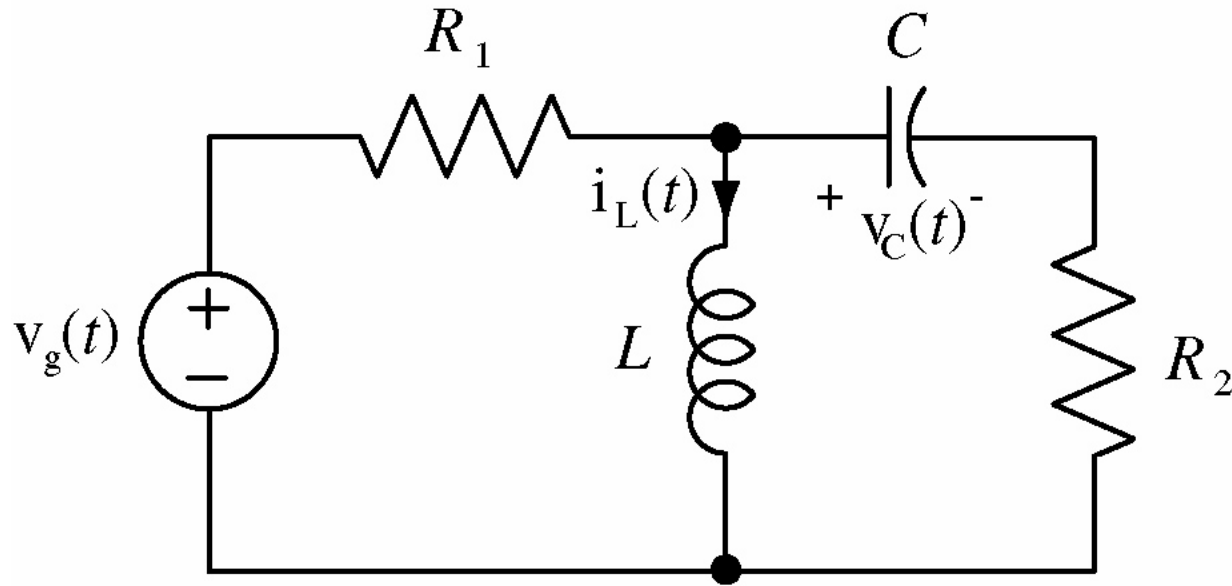# Laplace System Analysis

System Descriptions

- **Transfer functions** of continuous-time systems can be found from analysis of
  - Differential Equations
  - Block Diagrams
  - Circuit Diagrams

System Descriptions

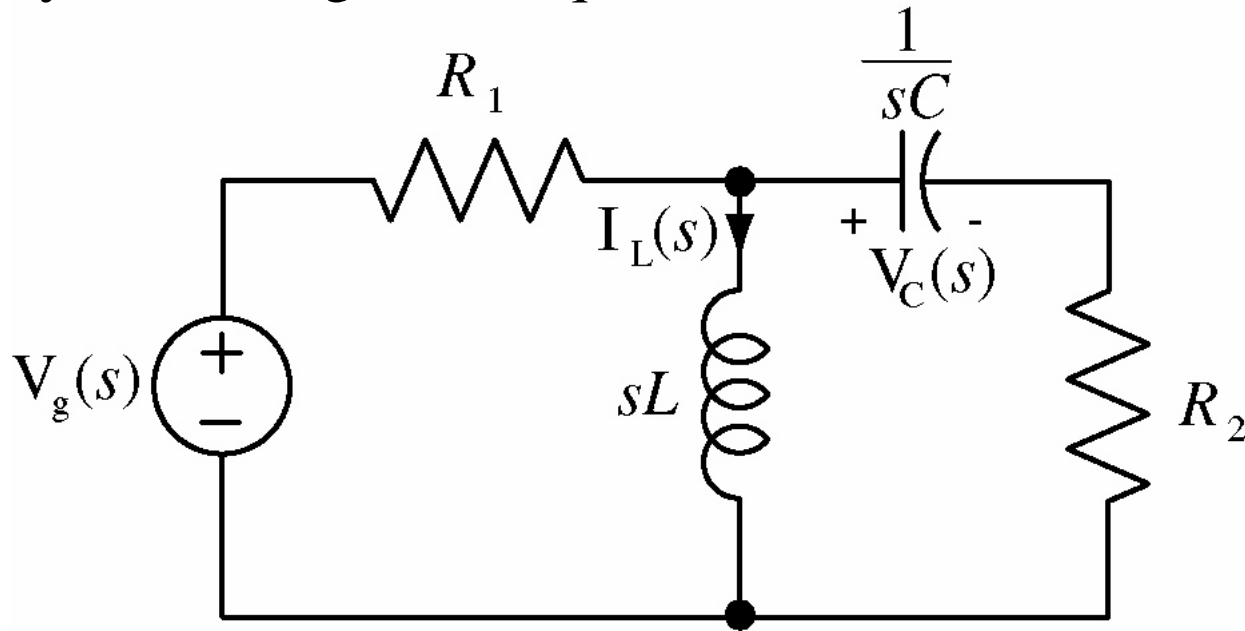A circuit can be described by a system of differential equations

$$-v_g(t) + R_1\left[i_L(t) + C\frac{d}{dt}(v_C(t))\right] + L\frac{d}{dt}(i_L(t)) = 0$$

$$-L\frac{d}{dt}(i_L(t)) + v_C(t) + R_2C\frac{d}{dt}(v_C(t)) = 0$$

System Descriptions

Using the Laplace transform, a circuit can be described by a system of algebraic equations

$$-V_g(s) + R_1\left[I_L(s) + sC\,V_C(s)\right] + sL\,I_L(s) = 0$$

$$-sL\,I_L(s) + V_C(s) + sR_2C\,V_C(s) = 0$$

System Descriptions

f(t) is the system excitation signal
and the velocity of mass, m_2,
$x'_2(t)$, is the system response signal, $y(t)$

A mechanical system can
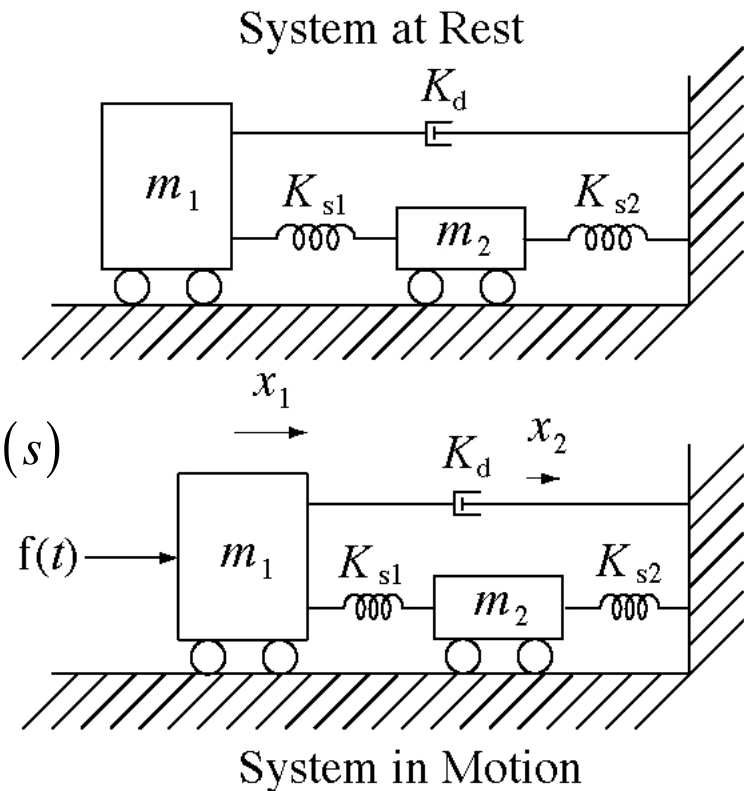be described by a system of differential equations

$$f(t) - K_d \, x'_1(t) - K_{s1}\left[x_1(t) - x_2(t)\right] = m_1 \, x''_1(t)$$

$$K_{s1}\left[x_1(t) - x_2(t)\right] - K_{s2} \, x_2(t) = m_2 \, x''_2(t)$$

or a system of algebraic equations.

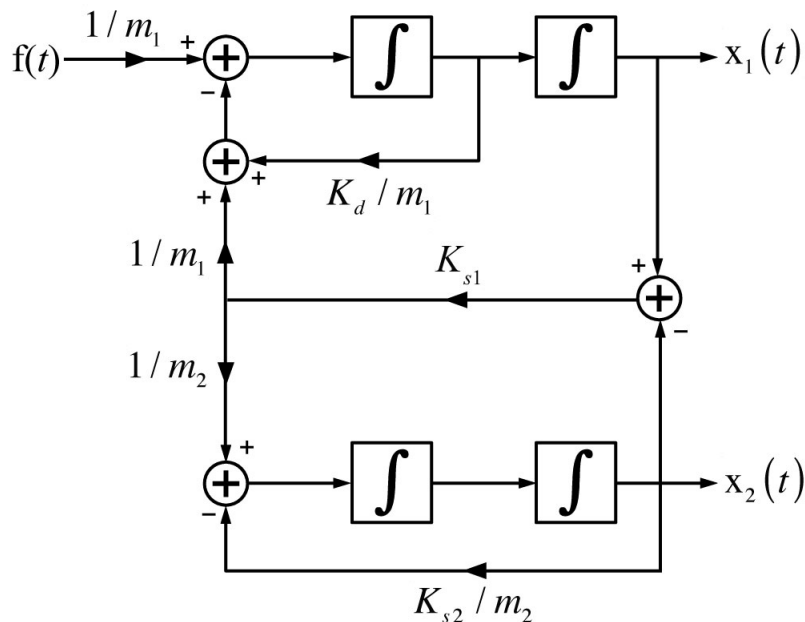$$F(s) - K_d s \, X_1(s) - K_{s1}\left[X_1(s) - X_2(s)\right] = m_1 s^2 \, X_1(s)$$

$$K_{s1}\left[X_1(s) - X_2(s)\right] - K_{s2} \, X_2(s) = m_2 s^2 \, X_2(s)$$
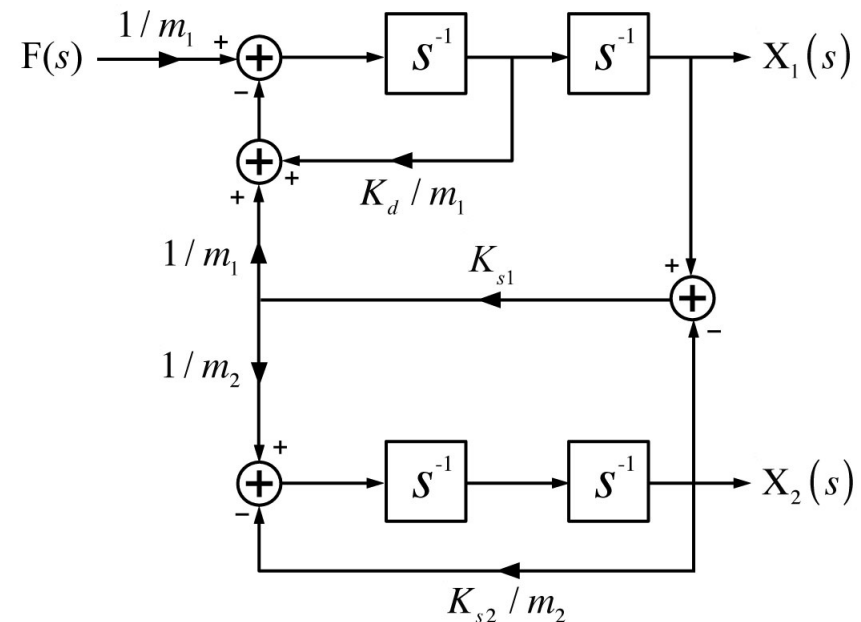

