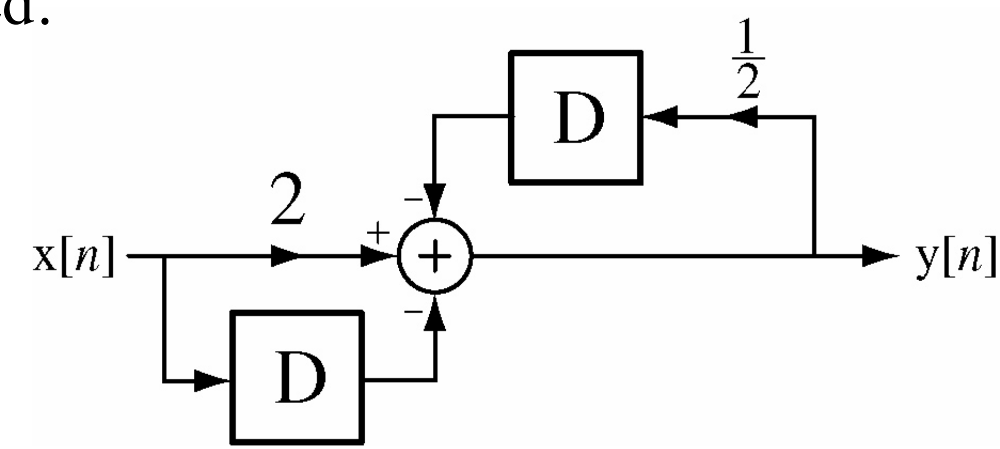
z Transform System Analysis

Block Diagrams and Transfer Functions

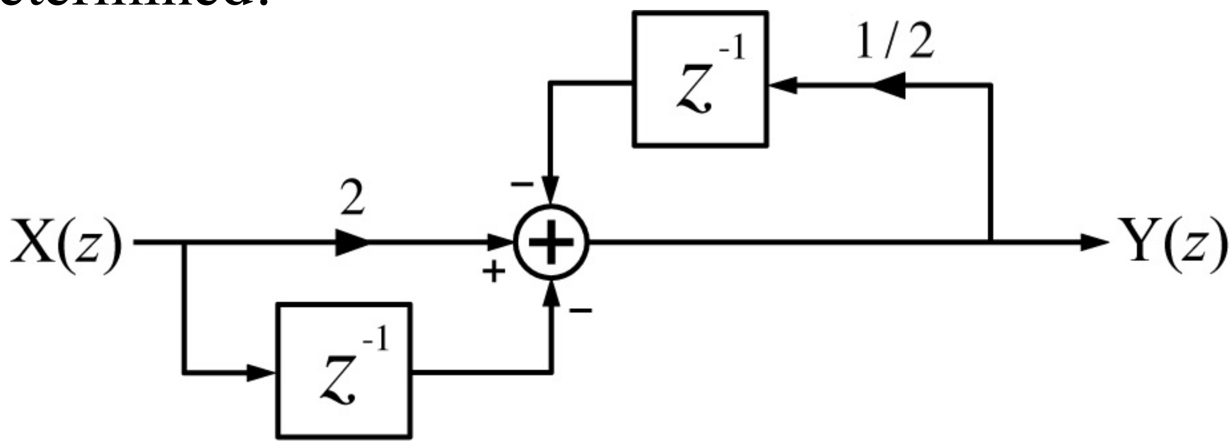
Just as with continuous-time systems, discrete-time systems are conveniently described by block diagrams and transfer functions can be determined from them. For example, from this discrete-time system block diagram the difference equation can be determined.



$$y[n] = 2\,x[n] - x[n-1] - (1/2)\,y[n-1]$$

Block Diagrams and Transfer Functions

From a z-domain block diagram the transfer function can be determined.



$$Y(z) = 2\,X(z) - z^{-1}\,X(z) - (1/2)z^{-1}\,Y(z)$$

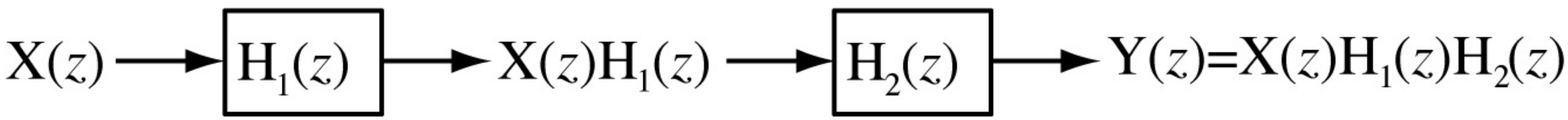
$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 - z^{-1}}{1 + (1/2)z^{-1}} = \frac{2z - 1}{z + 1/2}$$

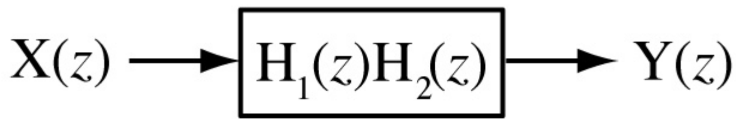
System Stability

A system is stable if its impulse response is **absolutely summable**. That requirement translates into the z-domain requirement that all the poles of the transfer function must lie in the **open** **interior of the unit circle**.

