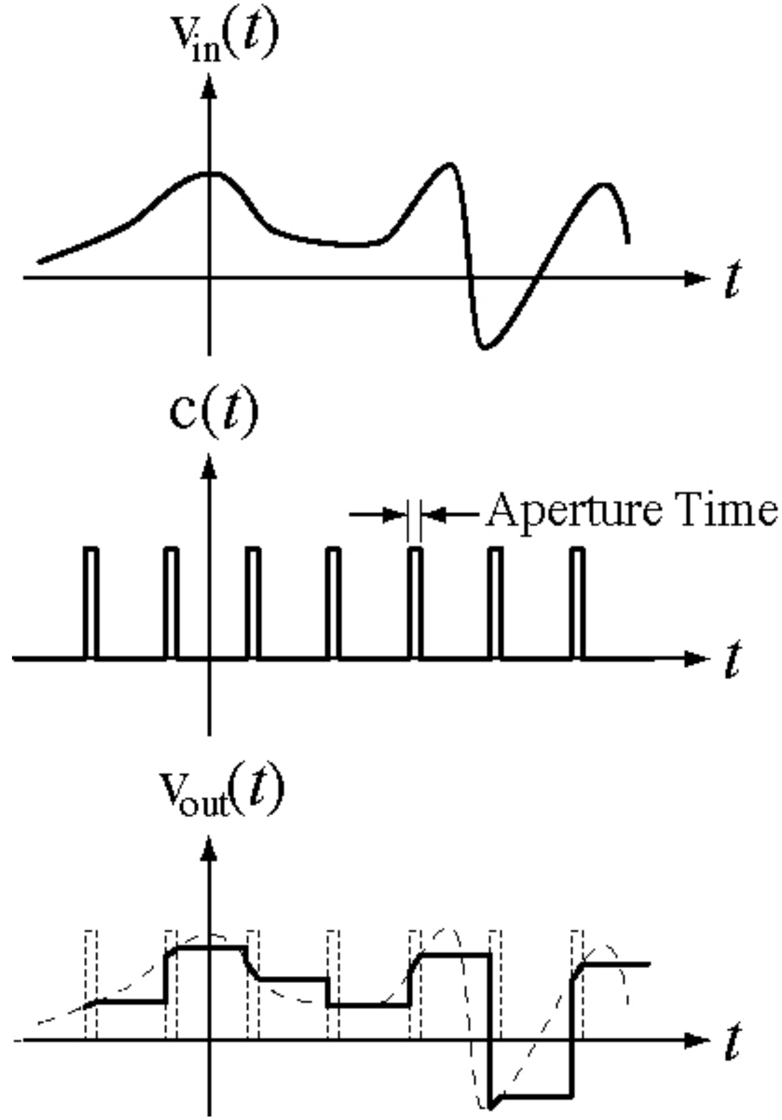
Sampling Methods

- Sampling is most commonly done with two devices, the **sample-and-hold** (**S/H**) and the **analog-to-digital-converter** (**ADC**)
- The S/H acquires a continuous-time signal at a point in time and holds it for later use
- The ADC converts continuous-time signal values at discrete points in time into numerical codes which can be stored in a digital system

Sampling Methods

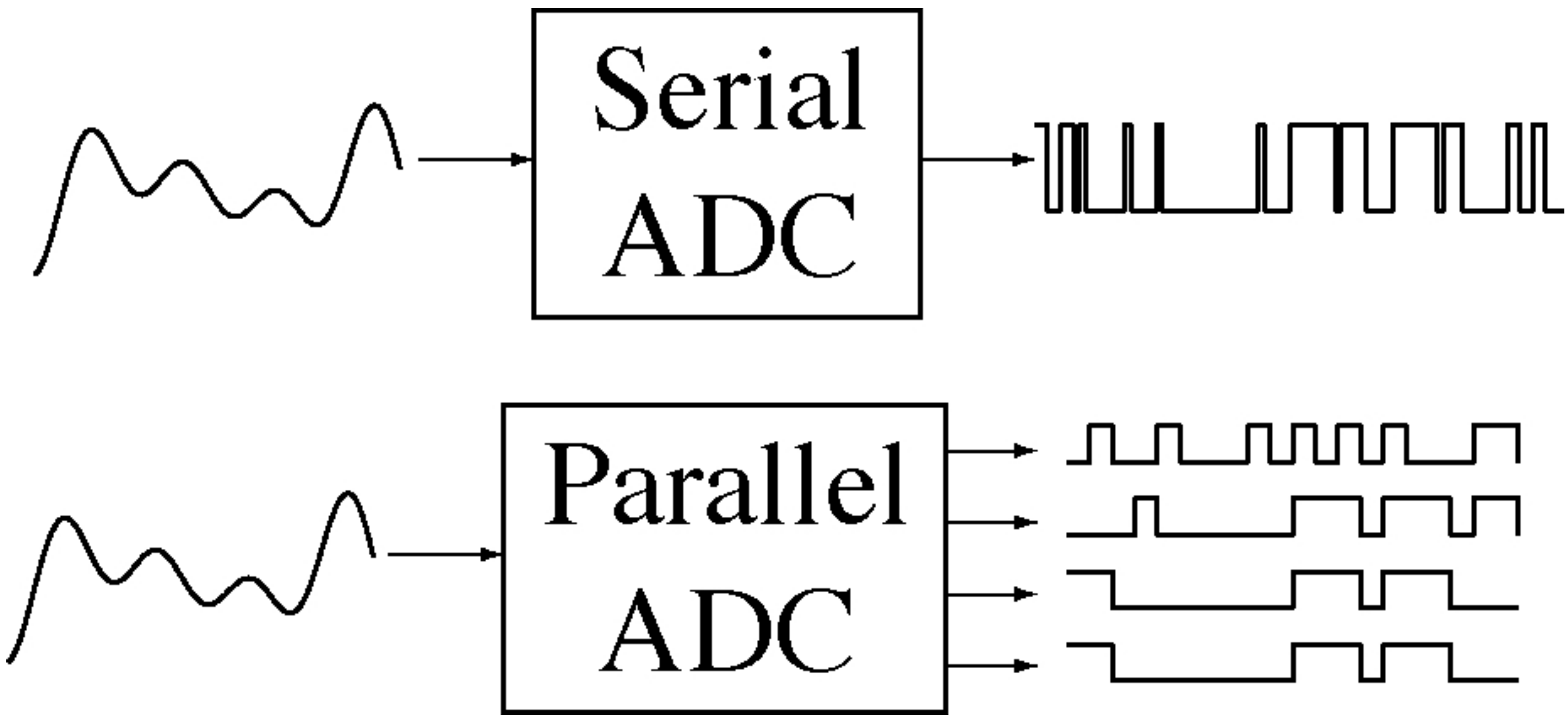
Sample-and-Hold

During the clock c(t) aperture time, the response of the S/H is the same as its excitation. At the end of that time, the response holds that value until the next aperture time.



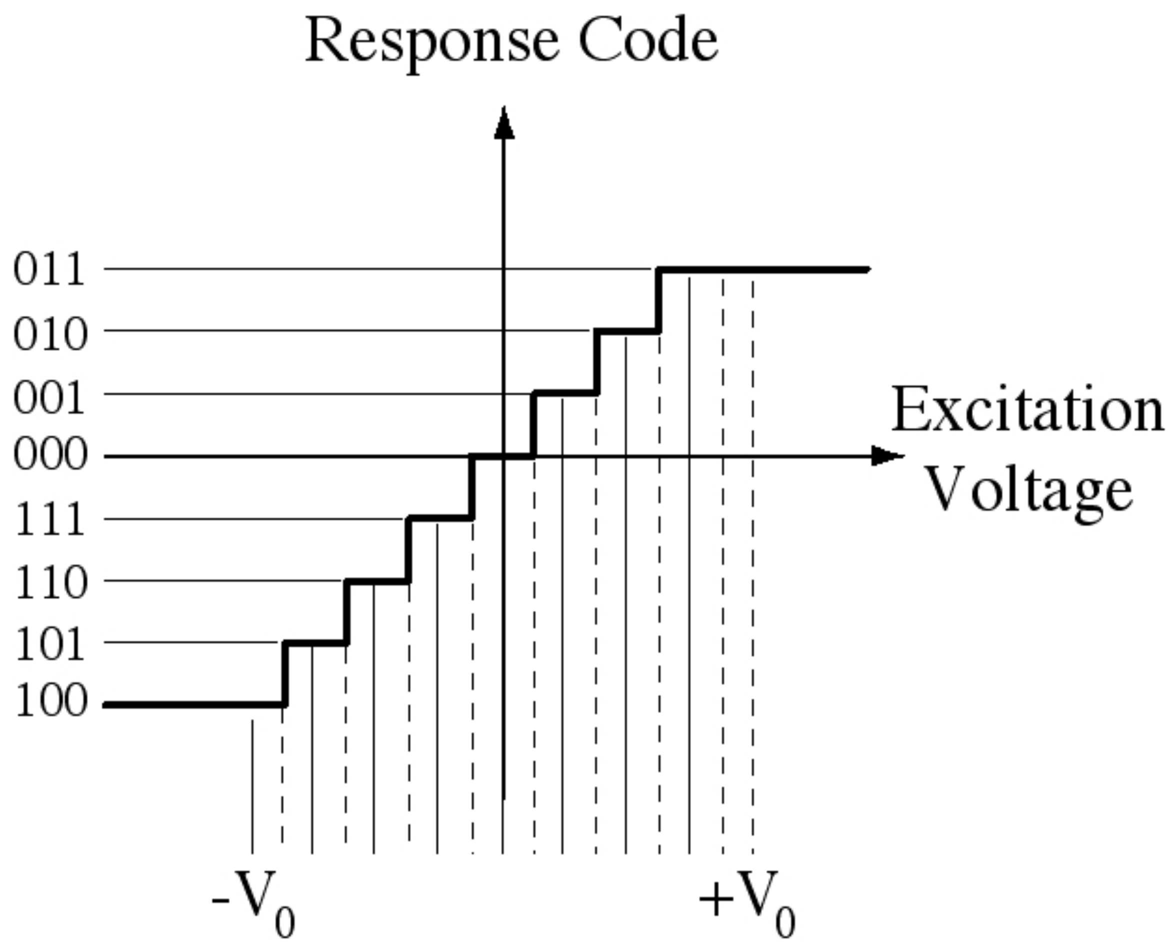
Sampling Methods

An ADC converts its input signal into a code. The code can be output serially or in parallel.