System at Rest

System in Motion

System Descriptions

The mechanical system can also be described by a block diagram.
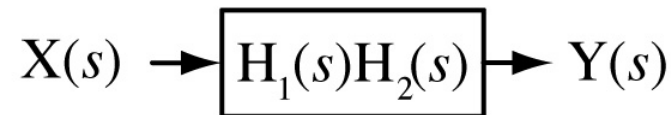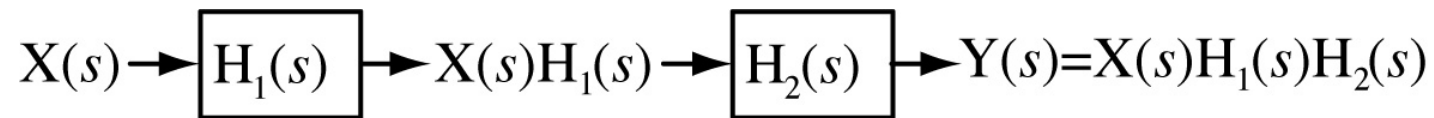
Time Domain

Frequency Domain

System Stability
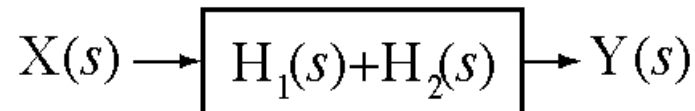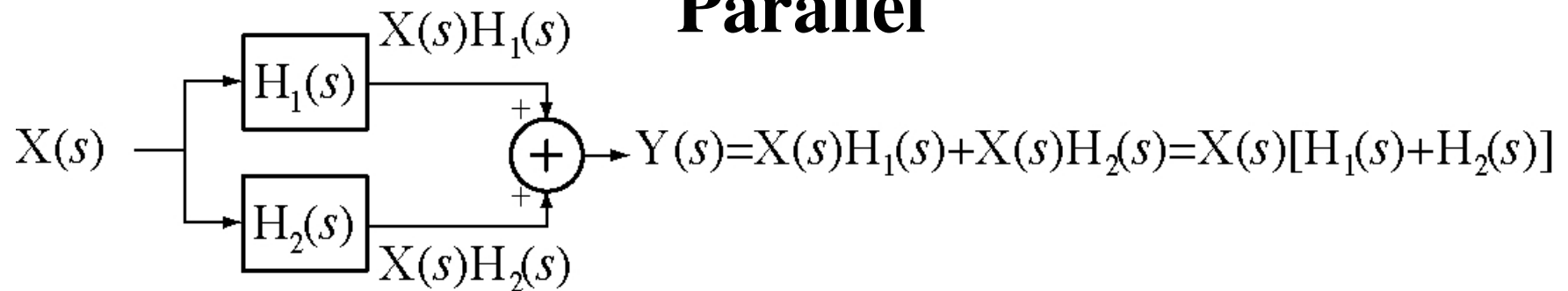
System **stability** is very important. A continuous-time LTI system is stable if its impulse response is **absolutely integrable**. This translates into the frequency domain as the requirement that all the poles of the system transfer function must lie in the <u>open</u> left half of the s plane (proven in the text). "Open left half-plane" means <u>not</u> including the ω axis.

System Interconnections

Cascade

$$X(s) \rightarrow \boxed{H_1(s)} \rightarrow X(s)H_1(s) \rightarrow \boxed{H_2(s)} \rightarrow Y(s)=X(s)H_1(s)H_2(s)$$

$$X(s) \rightarrow \boxed{H_1(s)H_2(s)} \rightarrow Y(s)$$

Parallel

$$X(s) \rightarrow \boxed{H_1(s)} \rightarrow X(s)H_1(s)$$
$$\boxed{H_2(s)} \rightarrow X(s)H_2(s)$$
$$\xrightarrow{+}{+} \bigoplus \rightarrow Y(s)=X(s)H_1(s)+X(s)H_2(s)=X(s)[H_1(s)+H_2(s)]$$

$$X(s) \rightarrow \boxed{H_1(s)+H_2(s)} \rightarrow Y(s)$$
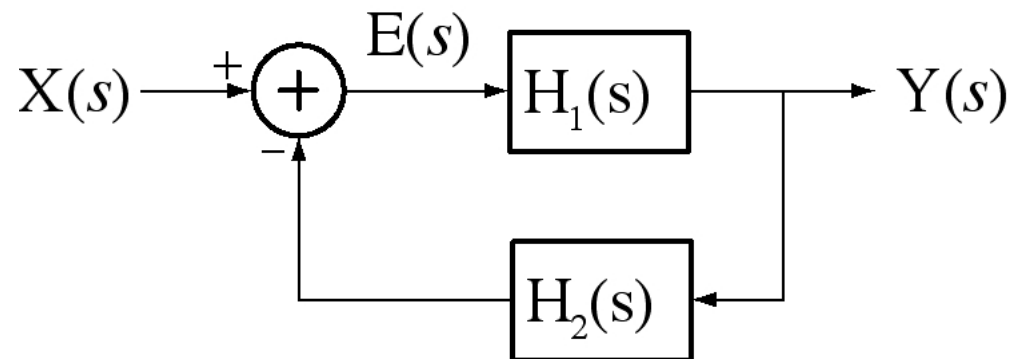
System Interconnections
Feedback

E(s) Error signal

H$_1(s)$ Forward path
transfer function of the
"plant"

H$_2(s)$ Feedback path
transfer function or the
"sensor".

$T(s) = H_1(s)H_2(s)$

Loop transfer function

$$E(s) = X(s) - H_2(s)Y(s)$$

$$Y(s) = H_1(s)E(s)$$
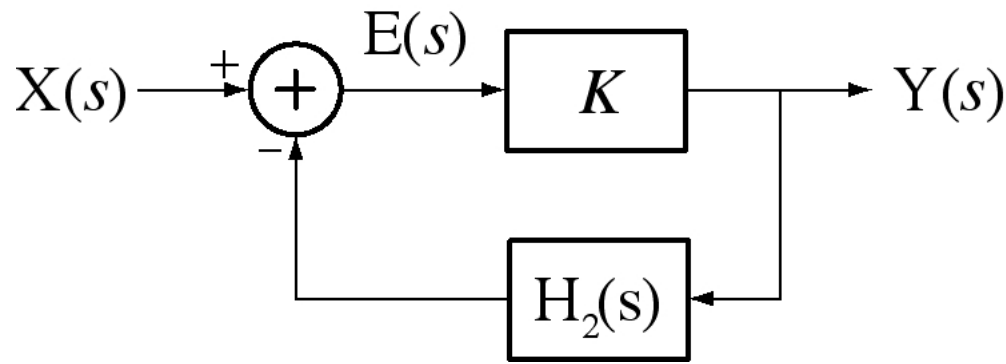
$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$

$$T(s) = H_1(s)H_2(s)$$

$$H(s) = \frac{H_1(s)}{1 + T(s)}$$

Analysis of Feedback Systems

Beneficial Effects

$$H(s) = \frac{K}{1 + K\,H_2(s)}$$

If K is large enough that $K\,H_2(s) \gg 1$ then $H(s) \approx \dfrac{1}{H_2(s)}$.

This means that the overall system is the approximate inverse of the system in the feedback path. This kind of system can be useful for reversing the effects of another system.

Analysis of Feedback Systems

A very important example of feedback systems is an electronic amplifier based on an operational amplifier
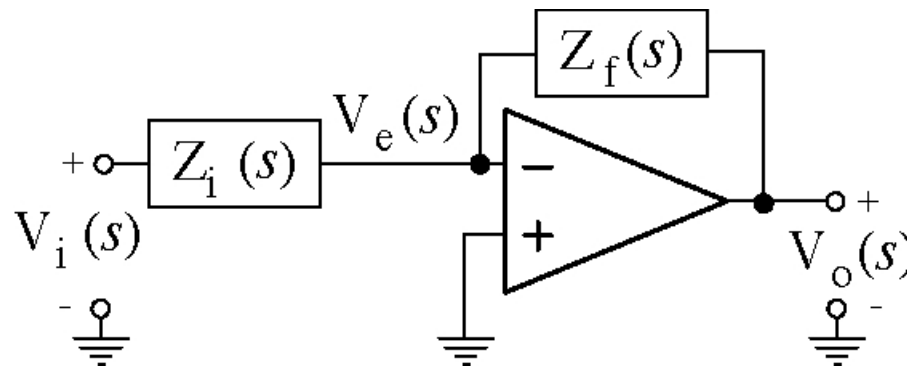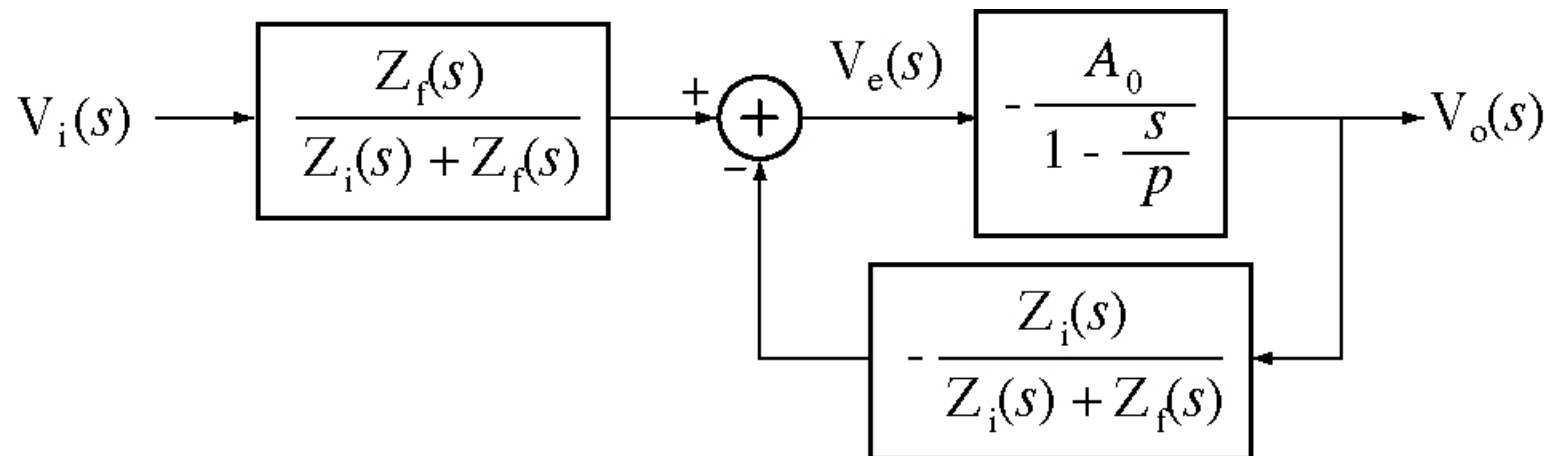

Let the operational amplifier gain be

$$H_1(s) = \frac{V_o(s)}{V_e(s)} = -\frac{A_0}{1 - s/p}$$

Analysis of Feedback Systems

The amplifier can be modeled as a feedback system with this block diagram.