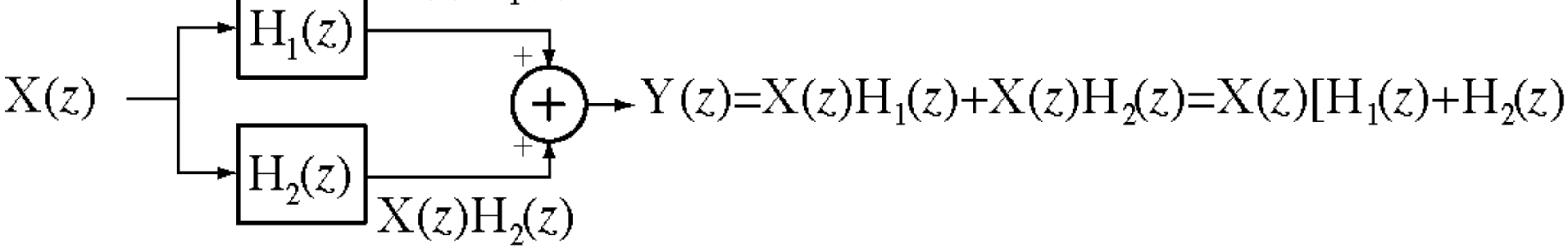
System Interconnections

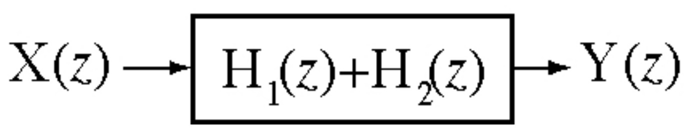
Cascade





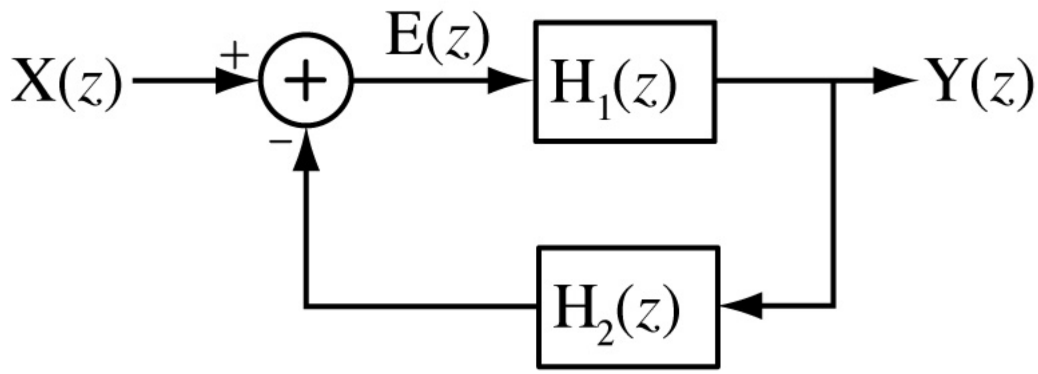
Parallel





System Interconnections

Feedback



$$\mathrm{H}(z) = \frac{\mathrm{Y}(z)}{\mathrm{X}(z)} = \frac{\mathrm{H}_1(z)}{1 + \mathrm{H}_1(z)\mathrm{H}_2(z)} = \frac{\mathrm{H}_1(z)}{1 + \mathrm{T}(z)}$$

$$\mathrm{T}(z) = \mathrm{H}_1(z)\mathrm{H}_2(z)$$

Responses to Standard Signals

If the system transfer function is $\mathrm{H}(z)=\dfrac{\mathrm{N}(z)}{\mathrm{D}(z)}$ the z transform

of the unit-sequence response is $\mathrm{H}_{-1}(z)=\dfrac{z}{z-1}\dfrac{\mathrm{N}(z)}{\mathrm{D}(z)}$ which

can be written in partial-fraction form as

$$\mathrm{Y}(z)=z\frac{\mathrm{N}_1(z)}{\mathrm{D}(z)}+\mathrm{H}(1)\frac{z}{z-1}$$

If the system is stable the **transient** term $z\dfrac{\mathrm{N}_1(z)}{\mathrm{D}(z)}$ dies out

and the **forced response** is $\mathrm{H}(1)\dfrac{z}{z-1}$.

Responses to Standard Signals

Let the system transfer function be $\mathrm{H}(z) = \dfrac{Kz}{z-p}$.

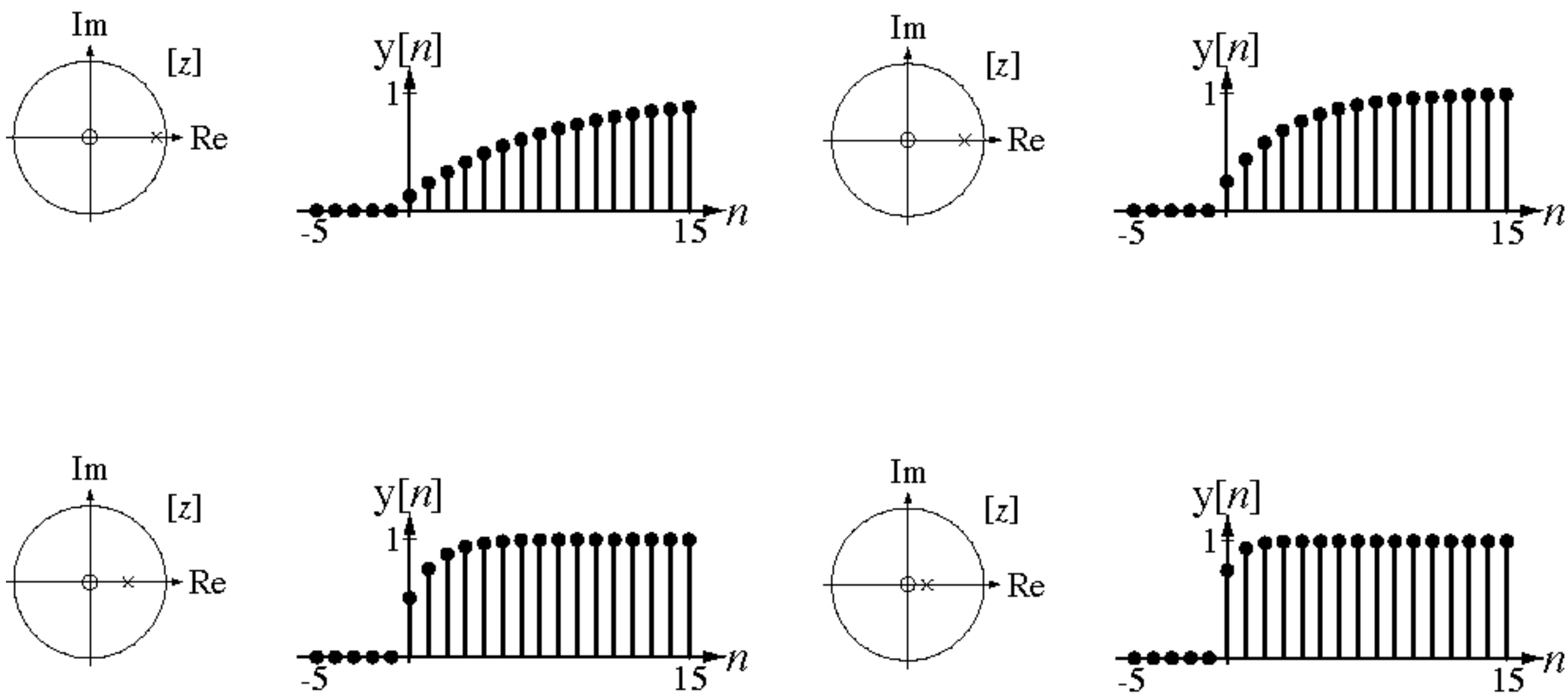
Then $\mathrm{Y}(z) = \dfrac{z}{z-1}\dfrac{Kz}{z-p} = \dfrac{K}{1-p}\left(\dfrac{z}{z-1} - \dfrac{pz}{z-p}\right)$

and $\mathrm{y}[n] = \dfrac{K}{1-p}\left(1-p^{n+1}\right)\mathrm{u}[n]$.

Let the constant K be $1-p$. Then $\mathrm{y}[n] = \left(1-p^{n+1}\right)\mathrm{u}[n]$.