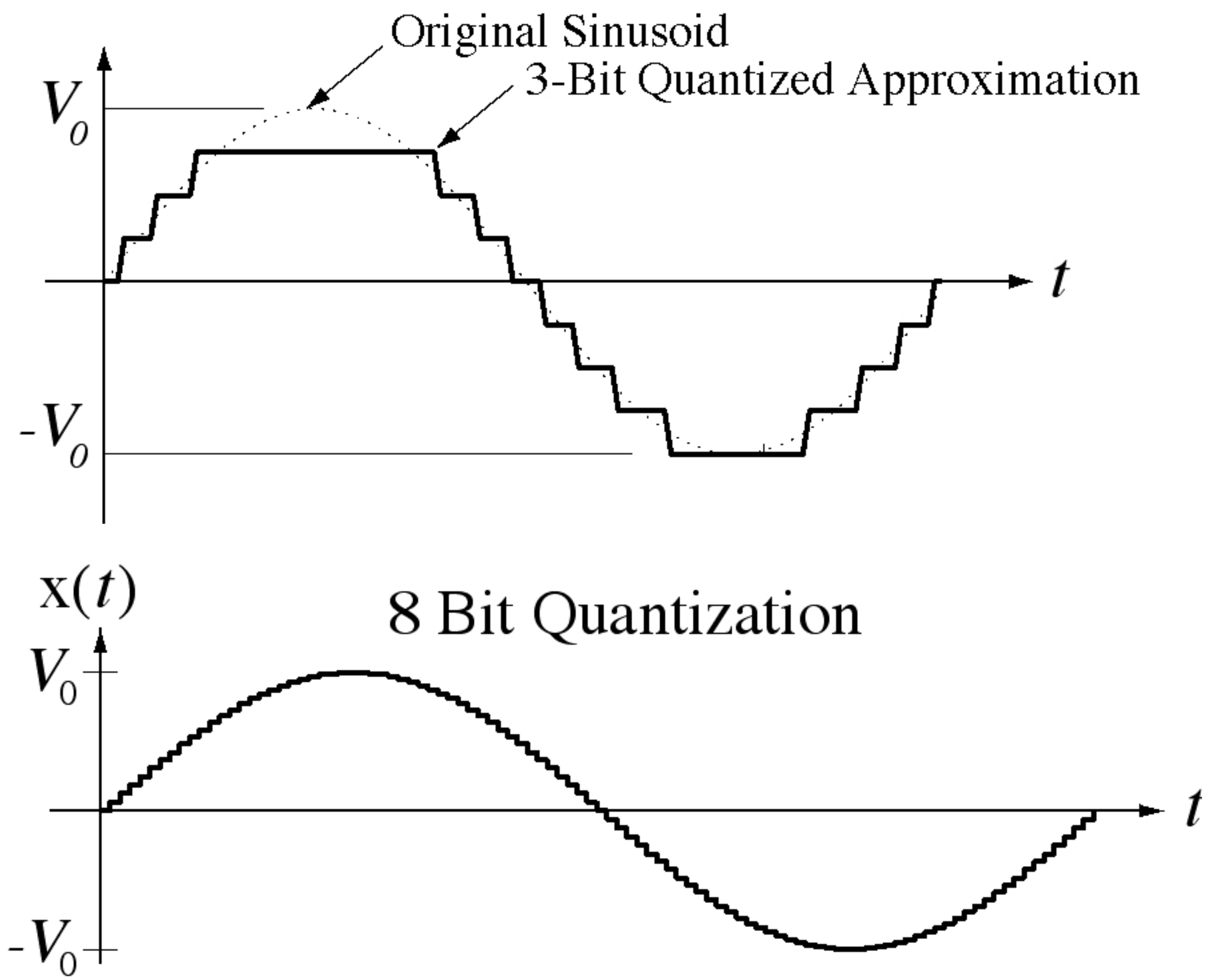


Sampling Methods

Excitation-Response Relationship for an ADC

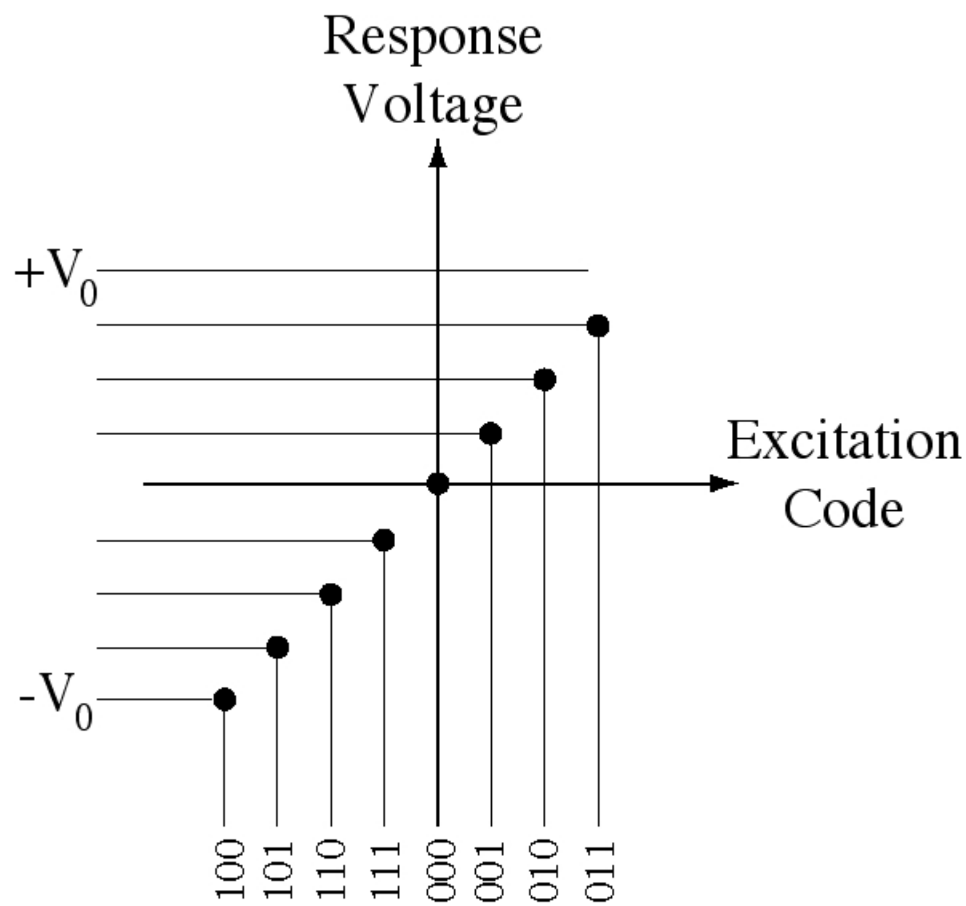


Sampling Methods



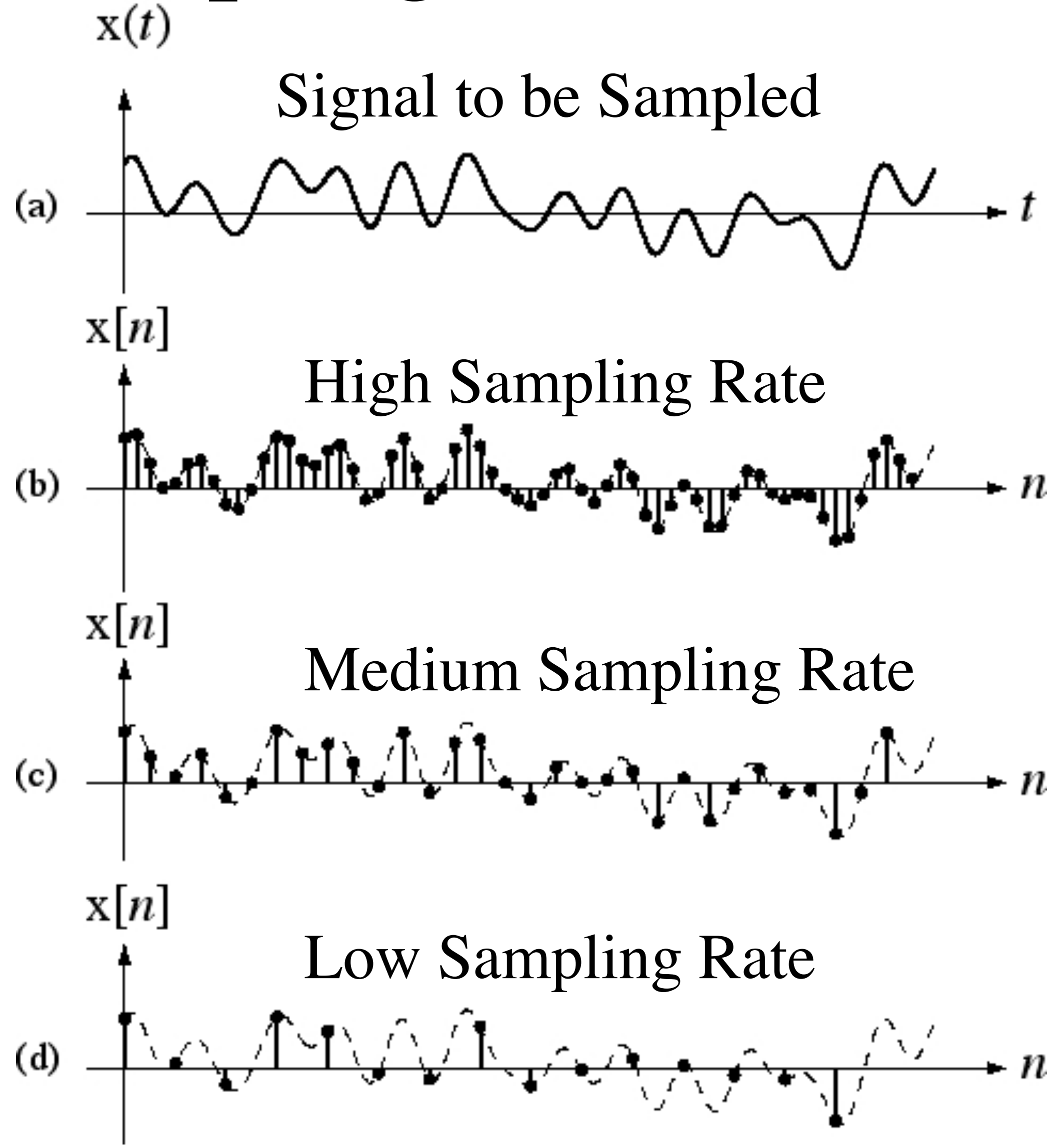
Sampling Methods

Encoded signal samples can be converted back into a CT signal by a **digital-to-analog converter** (**DAC**).



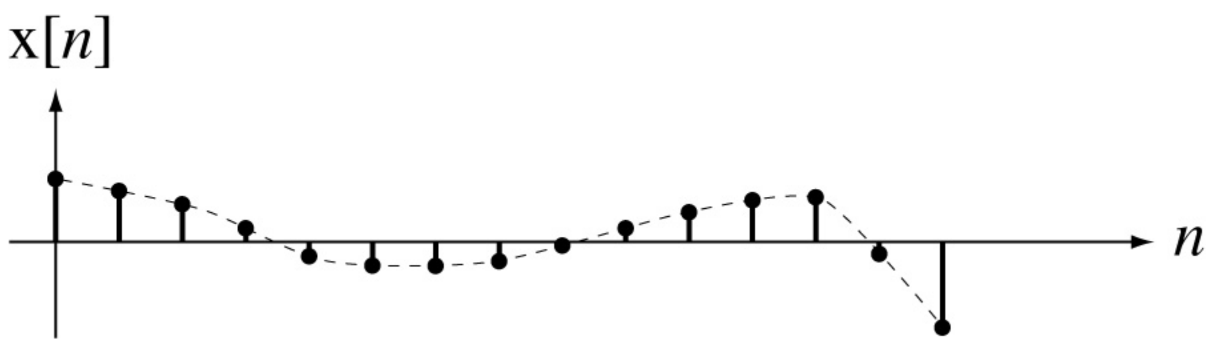
Sampling

The fundamental consideration in sampling theory is how fast to sample a signal to be able to reconstruct the signal from the samples.

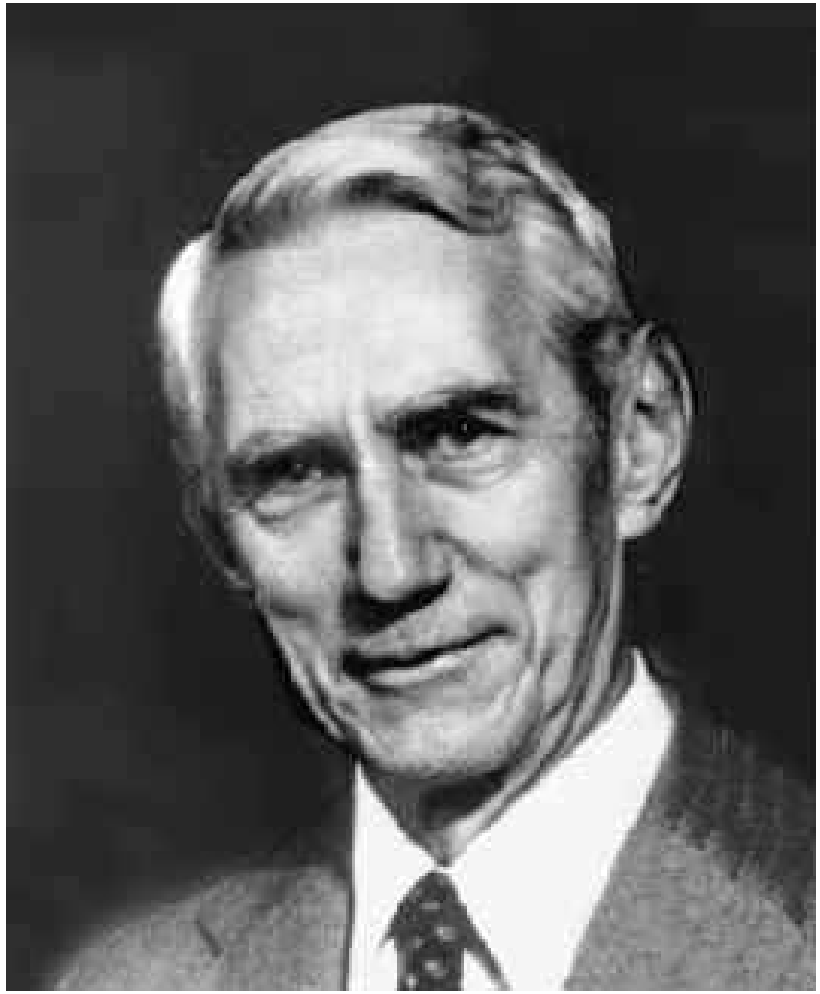


Sampling

The "low" sampling rate on the previous slide might be adequate on a signal that varies more slowly.



Claude Elwood Shannon

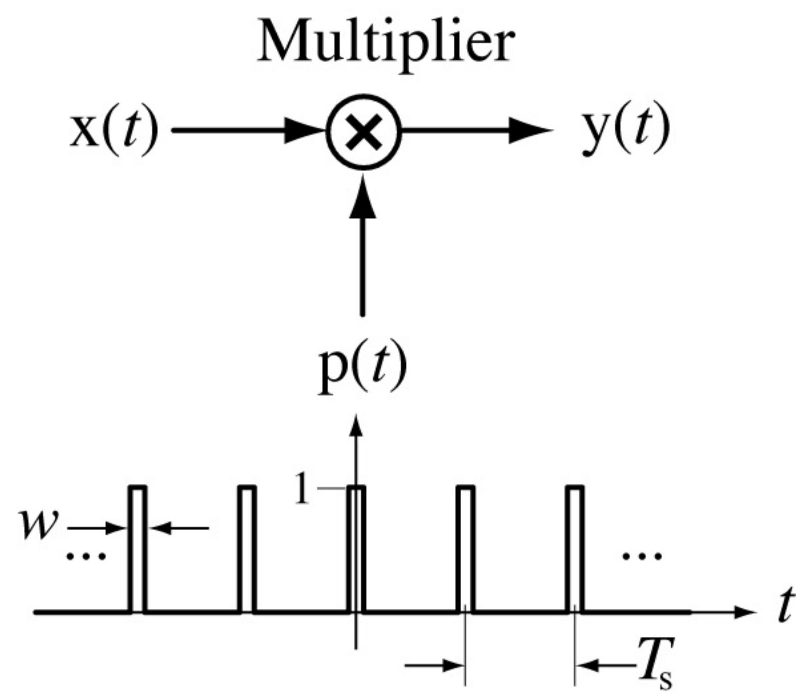


Pulse Amplitude Modulation

Consider an approximation to the ideal sampler, a pulse train $\mathrm{p}(t)$ multiplying a signal $\mathrm{x}(t)$ to produce a response $\mathrm{y}(t)$.

$$\mathrm{p}(t) = \mathrm{rect}(t/w) * \delta_{T_s}(t)$$

The average value of $\mathrm{y}(t)$ during each pulse is approximately the value of $\mathrm{x}(t)$ at the time of the center of the pulse. This is known as **pulse amplitude modulation**.