The overall gain can be written as

$$\frac{V_0(s)}{V_i(s)} = \frac{-A_0 Z_f(s)}{(1 - s/p + A_0)Z_i(s) + (1 - s/p)Z_f(s)}$$

Analysis of Feedback Systems

If the operational amplifier low-frequency gain A_0 is very large (which it usually is) then the overall amplifier gain reduces at low-frequencies to

$$\frac{V_0(s)}{V_i(s)} \cong -\frac{Z_f(s)}{Z_i(s)}$$

the gain formula based on an ideal operational amplifier.

Analysis of Feedback Systems

Let $Z_f(s) = 10$ kΩ and let $Z_i(s) = 1$ kΩ for a nominal gain of -10.

If $A_0 = 10^7$ and $p = -100$ then $H(-j100) = -9.999989 + j0.000011$

If $A_0 = 10^6$ and $p = -100$ then $H(-j100) = -9.99989 + j0.00011$

The change in overall system gain is about 0.001% for a change in open-loop gain of a factor of 10.

The half-power bandwidth of the operational amplifier itself is 15.9 Hz $(100/2\pi)$. The half-power bandwidth of the overall amplifier is approximately 14.5 MHz, an increase in bandwidth of a factor of approximately 910,000.

Analysis of Feedback Systems

Feedback can stabilize an unstable system. Let a forward-path transfer function be

$$H_1(s) = \frac{1}{s - p} \quad , \quad p > 0$$

This system is unstable because it has a pole in the right half-plane. If we then connect feedback with a transfer function K, a constant, the overall system gain becomes

$$H(s) = \frac{1}{s - p + K}$$

and, if $K > p$, the overall system is now stable.

Analysis of Feedback Systems

Feedback can make an unstable system stable but it can also make a stable system unstable. Even though all the poles of the forward and feedback systems may be in the open left half-plane, the poles of the overall feedback system can be in the right half-plane.

A familiar example of this kind of instability caused by feedback is a public address system. If the amplifier gain is set too high the system will go unstable and oscillate, usually with a very annoying high-pitched tone.
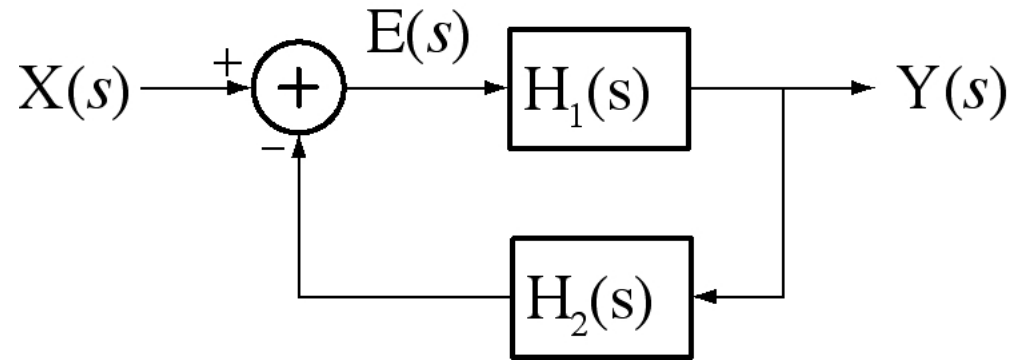
Analysis of Feedback Systems

As the amplifier gain is increased, any sound entering the microphone makes a stronger sound from the speaker until, at some gain level, the returned sound from the speaker is a large as the originating sound into the microphone. At that point the system goes unstable.
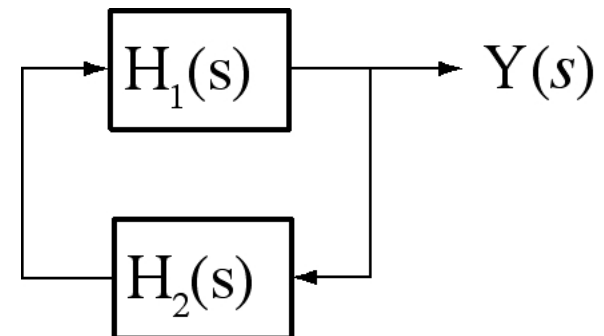
Public Address System
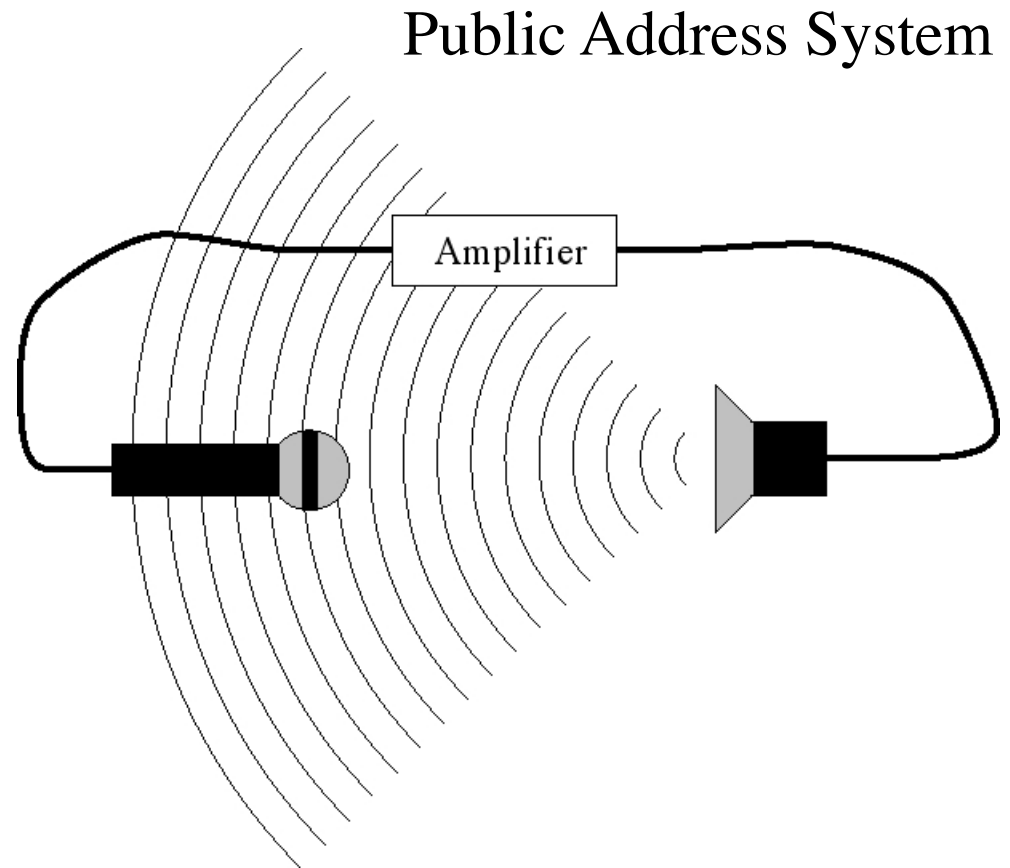
Analysis of Feedback Systems

Stable Oscillation Using Feedback
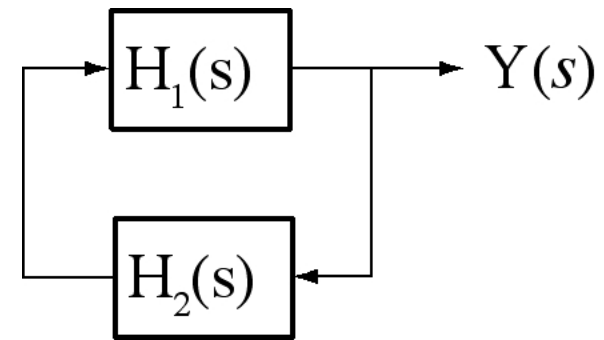
Prototype
Feedback
System

Feedback
System
Without
Excitation

Analysis of Feedback Systems
Stable Oscillation Using Feedback

Can the response be non-zero when the
excitation is zero? Yes, if the overall
system gain is infinite. If the system
transfer function has a pole pair on the
ω axis, then the transfer function is infinite
at the frequency of that pole pair and there can be a response without
an excitation. In practical terms the trick is to be sure the poles stay
on the ω axis. If the poles move into the left half-plane the response
attenuates with time. If the poles move into the right half-plane the
response grows with time (until the system goes non-linear).

Analysis of Feedback Systems
Stable Oscillation Using Feedback

A real example of a system that oscillates stably is a laser.
In a laser the forward path is an optical amplifier.

Pumped Laser Medium

Light In

Light Out
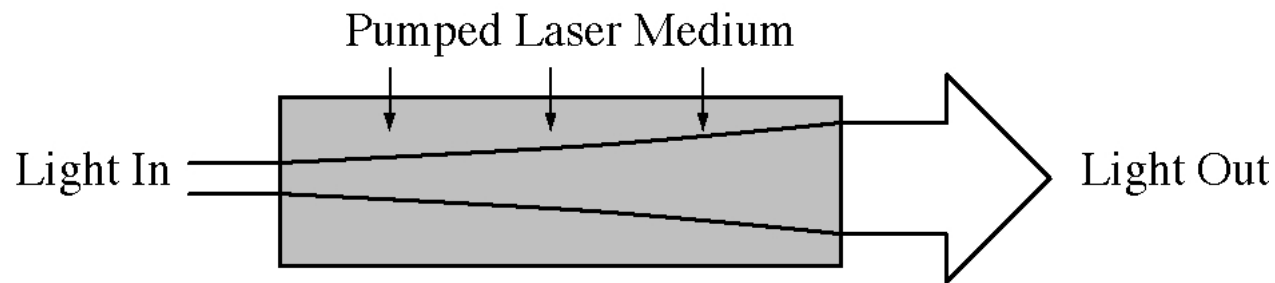
The feedback action is provided by putting mirrors at each
end of the optical amplifier.

