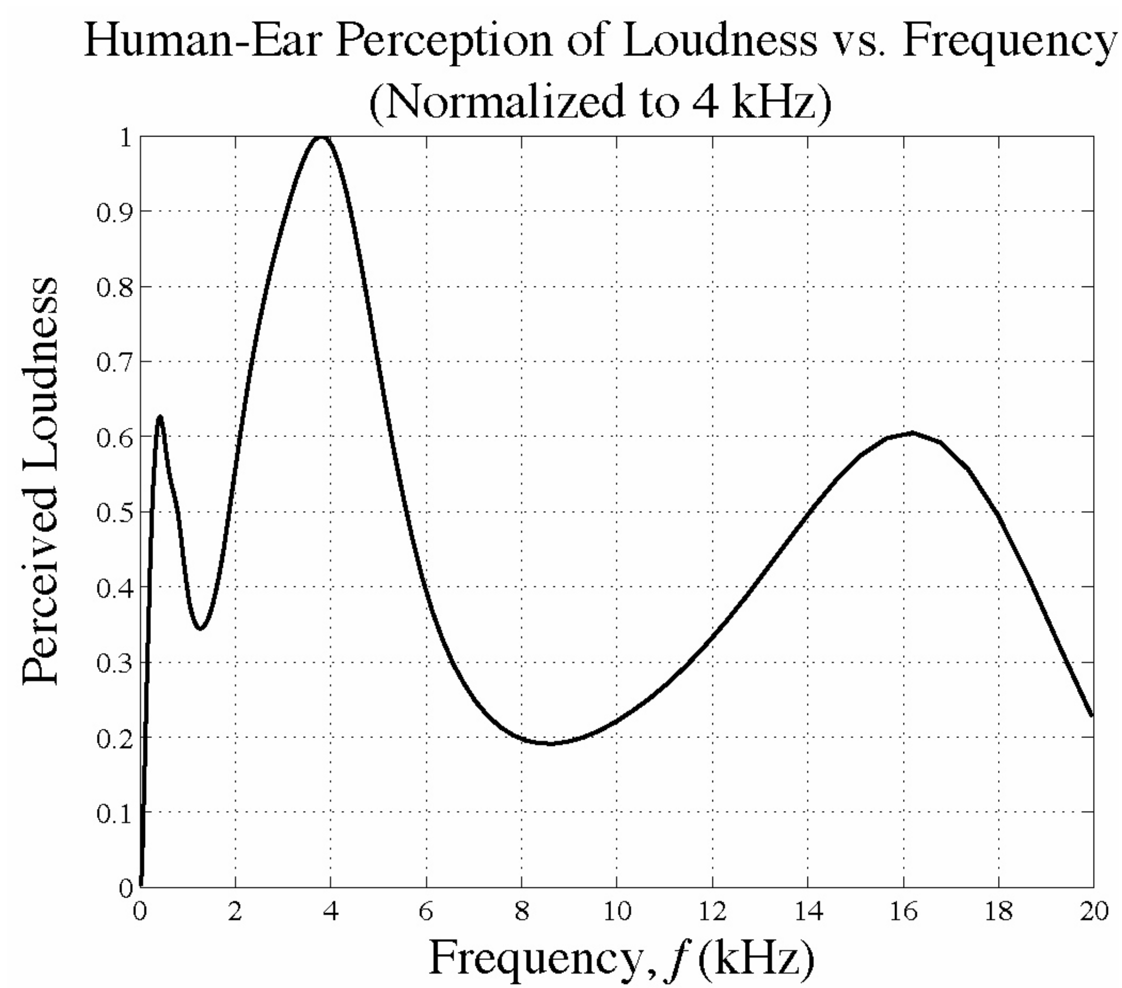
Frequency Response Analysis

Continuous Time

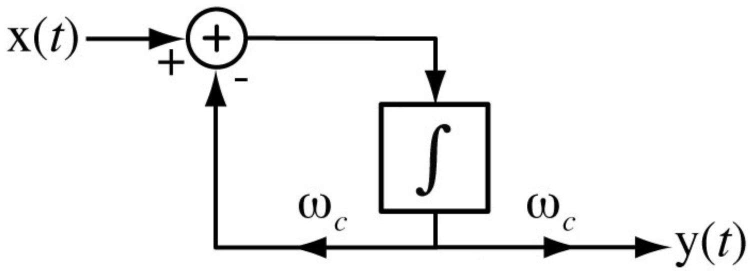
Frequency Response



Frequency Response

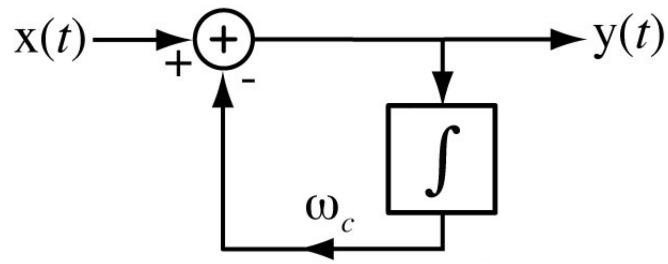
Lowpass Filter $\mathrm{H}(s) = \dfrac{\omega_c}{s + \omega_c}$

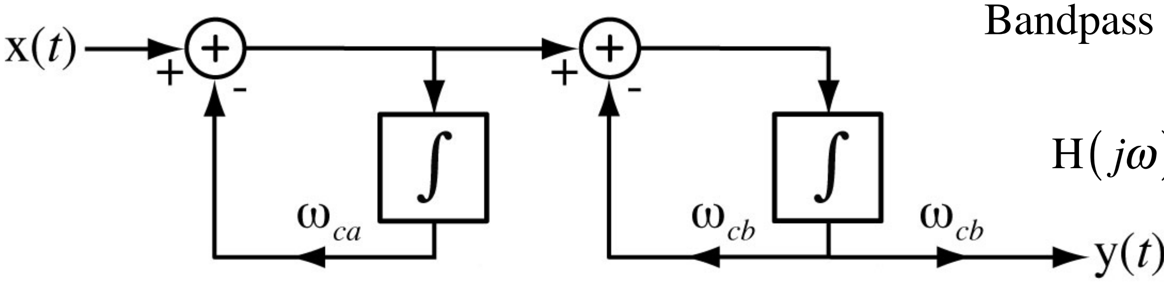
$$\mathrm{H}(j\omega) = \frac{\omega_c}{j\omega + \omega_c}$$



Highpass Filter $\mathrm{H}(s) = \dfrac{s}{s + \omega_c}$

$$\mathrm{H}(j\omega) = \frac{j\omega}{j\omega + \omega_c}$$





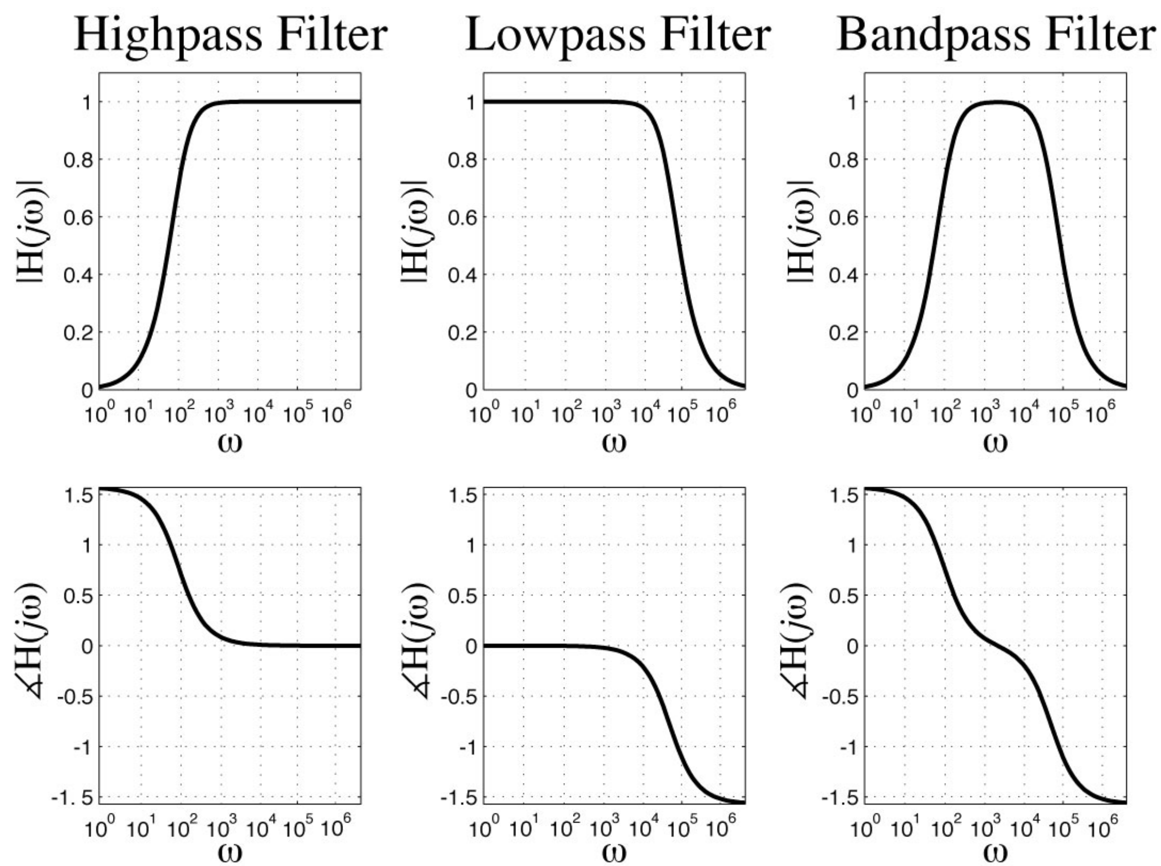
Bandpass Filter $\mathrm{H}(s) = \dfrac{\omega_{cb} s}{s + (\omega_{ca} + \omega_{cb}) s + \omega_{ca}\omega_{cb}}$

$$\mathrm{H}(j\omega) = \frac{j\omega\omega_{cb}}{(j\omega)^2 + j\omega(\omega_{ca} + \omega_{cb}) + \omega_{ca}\omega_{cb}}$$

(Cascade Connection of Lowpass and Highpass)

Frequency Response

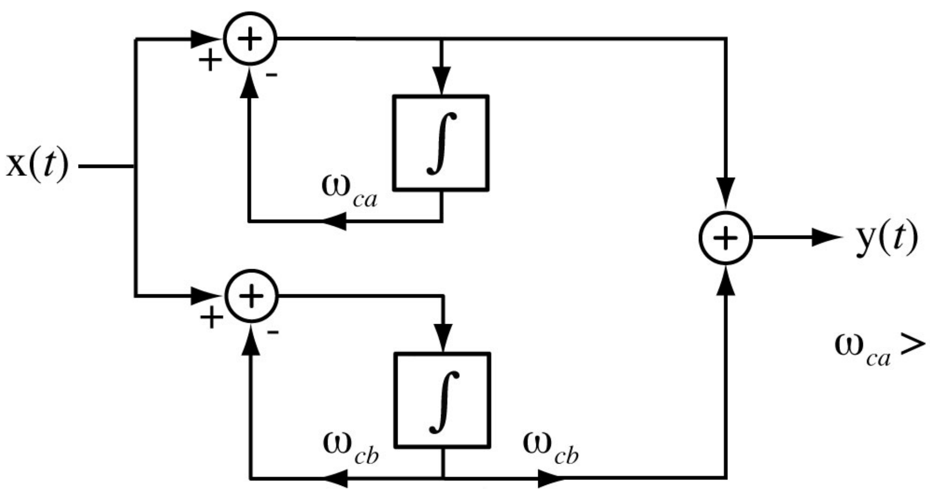
Frequency response magnitudes of the filters on the previous slide



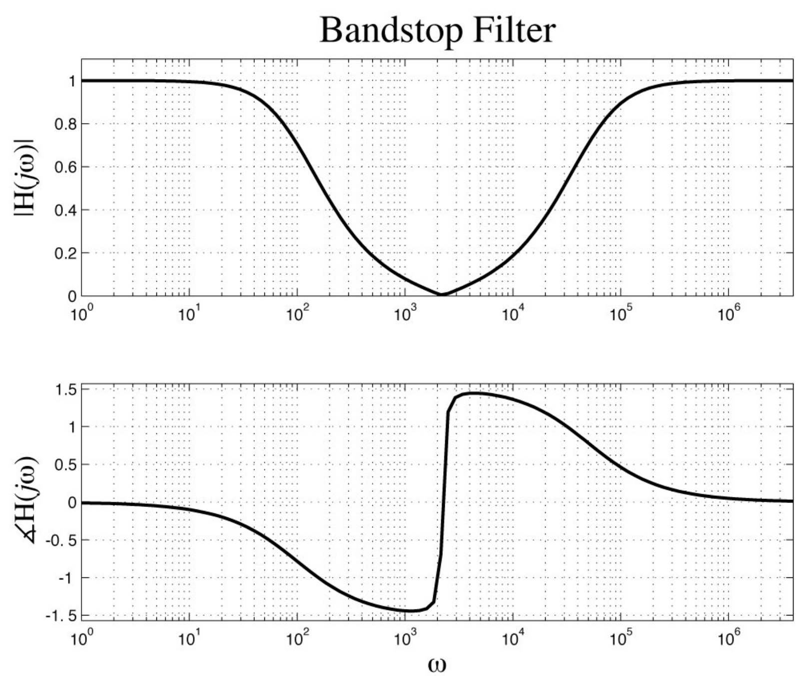
Frequency Response

Bandstop Filter $\mathrm{H}(s)=\dfrac{s^2+2\omega_{cb}s+\omega_{ca}\omega_{cb}}{s^2+(\omega_{ca}+\omega_{cb})s+\omega_{ca}\omega_{cb}}$

$$\mathrm{H}(j\omega)=\frac{(j\omega)^2+j2\omega\omega_{cb}+\omega_{ca}\omega_{cb}}{(j\omega)^2+j\omega(\omega_{ca}+\omega_{cb})+\omega_{ca}\omega_{cb}}$$

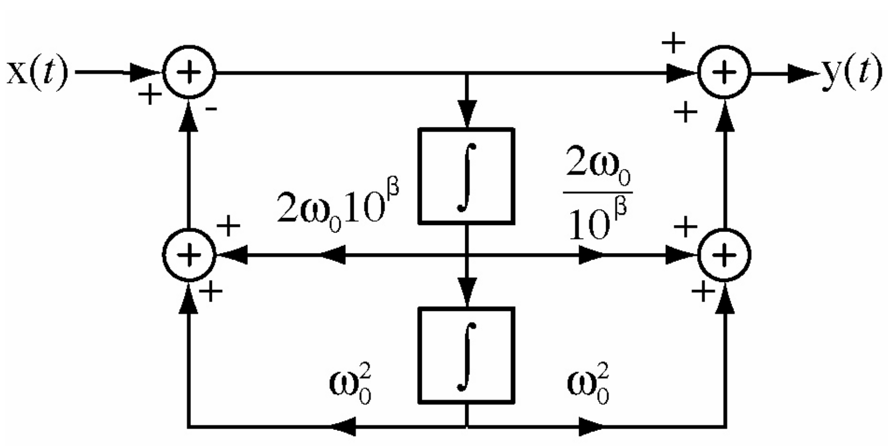


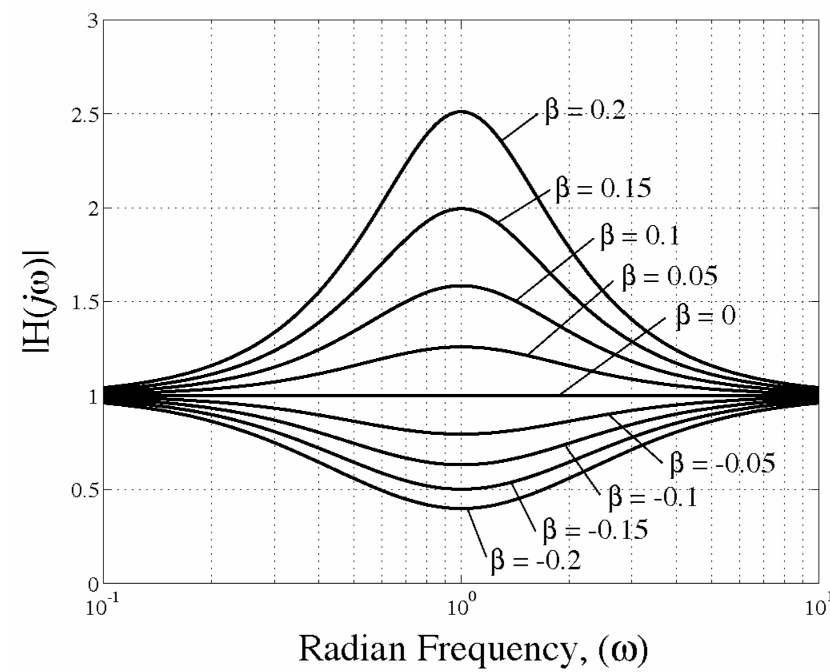
$\omega_{ca} > \omega_{cb}$



Frequency Response

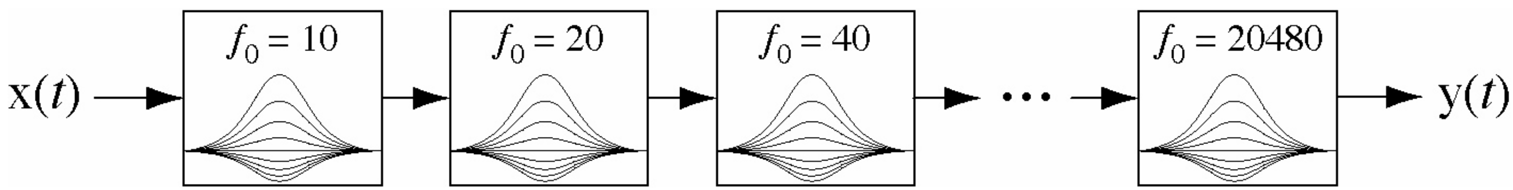
A **biquadratic** filter can be realized as a second-order system. Adjusting the parameter β changes the nature of the frequency response. It can emphasize or de-emphasize frequencies near its center frequency.

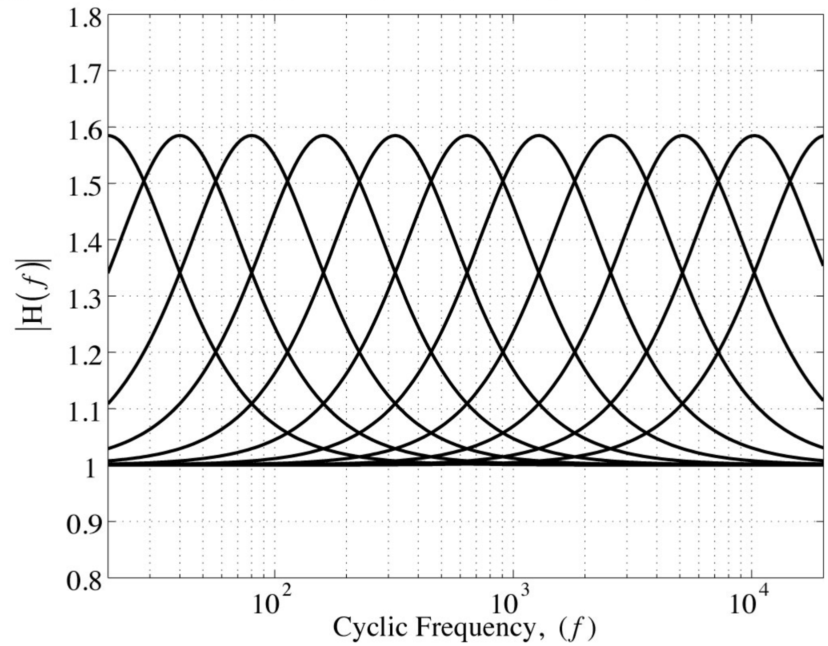




Frequency Response

A bank of cascaded biquadratic filters can be used as a graphic equalizer





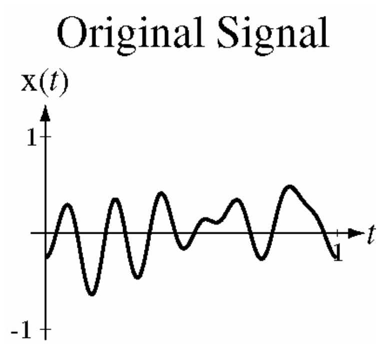
Ideal Filters

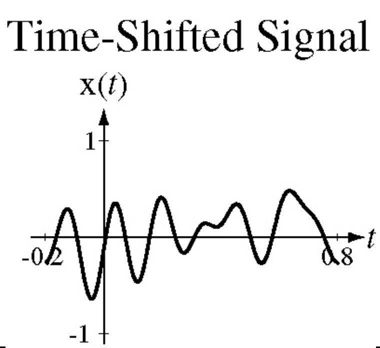
- **Filters** separate what is desired from what is not desired
- In the signals and systems context a filter separates signals in one frequency range from signals in another frequency range
- An **ideal filter** passes all signal power in its **passband** without distortion and completely blocks signal power outside its passband

Distortion

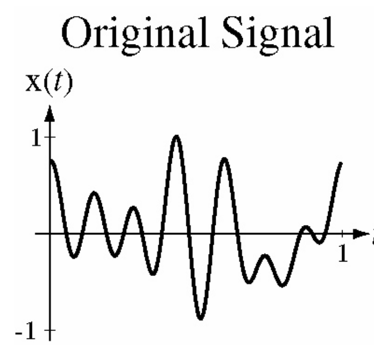
- **Distortion** is construed in signal analysis to mean “changing the shape” of a signal
- Multiplication of a signal by a constant (even a negative one) or shifting it in time do not change its shape