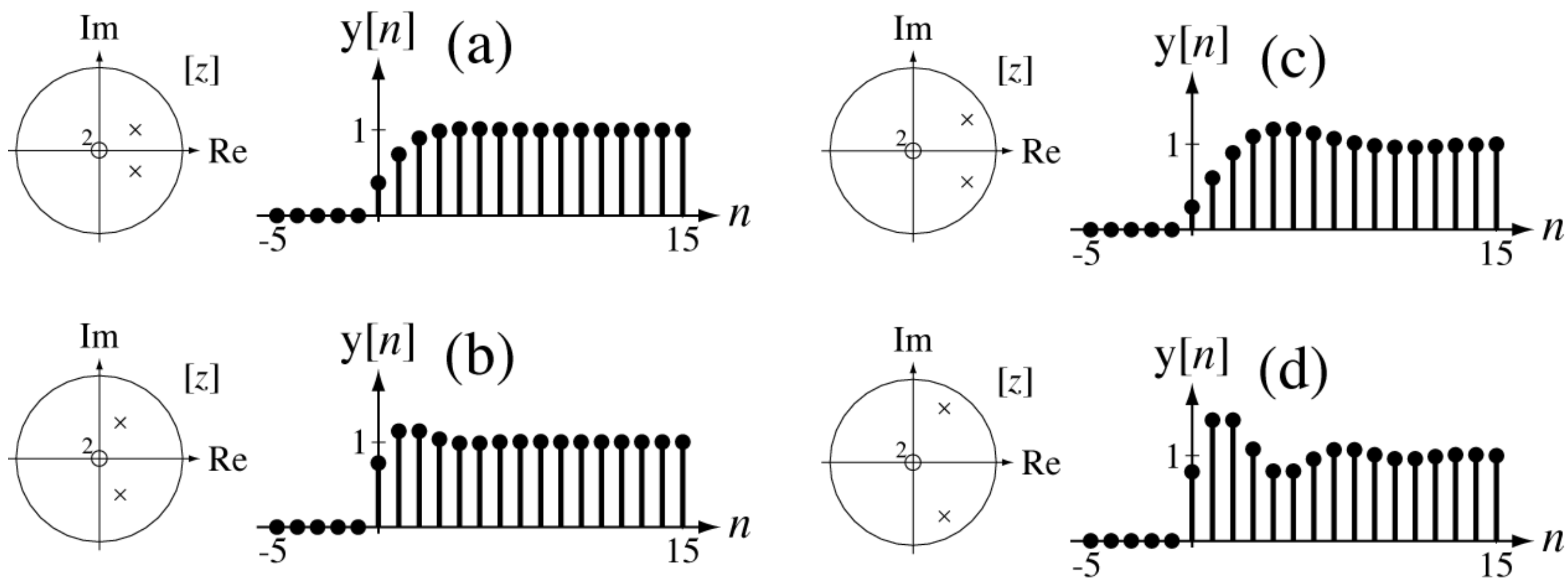
Responses to Standard Signals

Unit Sequence Response
One-Pole System



Responses to Standard Signals

Unit Sequence Response
Two-Pole System



Responses to Standard Signals

If the system transfer function is $\mathrm{H}(z) = \dfrac{\mathrm{N}(z)}{\mathrm{D}(z)}$ the z transform

of the response to a cosine applied at time $n = 0$ is

$$\mathrm{Y}(z) = \frac{\mathrm{N}(z)}{\mathrm{D}(z)} \frac{z\left[z - \cos(\Omega_0)\right]}{z^2 - 2z\cos(\Omega_0) + 1}$$

Let $p_1 = e^{j\Omega_0}$. Then the system response can be written as

$$\mathrm{y}[n] = \mathscr{Z}^{-1}\left(z \frac{\mathrm{N}_1(z)}{\mathrm{D}(z)}\right) + \left|\mathrm{H}(p_1)\right| \cos\left(\Omega_0 n + \angle \mathrm{H}(p_1)\right) \mathrm{u}[n]$$

and, if the system is stable, the forced response is

$$\left|\mathrm{H}(p_1)\right| \cos\left(\Omega_0 n + \measuredangle \mathrm{H}(p_1)\right) \mathrm{u}[n]$$

a sinusoid with, generally, different magnitude and phase.

z Transform - Laplace Transform Relationships

Let a signal x(t) be sampled to form

$$\mathrm{x}[n] = \mathrm{x}(nT_s)$$

and impulse sampled to form

$$\mathrm{x}_\delta(t) = \mathrm{x}(t)\delta_{T_s}(t)$$

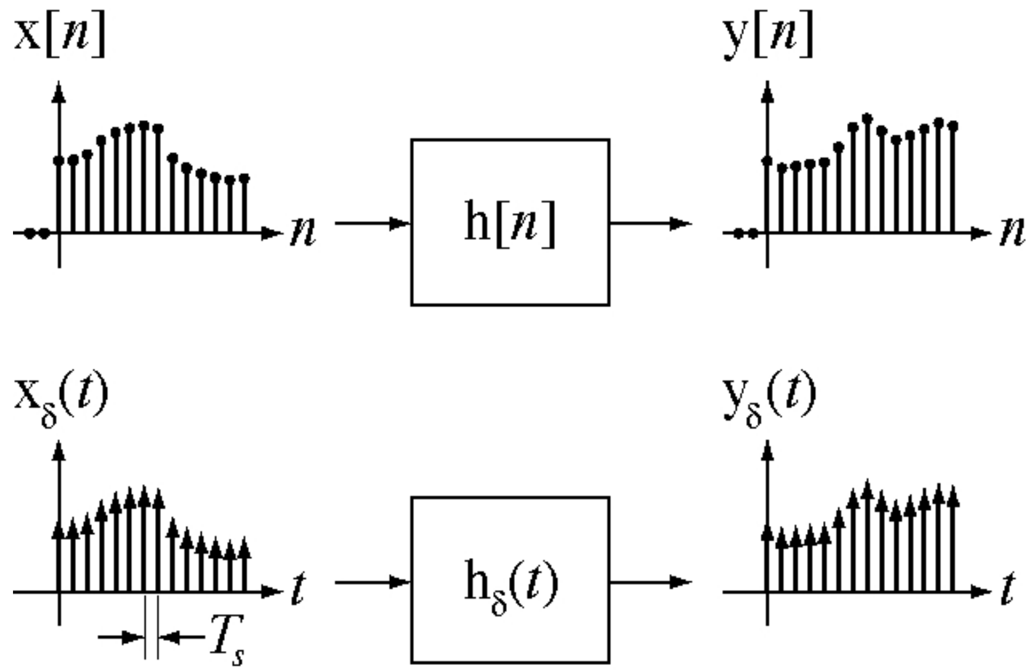
These two signals are equivalent in the sense that their impulse strengths are the same at corresponding times and the correspondence between times is $t = nT_s$.

z Transform - Laplace Transform Relationships

Let a discrete-time system have the impulse response $\text{h}[n]$ and let a continuous-time system have the impulse response

$$\text{h}_\delta(t) = \sum_{n=0}^{\infty} \text{h}[n]\delta(t - nT_s) \ .$$

If $\text{x}[n]$ is applied to the discrete-time system and $\text{x}_\delta(t)$ is applied to the continuous-time system, their responses will be equivalent in the sense that the impulse strengths are the same.