Pulse Amplitude Modulation

The response of the pulse modulator is

$$\mathrm{y}(t) = \mathrm{x}(t)\,\mathrm{p}(t) = \mathrm{x}(t)\left[\mathrm{rect}(t/w) * \delta_{T_s}(t)\right]$$

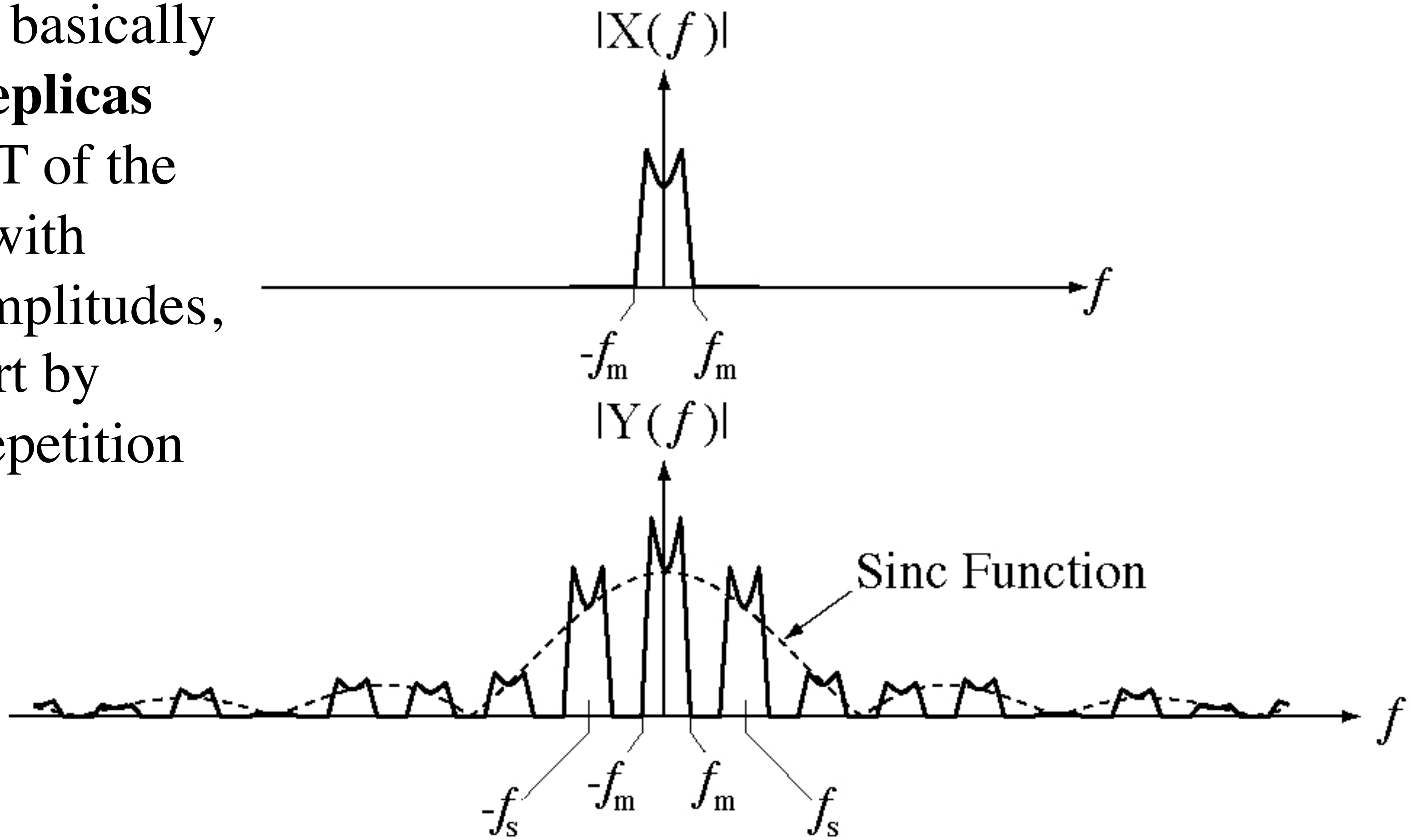
and its CTFT is

$$\mathrm{Y}(f) = wf_s \sum_{k=-\infty}^{\infty} \mathrm{sinc}(wkf_s)\,\mathrm{X}(f - kf_s)$$

where $f_s = 1/T_s$

Pulse Amplitude Modulation

The CTFT of the response is basically **multiple replicas** of the CTFT of the excitation with different amplitudes, spaced apart by the pulse repetition rate.



Pulse Amplitude Modulation

If the pulse train is modified to make the pulses have a constant area instead of a constant height, the pulse train becomes

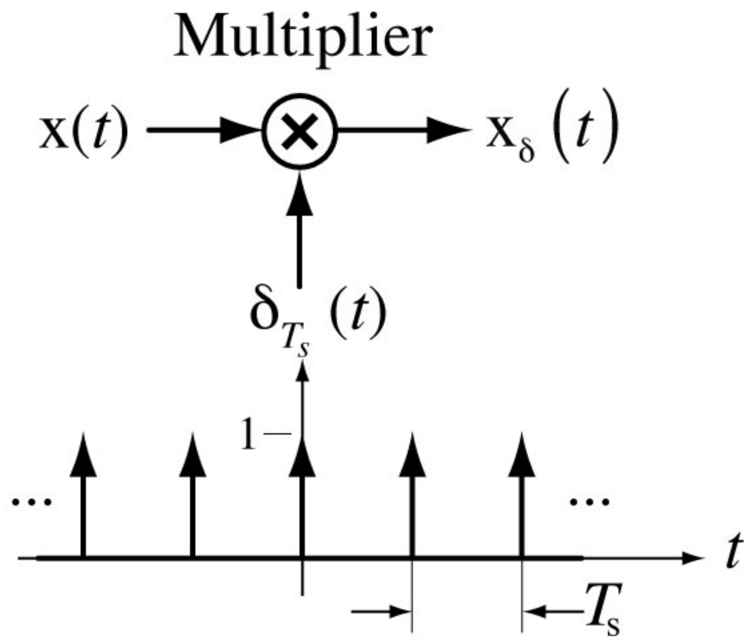
$$\mathrm{p}(t) = (1/w)\,\mathrm{rect}(t/w) * \delta_{T_s}(t)$$

and the CTFT of the modulated pulse train becomes

$$\mathrm{Y}(f) = f_s \sum_{k=-\infty}^{\infty} \mathrm{sinc}(wkf_s)\,\mathrm{X}(f - kf_s)$$

Pulse Amplitude Modulation

As the **aperture time** *w* of the pulses approaches zero the pulse train approaches a **periodic impulse** and the replicas of the original signal's spectrum all approach the same size. This limit is called **impulse sampling**.



Sampling vs. Impulse Sampling

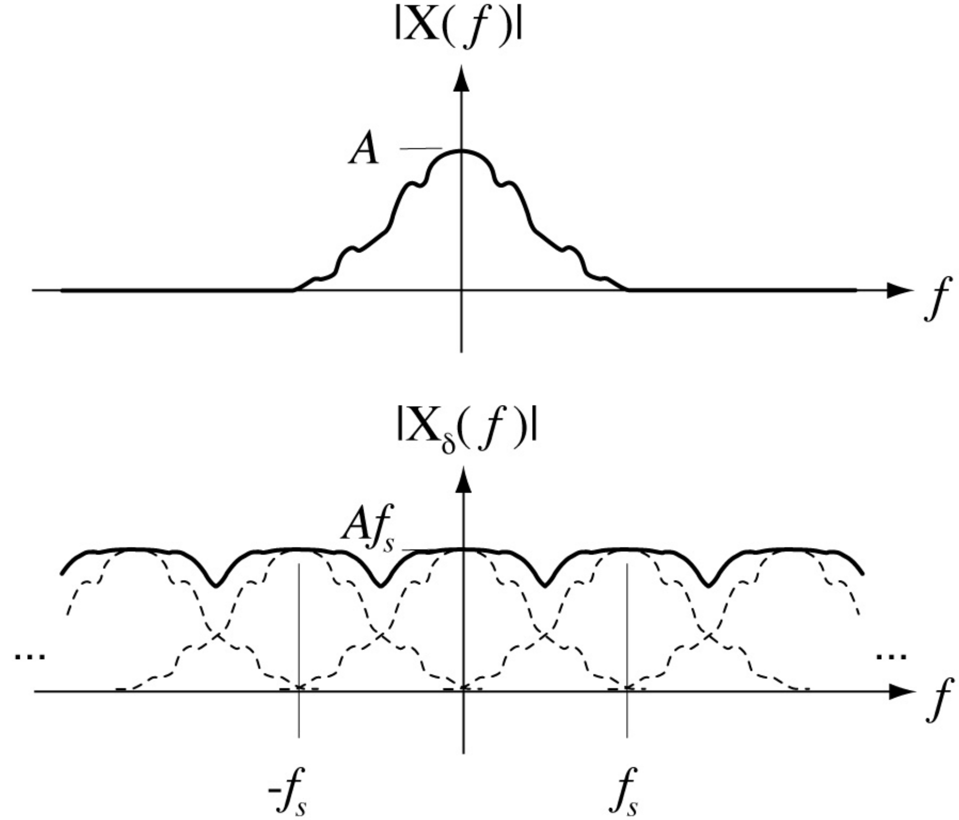
If we simply acquire the values of $\mathrm{x}(t)$ at the sampling times nT_s we form a discrete-time signal $\mathrm{x}[n]=\mathrm{x}(nT_s)$. This is known as sampling, in contrast to impulse sampling in which we form the continuous-time signal $\mathrm{x}_\delta(t)=\mathrm{x}(t)\delta_{T_s}(t)$. These are two different ways of conceiving the sampling process but they really contain the same information about the signal $\mathrm{x}(t)$. The two signals, $\mathrm{x}[n]$ and $\mathrm{x}_\delta(t)$, both consist only of impulses, discrete-time in one case and continuous-time in the other case, and the impulse strengths are the same for both at times that correspond through $t=nT_s$.

Aliasing

The CTFT of the impulse-sampled signal is

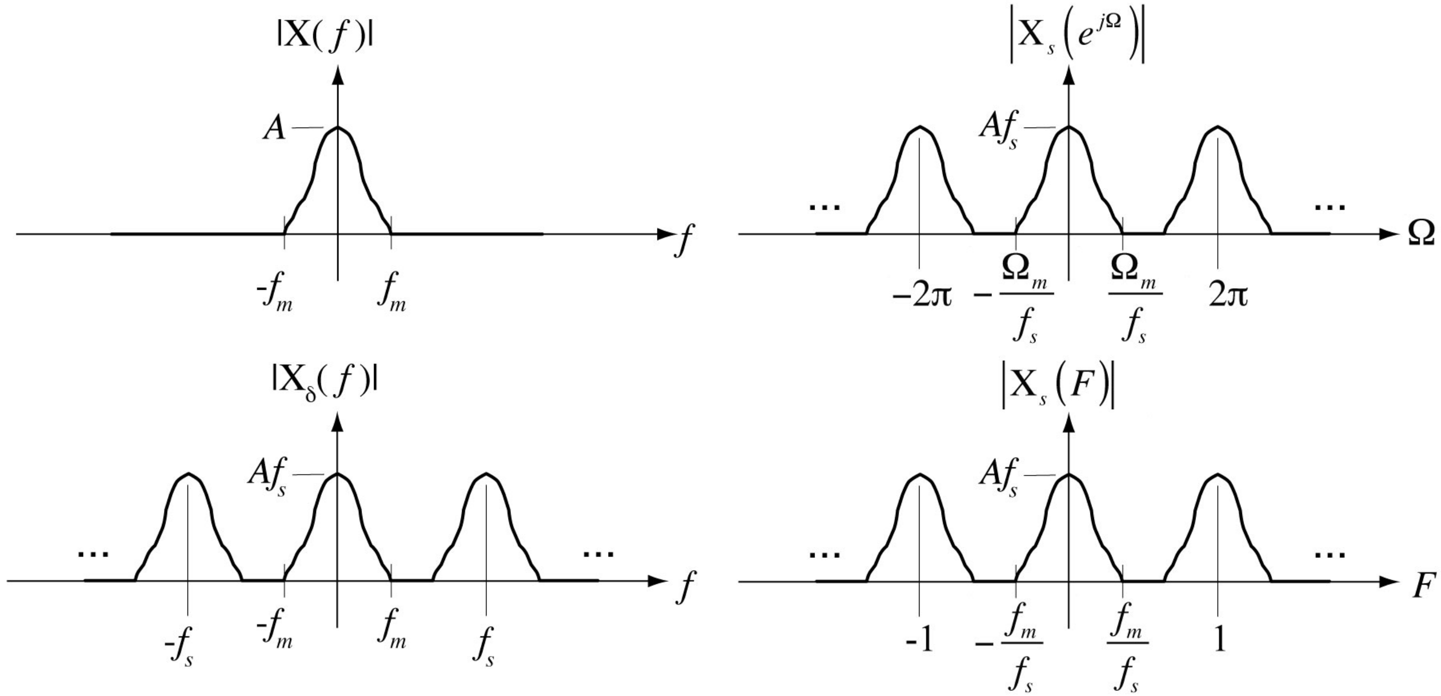
$$X_\delta(f) = X(f) * (1/T_s)\delta_{1/T_s}(f) = f_s\, X(f) * \delta_{f_s}(f)$$

If the sampling rate is less than twice the highest frequency of the original continuous-time signal, the replicas, called **aliases**, overlap.



Aliasing

If the CTFT of the original continuous-time signal is bandlimited and the sampling rate is more than twice the highest frequency in the signal, the aliases are separated and the original signal could be recovered by a lowpass filter that rejects the aliases.



The Sampling Theorem

If a continuous-time signal is sampled for all time at a rate f_s that is more than twice the bandlimit f_m of the signal, the original continuous-time signal can be recovered exactly from the samples.

The frequency $2f_m$ is called the Nyquist rate. A signal sampled at a rate less than the Nyquist rate is **undersampled** and a signal sampled at a rate greater than the Nyquist rate is **oversampled**.

Harry Nyquist

2/7/1889 - 4/4/1976

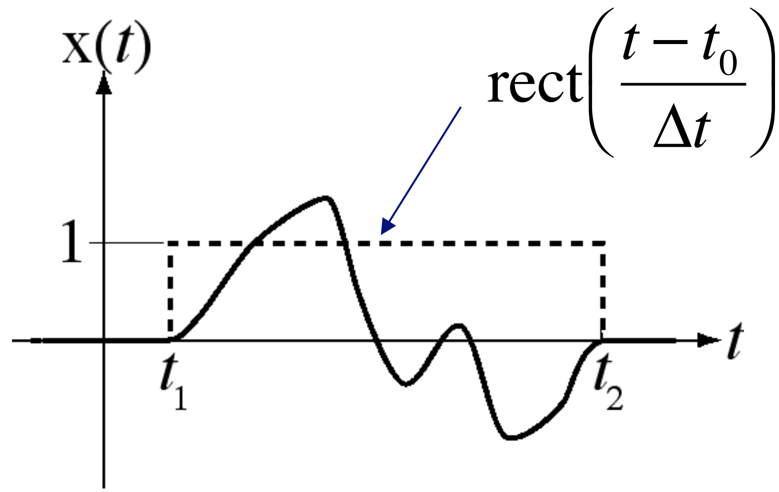
Timelimited and Bandlimited Signals

- The sampling theorem says that it is possible to sample a **bandlimited** signal at a rate sufficient to exactly reconstruct the signal from the samples.
- But it also says that the signal must be sampled for <u>all time</u>. This requirement holds even for signals that are **timelimited** (non-zero only for a finite time).

Timelimited and Bandlimited Signals

A signal that is timelimited cannot be bandlimited. Let $\text{x}(t)$ be a timelimited signal. Then

$$\text{x}(t) = \text{x}(t)\text{rect}\left(\frac{t - t_0}{\Delta t}\right)$$



The CTFT of $\text{x}(t)$ is

$$\text{X}(f) = \text{X}(f) * \Delta t \,\text{sinc}(\Delta t f)\, e^{-j2\pi f t_0}$$

Since this sinc function of f is not limited in f, anything convolved with it will also not be limited in f and cannot be the CTFT of a bandlimited signal.