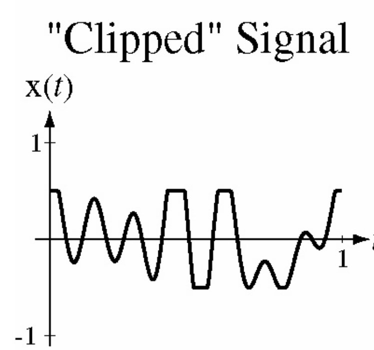
No Distortion





Distortion





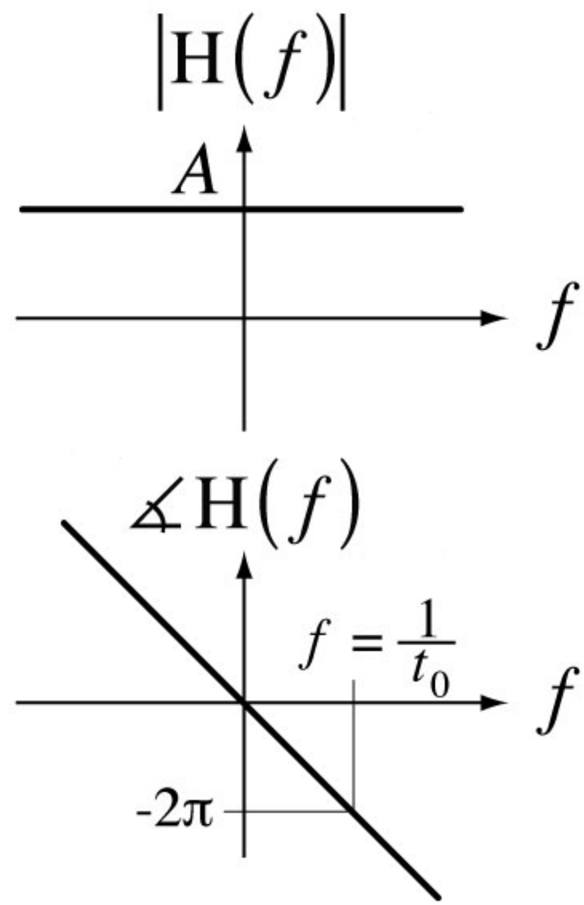
Distortion

Since a system can multiply by a constant or shift in time without distortion, a **distortionless system** would have an impulse response of the form

$$h(t) = A\delta(t - t_0)$$

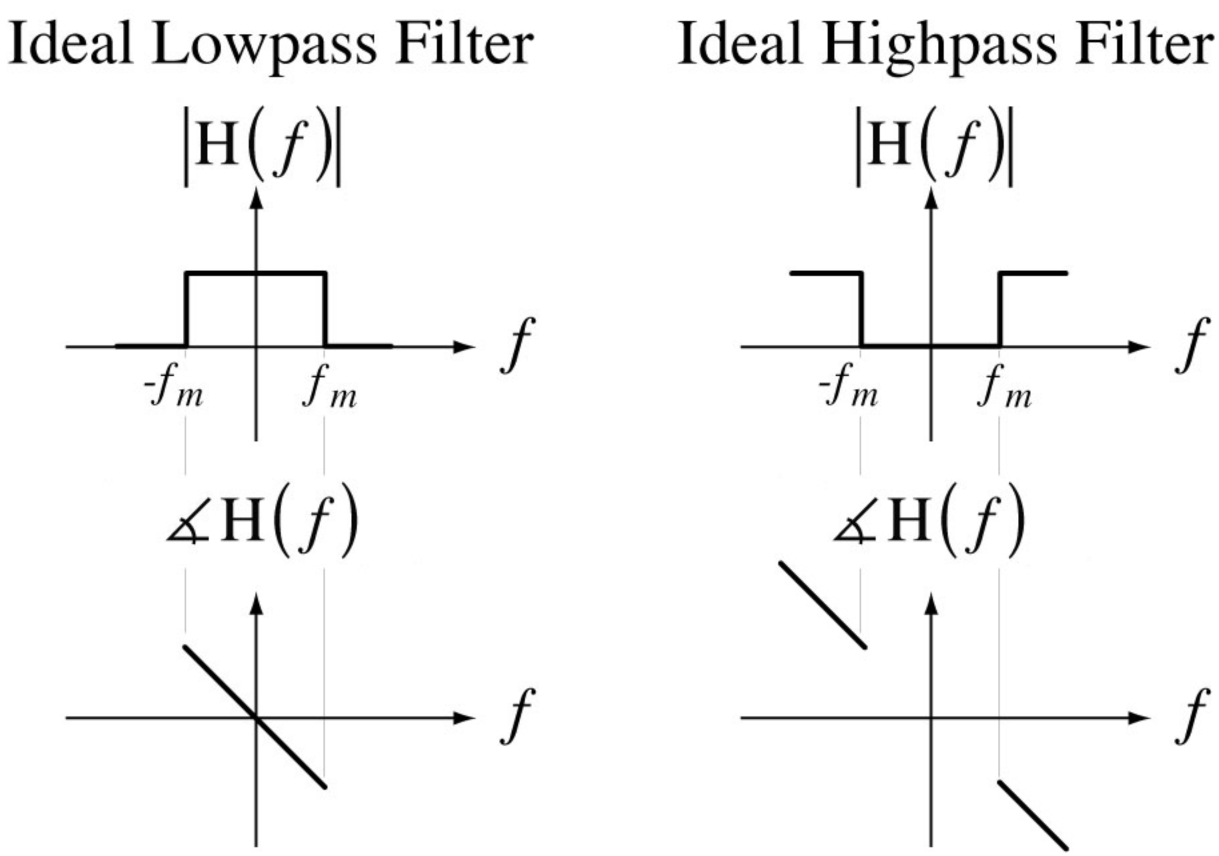
The corresponding frequency response is

$$H(f) = Ae^{-j2\pi f t_0}$$

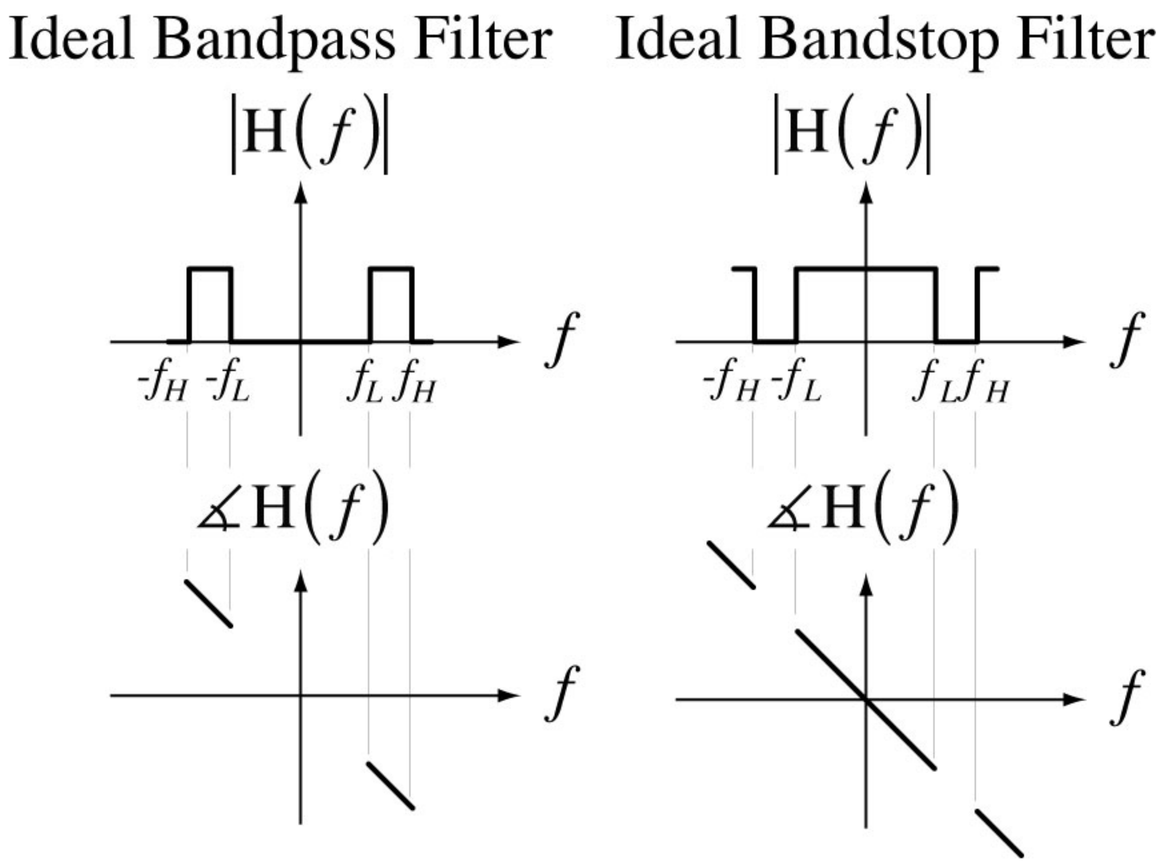


Filter Classifications

There are four commonly-used classification of filters, **lowpass**, **highpass**, **bandpass** and **bandstop**.



Filter Classifications

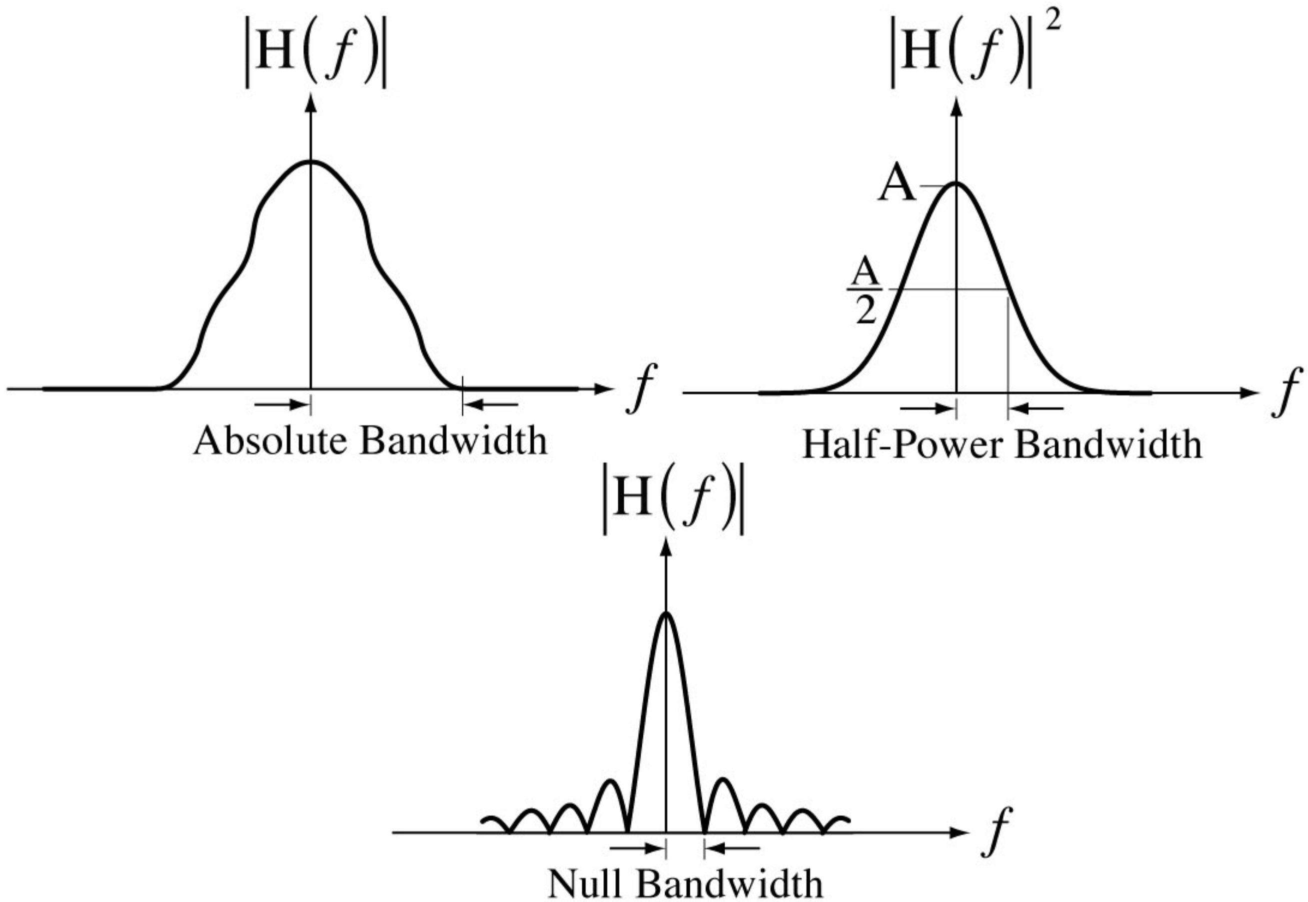


Bandwidth

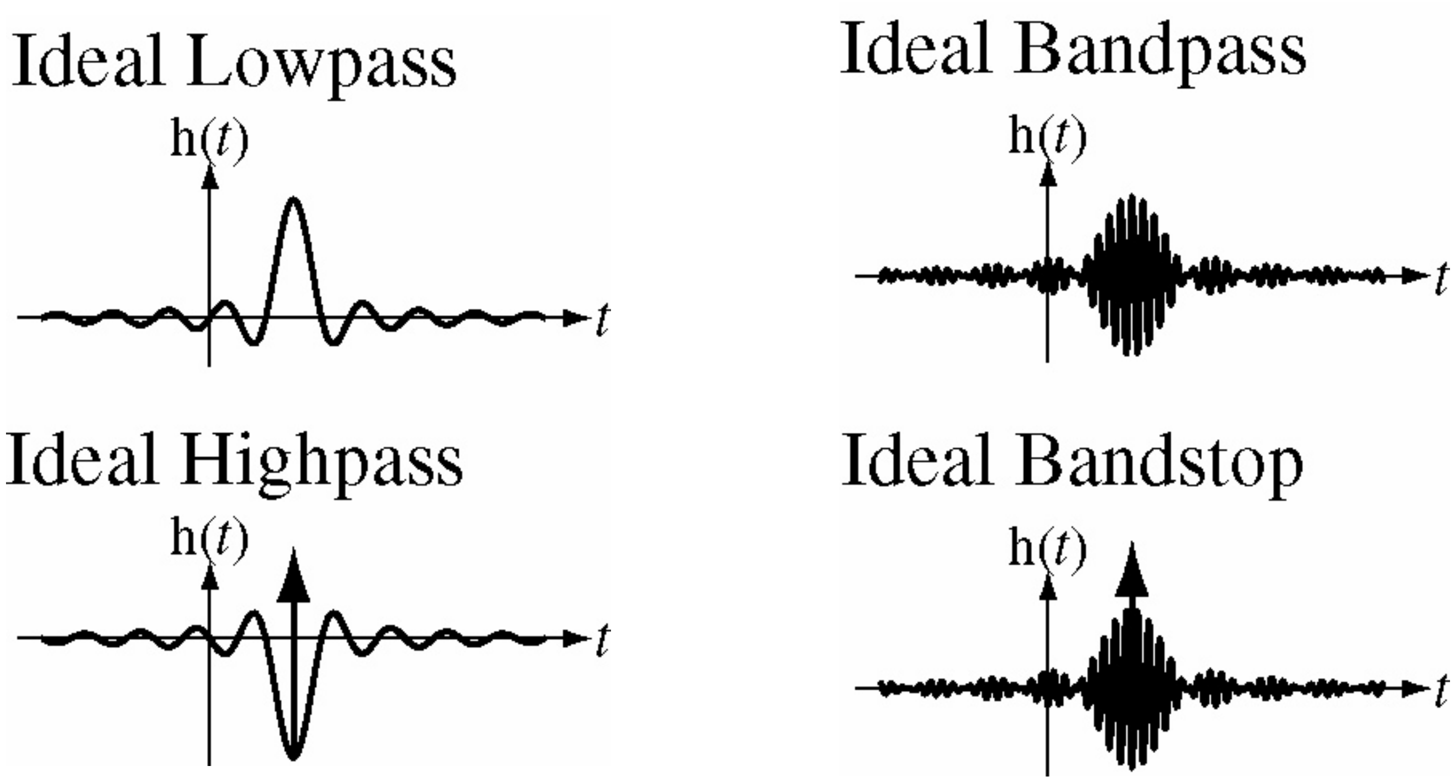
- **Bandwidth** generally means "a range of frequencies"
- This range could be the range of frequencies a filter passes or the range of frequencies present in a signal
- Bandwidth is traditionally construed to be range of frequencies in **positive** frequency space

Bandwidth

Common Bandwidth Definitions



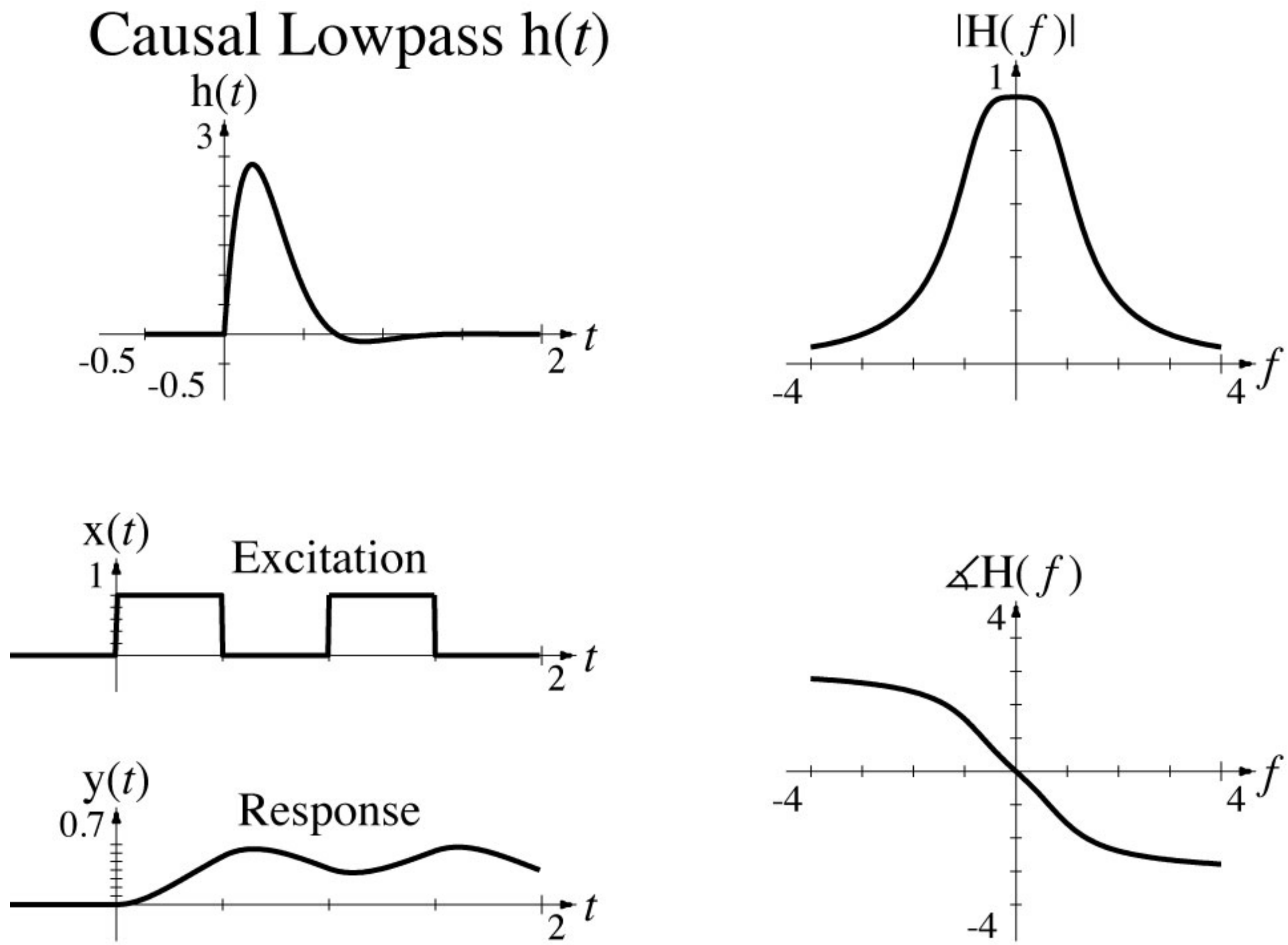
Impulse Responses of Ideal Filters



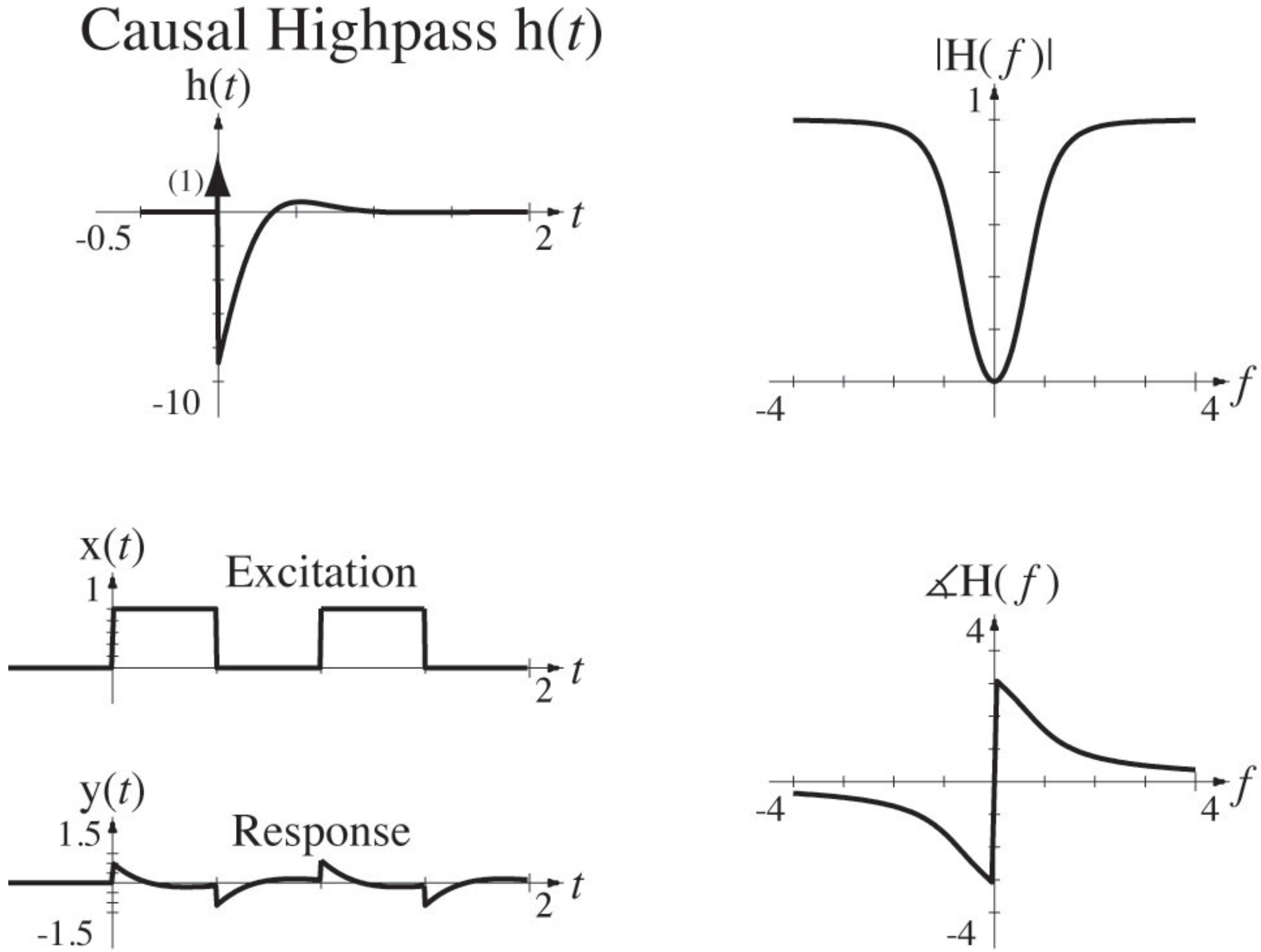
Impulse Response and Causality

- All the impulse responses of ideal filters contain sinc functions, alone or in combinations, which are infinite in extent
- Therefore all ideal-filter impulse responses begin before time $t = 0$
- This makes ideal filters **non-causal**
- Ideal filters cannot be physically realized, but they can be closely approximated