z Transform - Laplace Transform Relationships

The transfer function of the discrete-time system is

$$\mathrm{H}(z) = \sum_{n=0}^{\infty} \mathrm{h}[n] z^{-n}$$

and the transfer function of the continuous-time system is

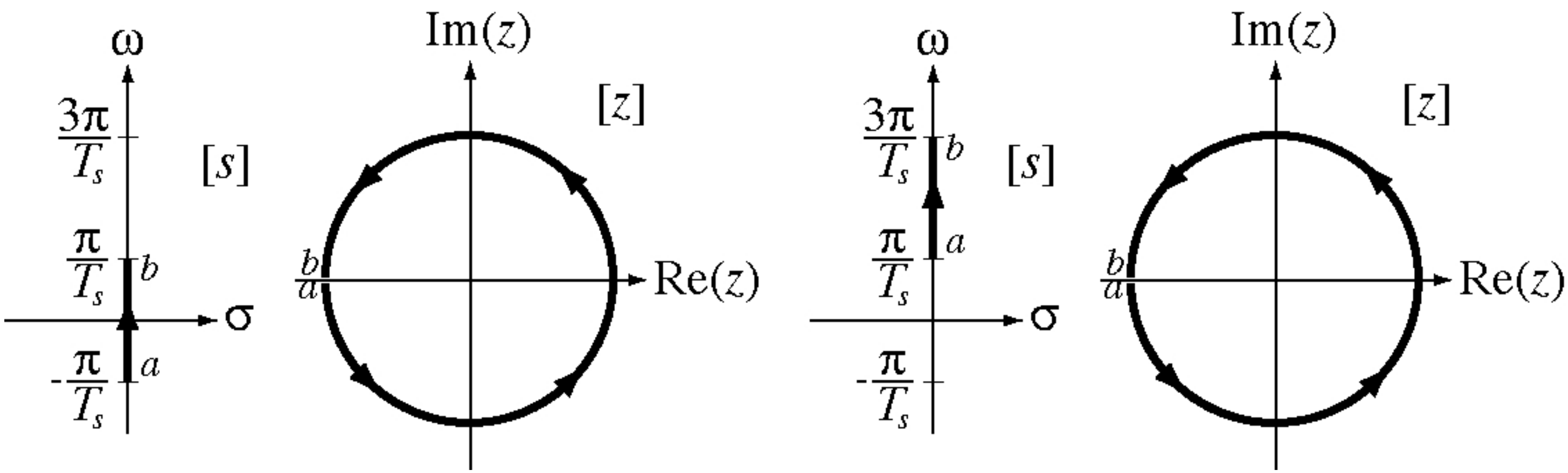
$$\mathrm{H}_{\delta}(s) = \sum_{n=0}^{\infty} \mathrm{h}[n] e^{-nT_s s}$$

The equivalence between them can be seen in the transformation

$$\mathrm{H}_{\delta}(s) = \mathrm{H}(z)\Big|_{z \to e^{sT_s}}$$

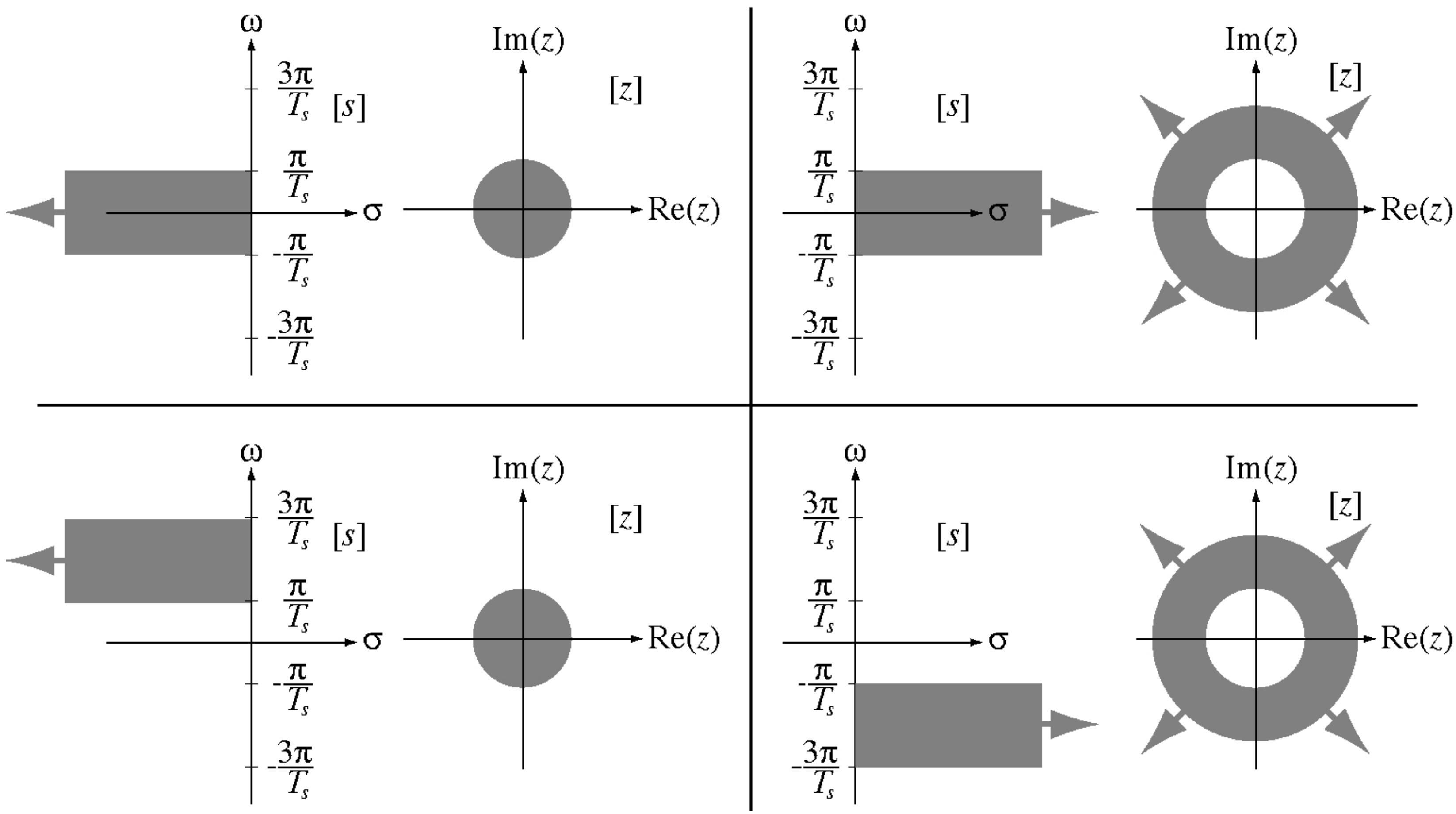
z Transform - Laplace Transform Relationships

The relationship $z = e^{sT_s}$ **maps** points in the s plane into corresponding points in the z plane.