Interpolation

The original continuous-time signal can be recovered (theoretically) from samples by a lowpass filter that passes the CTFT of the original continuous-time signal and rejects the aliases.

$$\underbrace{\mathrm{X}(f)}_{\substack{\text{CTFT of Original}\\ \text{Continuous-Time}\\ \text{Signal}}} = \underbrace{T_s \operatorname{rect}(f/2f_c)}_{\text{Ideal Lowpass Filter}} \times \underbrace{\mathrm{X}_\delta(f)}_{\substack{\text{CTFT of Impulse}\\ \text{Sampled Signal}}}$$

$$= T_s \operatorname{rect}(f/2f_c) \times f_s \mathrm{X}(f) * \delta_{f_s}(f)$$

Inverse transforming we get

$$\mathrm{x}(t) = \underbrace{T_s f_s}_{=1} 2 f_c \operatorname{sinc}(2 f_c t) * \underbrace{\mathrm{x}(t)(1/f_s)\delta_{T_s}(t)}_{=(1/f_s)\sum_{n=-\infty}^{\infty} \mathrm{x}(nT_s)\delta(t-nT_s)}$$

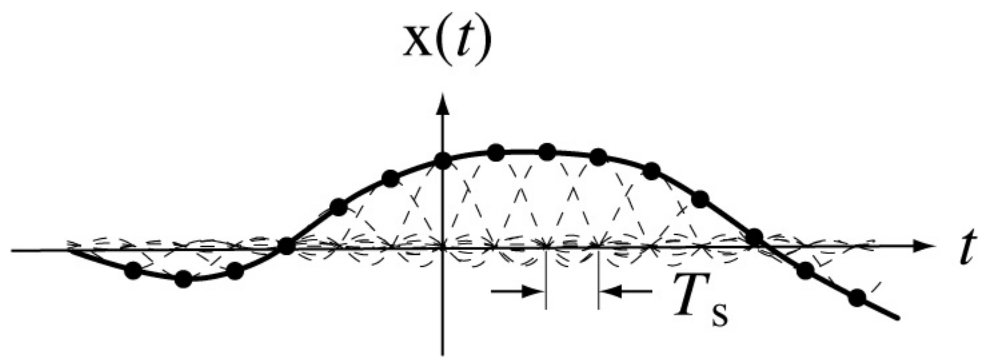
Interpolation

$$\mathrm{x}(t) = 2(f_c / f_s)\,\mathrm{sinc}(2 f_c t) * \sum_{n=-\infty}^{\infty} \mathrm{x}(nT_s)\,\delta(t - nT_s)$$

$$\mathrm{x}(t) = 2(f_c / f_s) \sum_{n=-\infty}^{\infty} \mathrm{x}(nT_s)\,\mathrm{sinc}\big(2 f_c (t - nT_s)\big)$$

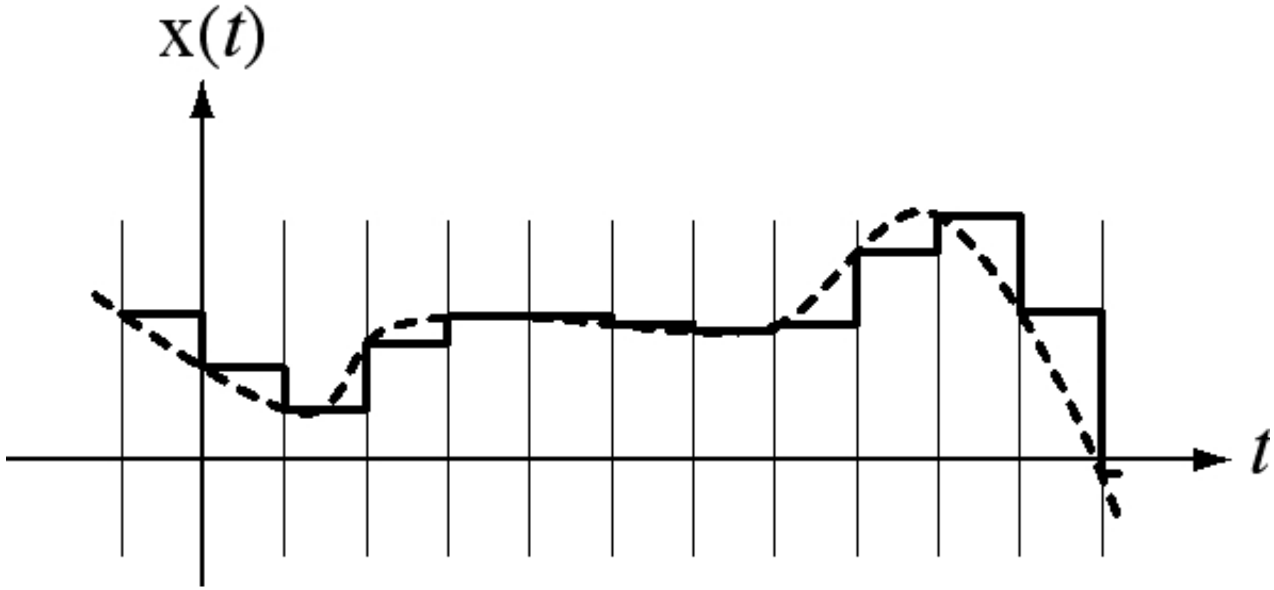
If $f_c = f_s / 2$

$$\mathrm{x}(t) = \sum_{n=-\infty}^{\infty} \mathrm{x}(nT_s)\,\mathrm{sinc}\big((t - nT_s)/T_s\big)$$



Practical Interpolation

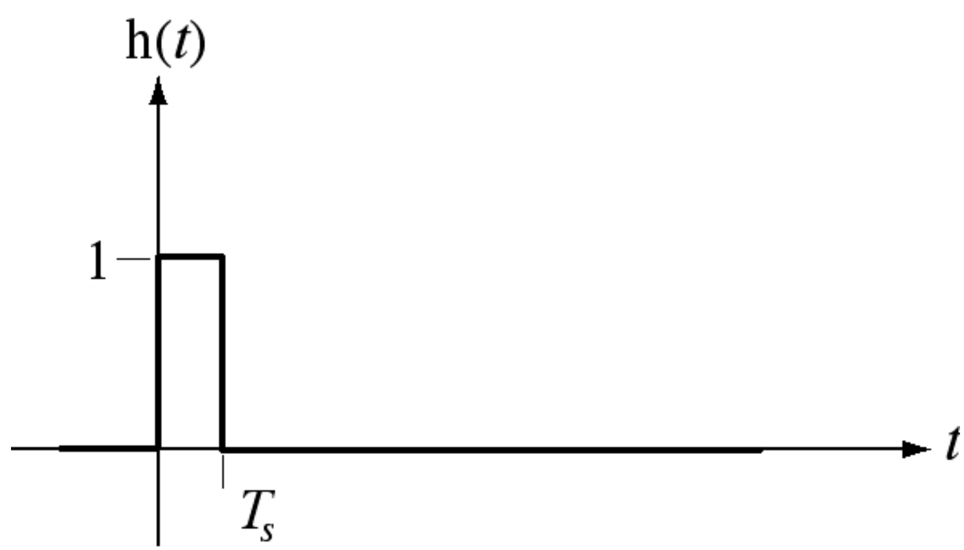
Sinc-function interpolation is theoretically perfect but it can never be done in practice because it requires samples from the signal for all time. Therefore real interpolation must make some compromises. Probably the simplest realizable interpolation technique is what a DAC does.



Practical Interpolation

The operation of a DAC can be mathematically modeled by a **zero - order hold** (**ZOH**), a device whose impulse response is a rectangular pulse whose width is the same as the time between samples.

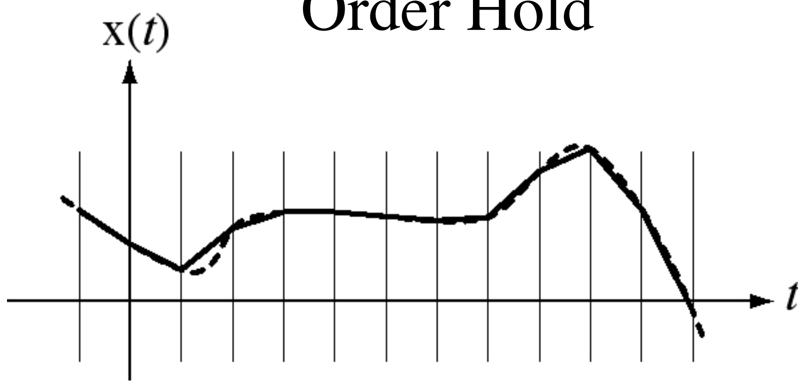
$$\mathrm{h}(t) = \begin{Bmatrix} 1 & , & 0 < t < T_s \\ 0 & , & \text{otherwise} \end{Bmatrix} = \mathrm{rect}\left(\frac{t - T_s/2}{T_s} \right)$$



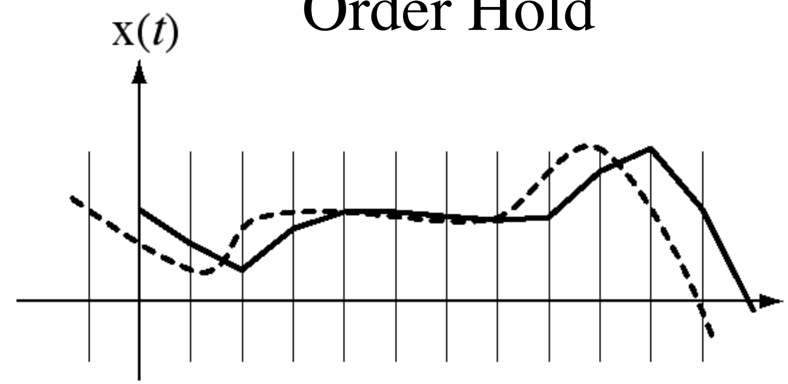
Practical Interpolation

A natural idea would be to simply draw straight lines between sample values. This cannot be done in real time because doing so requires knowledge of the next sample value before it occurs and that would require a non-causal system. If the reconstruction is delayed by one sample time, then it can be done with a causal system.

Non-Causal First-Order Hold

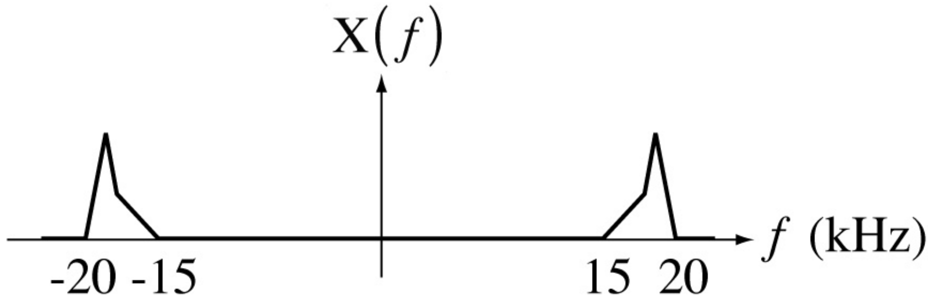


Causal First-Order Hold

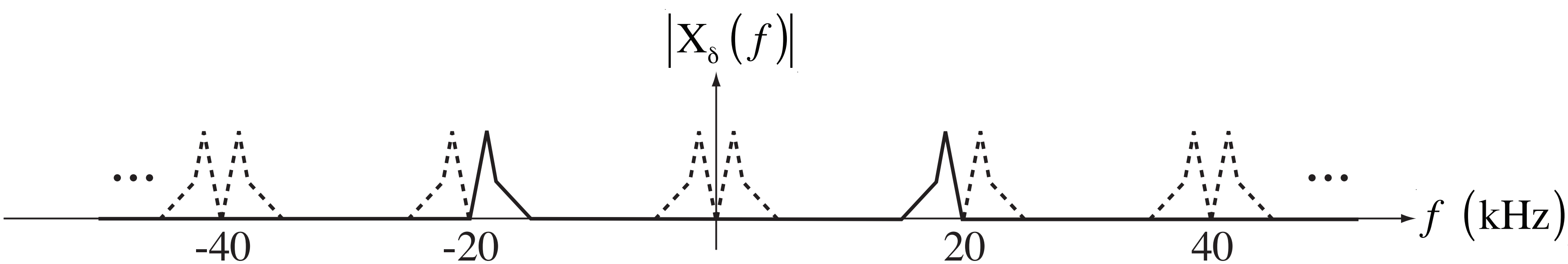


Sampling Bandpass Signals

CTFT of a bandpass signal



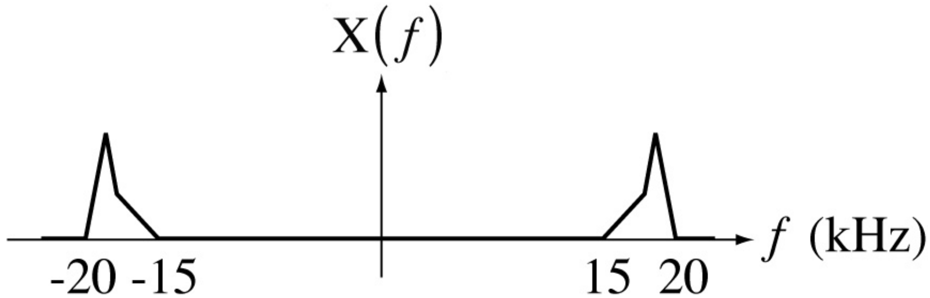
CTFT of that bandpass signal impulse sampled at 20 kHz



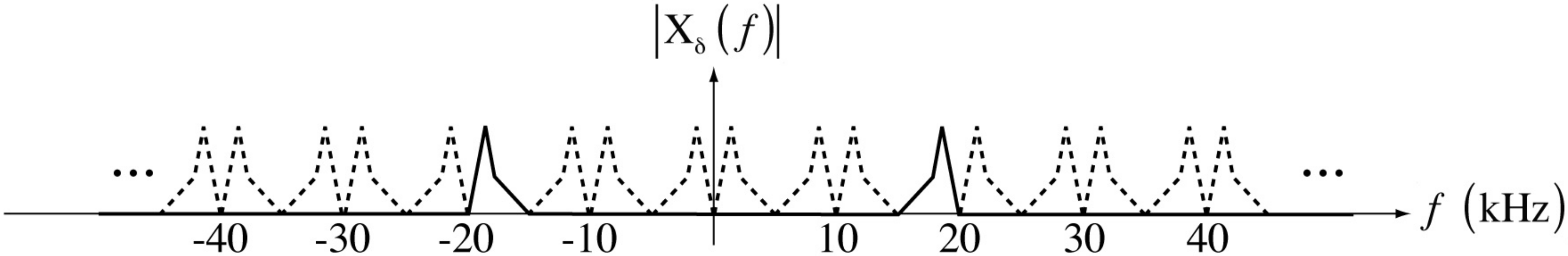
The original signal could be recovered by a bandpass filter even though the sampling rate is less than twice the highest frequency.