Mirror   Pumped Laser Medium   Mirror
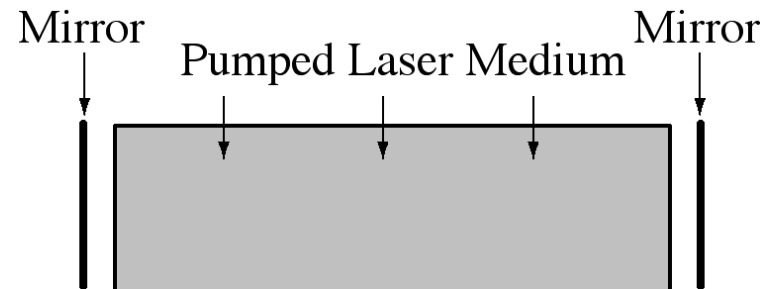
Analysis of Feedback Systems
Stable Oscillation Using Feedback

Laser action begins when a photon is spontaneously emitted from the pumped medium in a direction normal to the mirrors.
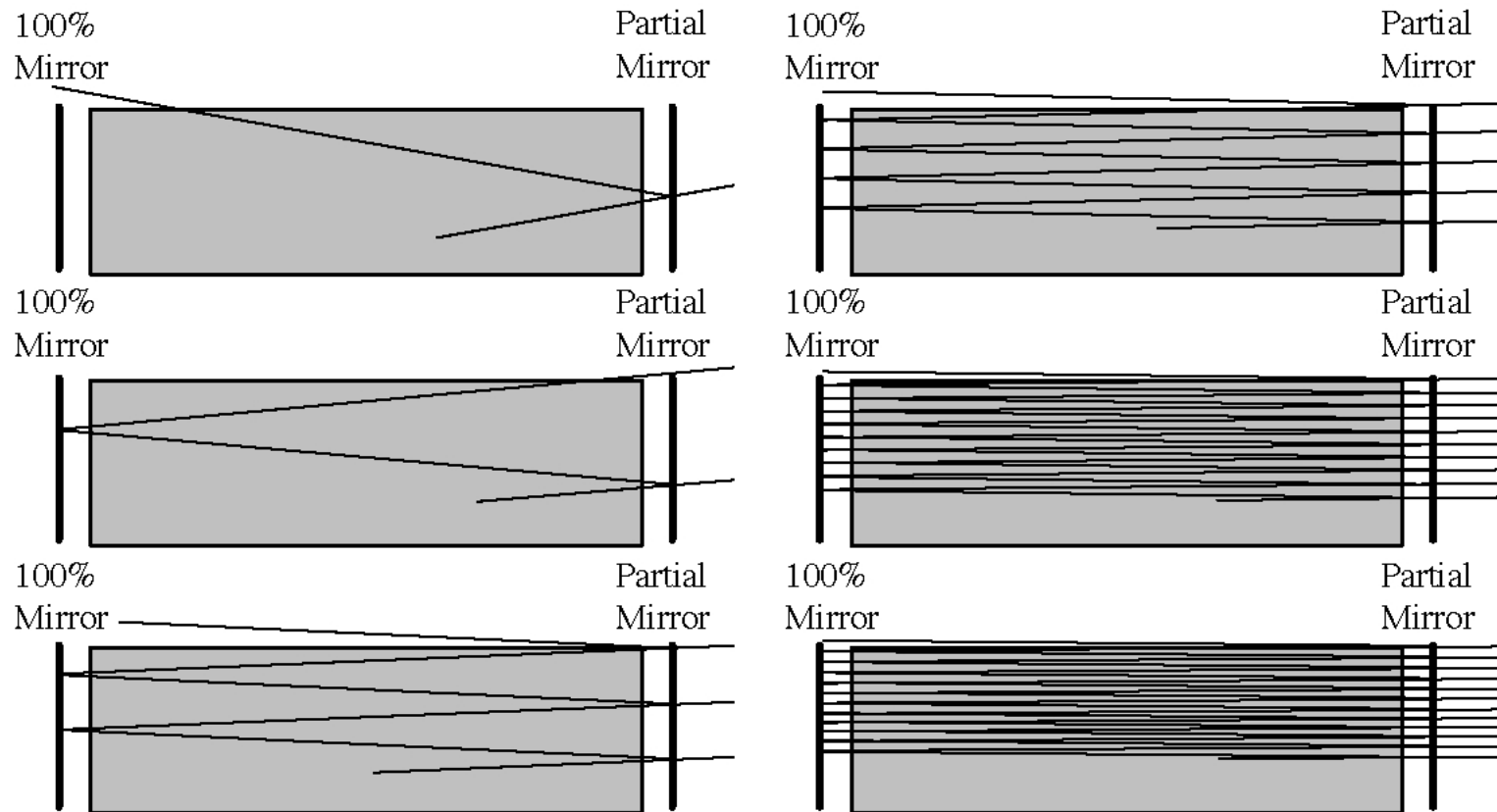
Analysis of Feedback Systems
Stable Oscillation Using Feedback

If the "round-trip" gain of the combination of pumped laser medium and mirrors is unity, sustained oscillation of light will occur. For that to occur the wavelength of the light must fit into the distance between mirrors an integer number of times.
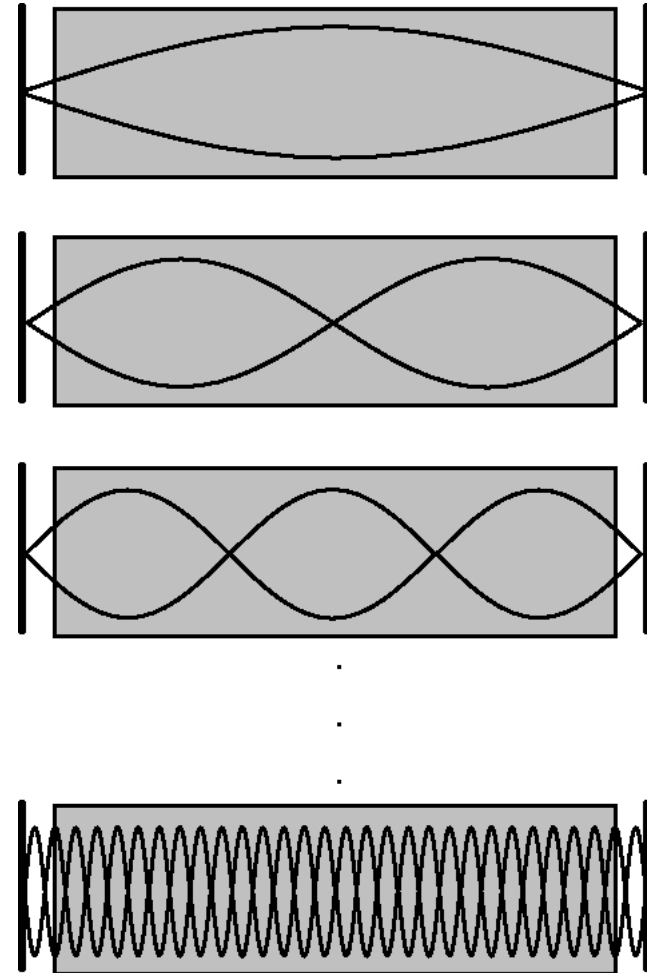
Analysis of Feedback Systems
Stable Oscillation Using Feedback

A laser can be modeled by a block diagram in which the K's represent the gain of the pumped medium or the reflection or transmission coefficient at a mirror, L is the distance between mirrors and c is the speed of light.
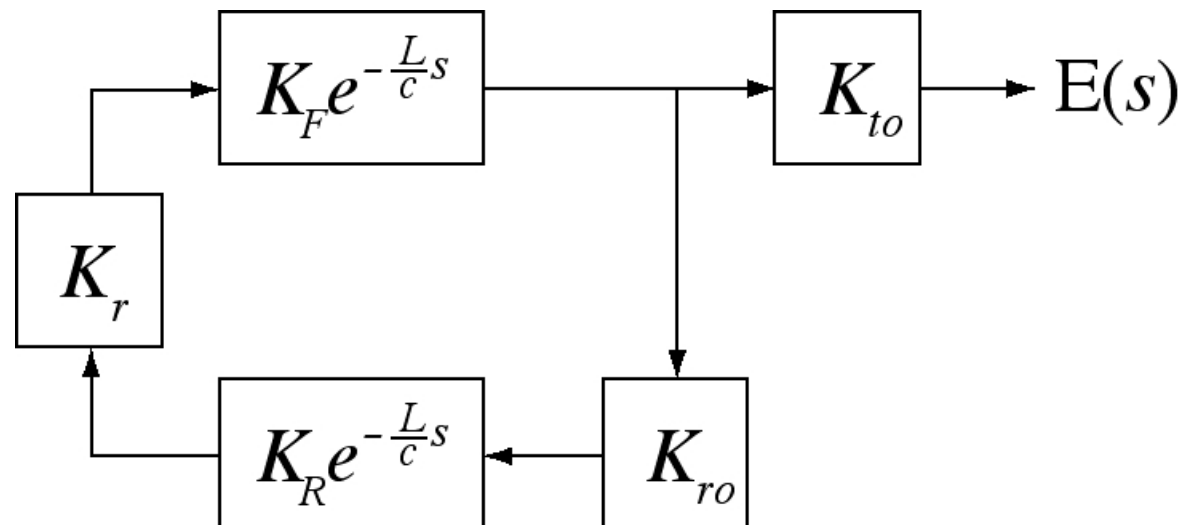
Analysis of Feedback Systems

Root Locus

Common Type of Feedback System
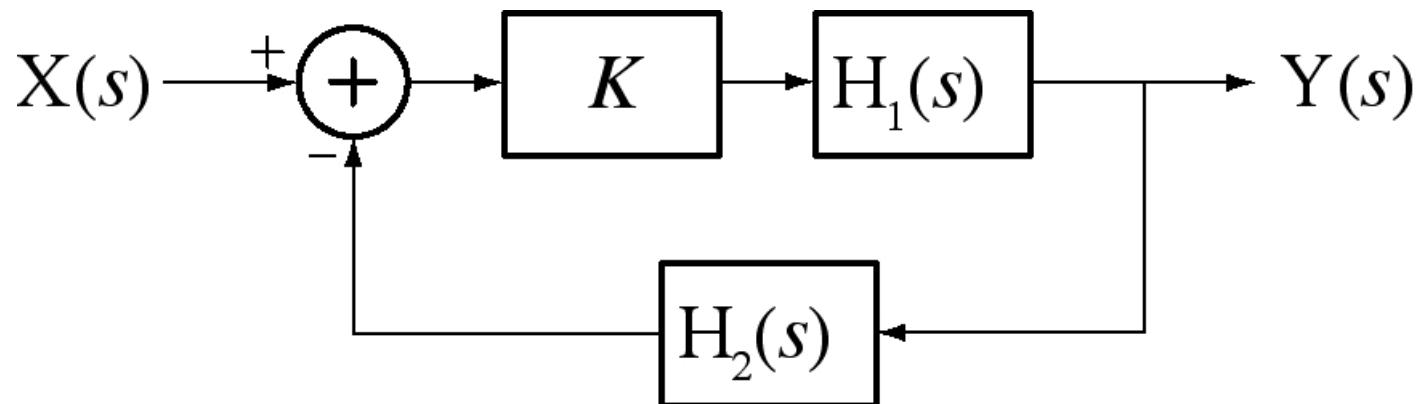
System Transfer Function $\qquad$ $H(s) = \dfrac{K\,H_1(s)}{1 + K\,H_1(s)\,H_2(s)}$

Loop Transfer Function $\qquad$ $T(s) = K\,H_1(s)\,H_2(s)$

Analysis of Feedback Systems
Root Locus

Poles of H(s) $\longrightarrow$ Zeros of 1 + T(s)