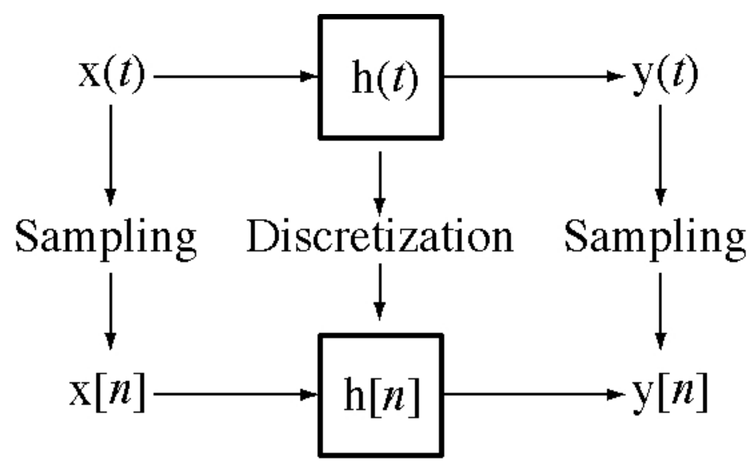
<u>Different</u> contours in the s plane map into the <u>same</u> contour in the z plane.

z Transform - Laplace Transform Relationships



Simulating Continuous-Time Systems with Discrete-time Systems

The ideal simulation of a continuous-time system by a discrete-time system would have the discrete-time system's excitation and response be samples from the continuous-time system's excitation and response. But that design goal is never achieved exactly in real systems at finite sampling rates.



Simulating continuous-time Systems with discrete-time Systems

One approach to simulation is to make the impulse response of the discrete-time system be a sampled version of the impulse response of the continuous-time system.

$$\mathrm{h}[n] = \mathrm{h}(nT_s)$$

With this choice, the response of the discrete-time system to a discrete-time unit impulse consists of samples of the response of the continuous-time system to a continuous-time unit impulse. This technique is called **impulse - invariant** design.

Simulating continuous-time Systems with discrete-time Systems

When $h[n] = h(nT_s)$ the impulse response of the discrete-time system is a sampled version of the impulse response of the continuous-time system but the unit discrete-time impulse is not a sampled version of the unit continuous-time impulse.

A continuous-time impulse cannot be sampled. First, as a practical matter the probability of taking a sample at exactly the time of occurrence of the impulse is zero. Second, even if the impulse were sampled at its time of occurrence what would the sample value be? The functional value of the impulse is not defined at its time of occurrence because the impulse is not an ordinary function.

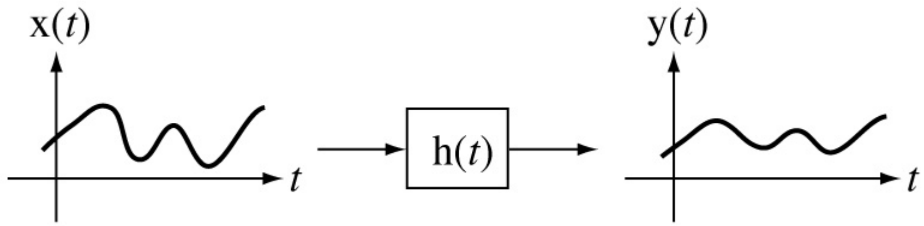
Simulating continuous-time Systems with discrete-time Systems

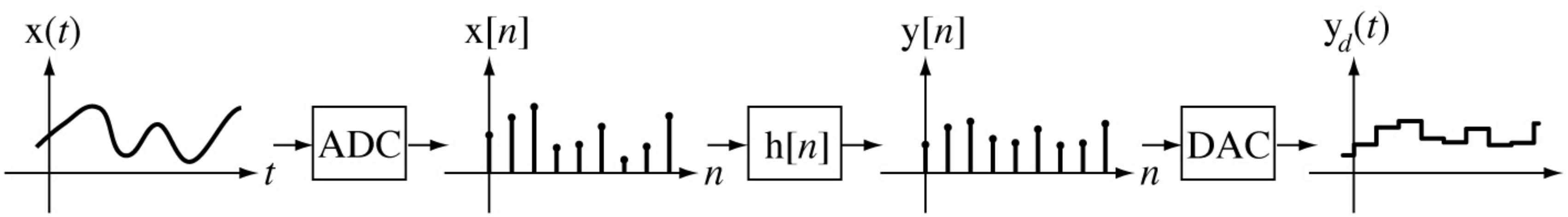
In impulse-invariant design, even though the impulse response is a sampled version of the continuous-time system's impulse response <u>that does not mean that the response to samples from any arbitrary excitation will be a sampled version of the continuous-time system's response to that excitation</u>.

All design methods for simulating continuous-time systems with discrete-time systems are approximations and whether or not the approximation is a good one depends on the design goals.

Sampled-Data Systems

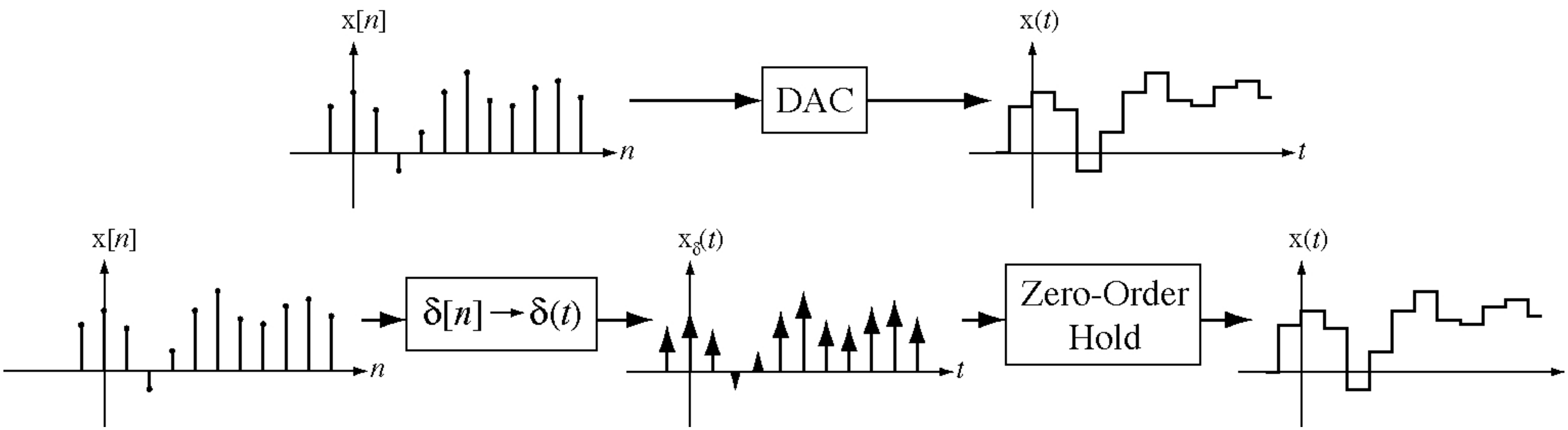
Real simulations of continuous-time systems by discrete-time systems usually sample the excitation with an ADC, process the samples and then produce a continuous-time signal with a DAC.