Sampling Bandpass Signals

CTFT of a bandpass signal

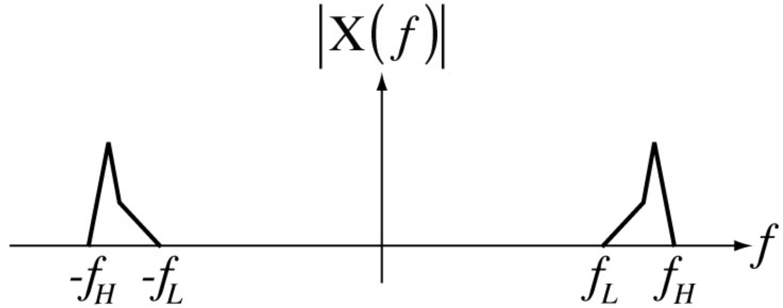


CTFT of that bandpass signal impulse sampled at 10 kHz



The original signal could still be recovered (barely) by an ideal bandpass filter even though the sampling rate is half of the highest frequency.

Sampling Bandpass Signals

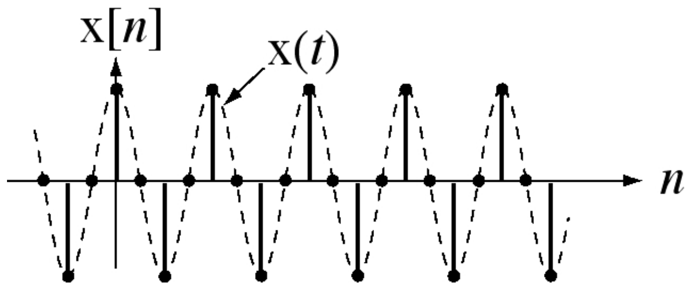


To be able to recover the original continuous-time signal from the samples $(k-1)f_s + (-f_L) < f_L \Rightarrow (k-1)f_s < 2f_L$ and $kf_s + (-f_H) > f_H \Rightarrow kf_s > 2f_H$. Combining and simplifying we arrive at at the general requirement for recovering the signal as

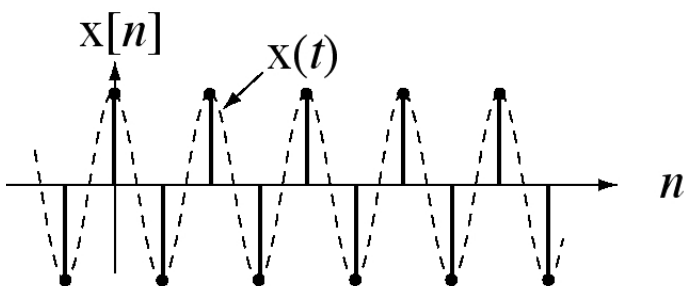
$$f_{s,\min} > \frac{2f_H}{\lfloor f_H / B \rfloor}$$

where B is the bandwidth $(f_H - f_L)$ and $\lfloor \cdot \rfloor$ means "greatest integer less than".

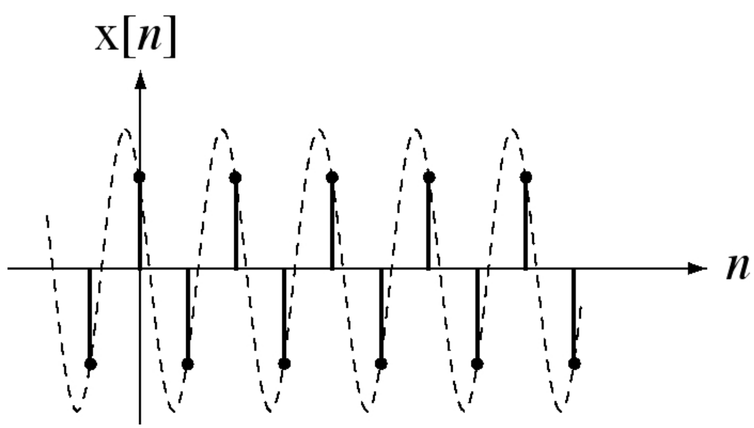
Sampling a Sinusoid



Cosine sampled at twice its Nyquist rate. Samples uniquely determine the signal.

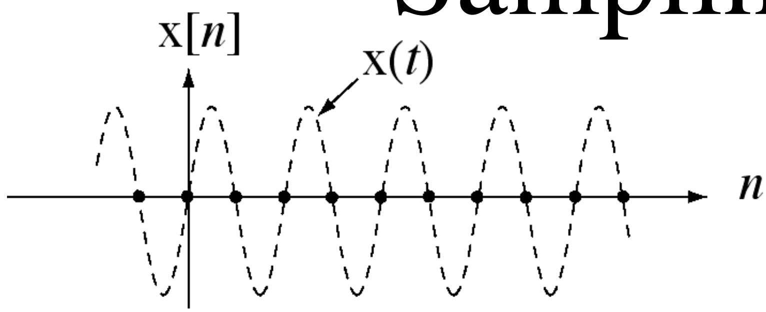


Cosine sampled at exactly its Nyquist rate. Samples do not uniquely determine the signal.



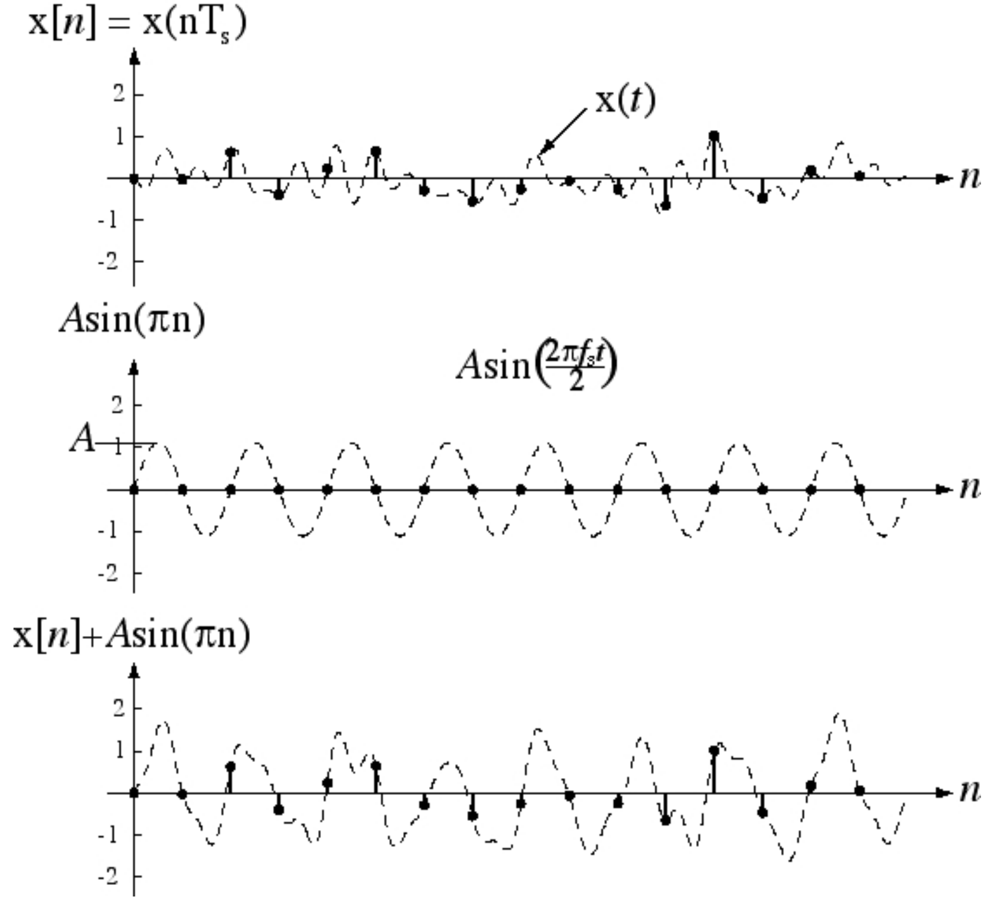
A different sinusoid of the same frequency with exactly the same samples as above.

Sampling a Sinusoid

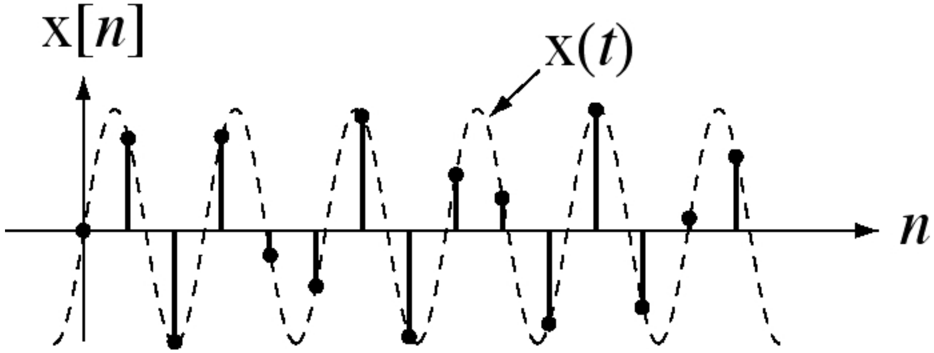


Sine sampled at its Nyquist rate. All the samples are zero.

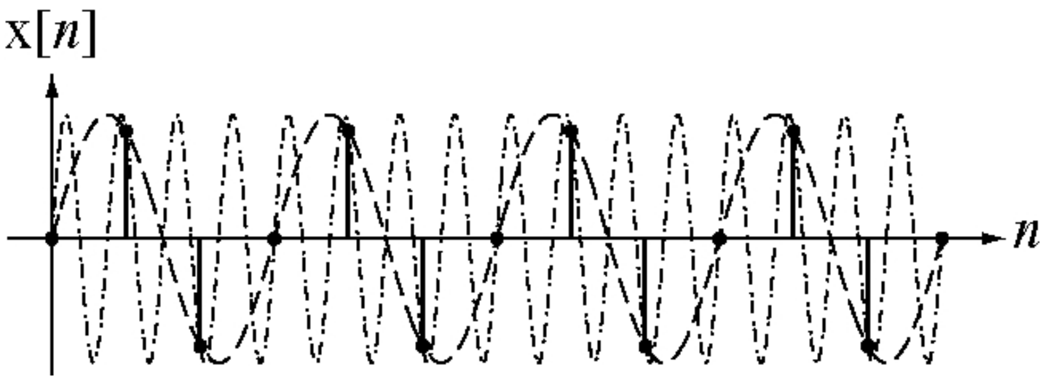
Adding a sine at the Nyquist frequency (half the sampling rate) to any signal does not change the samples.



Sampling a Sinusoid



Sine sampled slightly above its Nyquist rate



Two different sinusoids sampled at the same rate with the same samples

It can be shown that the samples from two sinusoids

$$x_1(t) = A\cos(2\pi f_0 t + \theta) \qquad x_2(t) = A\cos(2\pi(f_0 + kf_s)t + \theta)$$

taken at the rate f_s are the same for any integer value of k.

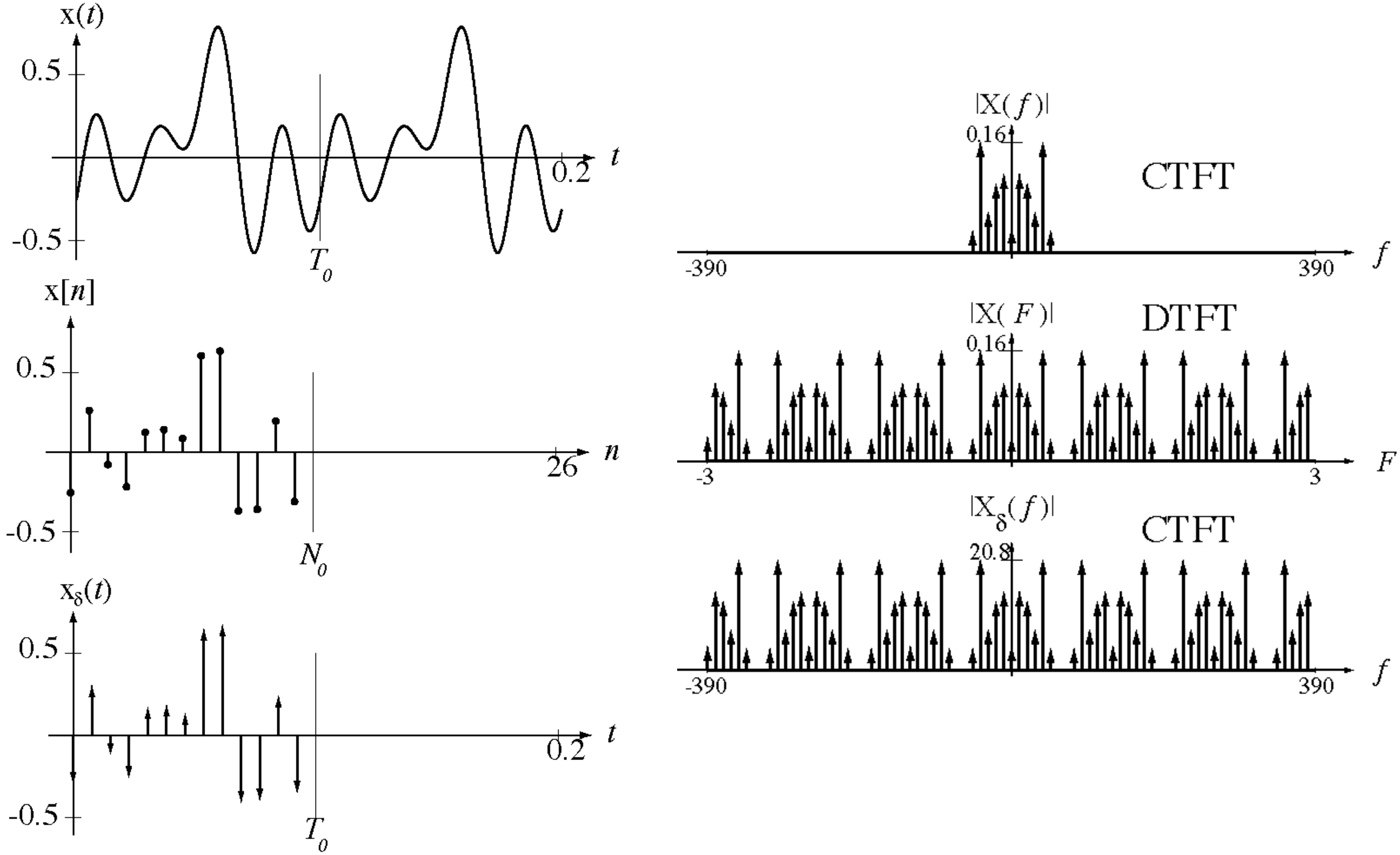
Bandlimited Periodic Signals

- If a signal is **bandlimited** it can be properly sampled according to the sampling theorem.
- If that signal is also **periodic** its CTFT consists only of impulses.
- Since it is bandlimited, there is a finite number of (non-zero) impulses.
- Therefore the signal can be exactly represented by a **finite set of numbers**, the impulse strengths.