T is of the form $\longrightarrow$ $T(s) = K\dfrac{P(s)}{Q(s)}$

Poles of H(s) $\longrightarrow$ Zeros of $1 + K\dfrac{P(s)}{Q(s)}$

Poles of H(s) $\longrightarrow$
$$Q(s) + K\,P(s) = 0$$
$$Q(s)/K + P(s) = 0$$

Analysis of Feedback Systems
Root Locus

K can range from zero to infinity. For K approaching zero, using $Q(s) + K P(s) = 0$, the poles of H are the same as the zeros of $Q(s) = 0$ which are the poles of T. For K approaching infinity, using $Q(s) / K + P(s) = 0$ the poles of H are the same as the zeros of $P(s) = 0$ which are the zeros of T. So the poles of H start on the poles of T and terminate on the zeros of T, some of which may be at infinity. The curves traced by these pole locations as K is varied are called the **root locus**.
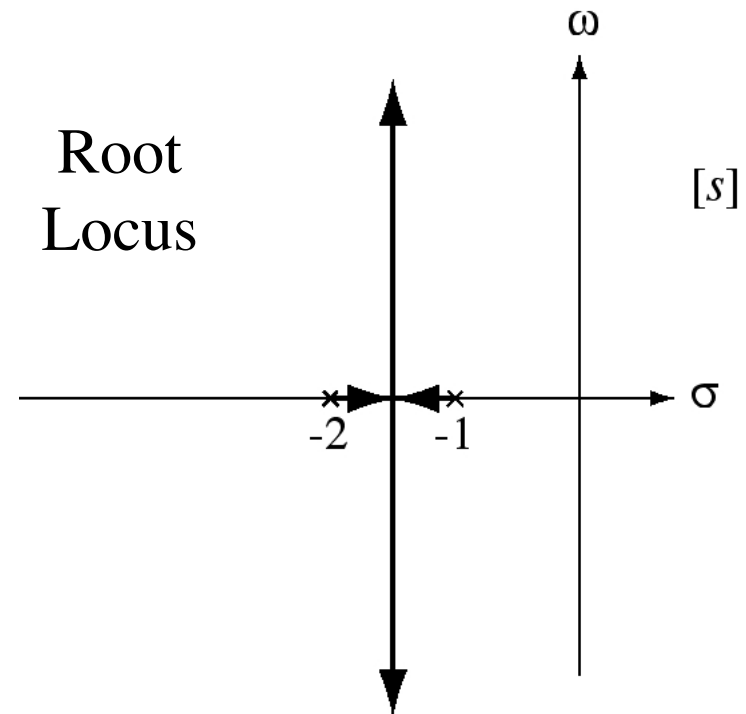
Analysis of Feedback Systems

Root Locus

Let $H_1(s) = \dfrac{K}{(s+1)(s+2)}$ and let $H_2(s) = 1$.

Then $T(s) = \dfrac{K}{(s+1)(s+2)}$

No matter how large K gets this system is stable because the poles always lie in the left half-plane (although for large K the system may be very underdamped).
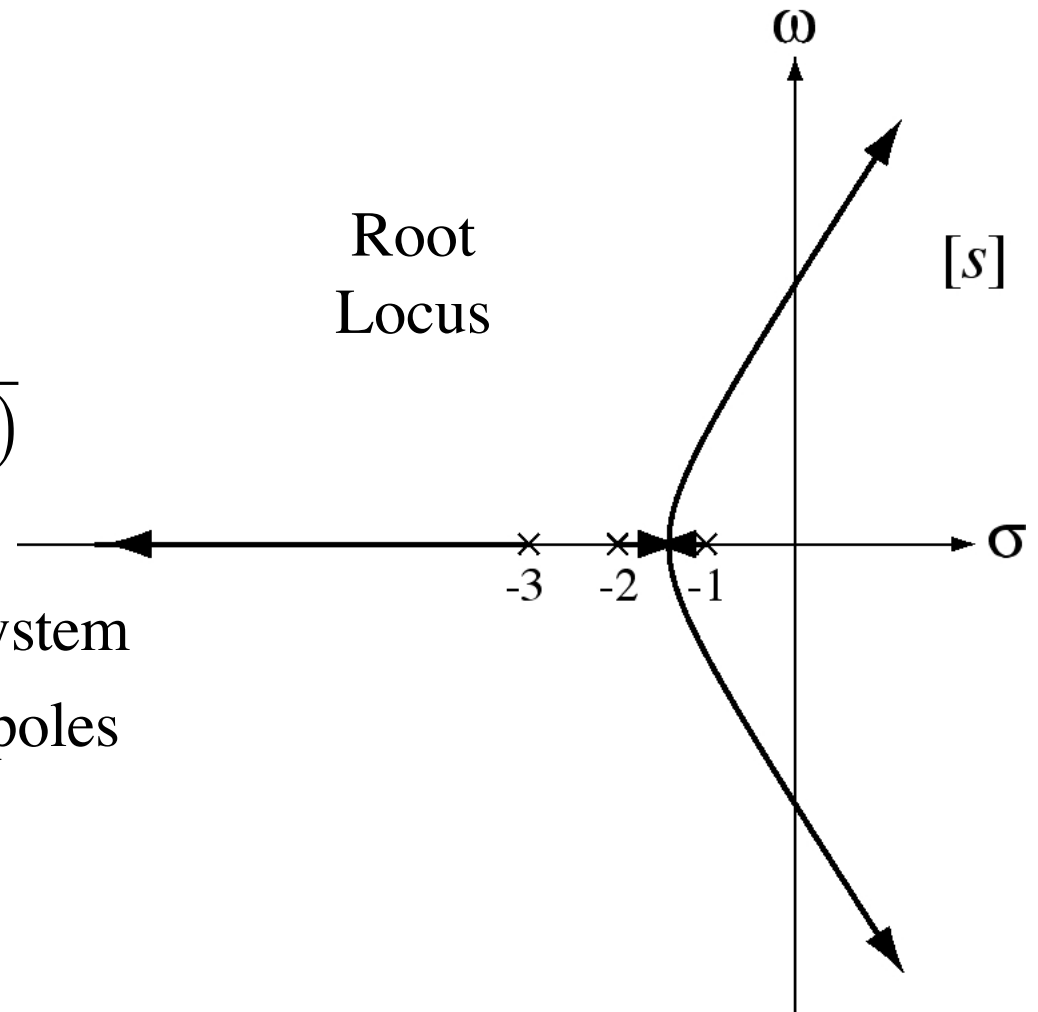
Root Locus

Analysis of Feedback Systems
Root Locus

Let $H_1(s) = \dfrac{K}{(s+1)(s+2)(s+3)}$

and let $H_2(s) = 1$

At some finite value of K the system becomes unstable because two poles move into the right half-plane.

Root Locus

$[s]$

ω

σ

-3   -2   -1

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Analysis of Feedback Systems
Root Locus

The behavior of the zeros of polynomial equations as their coefficients vary has been studied by mathematicians and they have formulated rules obeyed by the movement of these zeros. In our case, the zeros of the denominator polynomial of the transfer function are the system poles and the movement traces out the root locus.

Analysis of Feedback Systems

Root Locus

Rules for Drawing a Root Locus

1. The number of branches in a root locus is the greater of the degree of the numerator and the denominator of $T(s)$.

2. Each root-locus branch begins on a pole of $T(s)$ and terminates on a zero of $T(s)$. (Some zeros may be at infinity.)

3. Any portion of the real axis for which the sum of the number of real poles and/or real zeros lying to its right is odd, is a part of the root locus.

4. The root locus is symmetrical about the real axis.

Analysis of Feedback Systems
Root Locus

5. If the number of finite poles of $T(s)$ exceeds the number of finite zeros of $T(s)$ by an integer m then m branches of the root locus terminate on zeros of $T(s)$ which lie at infinity. Each of these branches approaches a straight-line asymptote and the angles of these asymptotes are at the angles, $(2k+1)\pi/m$, $k = 0,1,\cdots m-1$ with respect to the positive real axis. These asymptotes intersect on the real axis at the location,

$$\sigma = \frac{1}{m}\left(\sum \text{finite poles} - \sum \text{finite zeros}\right)$$

called the **centroid** of the root locus. (These are sums of <u>all</u> finite poles and <u>all</u> finite zeros, not just the ones on the real axis.)

Analysis of Feedback Systems
Root Locus

6. The breakaway or break-in points where the root locus branches

    intersect occur where $\dfrac{d}{ds}\left(\dfrac{1}{\mathrm{T}(s)}\right)=0$.

Analysis of Feedback Systems
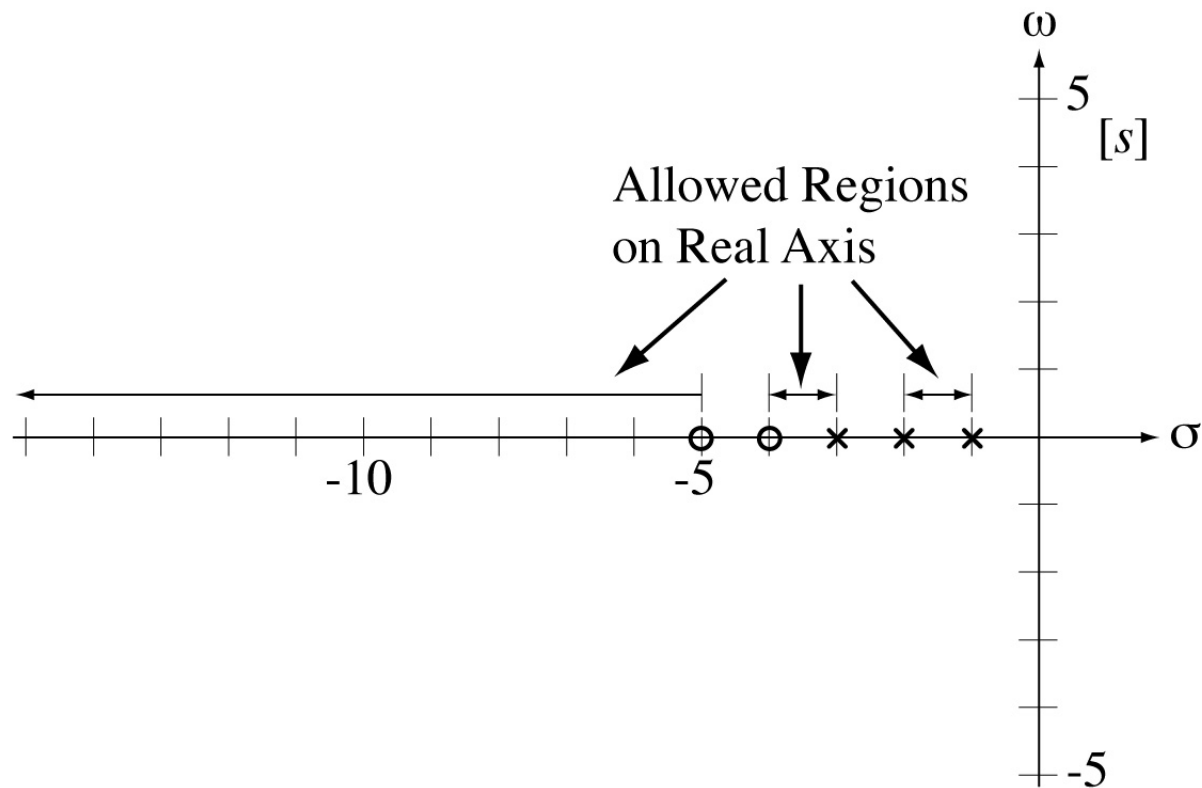
Root Locus Examples

Let a feedback system have a loop transfer function

$$T(s) = \frac{(s+4)(s+5)}{(s+1)(s+2)(s+3)}$$

It has three finite poles at $s = -1, -2$ and -3 and two finite zeros at $s = -4$ and -5. There are three root locus branches (Rule 1). The allowed regions on the real axis are $-2 < \sigma < -1$, $-4 < \sigma < -3$ and $\sigma < -5$ (Rule 3).