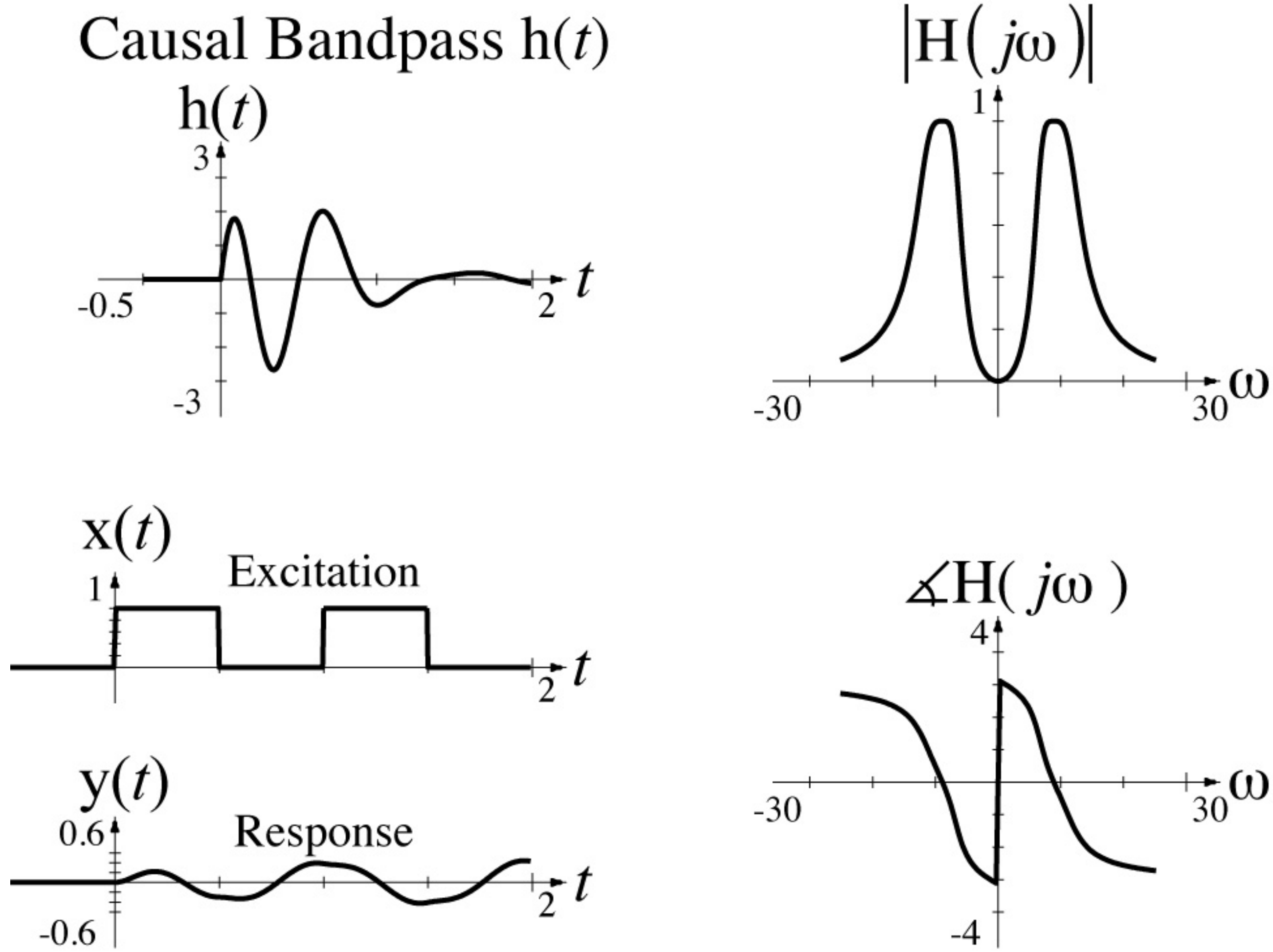
Impulse and Frequency Responses of Causal Filters



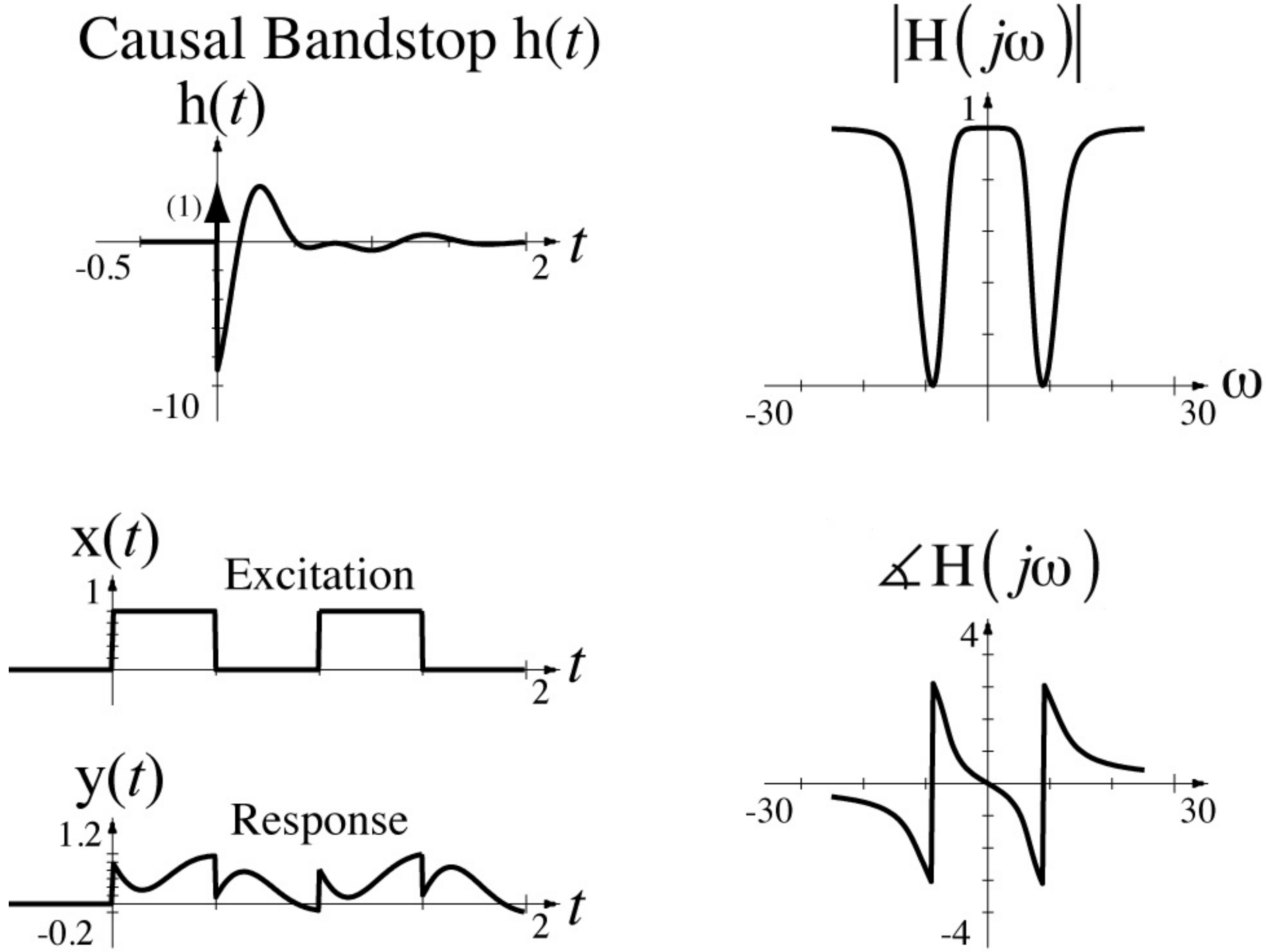
Impulse and Frequency Responses of Causal Filters



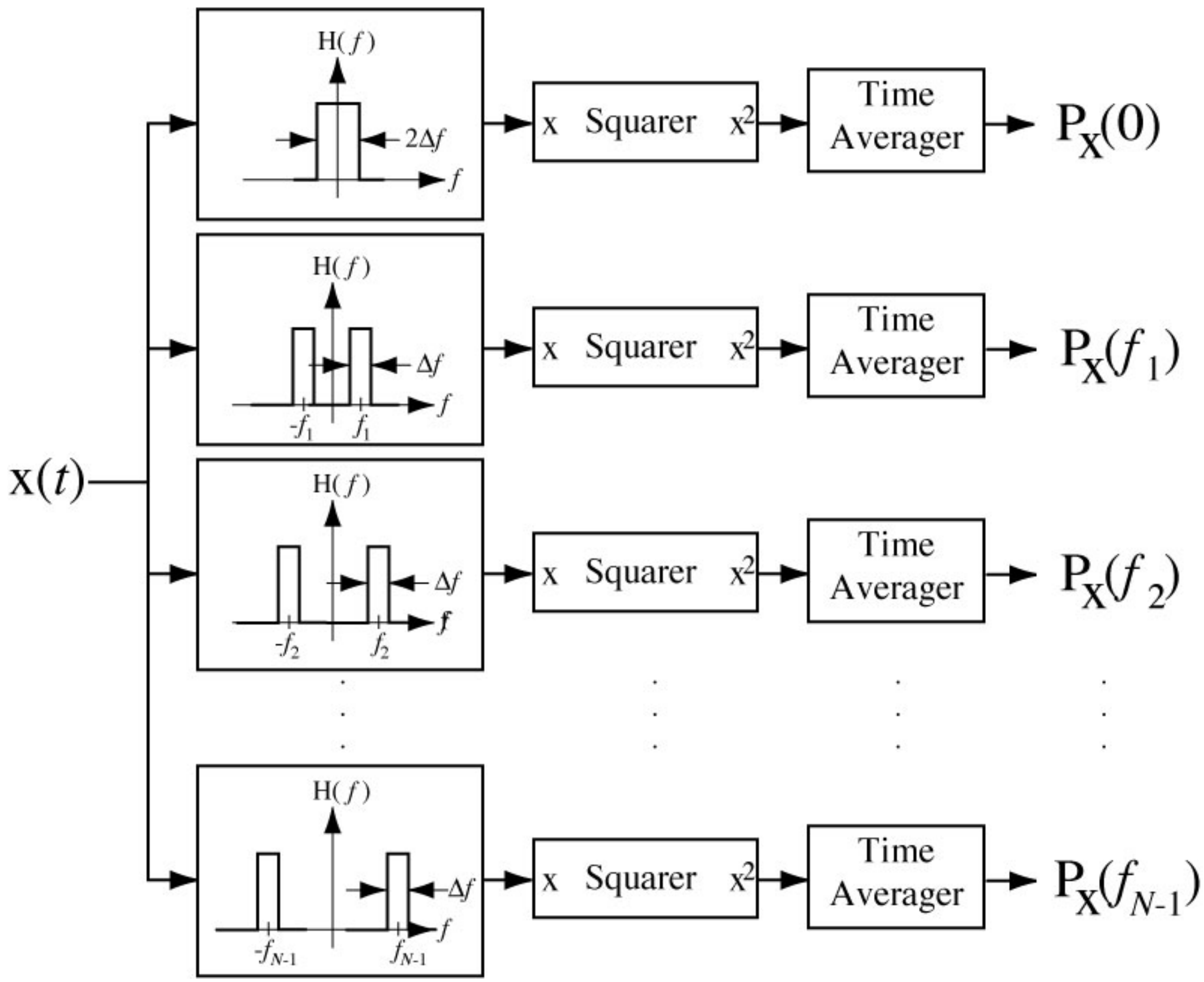
Impulse and Frequency Responses of Causal Filters



Impulse and Frequency Responses of Causal Filters

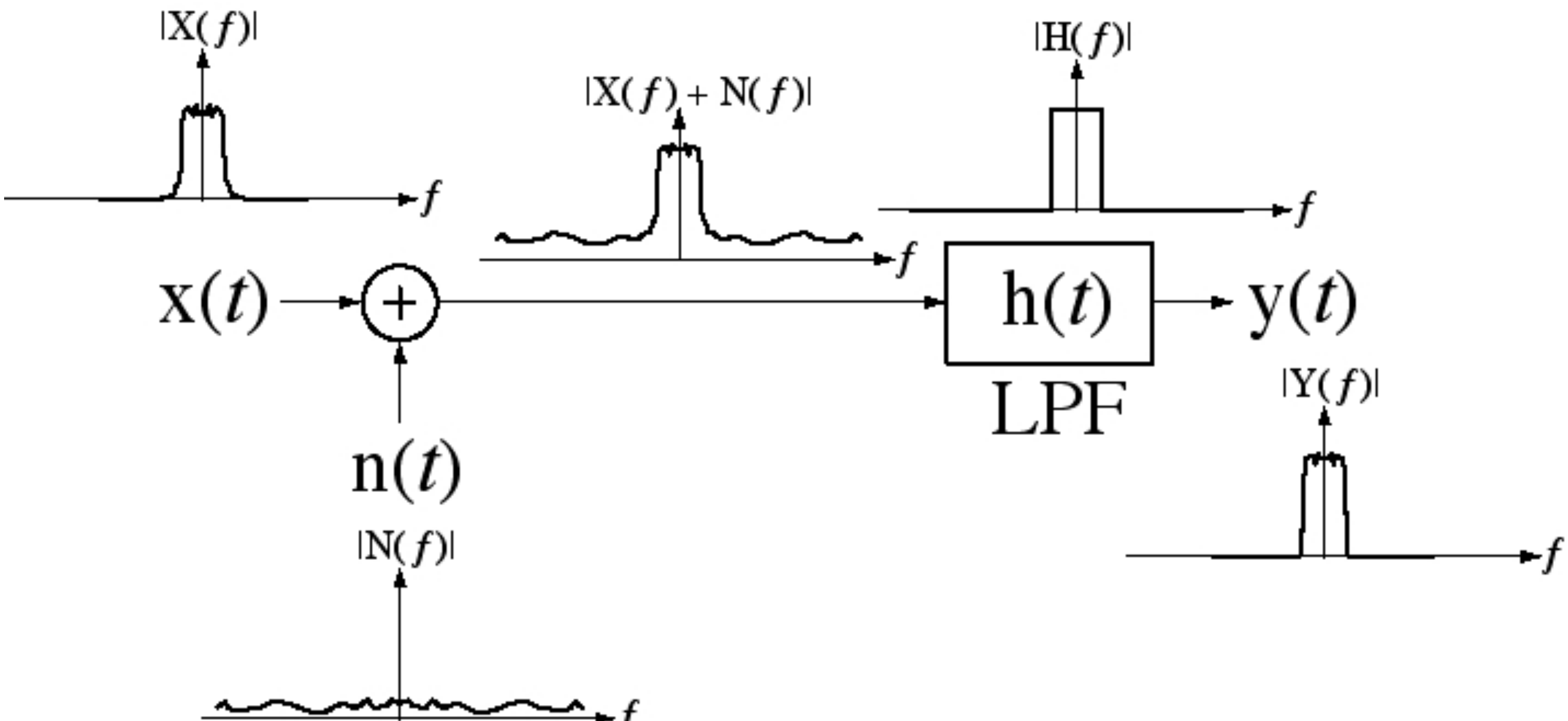


The Power Spectrum



Noise Removal

A very common use of filters is to remove **noise** from a signal. If the noise bandwidth is much greater than the signal bandwidth a large improvement in signal fidelity is possible.



The Decibel

The **bel** $(\mathbf{B})$(named in honor of Alexander Graham Bell)is defined as the common logarithm (base 10) of a power ratio. So if the excitation of a system is X and the response is Y, the power gain of the system is P_Y / P_X. Expressed in bels that would be

$$\left(P_\text{Y} / P_\text{X}\right)_\text{B} = \log_{10}\left(P_Y / P_X\right) = \log_{10}\left(Y^2 / X^2\right) = 2\log_{10}\left(Y / X\right)$$

Since the prefix deci means one-tenth, that same power ratio expressed in **decibels** $(\mathbf{dB})$would be

$$\left(P_\text{Y} / P_\text{X}\right)_\text{dB} = 10\log_{10}\left(P_Y / P_X\right) = 20\log_{10}\left(Y / X\right)$$

The Decibel

If a frequency response magnitude is the magnitude of the ratio of a system response to a system excitation

$$\left|\mathrm{H}(j\omega)\right| = \left|\frac{\mathrm{Y}(j\omega)}{\mathrm{X}(j\omega)}\right|$$

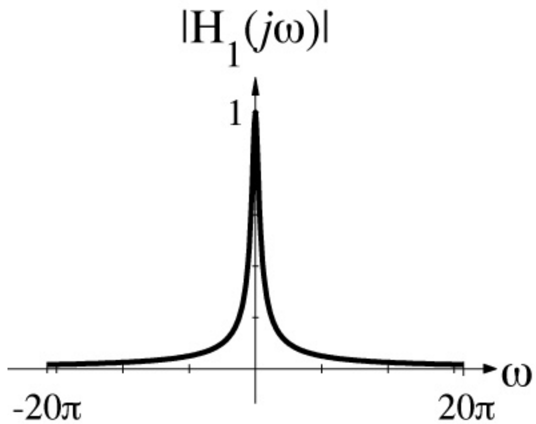
then that magnitude ratio, expressed in decibels, is

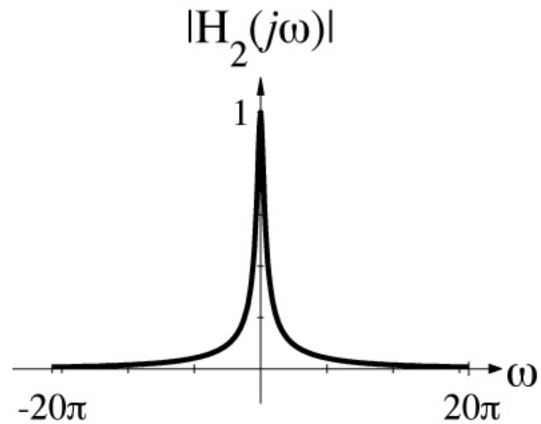
$$\left|\mathrm{H}(j\omega)\right|_{\mathrm{dB}} = 20\log_{10}\left|\mathrm{H}(j\omega)\right| = 20\log_{10}\left|\frac{\mathrm{Y}(j\omega)}{\mathrm{X}(j\omega)}\right| = \left|\mathrm{Y}(j\omega)\right|_{\mathrm{dB}} - \left|\mathrm{X}(j\omega)\right|_{\mathrm{dB}}$$

Log-Magnitude Frequency-Response Plots

Consider the two (different) transfer functions,

$$\mathrm{H}_1(j\omega) = \frac{1}{j\omega + 1} \quad \text{and} \quad \mathrm{H}_2(j\omega) = \frac{30}{30 - \omega^2 + j31\omega}$$

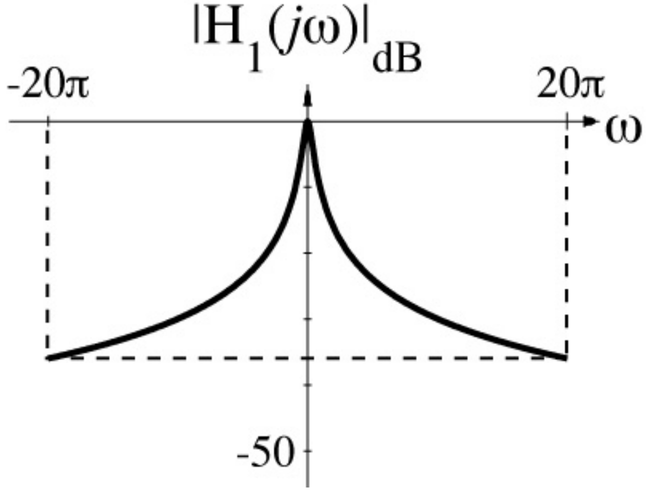


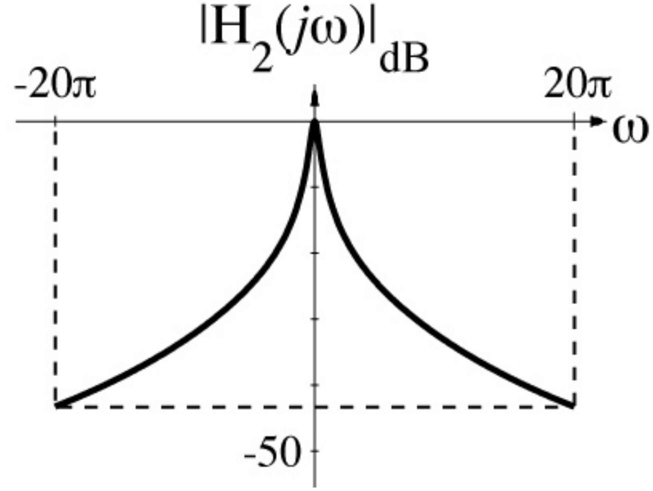


When plotted on this scale, these magnitude frequency response plots are indistinguishable.