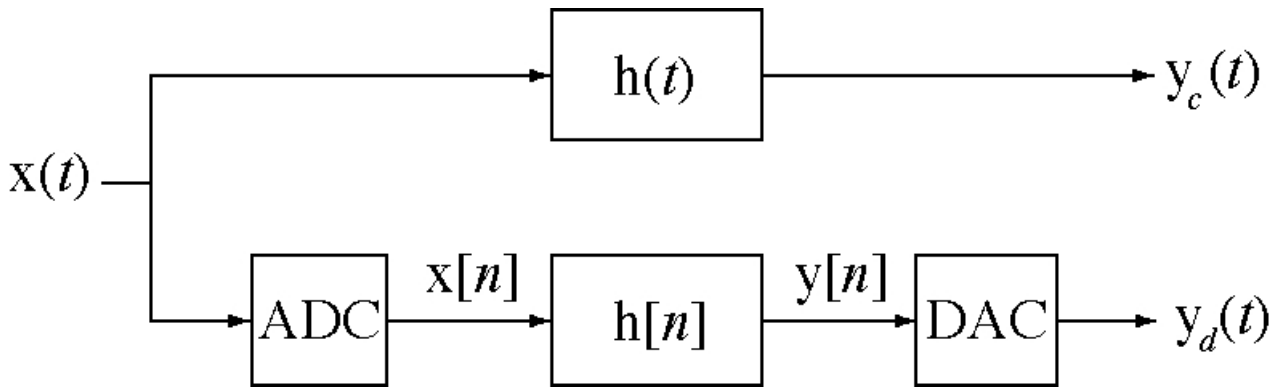
Sampled-Data Systems

An ADC simply samples a signal and produces numbers. A common way of **modeling** the action of a DAC is to imagine the discrete-time impulses in the discrete-time signal which drive the **DAC** are instead continuous-time impulses of the same strength and that the DAC has the impulse response of a **zero-order hold**.



Sampled-Data Systems

The desired equivalence between a continuous-time and a discrete-time system is illustrated below.



The design goal is to make $\mathrm{y}_d(t)$ look as much like $\mathrm{y}_c(t)$ as possible by choosing $\mathrm{h}[n]$ appropriately.

Sampled-Data Systems

Consider the response of the continuous-time system not to the actual signal x(t) but rather to an impulse-sampled version of it

$$\mathrm{x}_{\delta}(t)=\sum_{n=-\infty}^{\infty}\mathrm{x}(nT_s)\delta(t-nT_s)=\mathrm{x}(t)\delta_{T_s}(t)$$

The response is

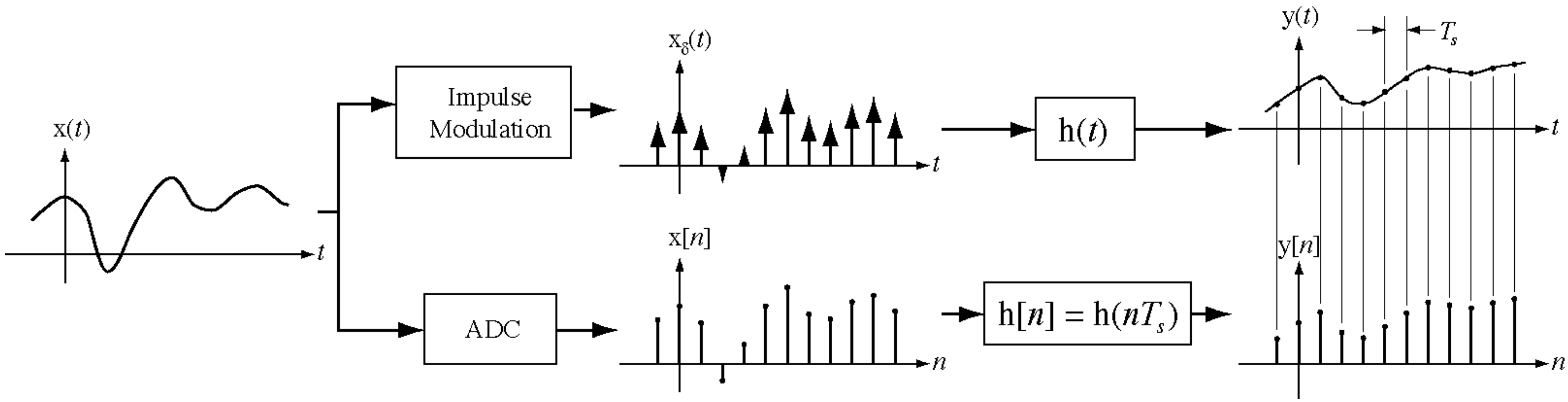
$$\mathrm{y}(t)=\mathrm{h}(t)*\mathrm{x}_{\delta}(t)=\mathrm{h}(t)*\sum_{m=-\infty}^{\infty}\mathrm{x}[m]\delta(t-mT_s)=\sum_{m=-\infty}^{\infty}\mathrm{x}[m]\mathrm{h}(t-mT_s)$$

where $\mathrm{x}[n]=\mathrm{x}(nT_s)$ and the response a the *n*th multiple of T_s is $\mathrm{y}(nT_s)=\sum_{m=-\infty}^{\infty}\mathrm{x}[m]\mathrm{h}((n-m)T_s)$.

The response of a discrete-time system with $\mathrm{h}[n]=\mathrm{h}(nT_s)$ to the excitation $\mathrm{x}[n]=\mathrm{x}(nT_s)$ is $\mathrm{y}[n]=\mathrm{x}[n]*\mathrm{h}[n]=\sum_{m=-\infty}^{\infty}\mathrm{x}[m]\mathrm{h}[n-m]$.

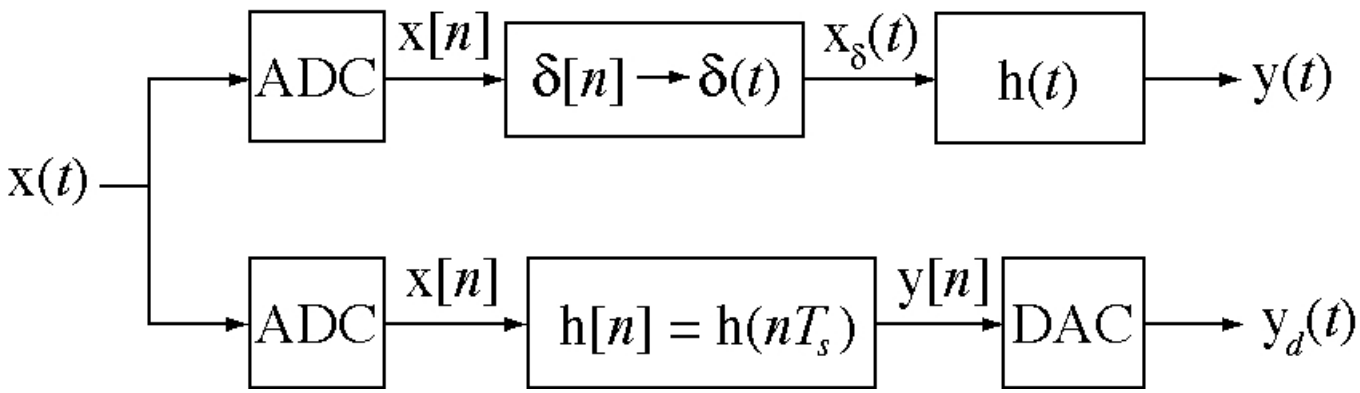
Sampled-Data Systems

The two responses are equivalent in the sense that the values at corresponding discrete-time and continuous-time times are the same.