Bandlimited Periodic Signals

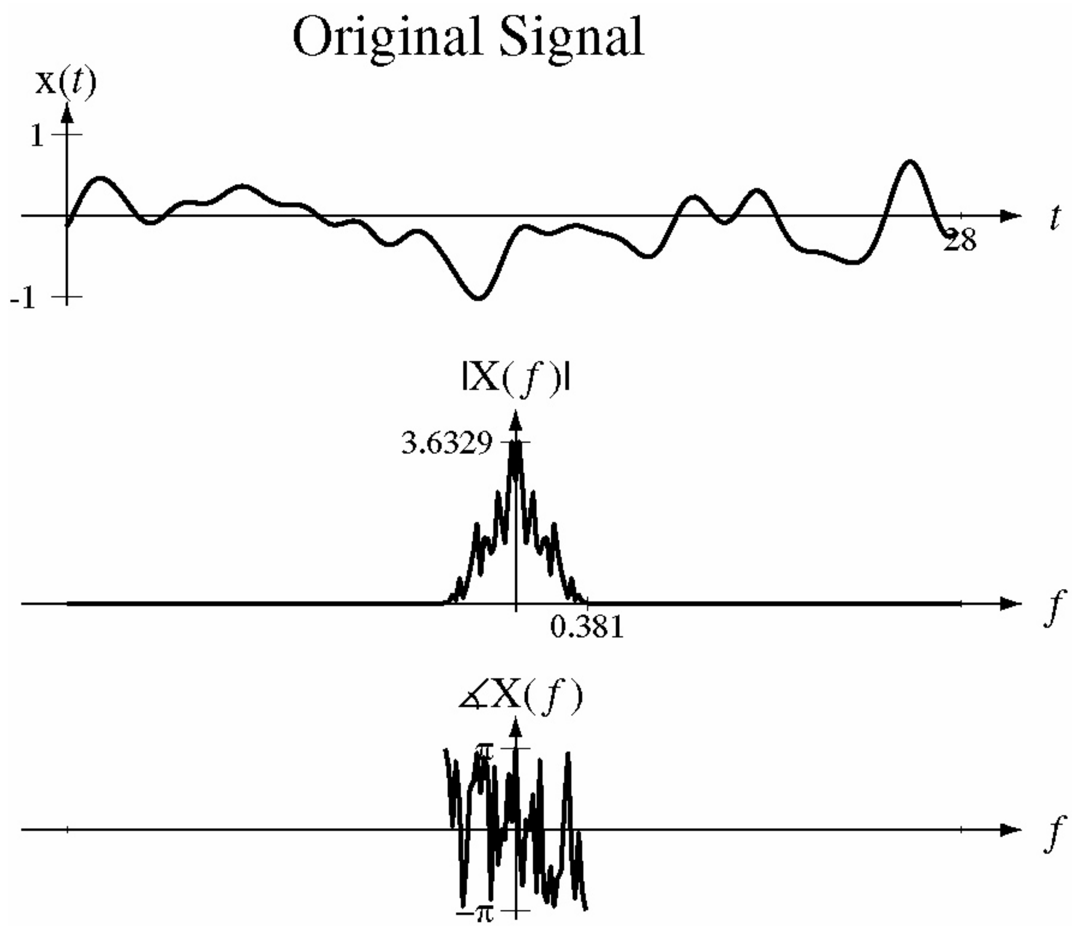
- If a **bandlimited** periodic signal is sampled above the Nyquist rate and at a rate which is an integer multiple of its fundamental frequency over exactly one fundamental period, that set of numbers is sufficient to completely describe it
- If the sampling continued, these same samples would be repeated in every fundamental period
- So the number of numbers needed to completely describe the signal is finite in both the time and frequency domains

Bandlimited Periodic Signals



CTFT-DFT Relationship

The relation between the CTFT of a continuous-time signal and the DFT of samples taken from it will be illustrated in the next few slides. Let an original continuous-time signal $x(t)$ be sampled N times at a rate f_s.



CTFT-DTFT Relationship

Let $\mathrm{x}(t)$ be a continuous-time signal and let

$$\mathrm{x}_\delta(t) = \mathrm{x}(t)\delta_{T_s}(t) = \sum_{n=-\infty}^{\infty} \mathrm{x}(nT_s)\delta(t - nT_s).$$ Also let $\mathrm{x}_s[n] = \mathrm{x}(nT_s)$.

Then $$\mathrm{X}_\delta(f) = \mathrm{X}(f) * f_s\delta_{f_s}(f) = \sum_{n=-\infty}^{\infty} \mathrm{x}(nT_s)e^{-j2\pi fnT_s}$$

and $$\mathrm{X}_\delta(f_sF) = f_s\sum_{k=-\infty}^{\infty} \mathrm{X}(f_s(F-k)) = \underbrace{\sum_{n=-\infty}^{\infty} \mathrm{x}_s[n]e^{-j2\pi nF}}_{=\mathrm{X}_s(F)}$$

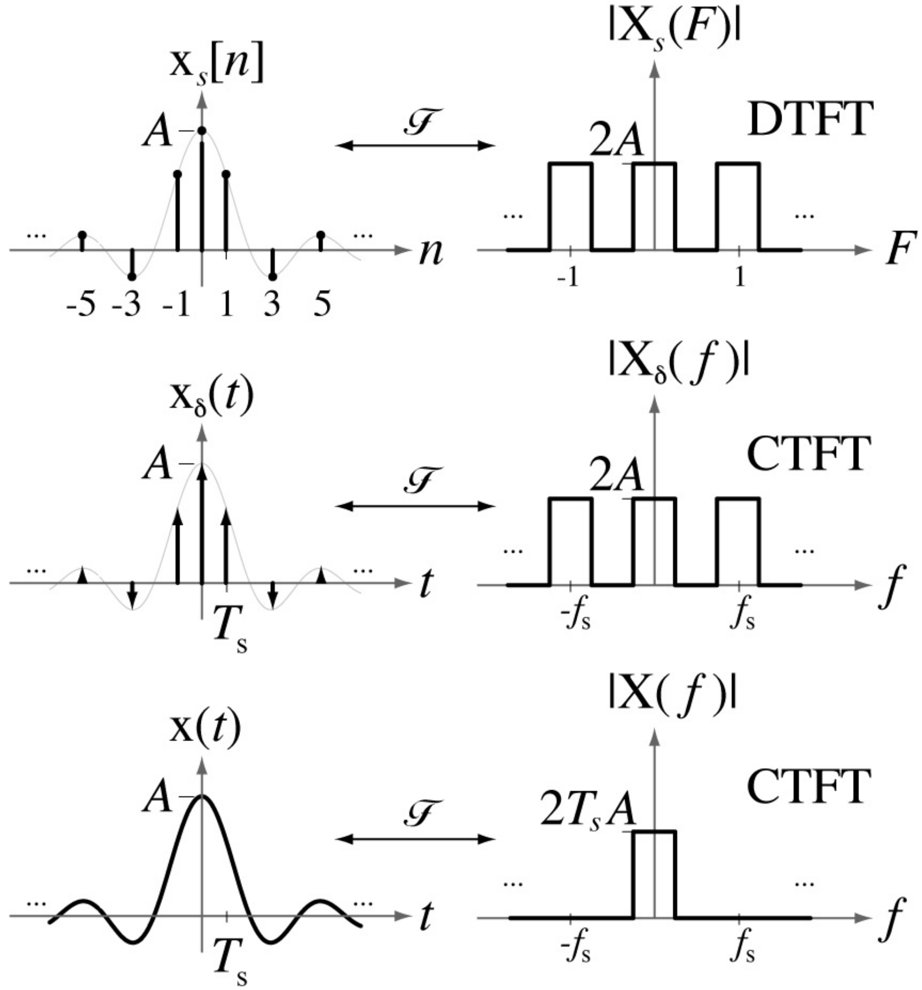
Summarizing, if $\mathrm{x}_\delta(t) = \mathrm{x}(t)\delta_{T_s}(t)$ and $\mathrm{x}_s[n] = \mathrm{x}(nT_s)$ then

$$\mathrm{X}_s(F) = \mathrm{X}_\delta(f_sF),\ \mathrm{X}_\delta(f) = \mathrm{X}_s(f/f_s) \text{ and } \mathrm{X}_s(F) = f_s\sum_{k=-\infty}^{\infty} \mathrm{X}(f_s(F-k))$$

$$\mathrm{X}_s(e^{j\Omega}) = \mathrm{X}_\delta(f_s\Omega/2\pi),\ \mathrm{X}_\delta(f) = \mathrm{X}_s(f/f_s) \text{ and } \mathrm{X}_s(e^{j\Omega}) = f_s\sum_{k=-\infty}^{\infty} \mathrm{X}(f_s(\Omega/2\pi - k))$$

CTFT-DTFT Relationship

Sampling in time corresponds to periodic repetition in frequency.



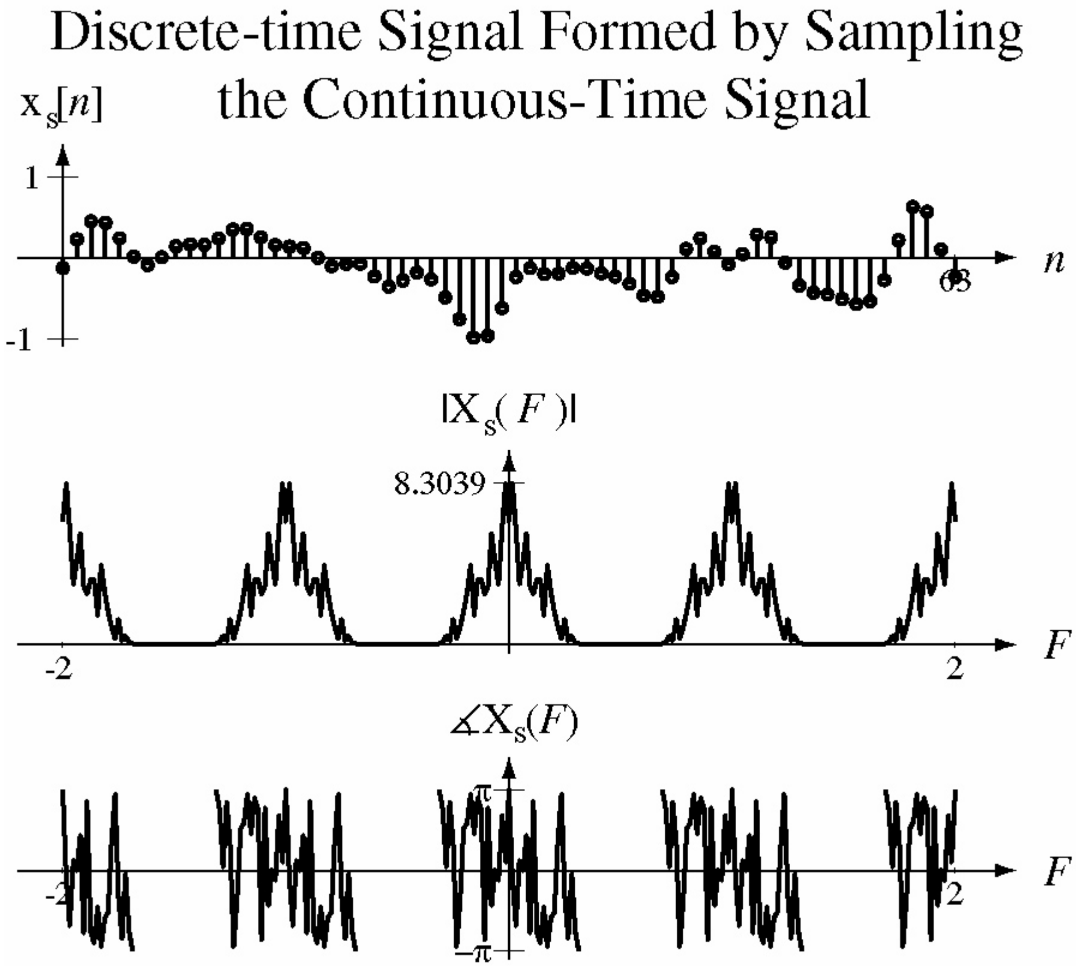
CTFT-DFT Relationship

The sampled signal is

$$x_s[n] = x(nT_s)$$

and its DTFT is

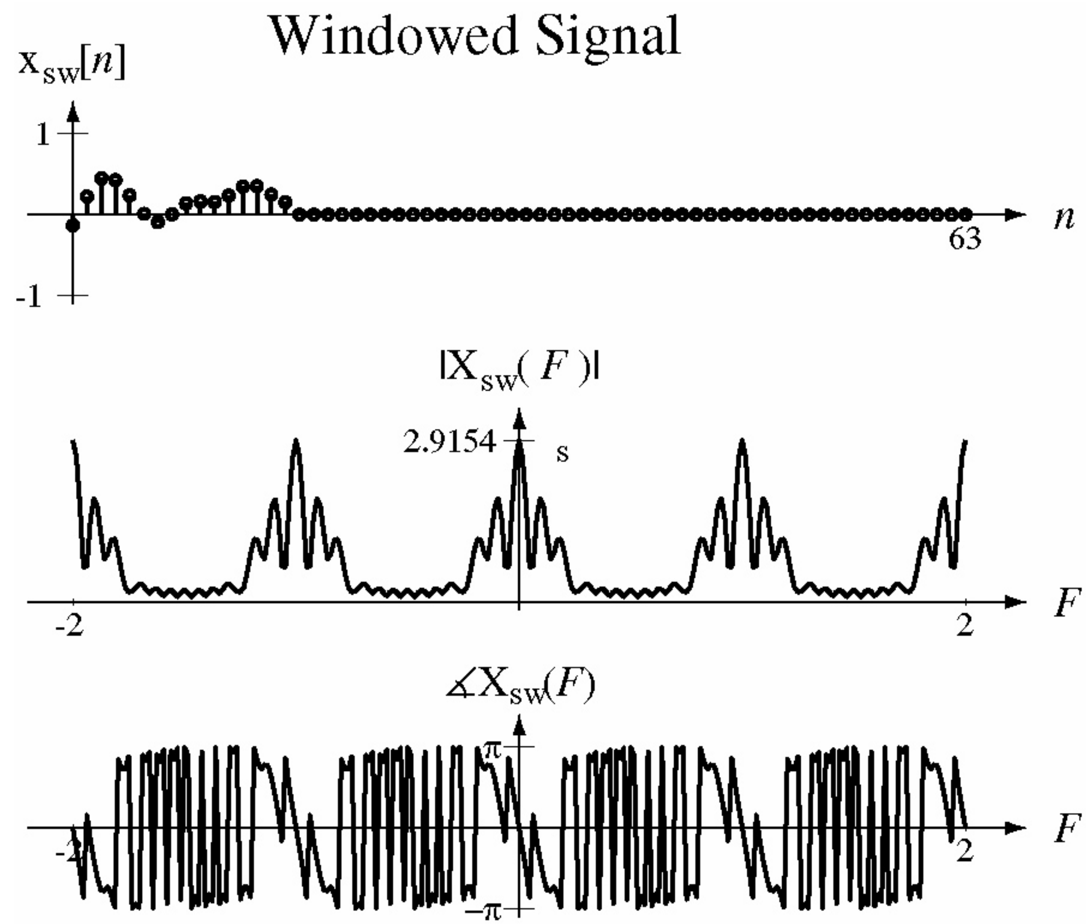
$$X_s(F) = f_s \sum_{n=-\infty}^{\infty} X\big(f_s(F-n)\big)$$