Log-Magnitude Frequency-Response Plots

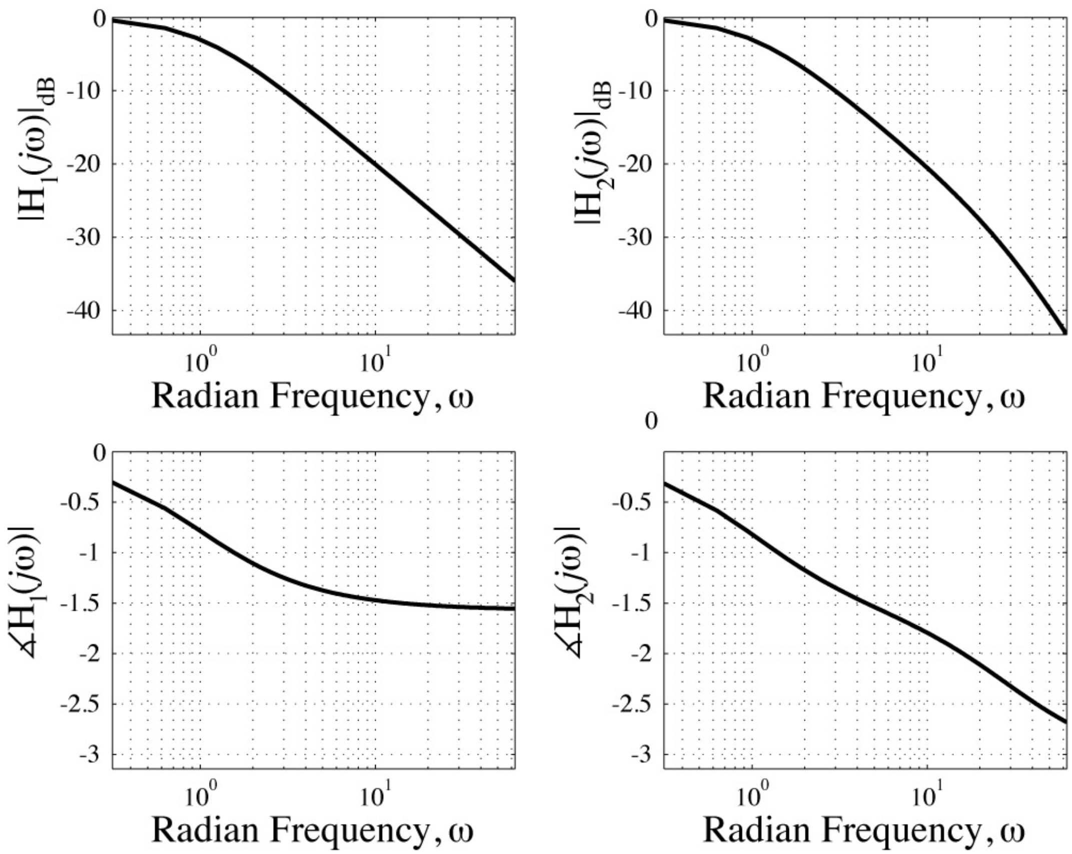
When the magnitude frequency responses are plotted on a **logarithmic scale** (in dB) the difference is visible.





Bode Diagrams

A magnitude-frequency-response Bode diagram is a graph of the frequency response magnitude in dB against a logarithmic frequency scale.



$$H_1(j\omega) = \frac{1}{j\omega + 1}$$

$$H_2(j\omega) = \frac{30}{30 - \omega^2 + j31\omega}$$

Bode Diagrams

Continuous-time LTI systems are described by equations of the general form,

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} \mathrm{y}(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} \mathrm{x}(t)$$

The corresonding transfer function is

$$\mathrm{H}(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + b_0}$$

Bode Diagrams

The transfer function can be written in the form

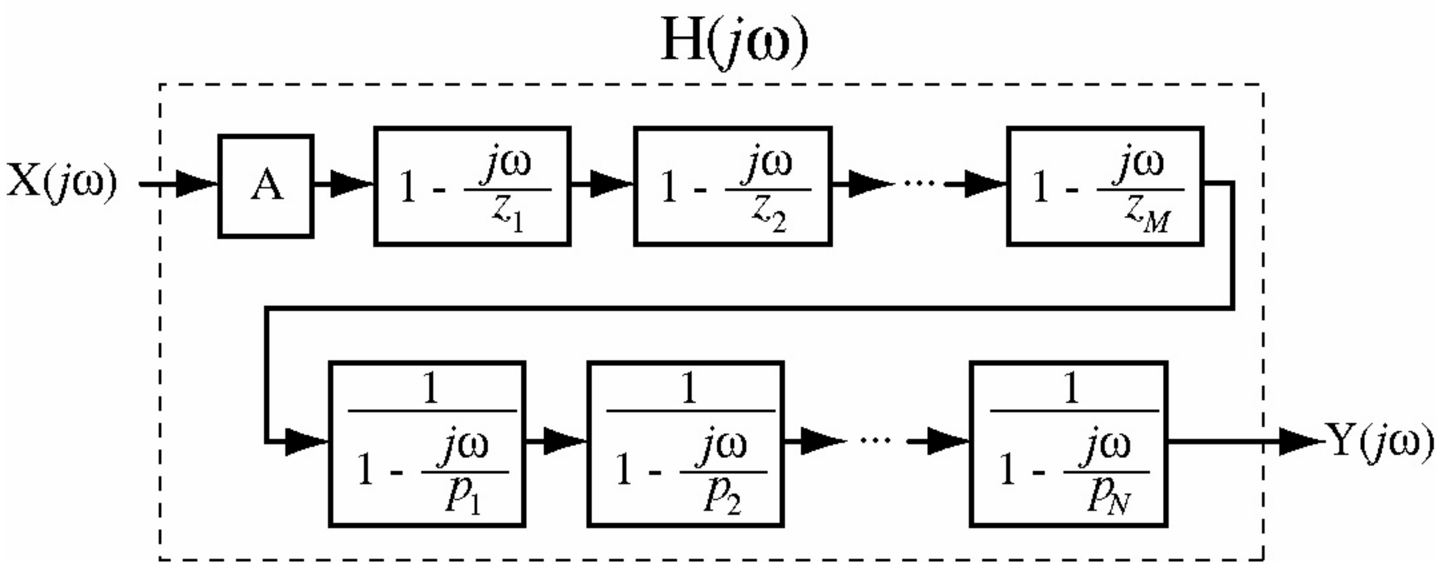
$$\mathrm{H}(s) = A\frac{(1 - s/z_1)(1 - s/z_2)\cdots(1 - s/z_M)}{(1 - s/p_1)(1 - s/p_2)\cdots(1 - s/p_N)}$$

where the z's are the values of s at which the frequency response goes to zero and the p's are the values of s at which the frequency response goes to infinity. These z's and p's are commonly referred to as the zeros and poles of the system. The frequency response is

$$\mathrm{H}(j\omega) = A\frac{(1 - j\omega/z_1)(1 - j\omega/z_2)\cdots(1 - j\omega/z_M)}{(1 - j\omega/p_1)(1 - j\omega/p_2)\cdots(1 - j\omega/p_N)}$$

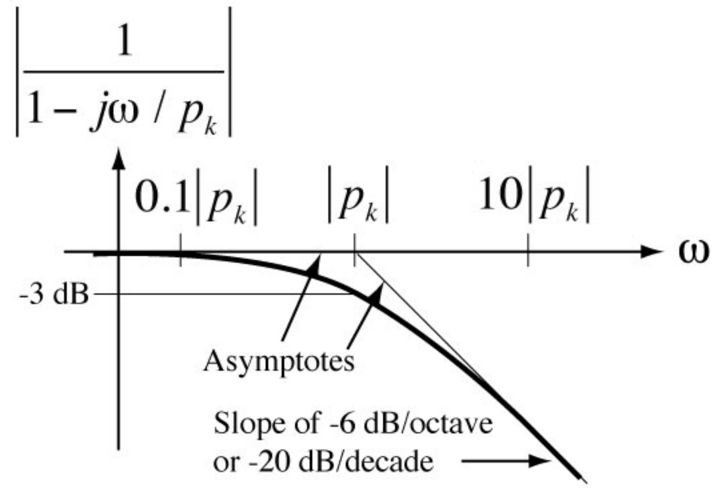
Bode Diagrams

From the factored form of the frequency response a system can be conceived as the cascade of simple systems, each of which has only one numerator factor or one denominator factor. Since the Bode diagram is logarithmic, multiplied frequency responses add when expressed in dB.



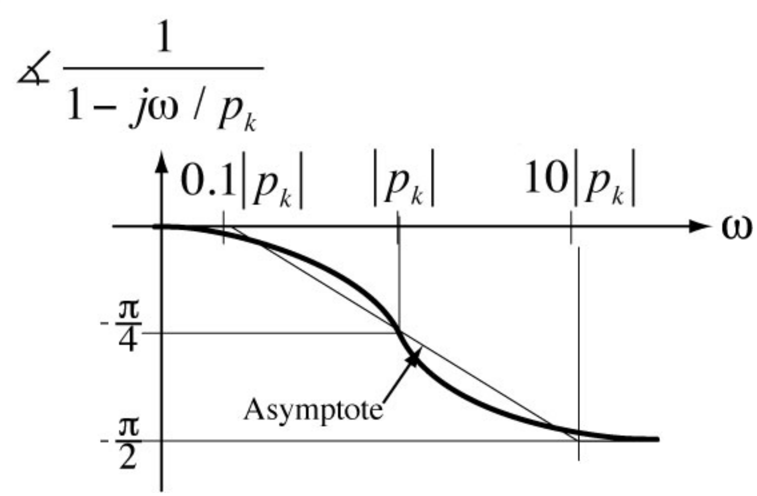
Bode Diagrams

System Bode diagrams are formed by adding the Bode diagrams of the simple systems which are in cascade. Each simple-system diagram is called a **component diagram**.



One Real Pole

$$\mathrm{H}(j\omega)=\frac{1}{1-j\omega/p_k}$$



Bode Diagrams

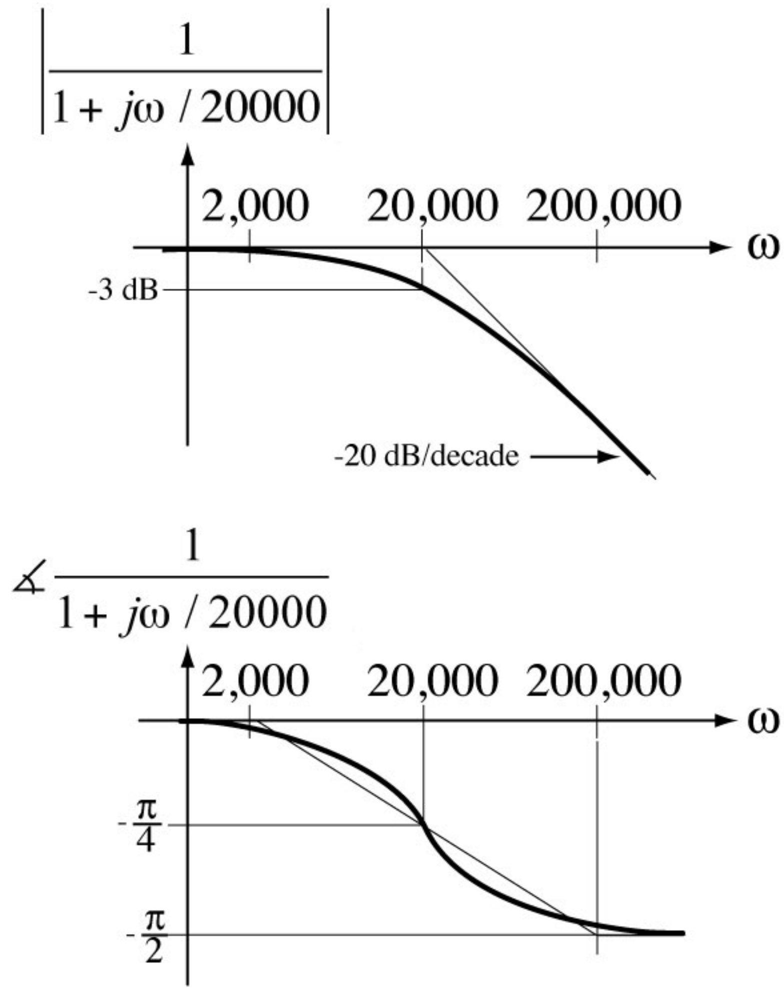
Let the frequency response of a lowpass filter be

$$H(j\omega) = \frac{1}{j50 \times 10^{-6}\omega + 1}$$

This can be written as

$$H(j\omega) = \frac{1}{1 - \frac{j\omega}{(-20{,}000)}}$$

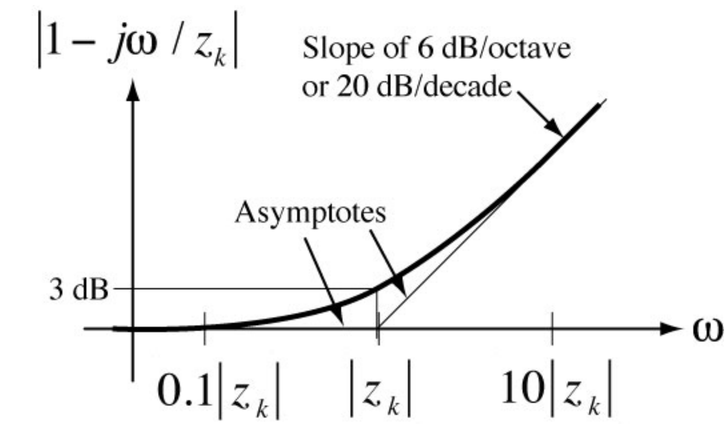
Its Bode diagram has one corner frequency at $\omega = 20{,}000$.

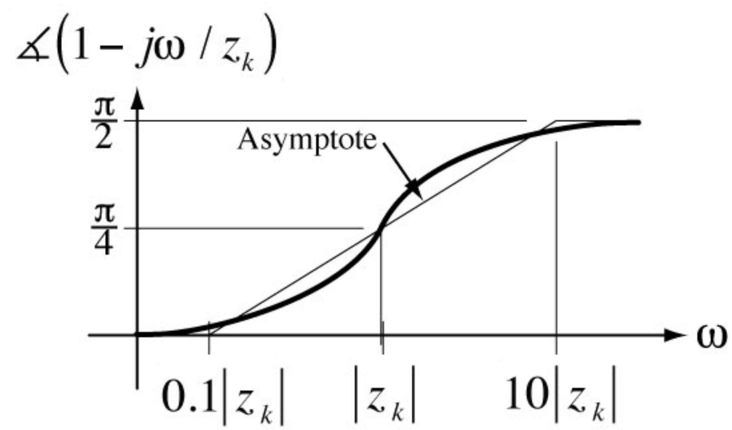


Bode Diagrams

One Real Zero

$$H(j\omega)=1-j\omega/z_k$$

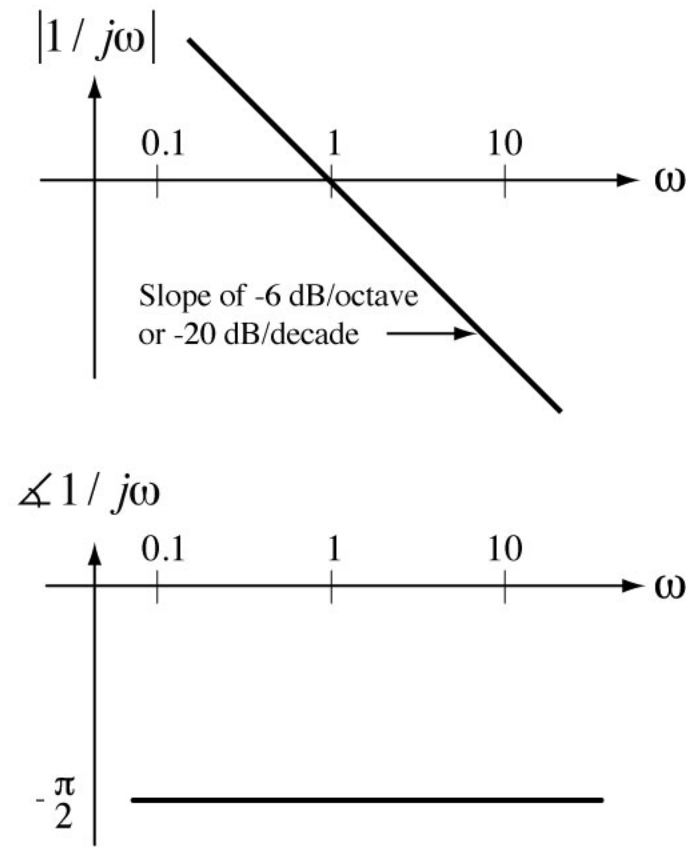




Bode Diagrams

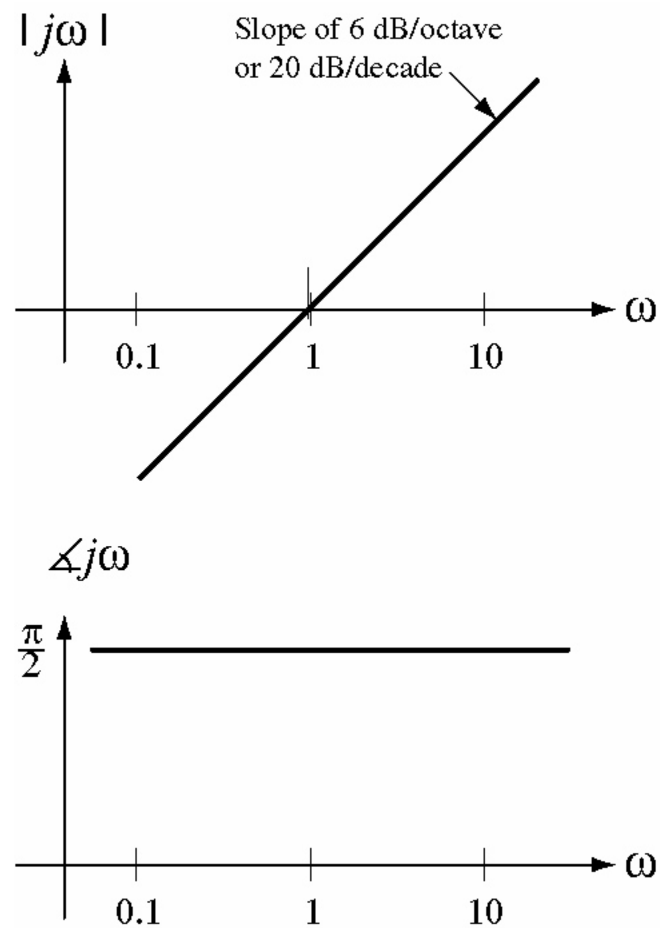
Integrator
(Pole at zero)

$$\mathrm{H}(j\omega) = 1 / j\omega$$



Bode Diagrams

Differentiator
(Zero at zero)



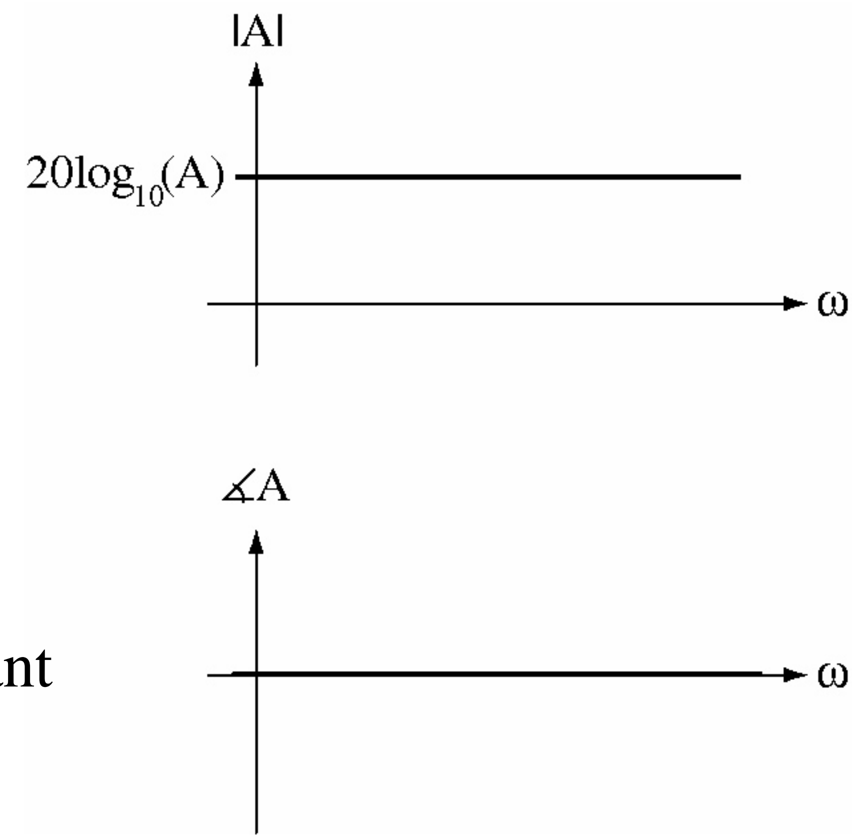
$$H(j\omega) = j\omega$$

Bode Diagrams

Frequency-Independent Gain

$$H(j\omega) = A$$

(This phase plot is for $A > 0$. If $A < 0$, the phase would be a constant π or $-\pi$ radians.)