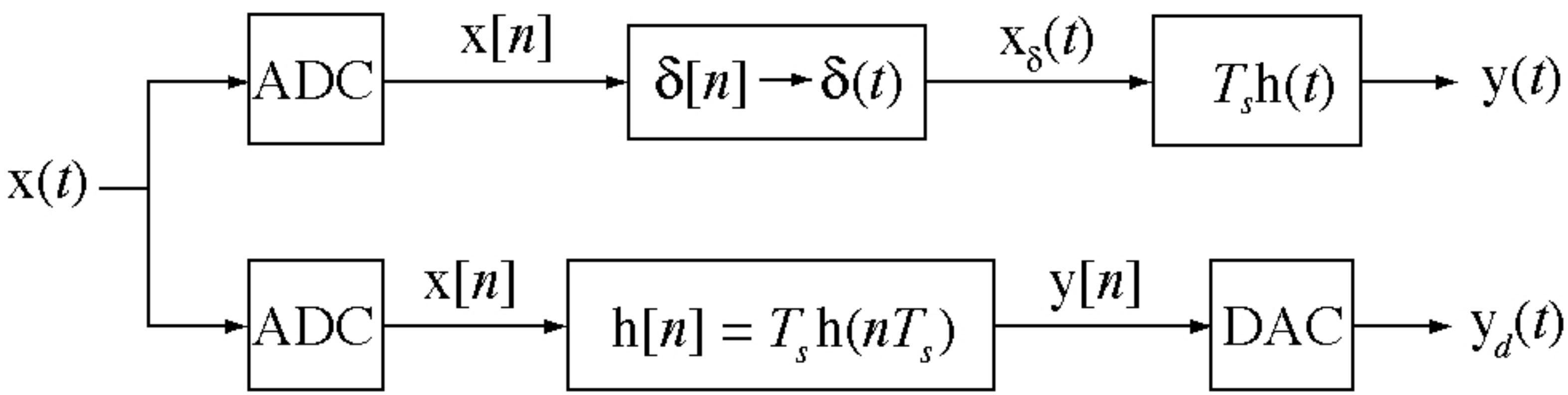


Sampled-Data Systems

Modify the continuous-time system to reflect the last analysis.



Then multiply the impulse responses of both systems by T_s



Sampled-Data Systems

In the modified continuous-time system

$$\mathrm{y}(t) = \mathrm{x}_\delta(t) * T_s\,\mathrm{h}(t) = \left[\sum_{n=-\infty}^{\infty} \mathrm{x}(nT_s)\delta(t - nT_s)\right] * \mathrm{h}(t)T_s = \sum_{n=-\infty}^{\infty} \mathrm{x}(nT_s)\mathrm{h}(t - nT_s)T_s$$

In the modified discrete-time system

$$\mathrm{y}[n] = \sum_{m=-\infty}^{\infty} \mathrm{x}[m]\mathrm{h}[n-m] = \sum_{m=-\infty}^{\infty} \mathrm{x}[m]T_s\,\mathrm{h}((n-m)T_s)$$

where $\mathrm{h}[n] = T_s\,\mathrm{h}(nT_s)$ and $\mathrm{h}(t)$ still represents the impulse response of the original continuous-time system. Now let T_s approach zero.

$$\lim_{T_s \to 0} \mathrm{y}(t) = \lim_{T_s \to 0} \sum_{n=-\infty}^{\infty} \mathrm{x}(nT_s)\mathrm{h}(t - nT_s)T_s = \int_{-\infty}^{\infty} \mathrm{x}(\tau)\mathrm{h}(t-\tau)\,d\tau$$

This is the response $\mathrm{y}_c(t)$ of the original continuous-time system.

Sampled-Data Systems

Summarizing, if the impulse response of the discrete-time system is chosen to be $T_s \, \mathrm{h}(nT_s)$ then, in the limit as the sampling rate approaches infinity, the response of the discrete-time system is exactly the same as the response of the continuous-time system.

Of course the sampling rate can never be infinite in practice. Therefore this design is an approximation which gets better as the sampling rate is increased.

Sampled-Data Systems

A continuous-time system is characterized by the transfer function

$$\mathrm{H}_s(s) = \frac{1}{s^2 + 40s + 300} \ .$$

Design a discrete-time system to approximate this continuous-time system. Use two different sampling rates $f_s = 10$ and $f_s = 100$ and compare step responses.

The impulse response of the continuous-time system is

$$\mathrm{h}(t) = \frac{1}{20}\left(e^{-10t} - e^{-30t}\right)\mathrm{u}(t) \ .$$

Sampled-Data Systems

The discrete-time impulse response is

$$\mathrm{h}[n] = \frac{T_s}{20}\left(e^{-10nT_s} - e^{-30nT_s}\right)\mathrm{u}[n]$$

and the transfer function is its z transform

$$\mathrm{H}_z(z) = \frac{T_s}{20}\left(\frac{z}{z - e^{-10T_s}} - \frac{z}{z - e^{-30T_s}}\right)$$