Analysis of Feedback Systems

Root Locus Examples

The root locus must begin on the poles at -1, -2 and -3 and terminate on zeros at -4, -5 and infinity (Rule 2).
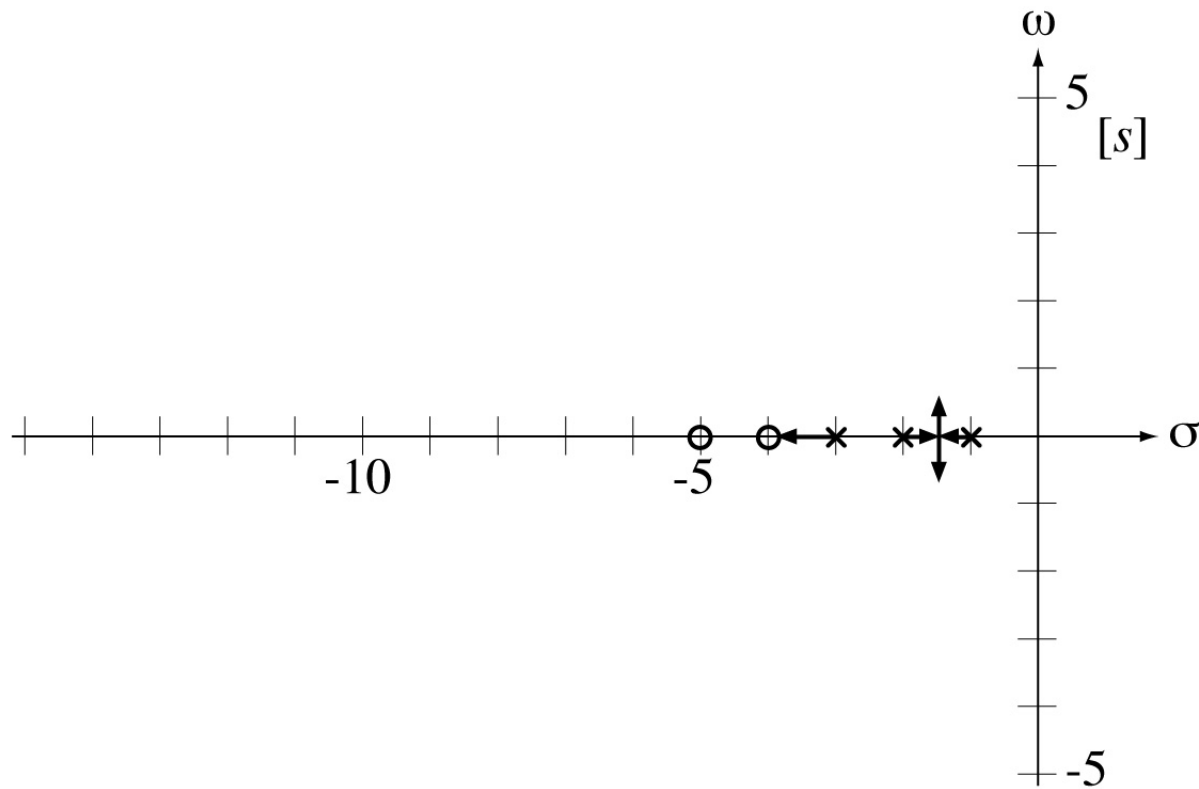
Analysis of Feedback Systems
Root Locus Examples

The two root locus branches beginning at -1 and -2 must move toward each other to stay in an allowed range (Rule 3). When they collide at a breakout point they both become complex and must be complex conjugates (Rule 4). The other branch beginning at -3 must move to the left to stay in an allowed range and can only terminate on the zero at -4 to maintain the symmetry of the root locus (Rules 3 and 4).

Analysis of Feedback Systems

Root Locus Examples

Analysis of Feedback Systems

Root Locus Examples

The other two root-locus branches must terminate on the zero at -5 and the zero at infinity. To maintain symmetry and approach the zeros in allowed regions, they must move to the allowed region on the real axis to the left of -5 (Rules 3 and 4). There is only one branch going to infinity and its angle is π radians as it should be (Rule 5). The breakout and break-in points are found by solving
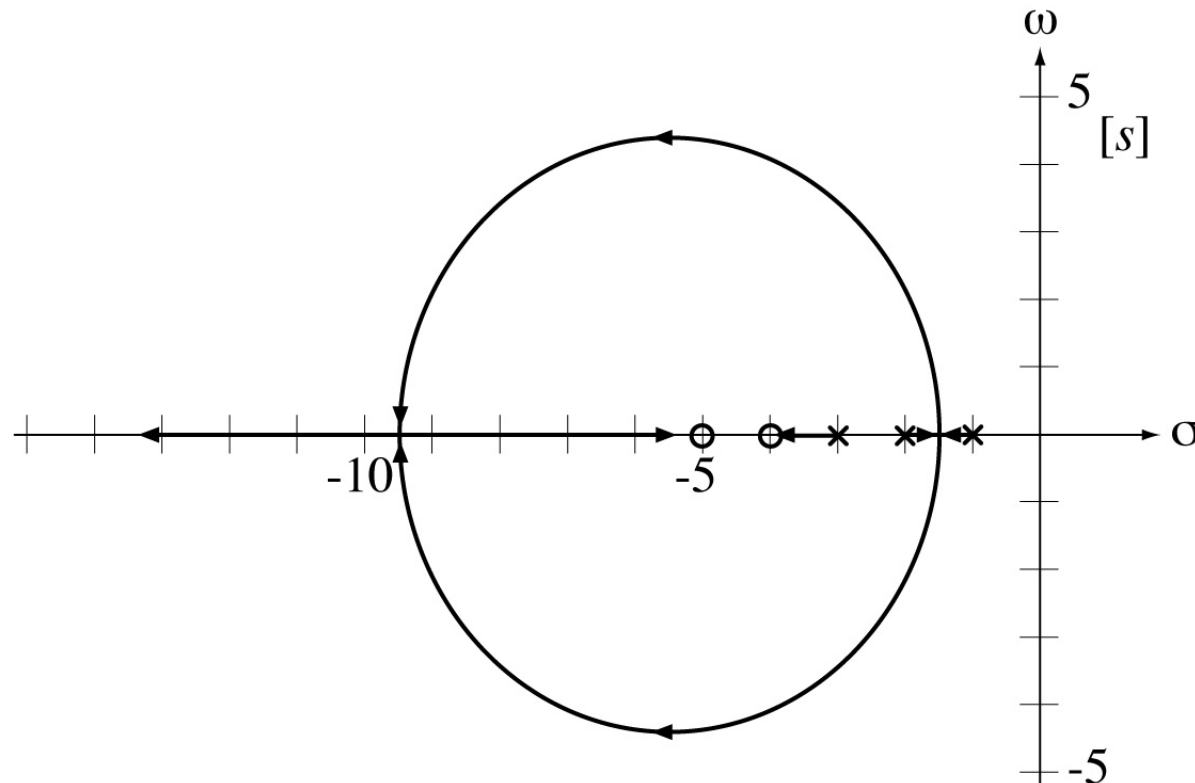
$$\frac{d}{ds}\left(\frac{1}{T(s)}\right) = \frac{d}{ds}\left[\frac{(s+1)(s+2)(s+3)}{(s+4)(s+5)}\right] = 0 \text{ (Rule 6)}.$$

The solutions are $s = -9.47, \ -4.34, \ -2.69$ and -1.5. So the breakout point is at -1.5 and the break-in point is at -9.47.
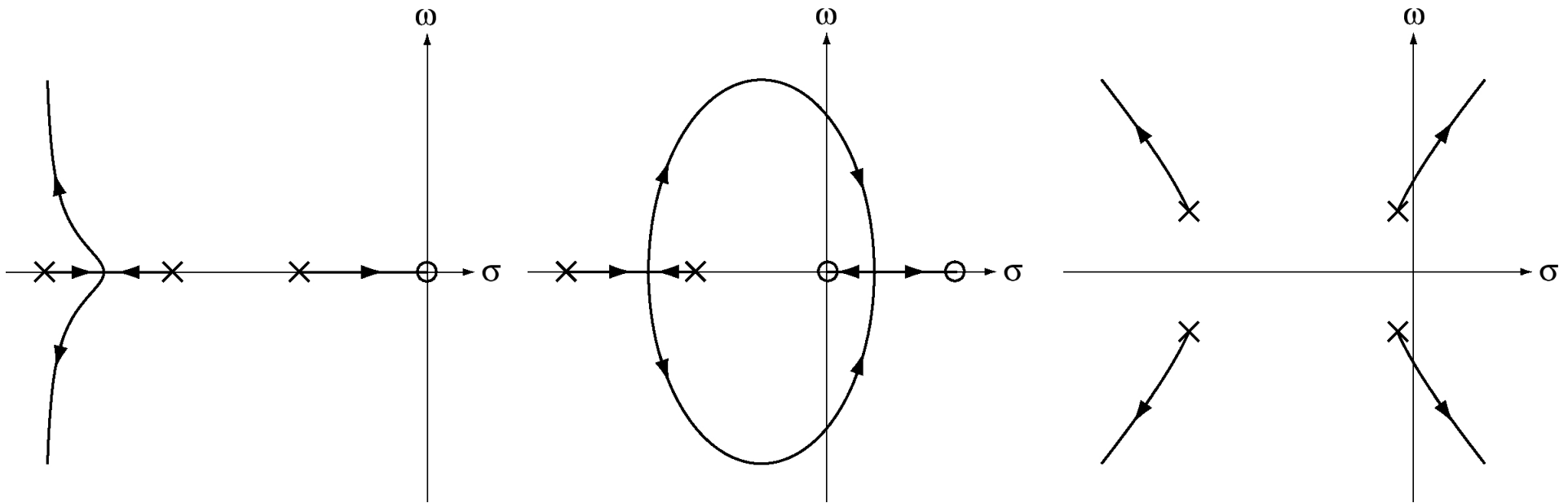
Analysis of Feedback Systems

Root Locus Examples

(The other solutions of $(d/ds)(1/\mathrm{T}(s))=0$ are the breakout and break-in points of the **complementary root locus** found by letting K approach <u>negative</u> infinity.)
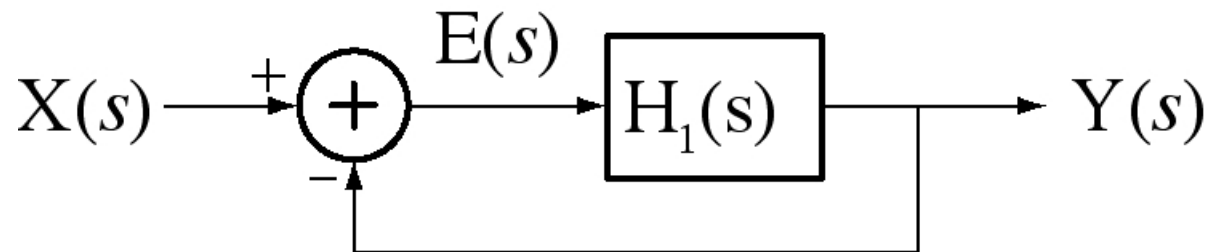
Analysis of Feedback Systems

Root Locus Examples

Analysis of Feedback Systems

Steady-State Tracking Errors in Unity-Gain Feedback Systems

A very common type of feedback system is the **unity**-**gain** feedback connection.