Bode Diagrams

Complex Pole Pair

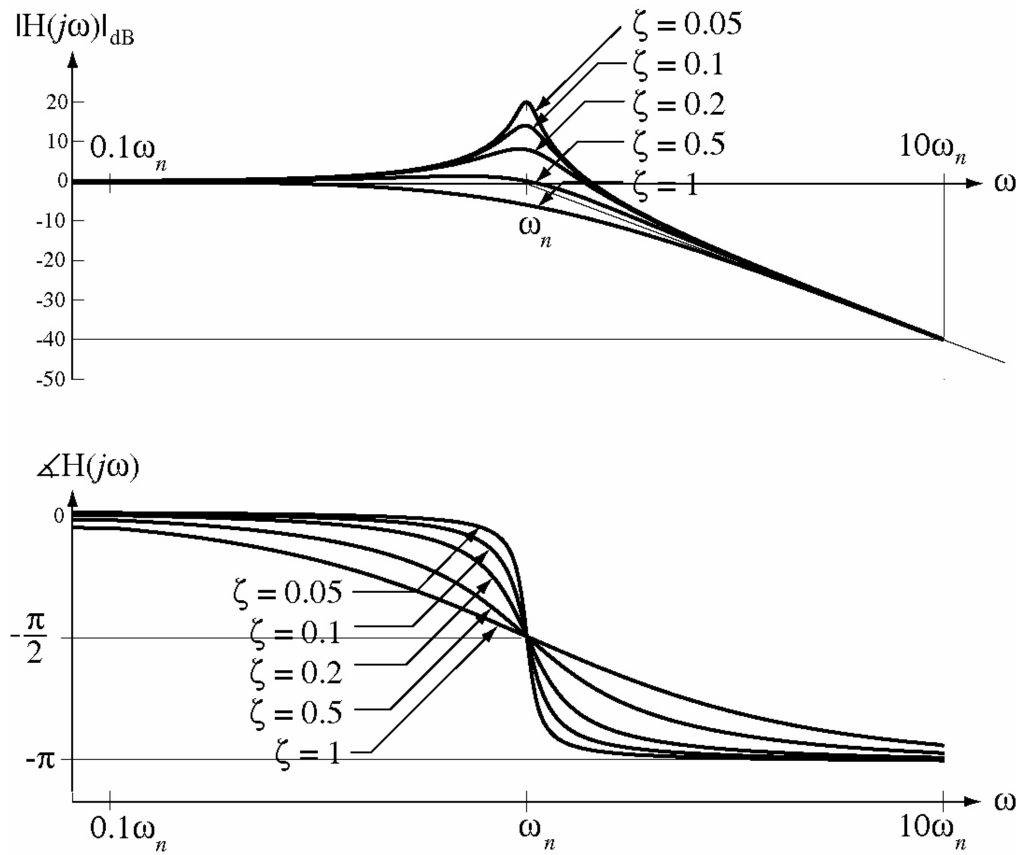
$$\mathrm{H}(j\omega)=\frac{1}{\left(1-\frac{j\omega}{p_1}\right)\left(1-\frac{j\omega}{p_2}\right)}=\frac{1}{1-j\omega\frac{2\,\mathrm{Re}(p_1)}{|p_1|^2}+\frac{(j\omega)^2}{|p_1|^2}}$$

The **natural radian frequency** ω_n is defined by

$$\omega_n^2 = p_1 p_2$$

The **damping ratio** ζ is defined by

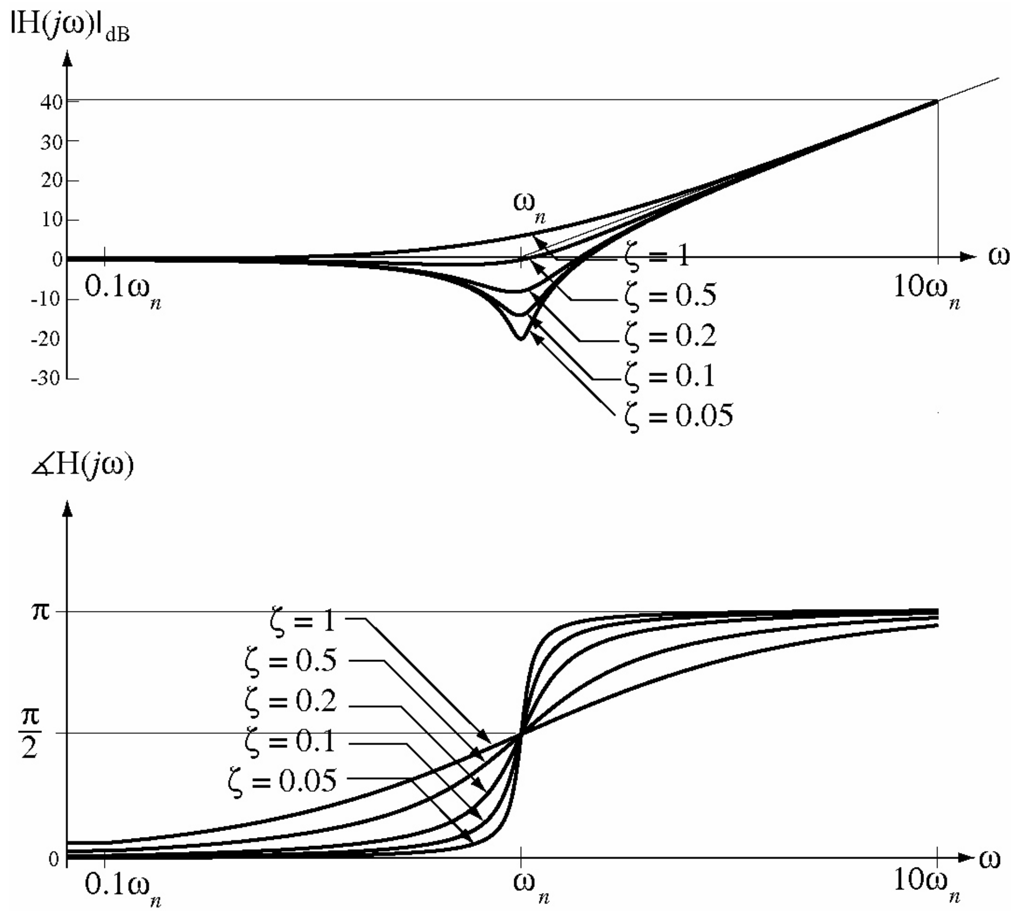
$$\zeta = -\frac{p_1+p_2}{2\sqrt{p_1 p_2}}$$



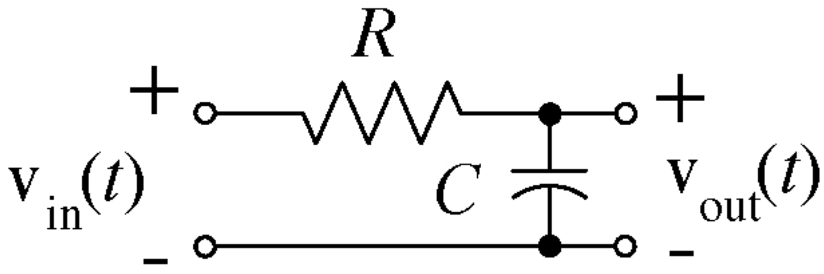
Bode Diagrams

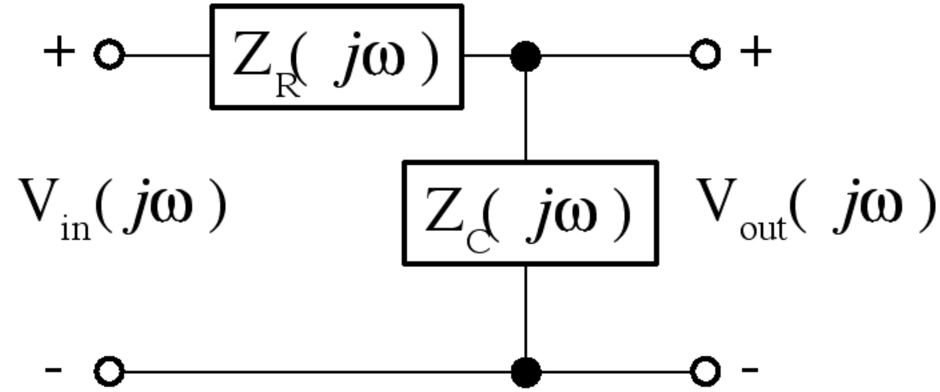
Complex Zero Pair

$$H(j\omega) = \left(1 - \frac{j\omega}{z_1}\right)\left(1 - \frac{j\omega}{z_2}\right) = 1 - j\omega\frac{2\,\mathrm{Re}(z_1)}{|z_1|^2} + \frac{(j\omega)^2}{|z_1|^2}$$



Practical Passive Filters

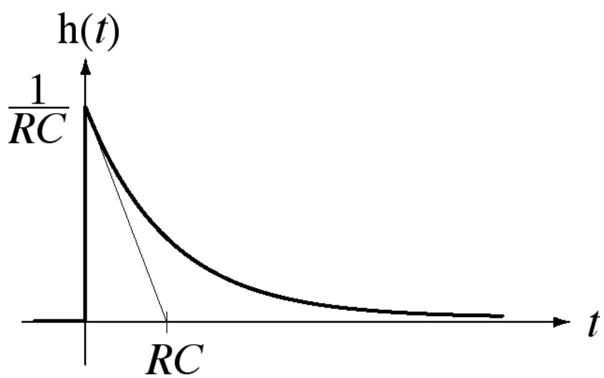


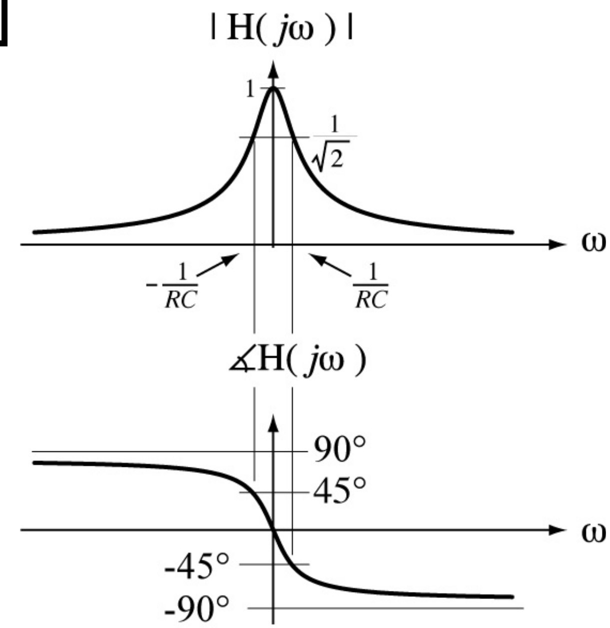


$$H(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)}$$

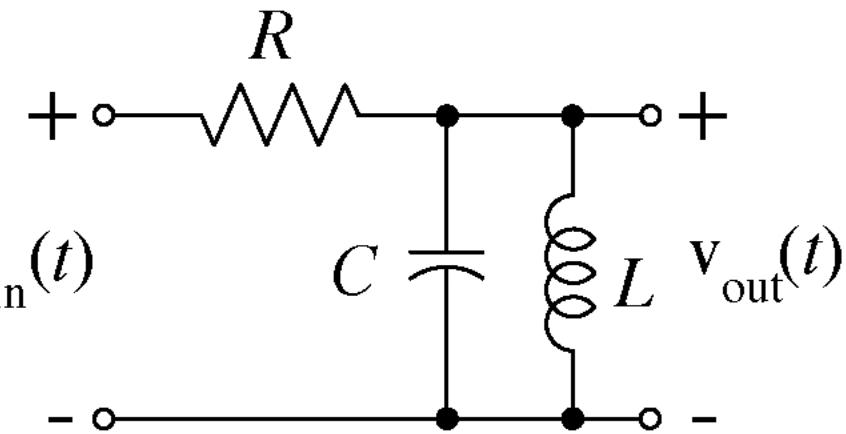
_RC_ Lowpass Filter

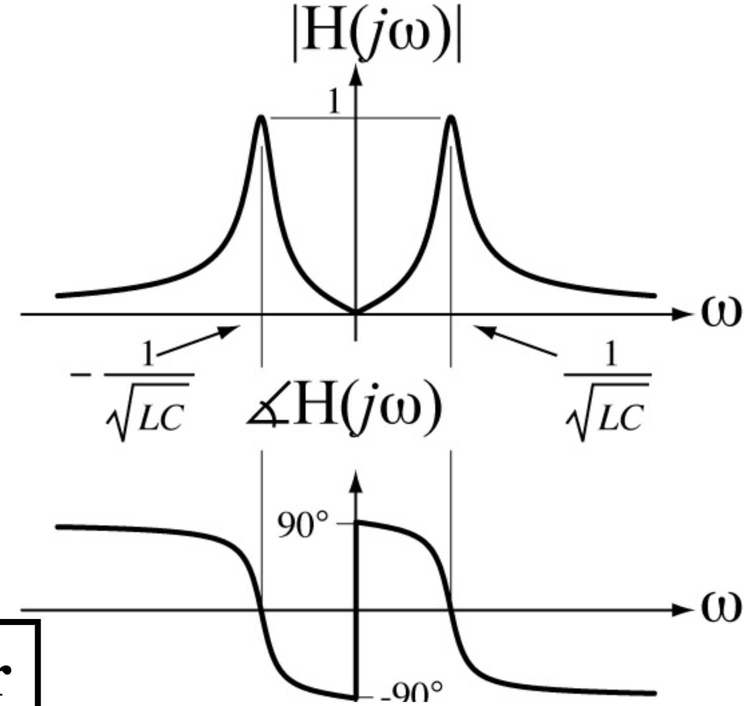
$$= \frac{Z_c(j\omega)}{Z_c(j\omega) + Z_R(j\omega)} = \frac{1}{j\omega RC + 1}$$



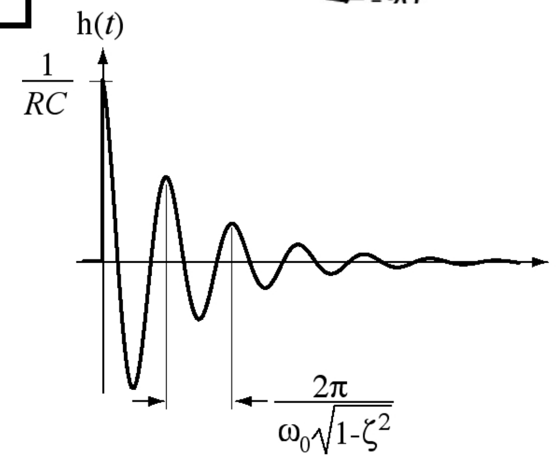


Practical Passive Filters





RLC Bandpass Filter

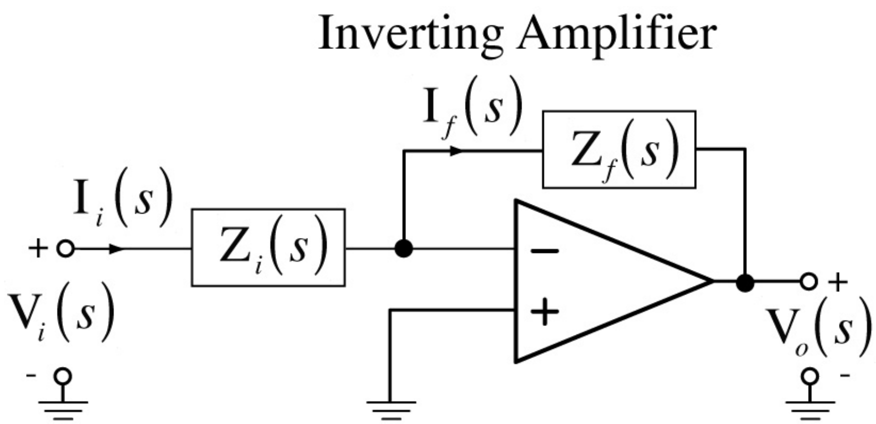


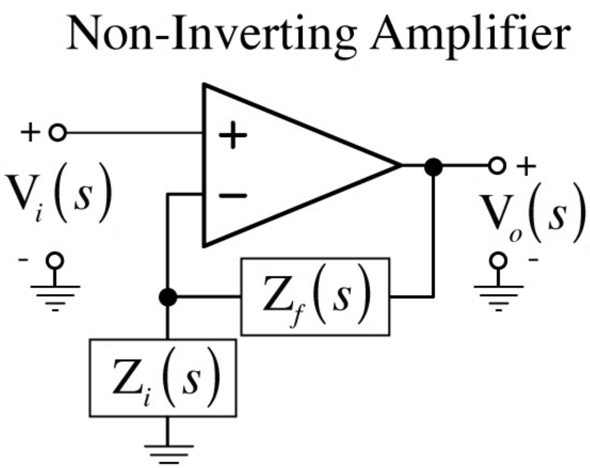
$$H(f) = \frac{V_{out}(f)}{V_{in}(f)} = \frac{j\dfrac{2\pi f}{RC}}{(j2\pi f)^2 + j\dfrac{2\pi f}{RC} + \dfrac{1}{LC}}$$

Practical Active Filters

Operational Amplifiers

The ideal operational amplifier has infinite input impedance, zero output impedance, infinite gain and infinite bandwidth.

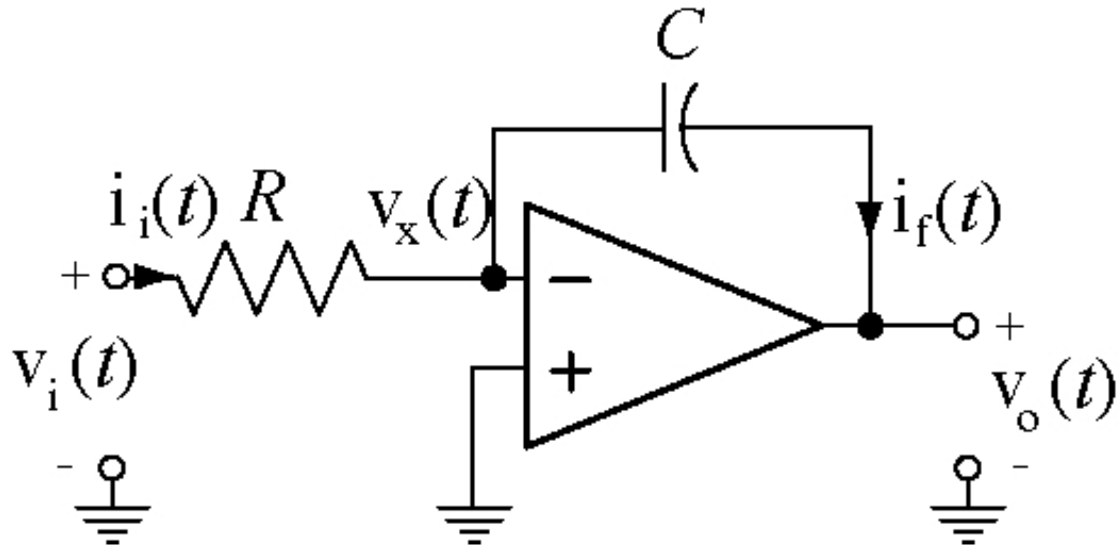




$$H(s) = \frac{V_o(s)}{V_i(s)} = -\frac{Z_f(s)}{Z_i(s)} \qquad H(s) = \frac{Z_f(s) + Z_i(s)}{Z_i(s)}$$

Practical Active Filters

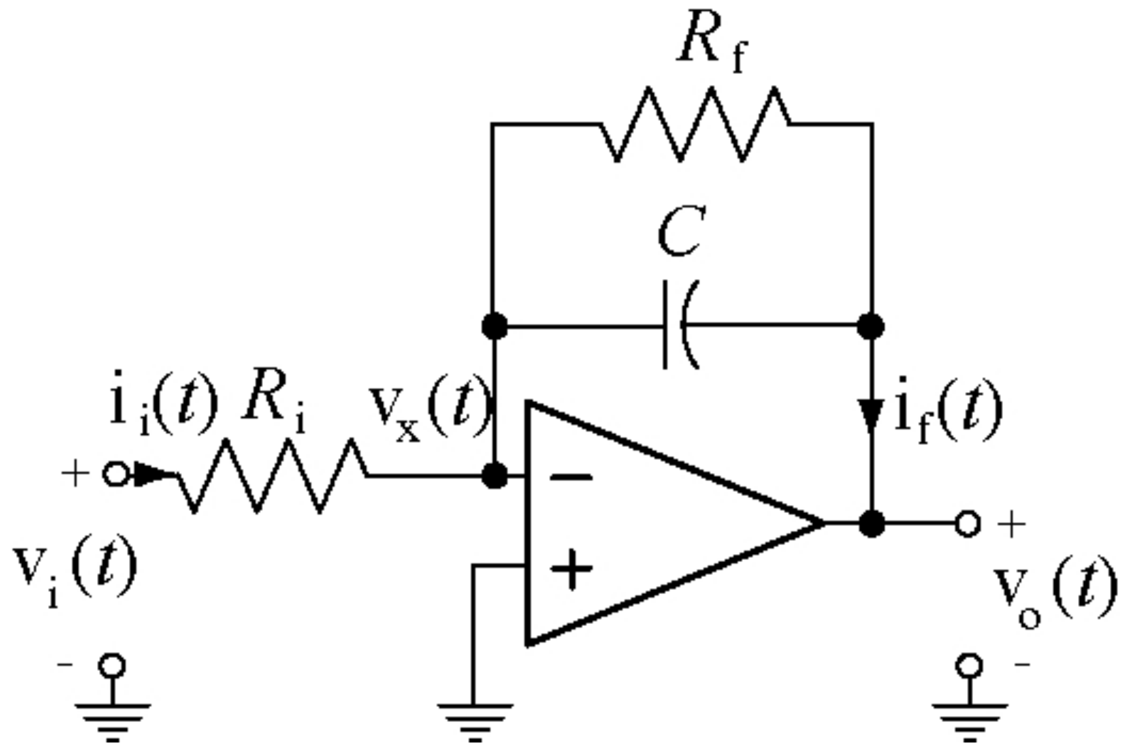
Active Integrator



$$V_o(f) = -\frac{1}{RC} \underbrace{\frac{V_i(f)}{j2\pi f}}_{\substack{\text{Fourier transform} \\ \text{of integral of } V_i(f)}}$$