The step response of the continuous-time system is

$$\mathrm{y}_c(t) = \frac{2 - 3e^{-10t} + e^{-30t}}{600}\mathrm{u}(t)$$

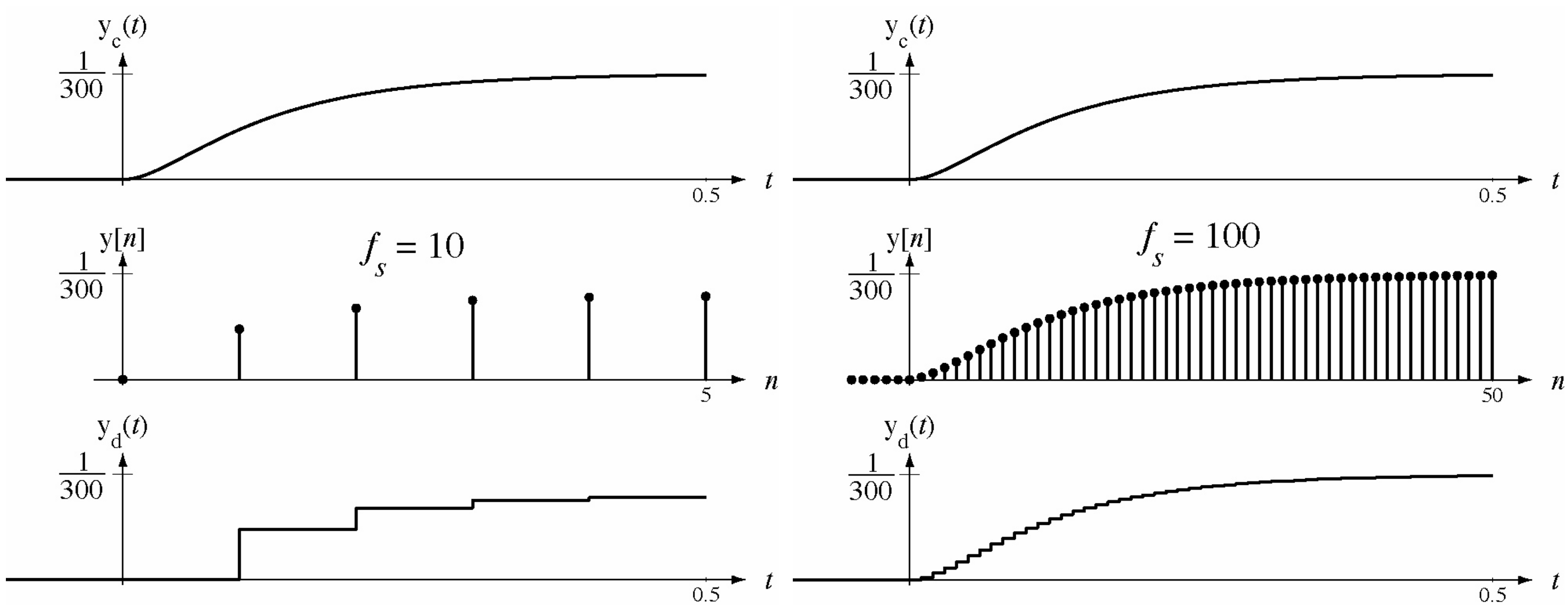
and the response of the discrete-time system to a unit sequence is

$$\mathrm{y}[n] = \frac{T_s}{20}\left[\frac{e^{-10T_s} - e^{-30T_s}}{\left(1 - e^{-10T_s}\right)\left(1 - e^{-30T_s}\right)} + \frac{e^{-10T_s}}{e^{-10T_s} - 1}e^{-10nT_s} - \frac{e^{-30T_s}}{e^{-30T_s} - 1}e^{-30nT_s}\right]\mathrm{u}[n]$$

Sampled-Data Systems

The response of the DAC is

$$y_d(t) = \sum_{n=0}^{\infty} y[n] \operatorname{rect}\left(\frac{t - T_s(n + 1/2)}{T_s}\right).$$

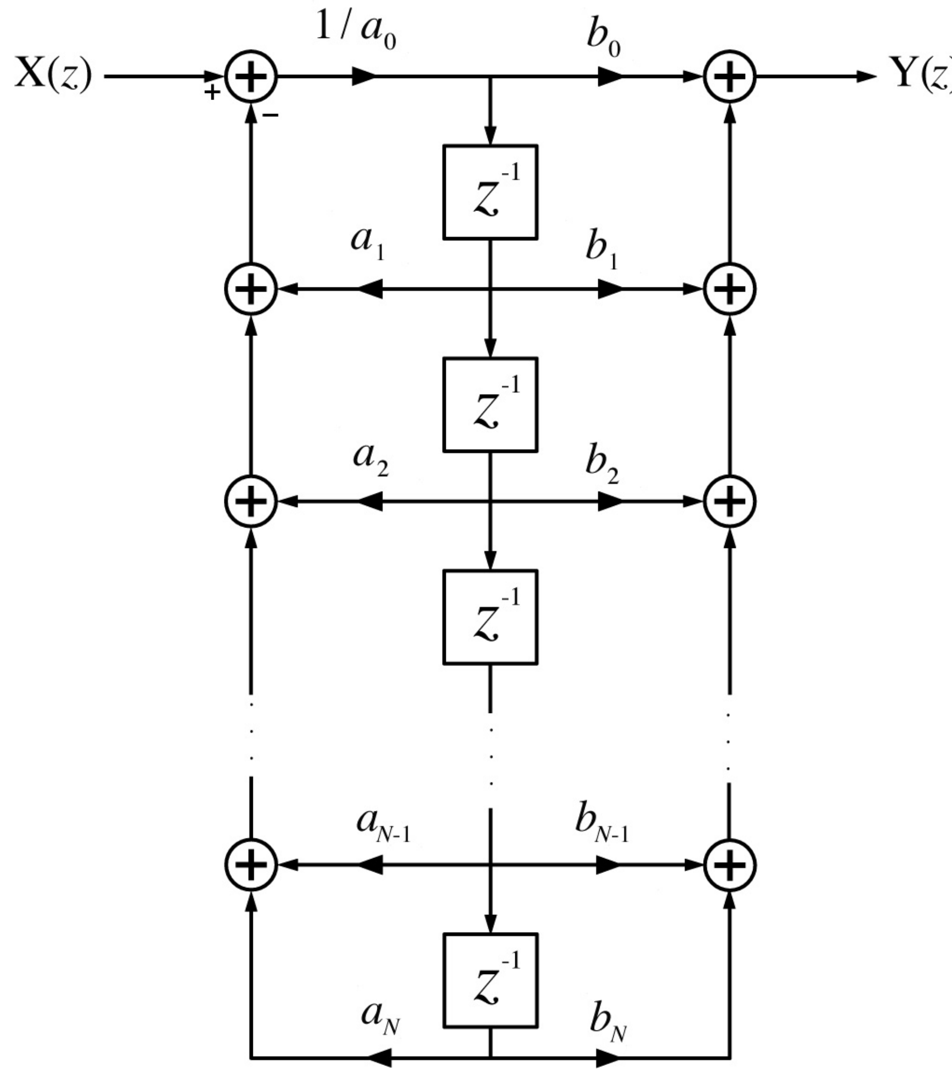


Standard Realizations

- Realization of a discrete-time system closely parallels the realization of a continuous-time system
- The basic forms, Direct Form II, cascade and parallel have the same structure
- A continuous-time system can be realized with integrators, summing junctions and multipliers
- A discrete-time system can be realized with delays, summing junctions and multipliers

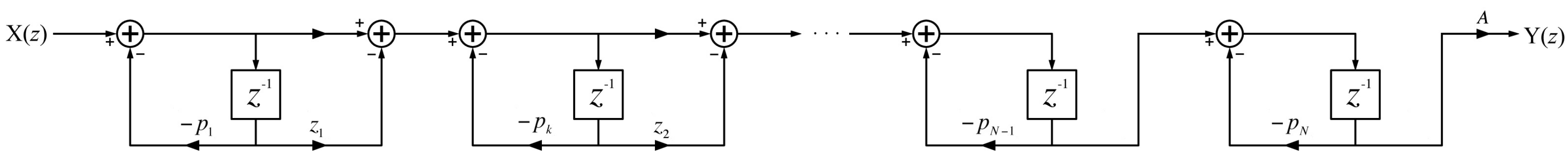
Standard Realizations

Direct Form II



Standard Realizations

Cascade



Standard Realizations

Parallel