The aim of this type of system is to make the response "track" the excitation. When the error signal is zero, the excitation and response are equal.

Analysis of Feedback Systems

Steady-State Tracking Errors in Unity-Gain Feedback Systems

The Laplace transform of the error signal is

$$E(s) = \frac{X(s)}{1 + H_1(s)}.$$

The steady-state value of this signal is (using the final-value theorem)

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} s\, E(s) = \lim_{s \to 0} s \frac{X(s)}{1 + H_1(s)}$$

If the excitation is the unit step $u(t)$ then the steady- state error is

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} \frac{1}{1 + H_1(s)}$$

Analysis of Feedback Systems

Steady-State Tracking Errors in Unity-Gain Feedback Systems

If the forward transfer function is in the common form,

$$H_1(s) = \frac{b_N s^N + b_{N-1} s^{N-1} + \cdots b_2 s^2 + b_1 s + b_0}{a_D s^D + a_{D-1} s^{D-1} + \cdots a_2 s^2 + a_1 s + a_0} \quad \text{then}$$

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} \frac{1}{1 + \dfrac{b_N s^N + b_{N-1} s^{N-1} + \cdots b_2 s^2 + b_1 s + b_0}{a_D s^D + a_{D-1} s^{D-1} + \cdots a_2 s^2 + a_1 s + a_0}} = \frac{a_0}{a_0 + b_0}$$

If $a_0 = 0$ and $b_0 \neq 0$ the steady-state error is zero and the forward transfer function can be written as

$$H_1(s) = \frac{b_N s^N + b_{N-1} s^{N-1} + \cdots b_2 s^2 + b_1 s + b_0}{s\left(a_D s^{D-1} + a_{D-1} s^{D-2} + \cdots a_2 s + a_1\right)} \quad \text{which has a pole at } s = 0.$$

Analysis of Feedback Systems

Steady-State Tracking Errors in Unity-Gain Feedback Systems

If the forward transfer function of a unity-gain feedback system has a pole at zero and the system is stable, the steady-state error with step excitation is zero. This type of system is called a "type 1" system (one pole at $s = 0$ in the forward transfer function). If there are no poles at $s = 0$, it is called a "type 0" system and the steady-state error with step excitation is non-zero.
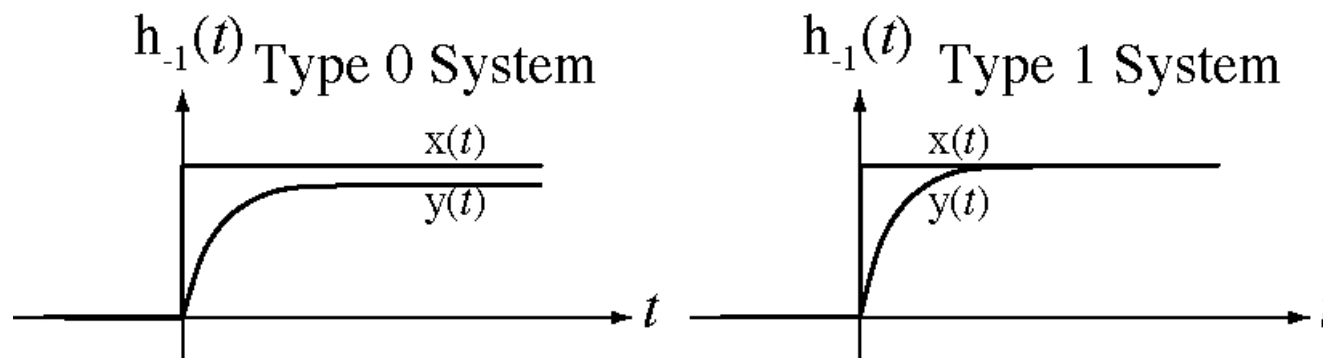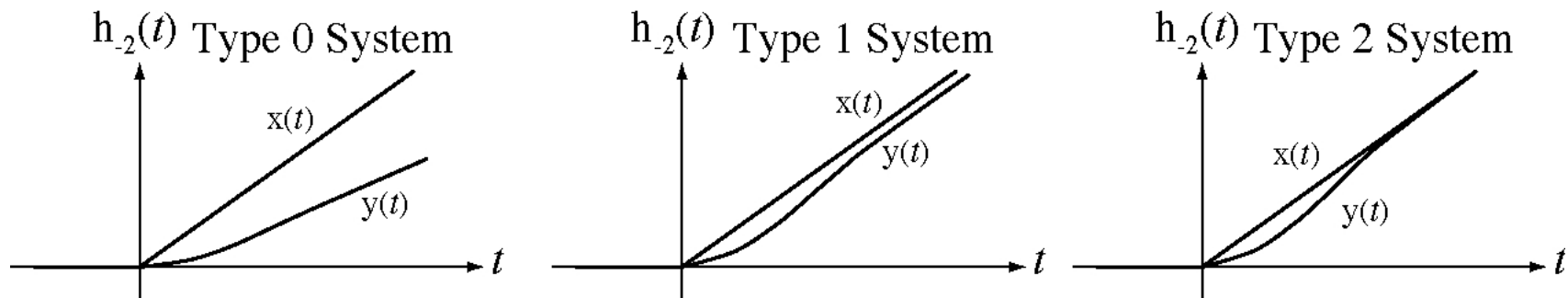
Analysis of Feedback Systems

Steady-State Tracking Errors in Unity-Gain Feedback Systems

The steady-state error with ramp excitation is

Infinite for a stable type 0 system

Finite and non-zero for a stable type 1 system

Zero for a stable type 2 system (2 poles at $s = 0$ in the forward transfer function)

$h_{-2}(t)$ Type 0 System

$h_{-2}(t)$ Type 1 System

$h_{-2}(t)$ Type 2 System

System Responses to Standard Signals
Unit Step Response

Let $H(s) = \dfrac{N(s)}{D(s)}$ be proper in s. Then the Laplace transform

of the unit-step response is

$$Y(s) = H_{-1}(s) = \frac{N(s)}{s\,D(s)} = \frac{N_1(s)}{D(s)} + \frac{H(0)}{s}$$

If the system is stable, the inverse Laplace transform of $\dfrac{N(s)}{D(s)}$

is called the transient response and the forced response is $\dfrac{H(0)}{s}$.

System Responses to Standard Signals

Let $H(s) = \dfrac{N(s)}{D(s)}$ be proper in s. Then the Laplace transform

of a general excitation is $X(s)$ and the Laplace transform of

the response is

$$Y(s) = \frac{N(s)}{D(s)}X(s) = \frac{N(s)}{D(s)}\frac{N_x(s)}{D_x(s)} = \underbrace{\frac{N_1(s)}{D(s)}}_{\substack{\text{same poles} \\ \text{as system}}} + \underbrace{\frac{N_{x1}(s)}{D_x(s)}}_{\substack{\text{same poles} \\ \text{as excitation}}}$$
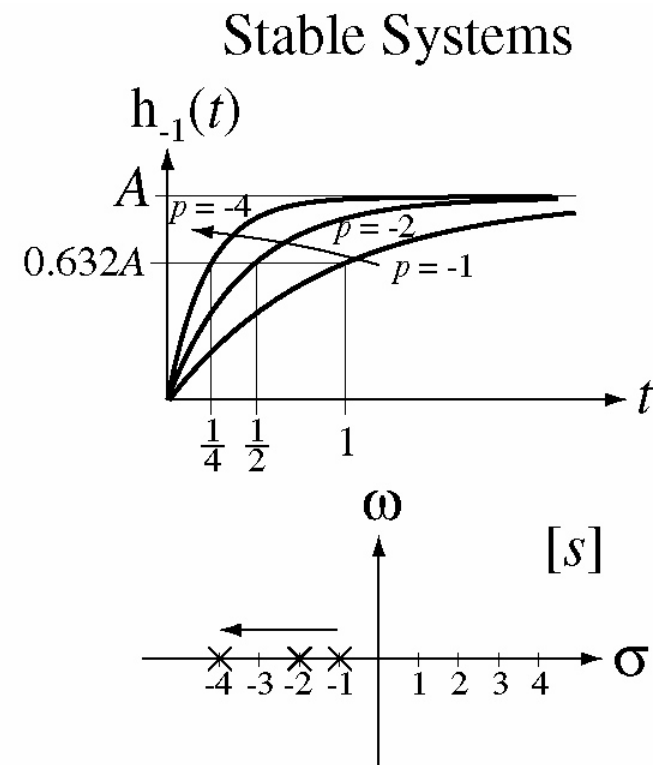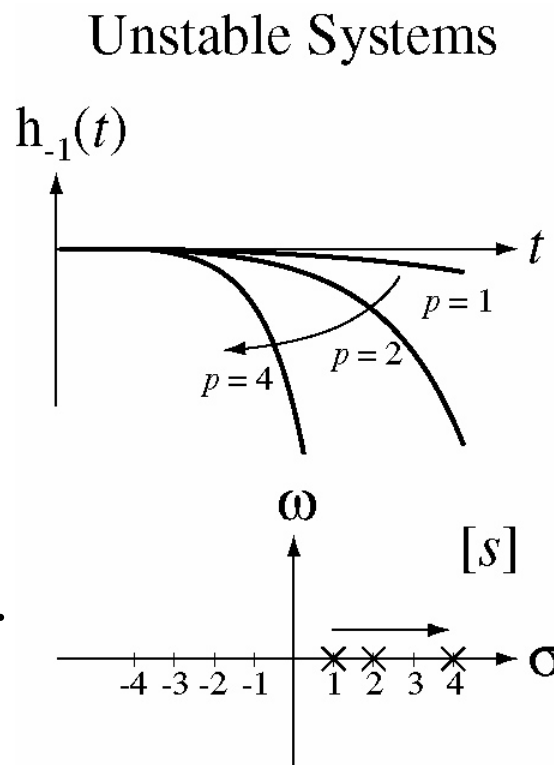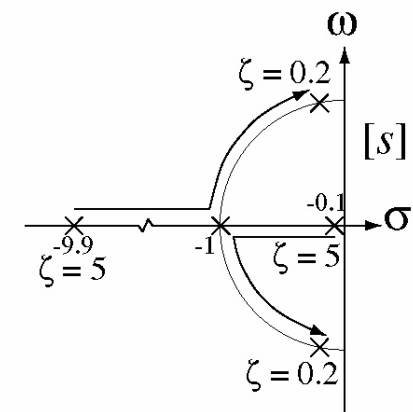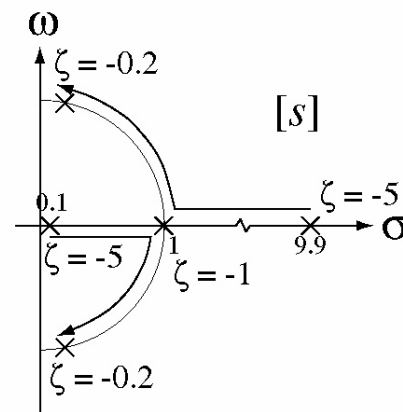
System Responses to Standard Signals
Unit Step Response

Let $\mathrm{H}(s) = \dfrac{A}{1 - s/p}$.