Practical Active Filters

Active *RC* Lowpass Filter

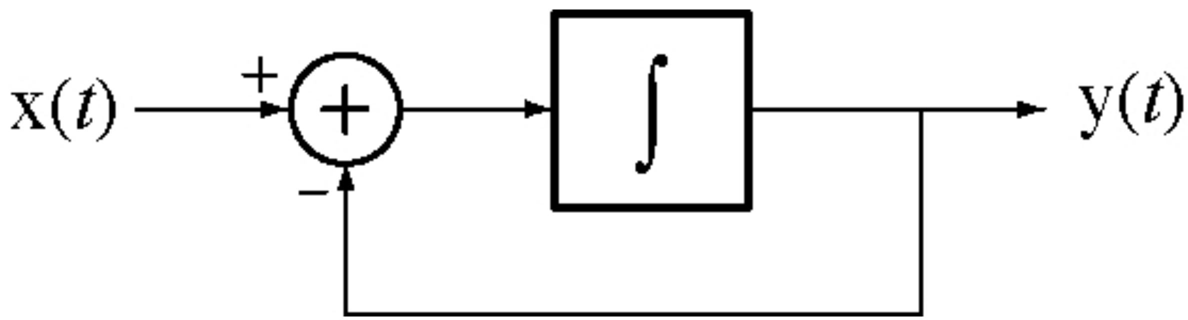


$$\frac{V_o(f)}{V_i(f)} = -\frac{R_f}{R_i}\frac{1}{j2\pi fCR_f + 1}$$

Practical Active Filters

Lowpass Filter

An integrator with feedback is a lowpass filter.



$$y'(t) + y(t) = x(t)$$

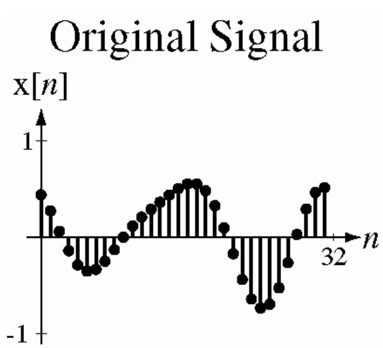
$$H(j\omega) = \frac{1}{j\omega + 1}$$

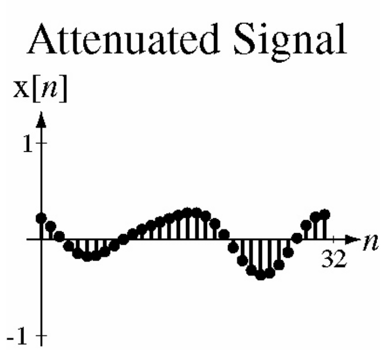
Discrete Time

Distortion

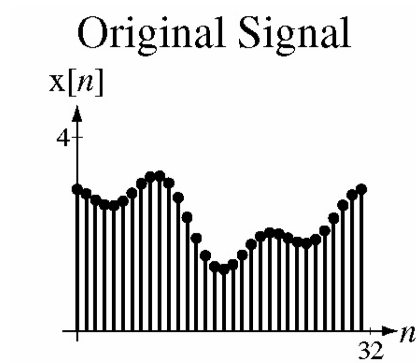
- **Distortion** means the same thing for discrete-time signals as it does for continuous-time signals, changing the shape of a signal

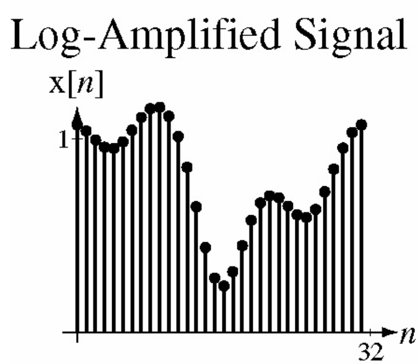
No Distortion





Distortion





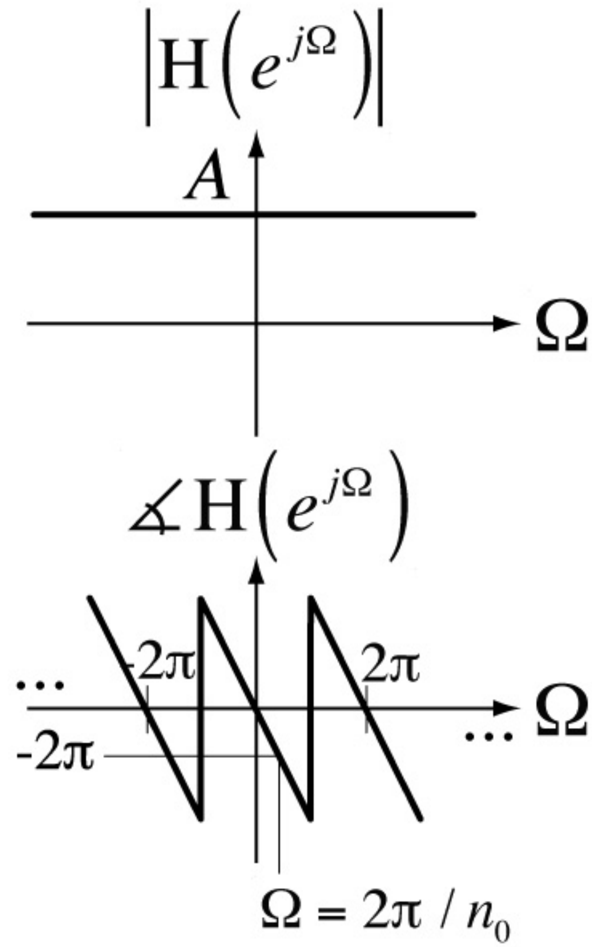
Distortion

A distortionless system would have an impulse response of the form,

$$\mathrm{h}[n] = A\delta[n - n_0]$$

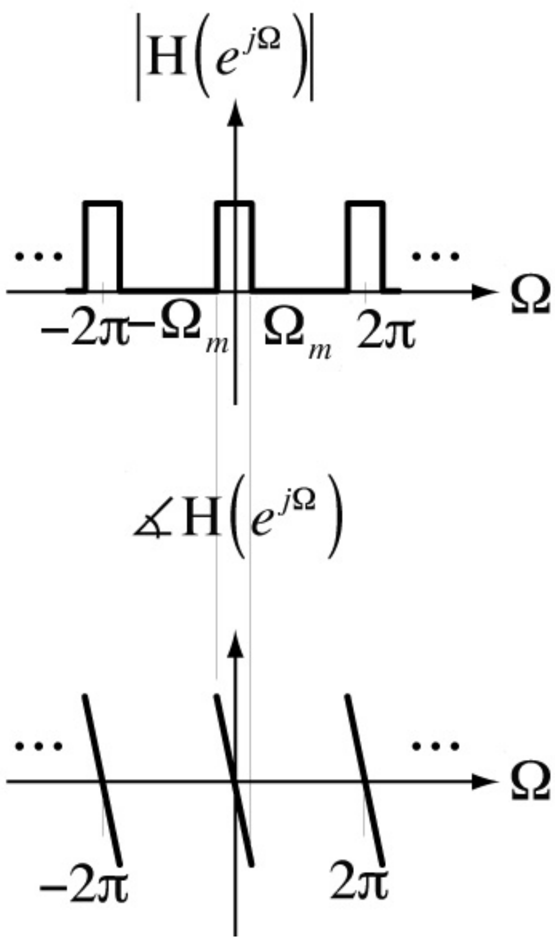
The corresponding transfer function is

$$\mathrm{H}\left(e^{j\Omega}\right) = Ae^{-j\Omega n_0}$$

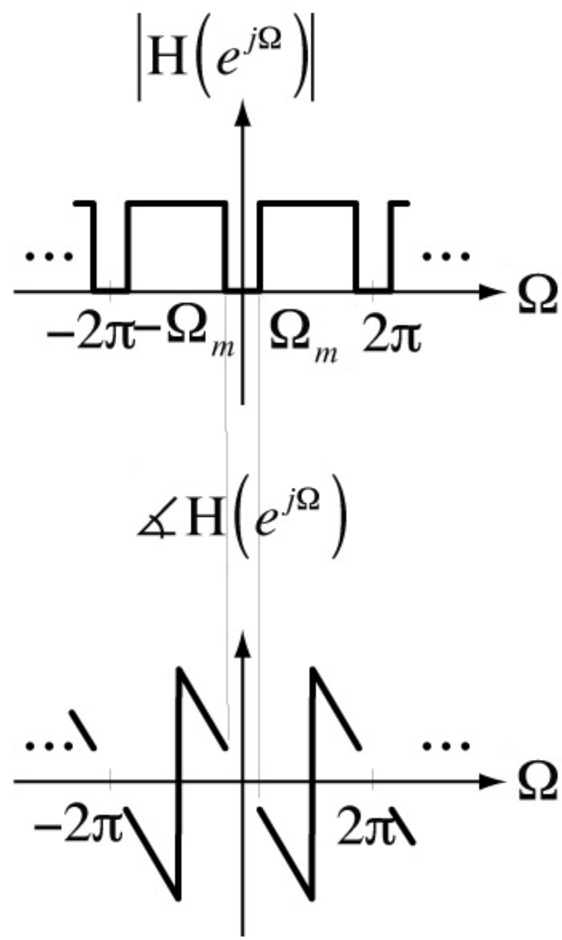


Filter Classifications

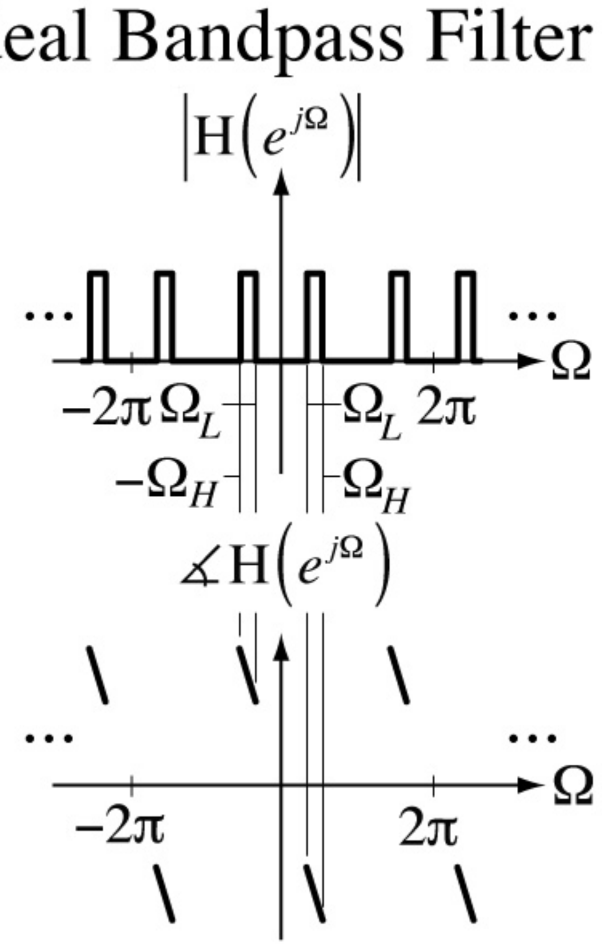
Ideal Lowpass Filter

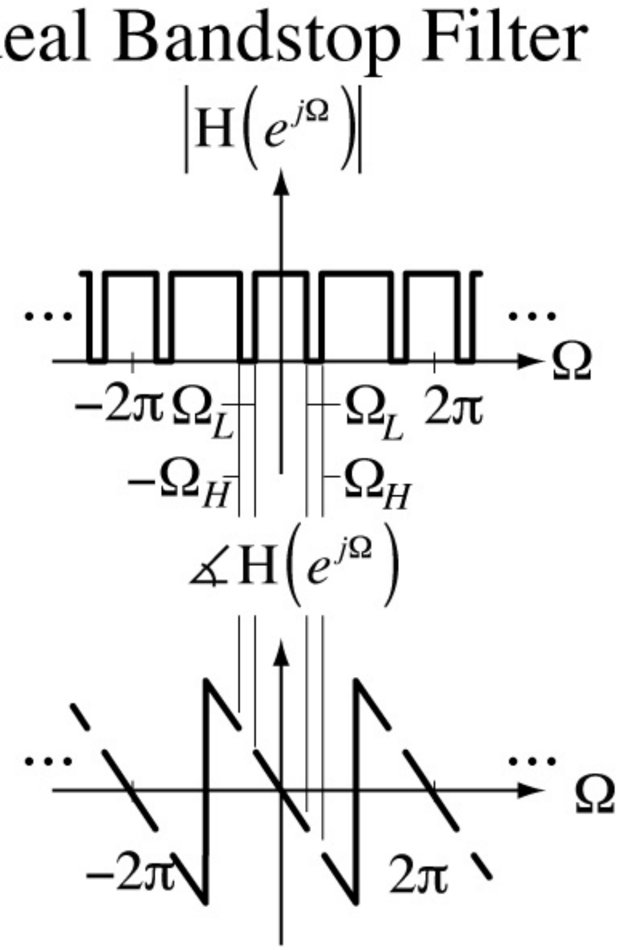


Ideal Highpass Filter

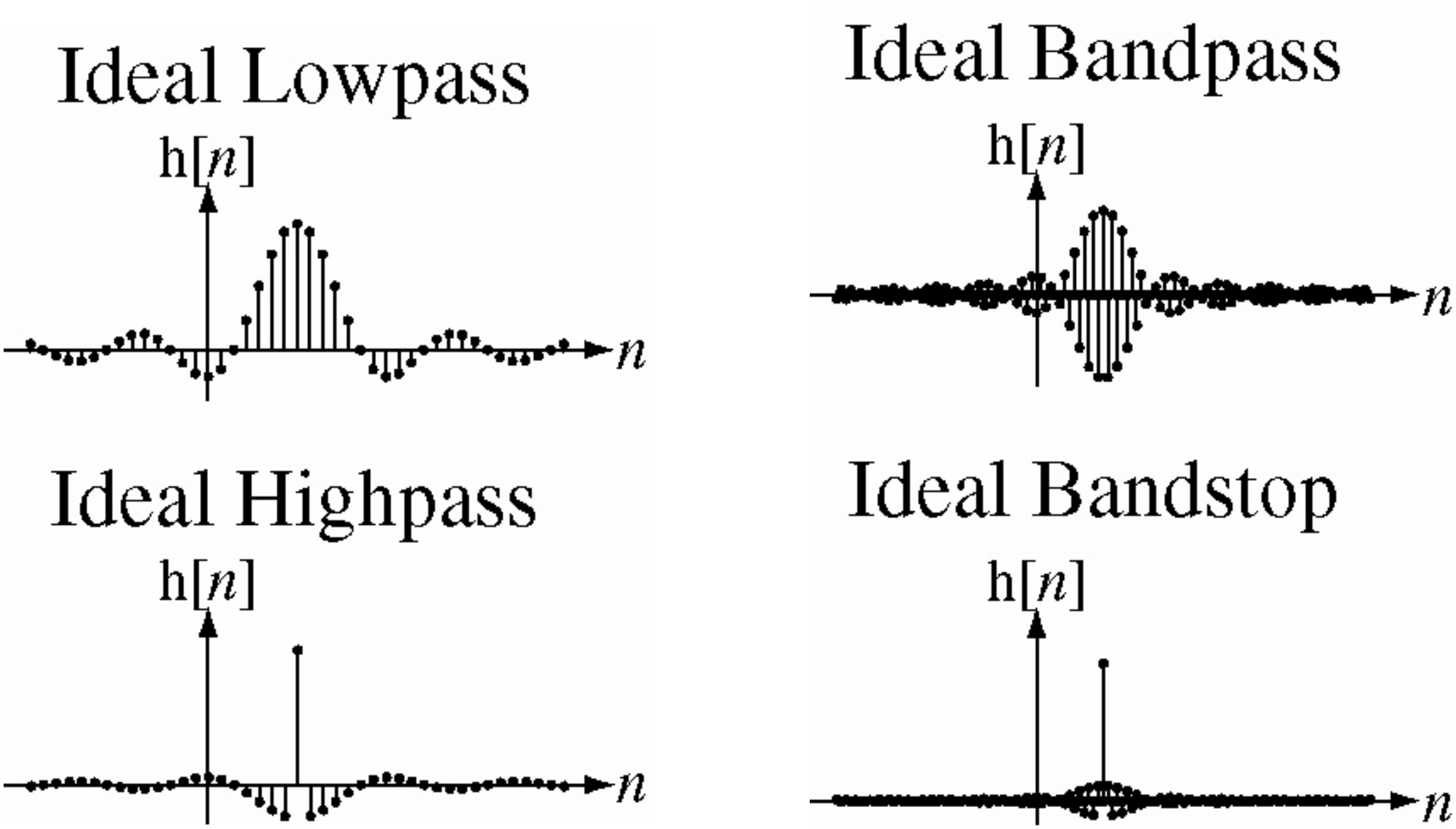


Filter Classifications





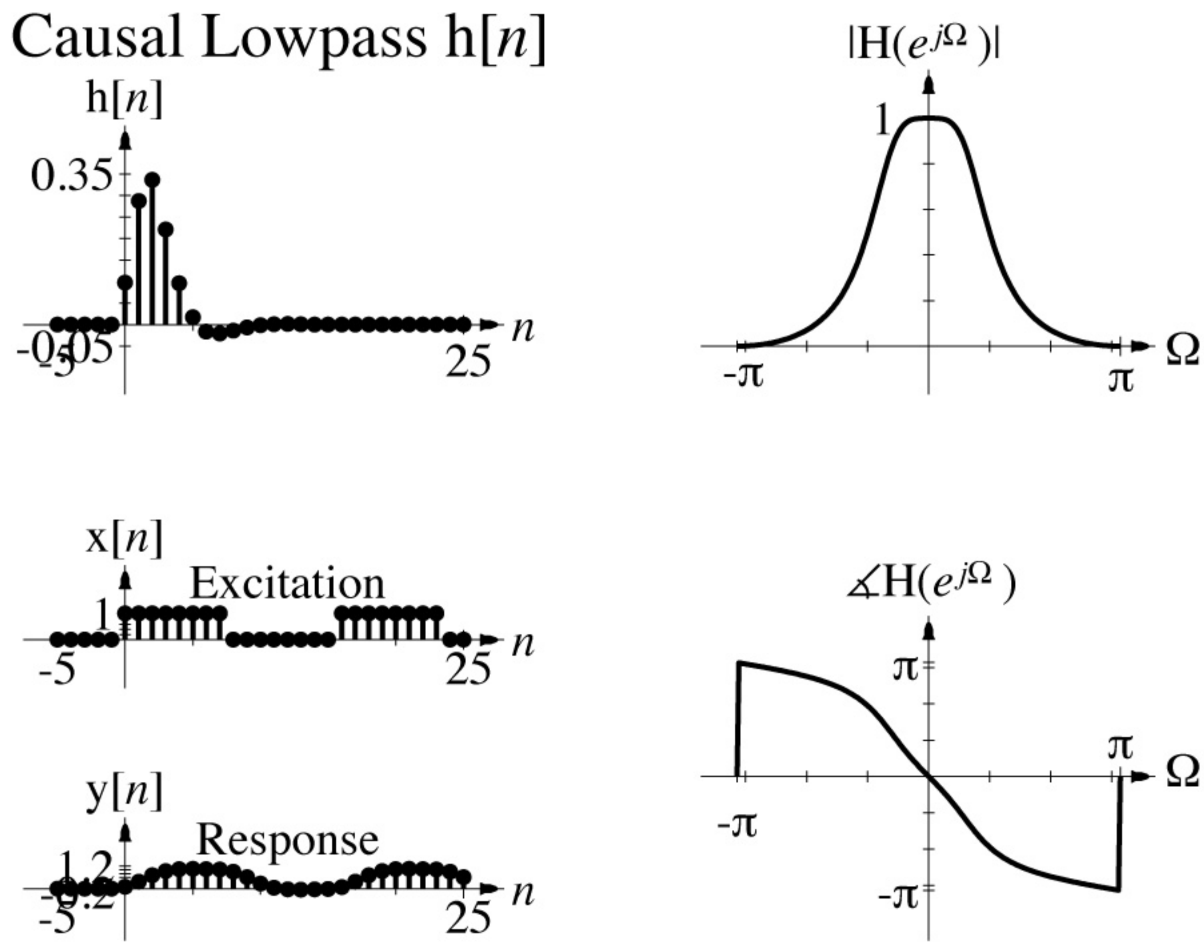
Impulse Responses of Ideal Filters



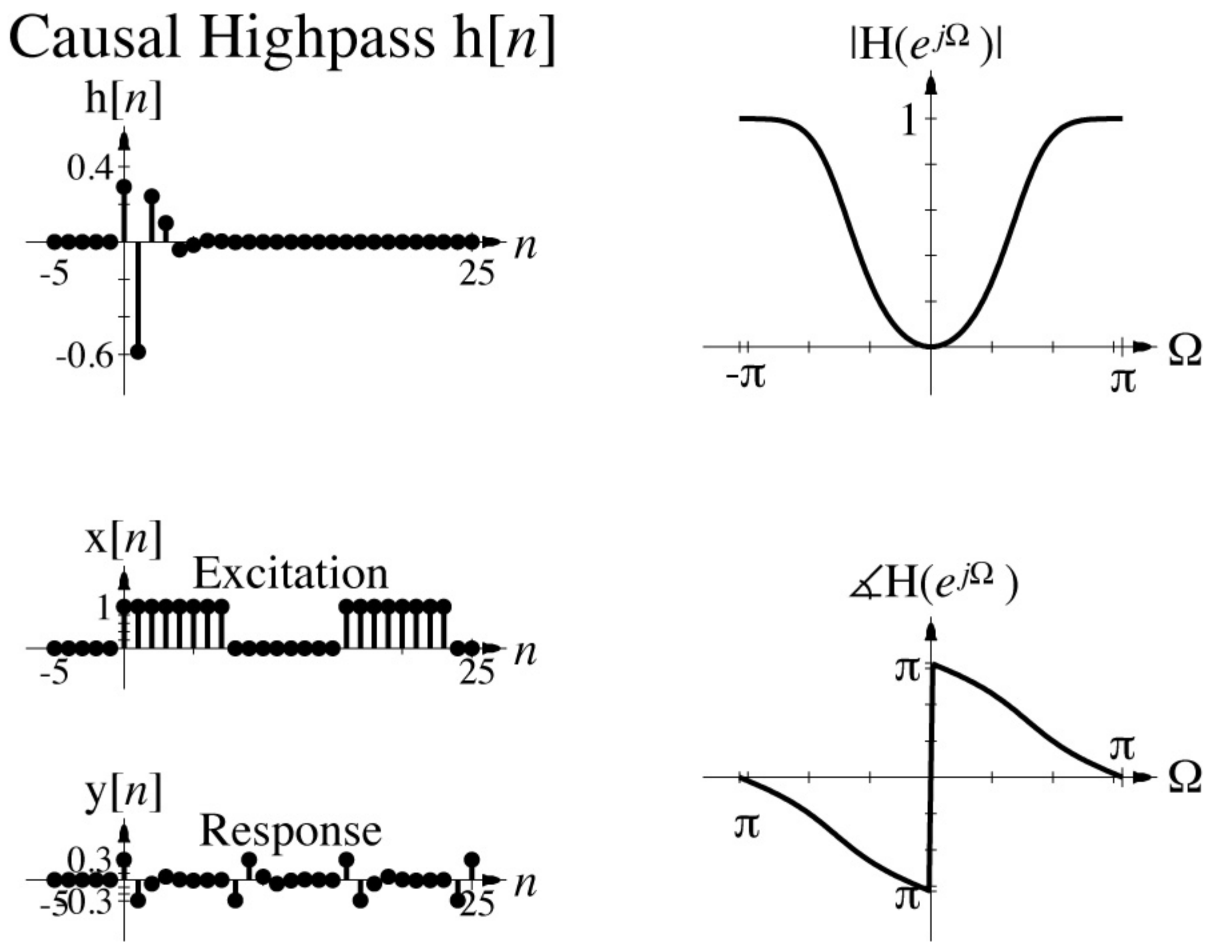
Impulse Response and Causality

- Discrete-time ideal filters are **non-causal** for the same reason that continuous-time ideal filters are non-causal