CTFT-DFT Relationship

Only N samples are taken. If the first sample is taken at time $t = 0$ (the usual assumption) that is equivalent to multiplying the sampled signal by the **window** function

$$\mathrm{w}[n] = \begin{cases} 1 & , \ 0 \le n < N \\ 0 & , \ \text{otherwise} \end{cases}$$



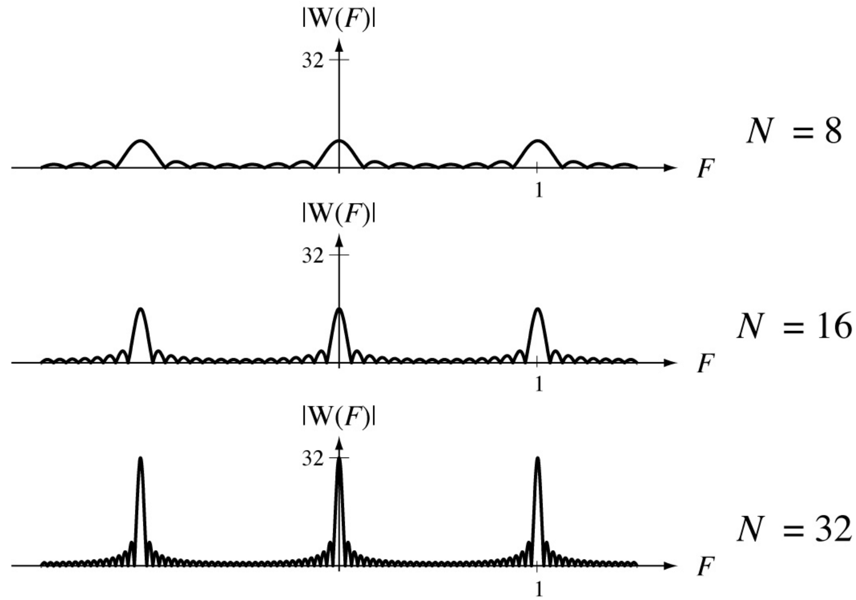
CTFT-DFT Relationship

The DTFT of $\mathrm{x}_{sw}[n]$ is the periodic convolution of $\mathrm{X}_s(F)$ with $\mathrm{W}(F)$.

$$\mathrm{X}_{sw}(F) = \mathrm{W}(F) \circledast \mathrm{X}_s(F) \quad , \quad \mathrm{W}(F) = e^{-j\pi F(N-1)} N \,\mathrm{drcl}(F,N)$$

$$\mathrm{X}_{sw}(F) = f_s \left[e^{-j\pi F(N-1)} N \,\mathrm{drcl}(F,N) \right] * \mathrm{X}(f_s F)$$

DTFT of the Window, w[n]



Sampling in Frequency

Let $\mathrm{x}[n]$ be an aperiodic function with DTFT $\mathrm{X}(F)$ and let $\mathrm{x}_p[n]$ be a periodic extension of $\mathrm{x}[n]$ with period N_p such that $\mathrm{x}_p[n] = \sum_{m=-\infty}^{\infty} \mathrm{x}[n - mN_p] = \mathrm{x}[n] * \delta_{N_p}[n]$. Then

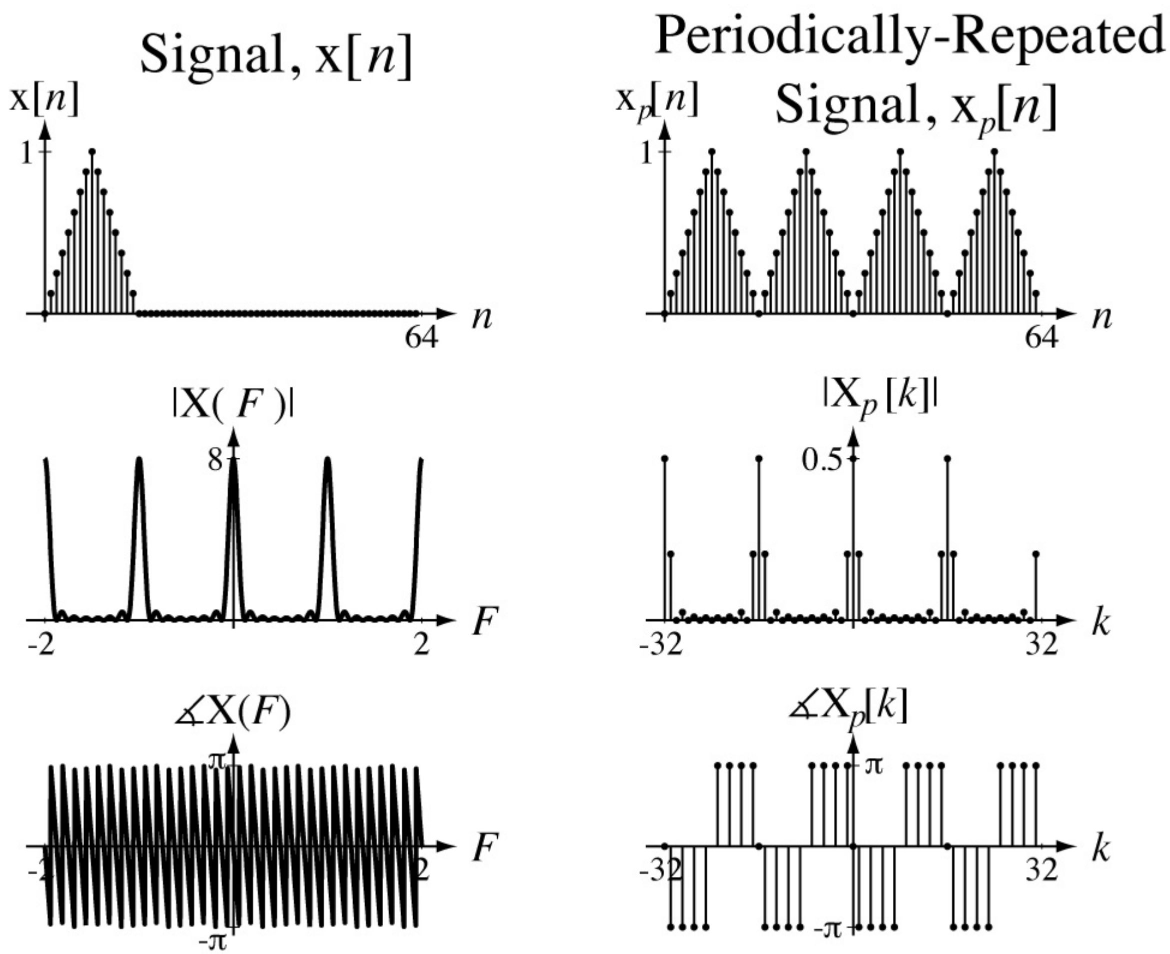
$$\mathrm{X}_p(F) = \mathrm{X}(F)(1/N_p)\delta_{1/N_p}(F) = (1/N_p)\sum_{k=-\infty}^{\infty} \mathrm{X}(k/N_p)\delta(F - k/N_p)$$

and $\mathrm{X}_p[k] = \mathrm{X}(k/N_p)$. Now let $\mathrm{x}_{swp}[n] = \sum_{m=-\infty}^{\infty} \mathrm{x}_{sw}[n - mN]$ with period N. Then $\mathrm{X}_{swp}[k] = \mathrm{X}_{sw}(k/N)$, k an integer and

$$\mathrm{X}_{swp}[k] = f_s\left[e^{-j\pi F(N-1)} N \,\mathrm{drcl}(F,N) * \mathrm{X}(f_s F)\right]_{F \to k/N}.$$

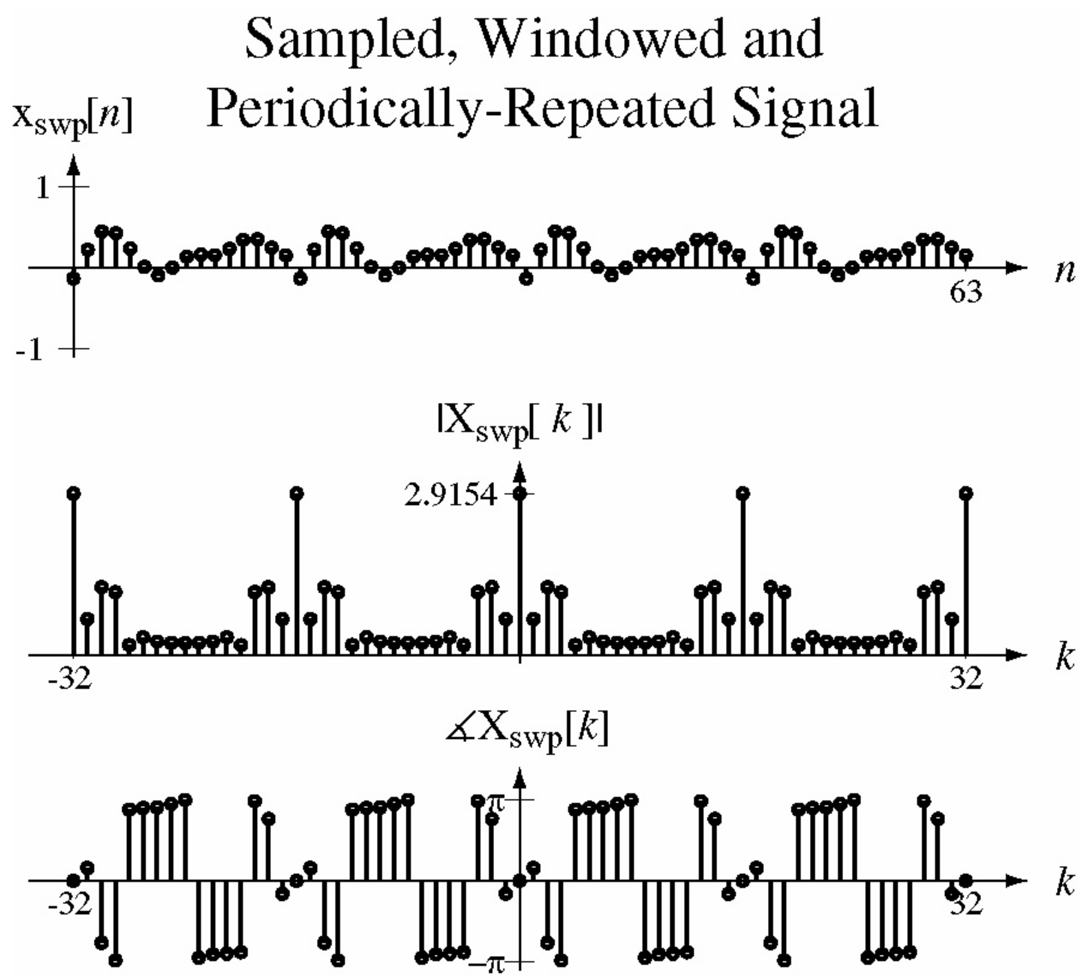
Sampling in Frequency

Sampling in frequency corresponds to periodic repetition in time.



CTFT-DFT Relationship

The last step in the process is to periodically repeat the time-domain signal. The corresponding effect in the frequency domain is sampling. Then there are two periodic impulse signals which are related to each other through the DFT.



CTFT-DFT Relationship

The original signal and the final signal are related by

$$\mathrm{X}_{swp}[k] = f_s \Big[\underbrace{e^{-j\pi F(N-1)} N \,\mathrm{drcl}(F,N)}_{\mathrm{W}(F)} * \mathrm{X}(f_s F) \Big]_{F \to k/N}$$

In words, the CTFT of the original signal is transformed by replacing f with $f_s F$. That result is convolved with the DTFT of the window function. Then that result is transformed by replacing F by k / N. Then that result is multiplied by f_s.

CTFT-DFT Relationship

In moving from the CTFT of a continuous-time signal to the DFT of samples of the continuous-time signal taken over a finite time, we do the following.

In the time domain

1. Sample the continuous time signal,
2. Window the samples by multiplying them by a window function,

and 3. Periodically repeat the non-zero samples from step 2.

In the frequency domain

1. Find the DTFT of the sampled signal which is a scaled-and-periodically-repeated version of the CTFT of the original signal.
2. Periodically convolve the DTFT of the sampled signal with the DTFT of the window function,

and 3. Sample in frequency the result of step 2.

Approximating the CTFT with the DFT

If $\mathrm{x}(t)$ is a causal energy signal then its CTFT can be approximated at discrete frequencies kf_s / N, k an integer, by

$$\mathrm{X}(kf_s / N) \cong T_s \sum_{n=0}^{N-1} \mathrm{x}(nT_s) e^{-j2\pi kn/N} \cong T_s \times \mathcal{DFT}\left(\mathrm{x}(nT_s)\right) \quad , \quad |k| << N$$

where N is an integer and NT_s covers all or most of the energy of $\mathrm{x}(t)$.

Approximating the Inverse CTFT with the DFT

If $X(kf_s / N)$ is known in the range $-N << -k_{\max} \le k \le k_{\max} << N$ and if the magnitude of $X(kf_s / N)$ is negligible outside that range then the inverse CTFT of X can be approximated by

$$x(nT_s) \cong (1/T_s) \times \mathcal{DFT}^{-1}\left(X_{ext}(kf_s / N)\right)$$

where

$$X_{ext}(kf_s / N) = \begin{cases} X(kf_s / N) & , \; -k_{\max} \le k \le k_{\max} \\ 0 & , \; k_{\max} < |k| \le N/2 \end{cases}$$

and

$$X_{ext}(kf_s / N) = X_{ext}\left((k + mN) f_s / N\right)$$

Approximating the DTFT with the DFT

If $\mathrm{x}[n]$ is a causal energy signal its DTFT at discrete cyclic frequency values k/N can be computed by

$$\mathrm{X}(F)_{F \to k/N} = \mathrm{X}(k/N) \cong \mathcal{DFT}\left(\mathrm{x}[n]\right)$$

or at discrete radian frequencies by

$$\mathrm{X}\left(e^{j\Omega}\right)_{\Omega \to 2\pi k/N} = \mathrm{X}\left(e^{j2\pi k/N}\right) \cong \mathcal{DFT}\left(\mathrm{x}[n]\right).$$

If $\mathrm{x}[n]$ is also time limited to a discrete time $n_{\max} < N$, the computed DTFT is exact at those frequency values.