Then the unit-step response is

$$\mathrm{h}_{-1}(t) = A\left(1 - e^{pt}\right)\mathrm{u}(t).$$

Unstable Systems

$\mathrm{h}_{-1}(t)$

$p = 1$

$p = 2$

$p = 4$

ω

$[s]$

σ

-4 -3 -2 -1   1 2 3 4

Stable Systems

$\mathrm{h}_{-1}(t)$

A

$p = -4$

$p = -2$

$p = -1$

$0.632A$

$\frac{1}{4}$ $\frac{1}{2}$ 1

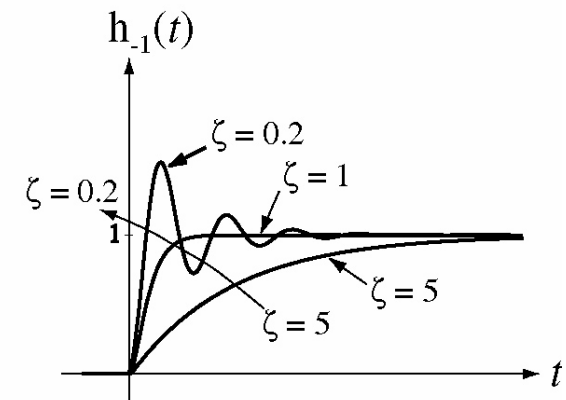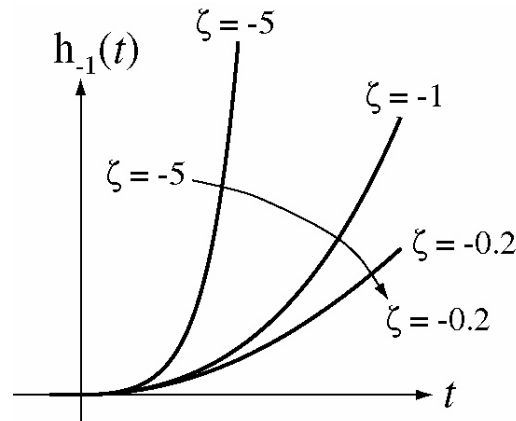ω

$[s]$

σ

-4 -3 -2 -1   1 2 3 4

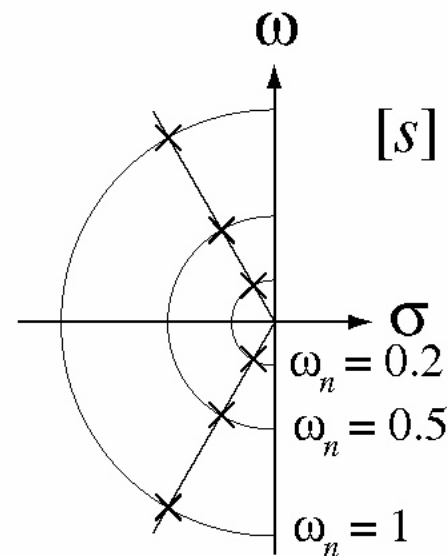# System Responses to Standard Signals
Unit Step Response

Let
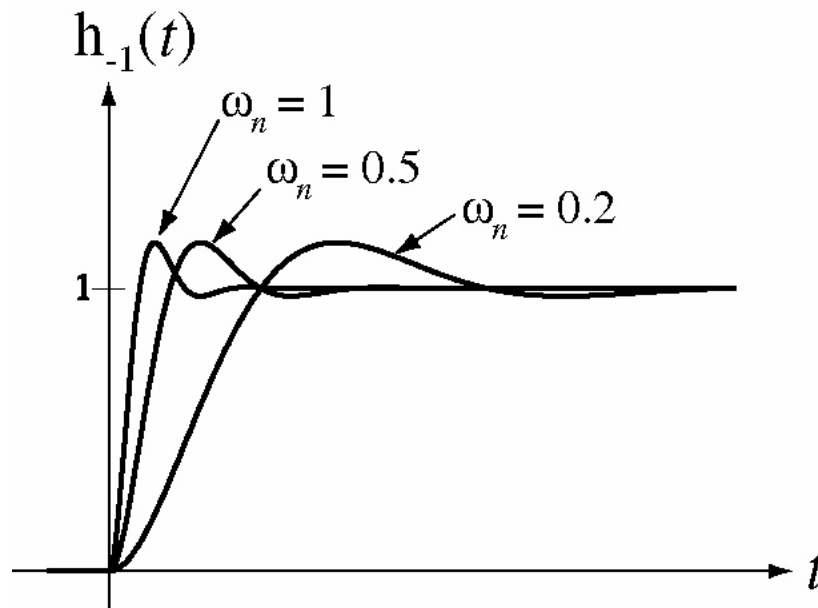
$$H(s) = \frac{A\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

System Responses to Standard Signals
Unit Step Response

Let $H(s) = \dfrac{A\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$

System Responses to Standard Signals

Let $H(s) = \dfrac{N(s)}{D(s)}$ be proper in s. If the excitation is a unit-amplitude

cosine applied to the system at time $t = 0$, the response is

$$Y(s) = \frac{N(s)}{D(s)} \frac{s}{s^2 + \omega_0^2}$$

which can be reduced and inverse Laplace transformed into

$$y(t) = \mathscr{L}^{-1}\left(\frac{N_1(s)}{D(s)} \right) + \left| H(j\omega_0) \right| \cos\left(\omega_0 t + \measuredangle H(j\omega_0) \right) u(t)$$

If the system is stable, the steady-state response is a sinusoid of

same frequency as the excitation but, generally a different

magnitude and phase.

Standard Realizations of Systems

There are multiple ways of drawing a system block diagram
corresponding to a given transfer function of the form

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{N} b_k s^k}{\sum_{k=0}^{N} a_k s^k} = \frac{b_N s^N + b_{N-1} s^{N-1} + \cdots + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0}$$

(Here the numerator and denominator are both assumed to be
of order N. If the numerator order is less than N, some of the b's
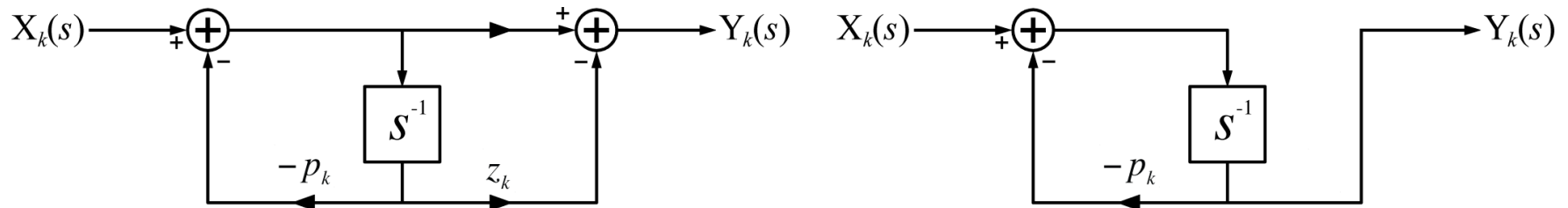are zero.)

Standard Realizations of Systems
Cascade Form

The transfer function can be factored into the form,

$$H(s) = A \frac{s - z_1}{s - p_1} \frac{s - z_2}{s - p_2} \cdots \frac{s - z_M}{s - p_M} \frac{1}{s - p_{M+1}} \frac{1}{s - p_{M+2}} \cdots \frac{1}{s - p_N}$$

and each factor can be realized in a small Direct Form II subsystem of either of the two forms
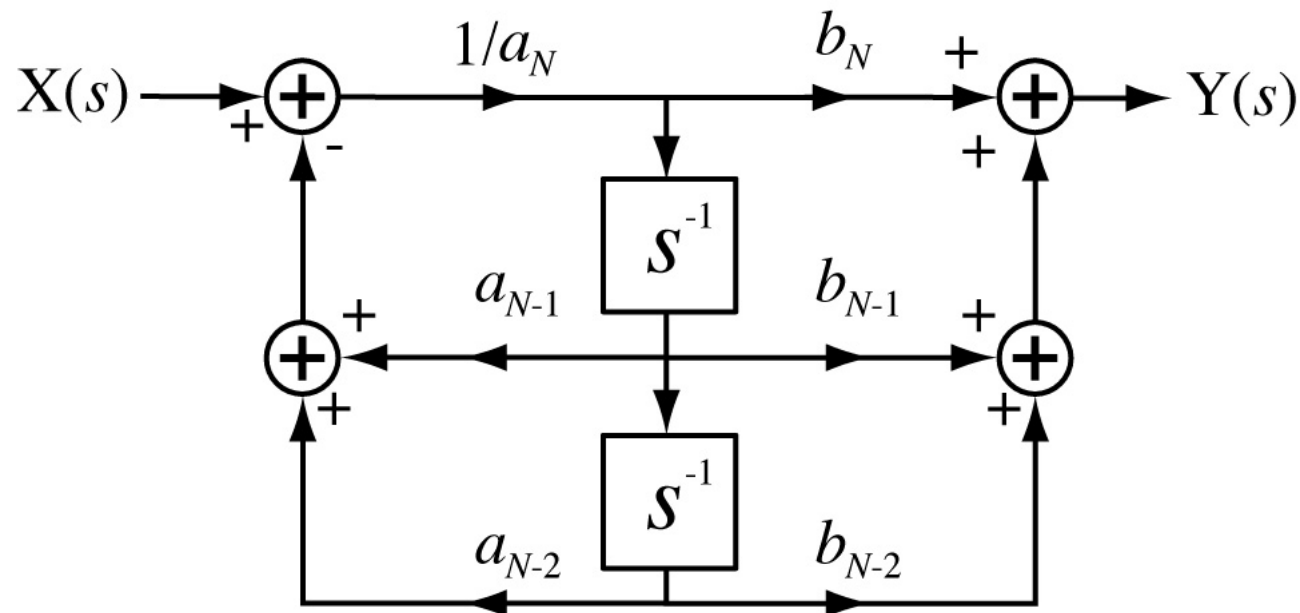

and these subsystems can then be cascade connected.

Standard Realizations of Systems
Cascade Form

A problem that arises in the cascade form is that some poles or zeros may be <u>complex</u>. In that case, a complex conjugate pair can be combined into one **second-order subsystem** of the form

Standard Realizations of Systems
Parallel Form

The transfer function can be expanded in partial fractions of the form

$$H(s) = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \cdots + \frac{K_N}{s - p_N}$$

Each of these terms describes a subsystem. When all the subsystems are connected in parallel the overall system is realized.

Standard Realizations of Systems
Parallel Form