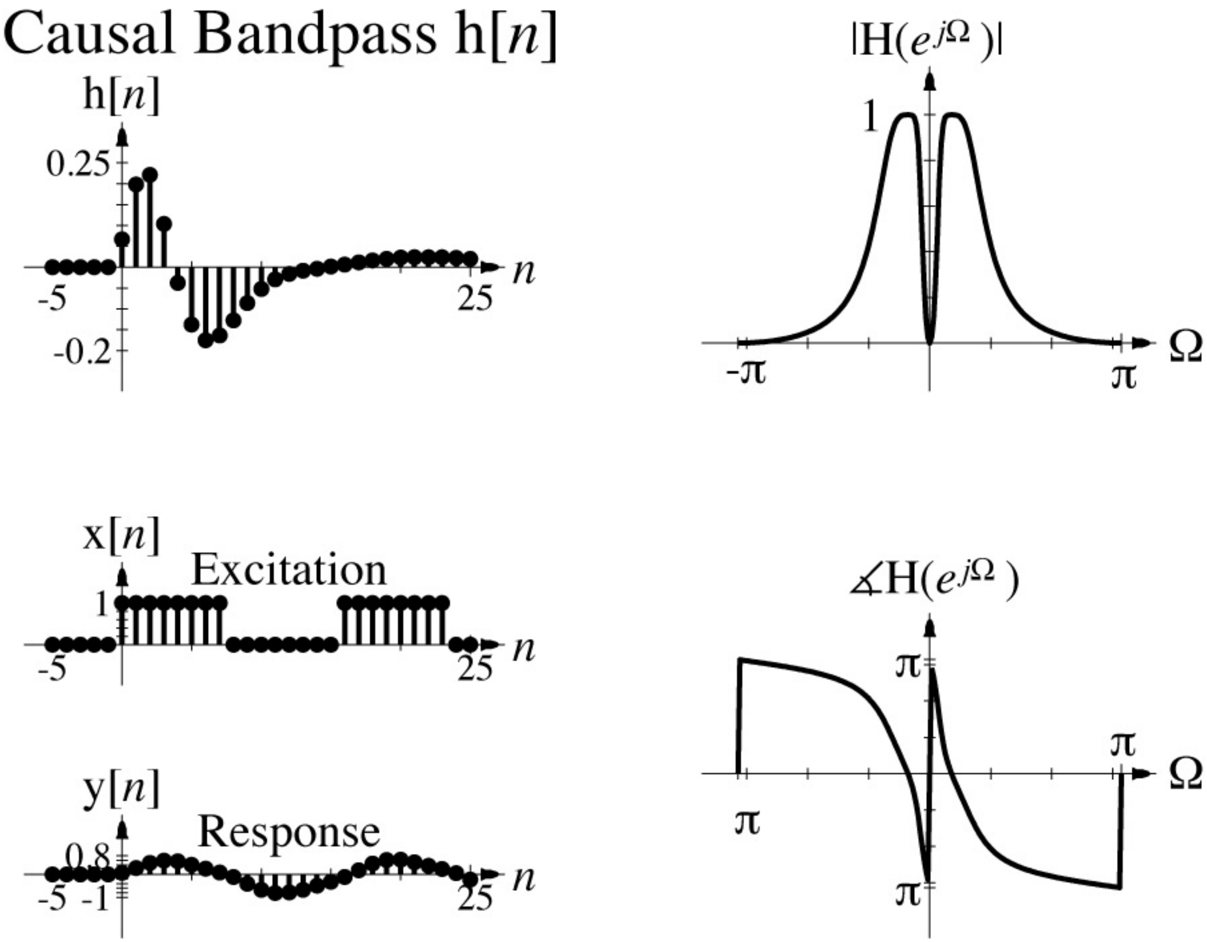
Impulse and Frequency Responses of Causal Filters



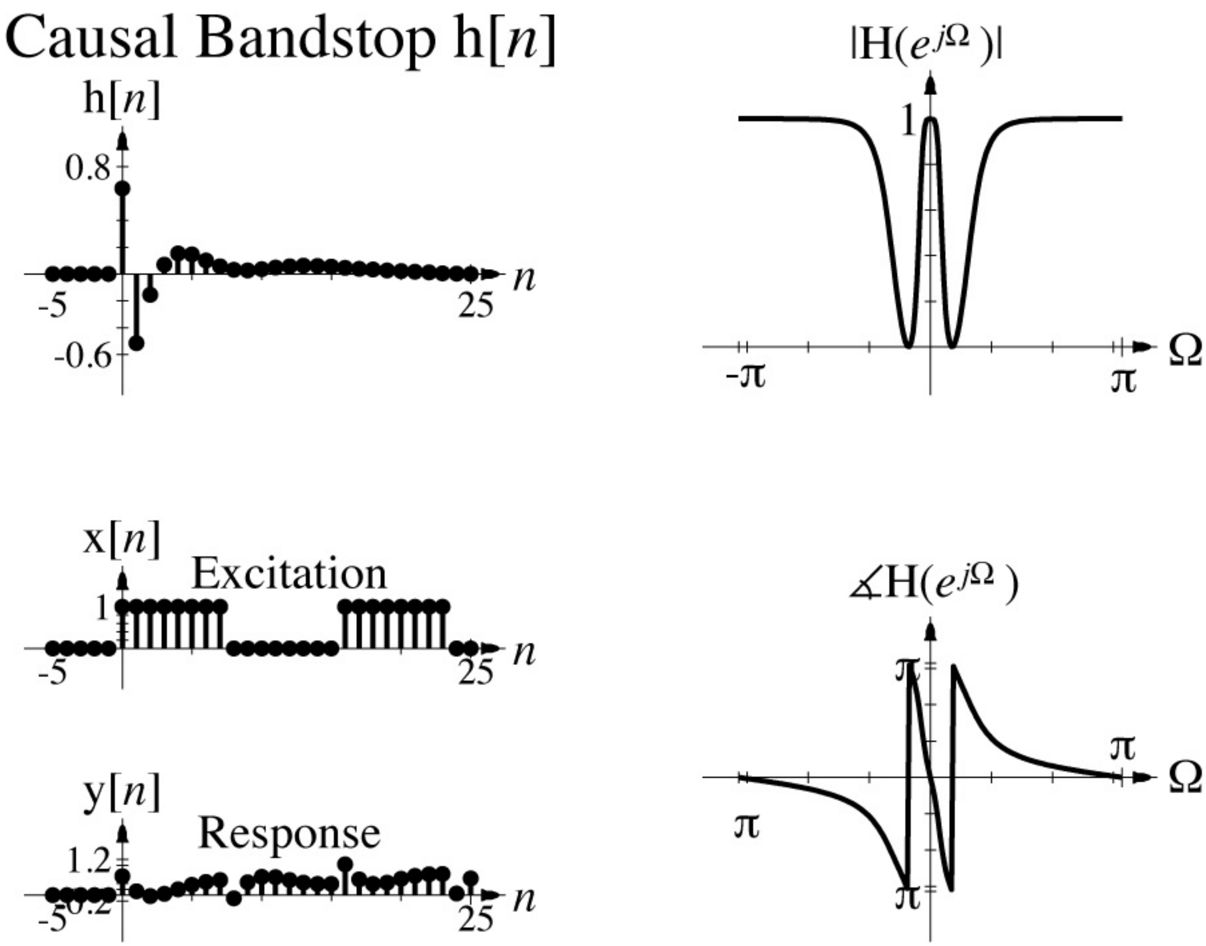
Impulse and Frequency Responses of Causal Filters



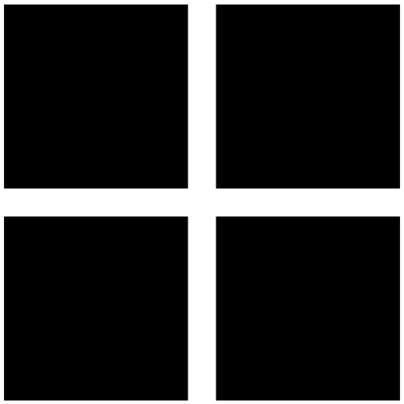
Impulse and Frequency Responses of Causal Filters



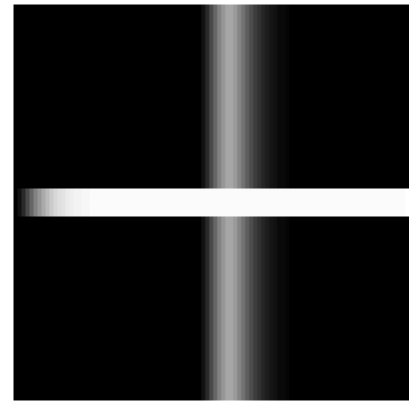
Impulse and Frequency Responses of Causal Filters

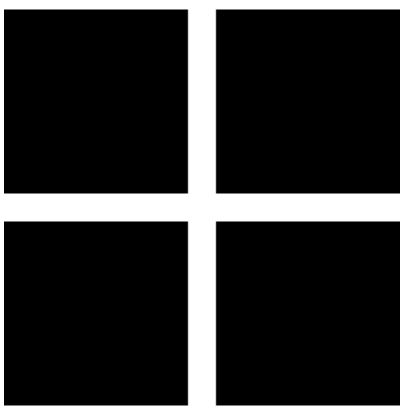


Two-Dimensional Filtering of Images

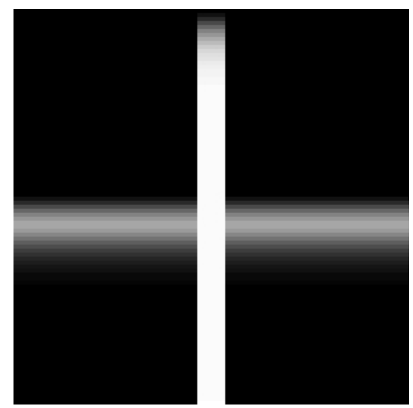


Causal Lowpass
Filtering
of Rows in
an Image

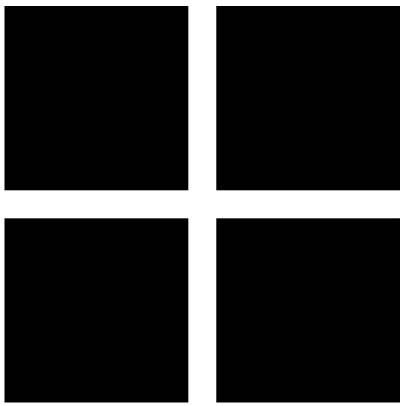




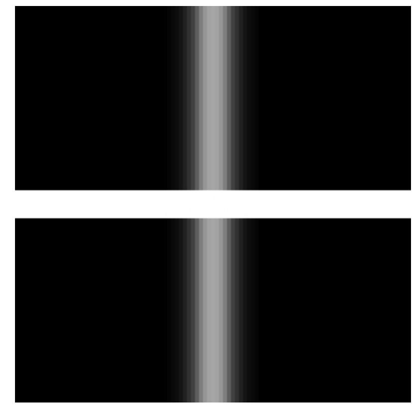
Causal Lowpass
Filtering
of Columns in
an Image

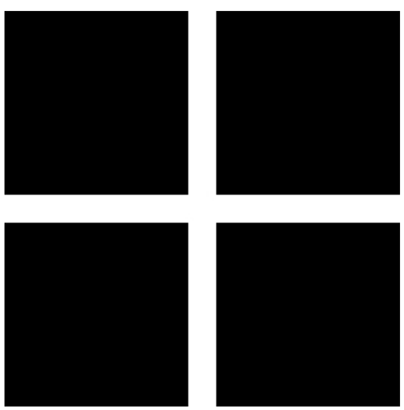


Two-Dimensional Filtering of Images

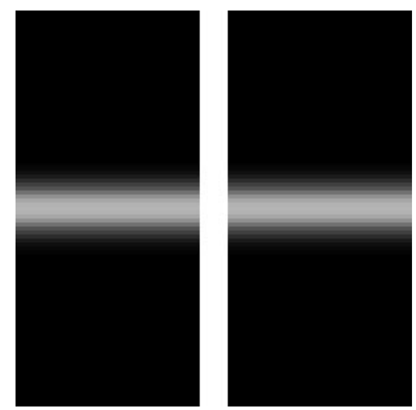


“Non-Causal” Lowpass Filtering of Rows in an Image

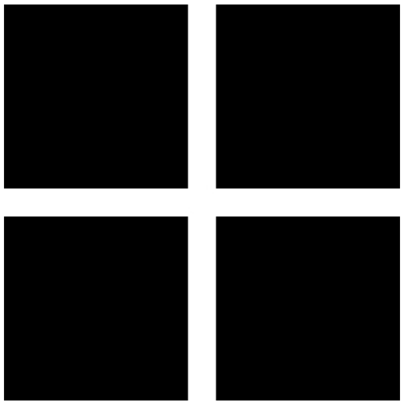




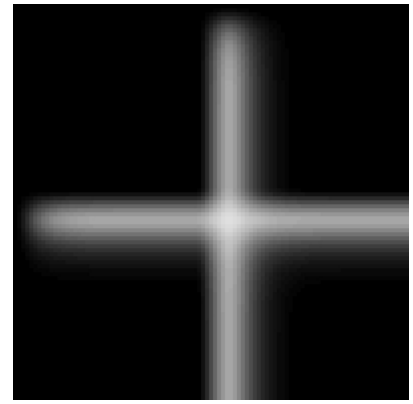
“Non-Causal” Lowpass Filtering of Columns in an Image