Approximating Continuous-Time Aperiodic Convolution with the DFT

If $\mathrm{x}(t)$ and $\mathrm{h}(t)$ are both aperiodic energy signals and $\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t)$ their aperiodic convolution at times nT_s can be approximated by

$$\mathrm{y}(nT_s) \cong T_s \times \mathcal{DFT}^{-1}\left(\mathcal{DFT}\left(\mathrm{x}(nT_s)\right) \times \mathcal{DFT}\left(\mathrm{h}(nT_s)\right)\right)$$

for $|n| << N$.

Approximating Continuous-Time Periodic Convolution with the DFT

If $\mathrm{x}(t)$ and $\mathrm{h}(t)$ are both periodic signals with common period T sampled N times at a rate which is an integer multiple of their fundamental periods and above the Nyquist rate and $\mathrm{y}(t) = \mathrm{x}(t) \circledast \mathrm{h}(t)$ their periodic convolution at times nT_s can be approximated by

$$\mathrm{y}(nT_s) \cong T_s \times \mathcal{DFT}^{-1}\left(\mathcal{DFT}\left(\mathrm{x}(nT_s)\right) \times \mathcal{DFT}\left(\mathrm{h}(nT_s)\right)\right).$$

Approximating Discrete-Time Aperiodic Convolution with the DFT

If $\mathrm{x}[n]$ and $\mathrm{h}[n]$ are both energy signals and most or all of their energy occurs in the time range $0 \le n < N$ and $\mathrm{y}[n] = \mathrm{x}[n] * \mathrm{h}[n]$ then

$$\mathrm{y}[n] \cong \mathcal{DFT}^{-1}\left(\mathcal{DFT}\left(\mathrm{x}[n]\right) \times \mathcal{DFT}\left(\mathrm{h}[n]\right)\right)$$

for $|n| << N$.

Discrete-Time Periodic Convolution with the DFT

If $\mathrm{x}[n]$ and $\mathrm{h}[n]$ are both periodic signals with common period N and $\mathrm{y}[n] = \mathrm{x}[n] \circledast \mathrm{h}[n]$ their periodic convolution at times n can be computed by

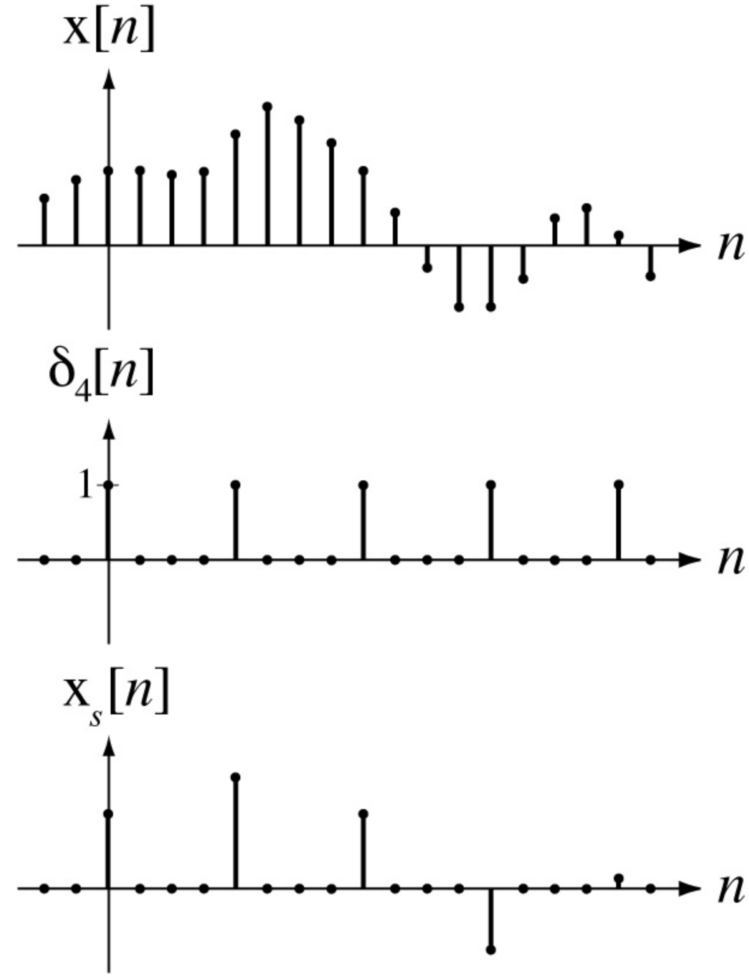
$$\mathrm{y}[n] = \mathcal{DFT}^{-1}\left(\mathcal{DFT}\left(\mathrm{x}[n]\right) \times \mathcal{DFT}\left(\mathrm{h}[n]\right)\right)$$

using N points in the DFT, and the computation is exact.

Discrete-Time Sampling

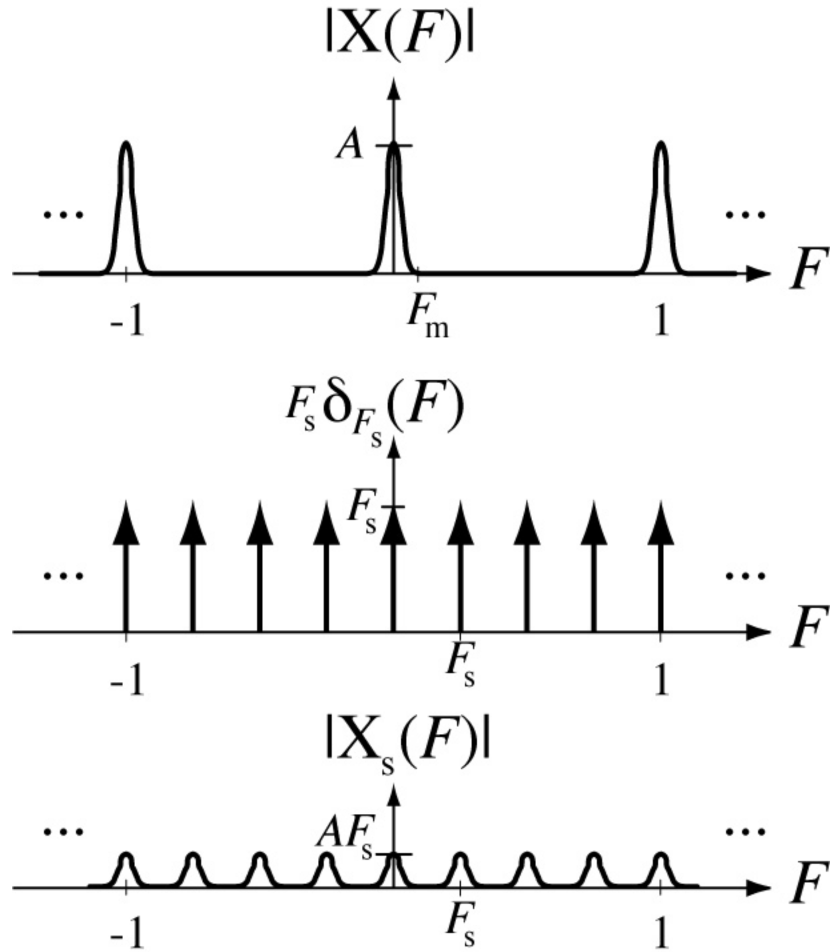
A discrete-time signal $\mathrm{x}[n]$ is sampled by multiplying it by a discrete-time periodic impulse to form $\mathrm{x}_s[n]$. The time between samples is the period of the periodic impulse N_s.

$$\mathrm{x}_s[n] = \mathrm{x}[n]\delta_{N_s}[n]$$



Discrete-Time Sampling

Aliases appear in the DTFT of the sampled signal and, if they do not overlap, the original signal can be recovered from the samples. The minimum sampling rate for recovering the signal is $2F_m$, twice the highest discrete-time cyclic frequency in the signal.



Discrete-Time Sampling

The original signal can be recovered from the samples by interpolation using a lowpass digital filter.

$$\mathrm{X}(F) = \underbrace{\mathrm{X}_s(F)}_{\substack{\text{DTFT of}\\ \text{Sampled}\\ \text{Signal}}} \times \underbrace{(1/F_s)\,\mathrm{rect}(F/2F_c) * \delta_1(F)}_{\text{Lowpass Digital Filter}}$$

A discrete-time sinc function is the ideal interpolating function.

$$\mathrm{x}[n] = \mathrm{x}_s[n] * (2F_c/F_s)\,\mathrm{sinc}(2F_c n)$$

Discrete-Time Sampling

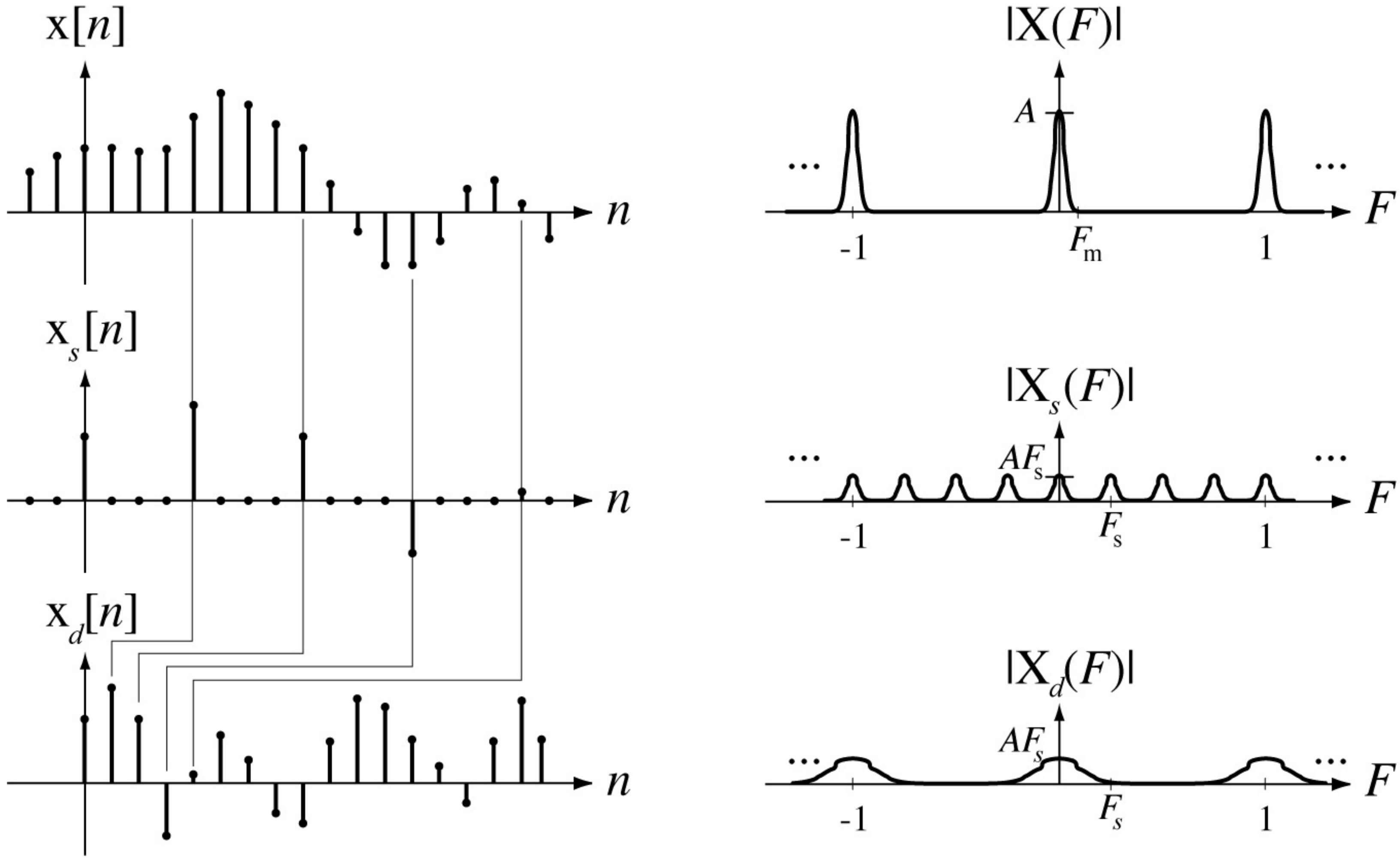
When a discrete-time signal is sampled, all the values of the signal not at the sample times are set to zero. For efficient transmission of the sampled signal these zero values are omitted and only the sample values are transmitted. This is decimation or **downsampling**. The decimated signal is $x_d[n] = x_s[N_s n] = x[N_s n]$. The DTFT of the decimated signal is $X_d(F) = \sum_{n=-\infty}^{\infty} x_d[n] e^{-j2\pi Fn} = \sum_{n=-\infty}^{\infty} x_s[N_s n] e^{-j2\pi Fn}$.

Let $m = N_s n$. Then

$$X_d(F) = \sum_{\substack{m=-\infty \\ m=\text{integer} \\ \text{multiple of } N_s}}^{\infty} x_s[m] e^{-j2\pi Fm/N_s} = X_s(F/N_s)$$

Decimation in time corresponds to expansion in frequency by a factor of N_s.

Discrete-Time Sampling



Discrete-Time Sampling

The opposite of decimation is interpolation or **upsampling** which is used to restore the original signal from the sampled-and-decimated signal. Let the decimated signal be $\mathrm{x}[n]$. Then the upsampled signal is

$$\mathrm{x}_s[n] = \begin{cases} \mathrm{x}[n/N_s] & , \; n/N_s \text{ an integer} \\ 0 & , \; \text{otherwise} \end{cases}$$

The zeros that were removed in decimation are restored. The corresponding effect in the frequency domain of this expansion in the time domain is compression by a factor of N_s, $\mathrm{X}_s(F) = \mathrm{X}(N_s F)$.

Discrete-Time Sampling

The next step is to lowpass filter the time-expanded signal $\mathrm{x}_s[n]$ to form $\mathrm{x}_i[n]$.

$$\mathrm{X}_i(F) = \underbrace{\mathrm{X}_s(F)}_{\substack{\text{DTFT of} \\ \text{Time-} \\ \text{Expanded} \\ \text{Signal}}} \times \underbrace{\mathrm{rect}(N_s F) * \delta_1(F)}_{\text{Lowpass Filter}}$$

In the time domain

$$\mathrm{x}_i[n] = \mathrm{x}_s[n] * (1/N_s)\,\mathrm{sinc}(n/N_s).$$

Except for a gain factor, this is the same as the original signal that was first sampled.

Discrete-Time Sampling