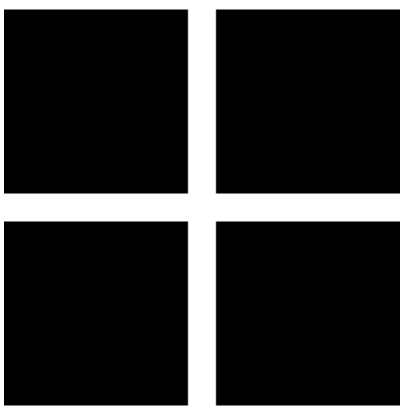


Two-Dimensional Filtering of Images

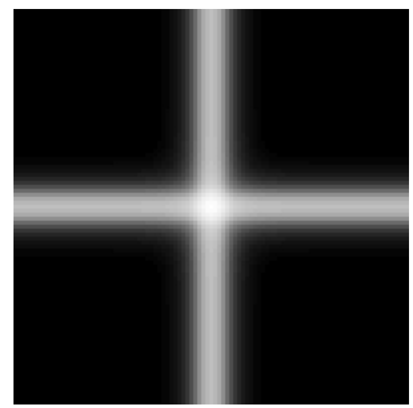


Causal Lowpass Filtering of Rows and Columns in an Image



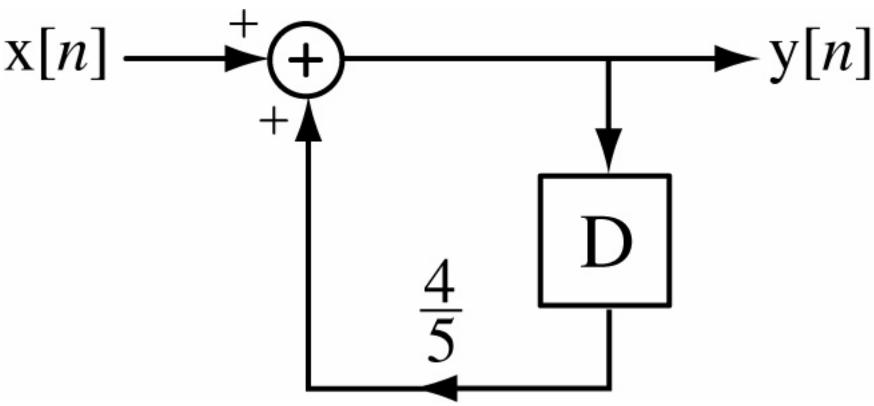


“Non-Causal” Lowpass Filtering of Rows and Columns in an Image



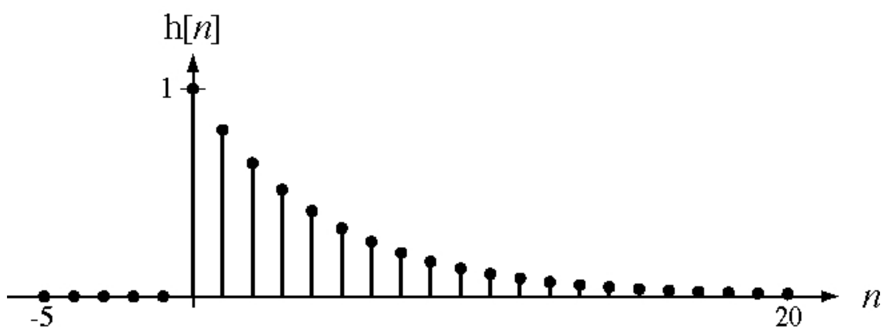
Discrete-Time Filters

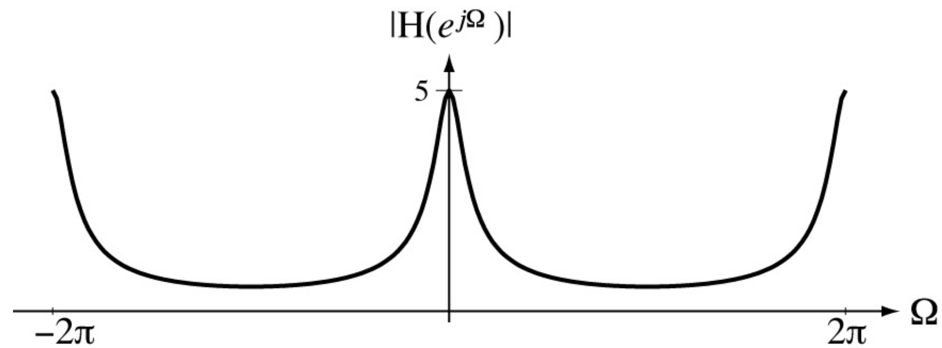
Lowpass Filter

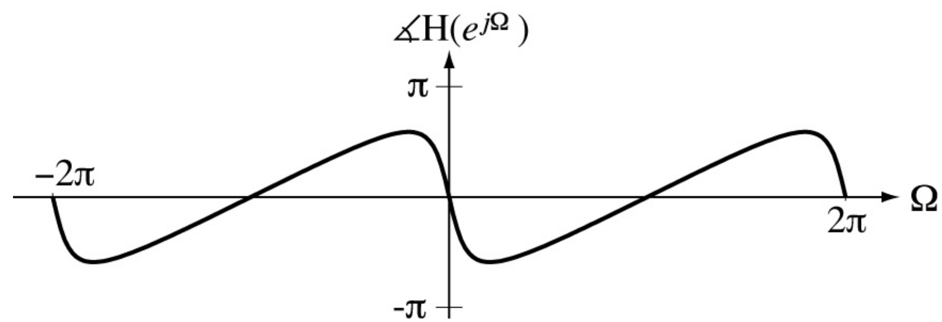


$$\mathrm{H}\left(e^{j\Omega}\right)=\frac{e^{j\Omega}}{e^{j\Omega}-0.8}$$

$$\mathrm{h}[n]=(4/5)^{n}\,\mathrm{u}[n]$$

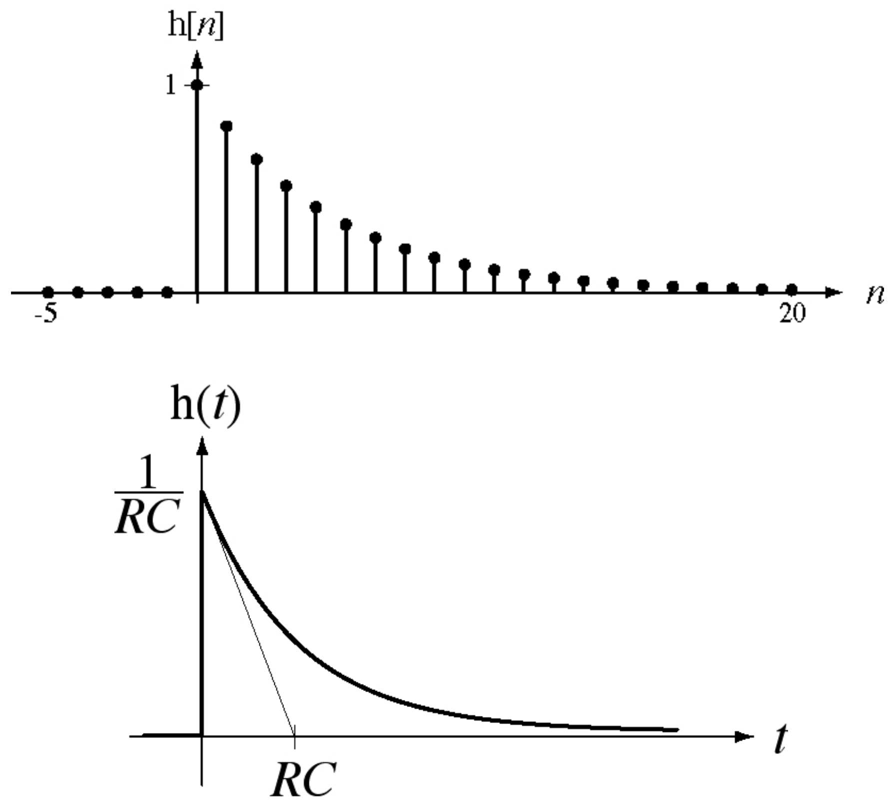






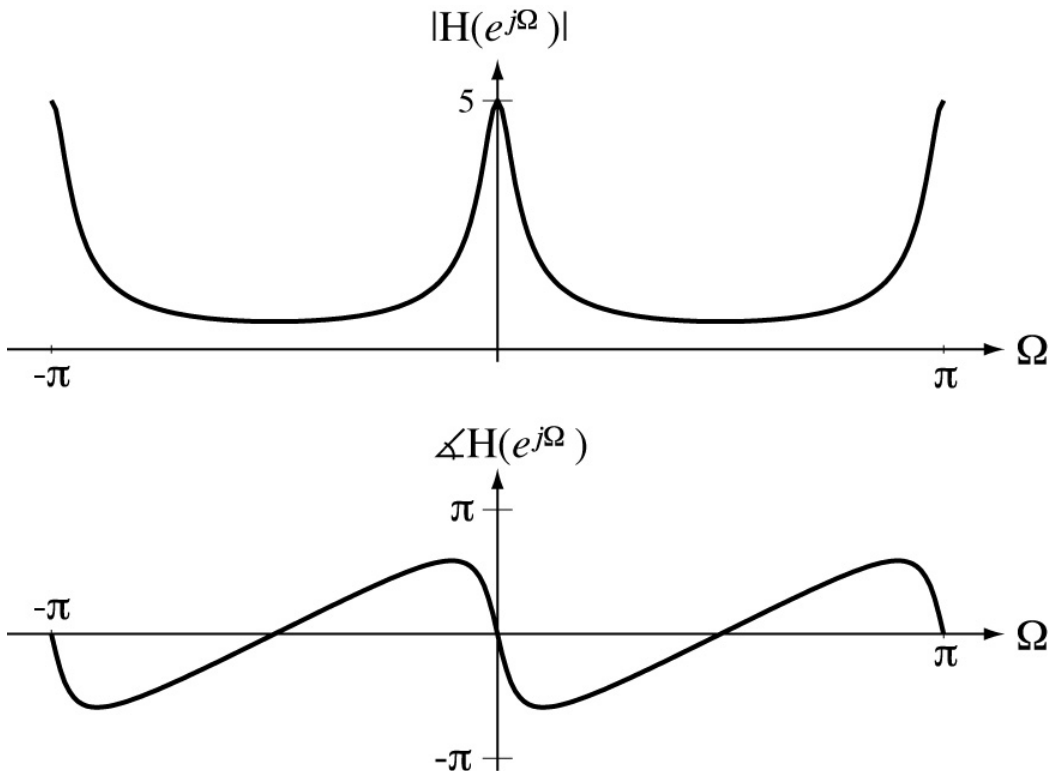
Discrete-Time Filters

Comparison of a discrete-time lowpass filter impulse response with an *RC* passive lowpass filter impulse response

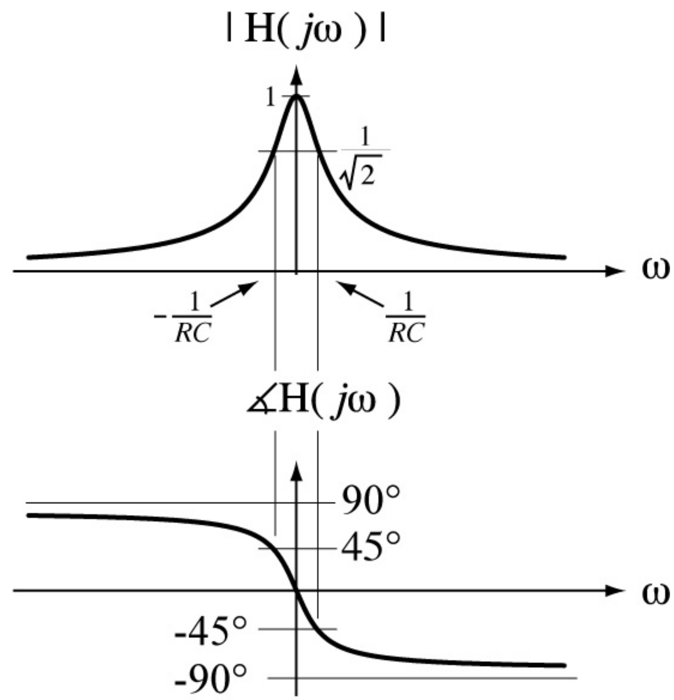


Discrete-Time Filters

Discrete-time Lowpass Filter Frequency Response

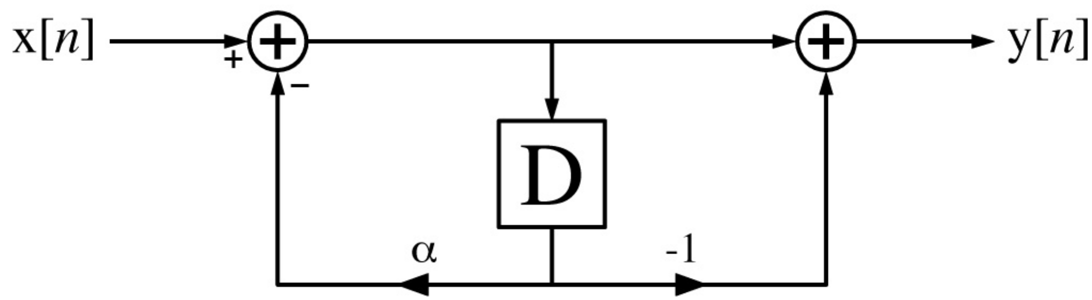


***RC* Lowpass Filter Frequency Response**

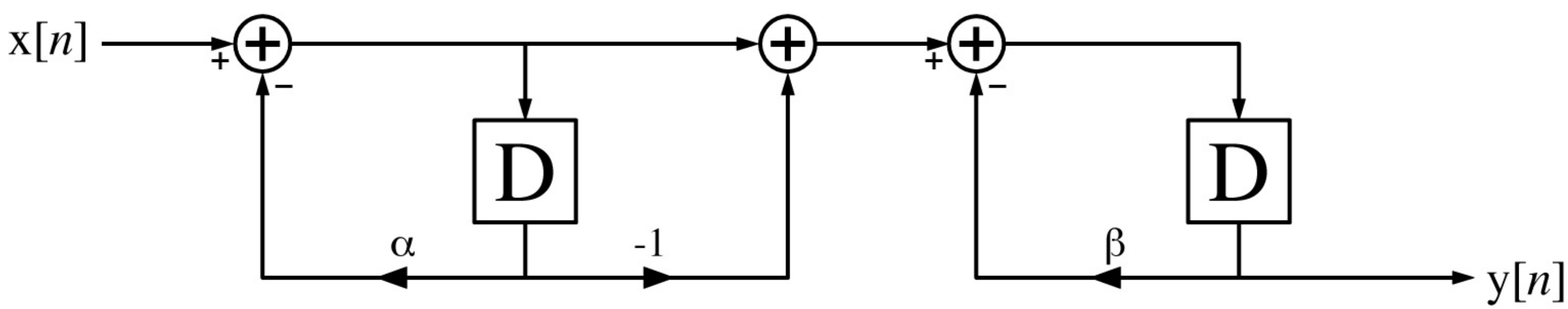


Discrete-Time Filters

Highpass

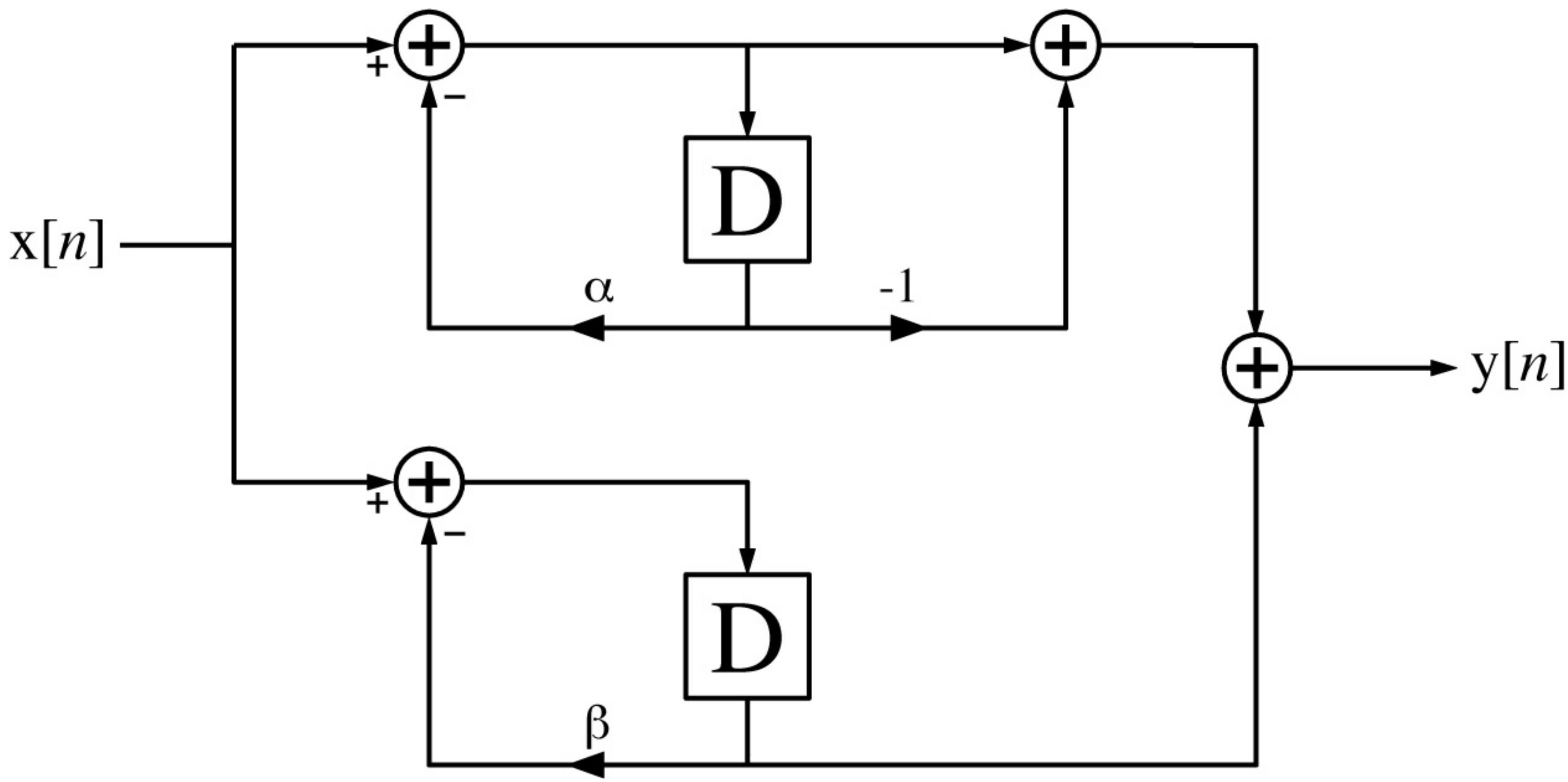


Bandpass



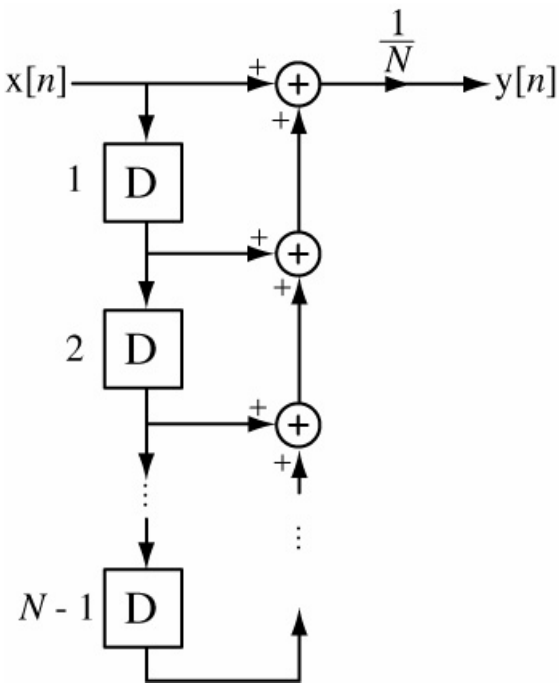
Discrete-Time Filters

Bandstop



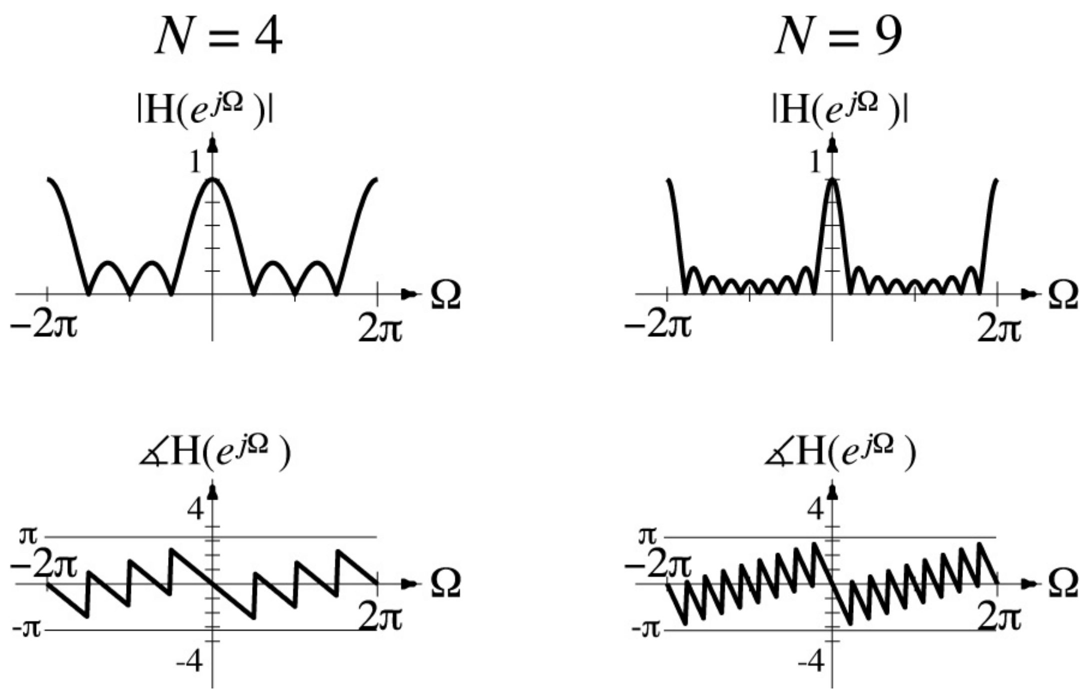
Discrete-Time Filters

Moving-Average Filter



$$H(e^{j\Omega}) = \frac{e^{-j(N-1)\Omega/2}}{N}\frac{\sin(N\Omega/2)}{\sin(\Omega/2)}$$

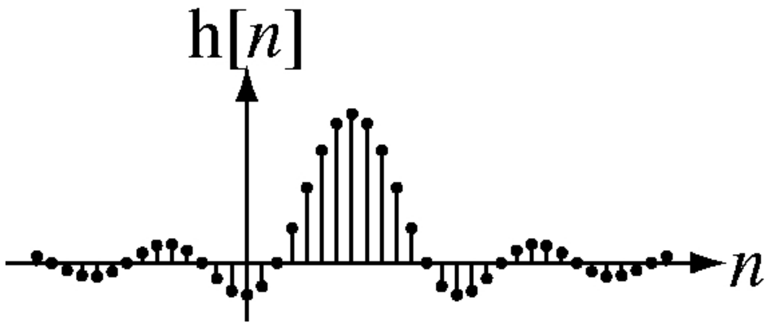
$$= e^{-j(N-1)\Omega/2}\,\mathrm{drcl}(\Omega/2\pi, N)$$



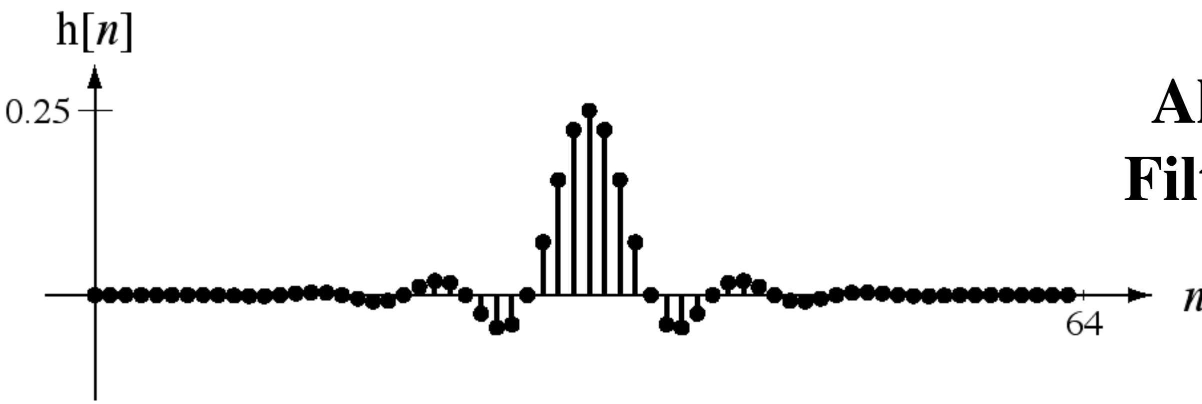
$$h[n] = (u[n] - u[n-N])/N$$

Always Stable

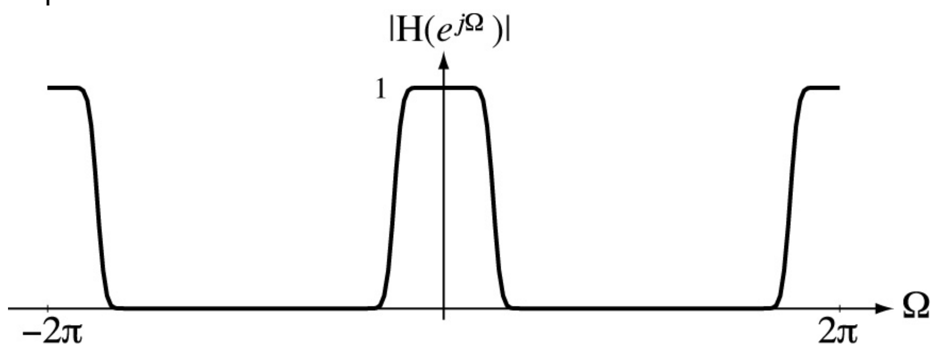
Discrete-Time Filters



Ideal Lowpass Filter Impulse Response



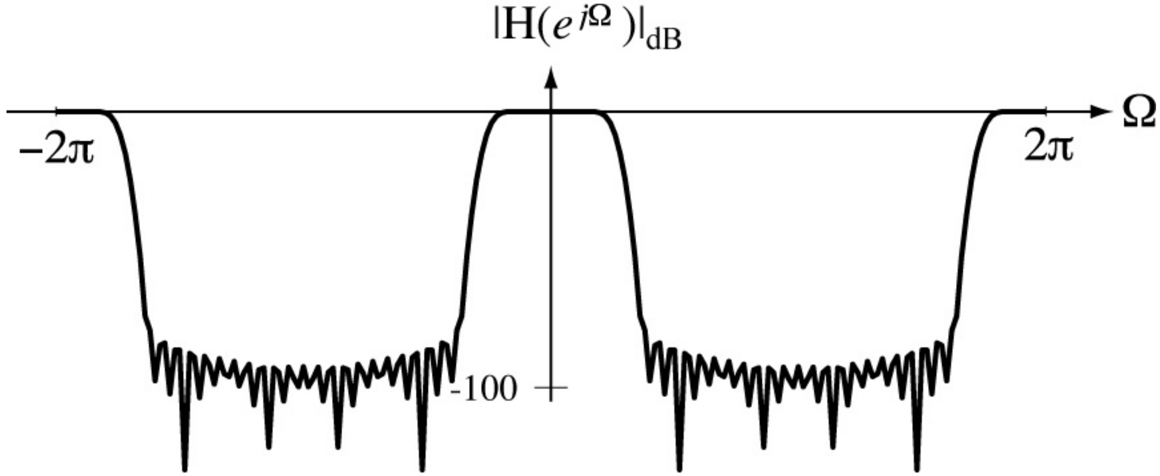
Almost-Ideal Lowpass Filter Impulse Response



Almost-Ideal Lowpass Filter Magnitude Frequency Response

Discrete-Time Filters

Almost-Ideal Lowpass Filter Magnitude Frequency Response in dB



Advantages of Discrete-Time Filters

- They are almost insensitive to environmental effects
- Continuous-time filters at low frequencies may require very large components, discrete-time filters do not
- Discrete-time filters are often **programmable** making them easy to modify
- Discrete-time signals can be stored indefinitely on magnetic media, stored continuous-time signals degrade over time
- Discrete-time filters can handle multiple signals by **multiplexing** them