

# State Space Analysis



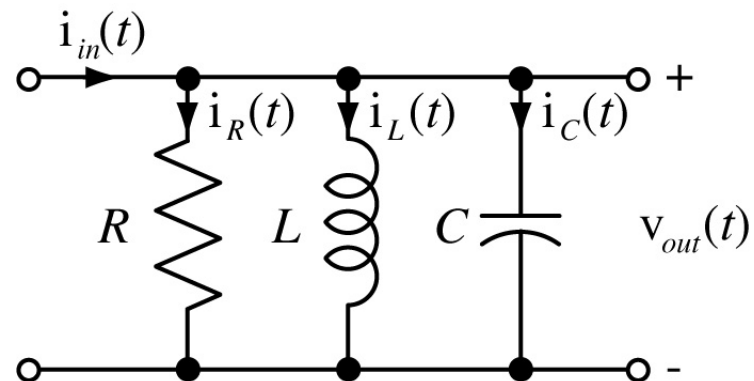
# Continuous Time

# State Variables

Every system has an **order**. If it is described by a single differential equation, the order of the system is the same as the order of the differential equation. If it is described by a system of differential equations, the order of the system is the sum of the orders of the differential equations. **State variables** are a set of variables which are sufficient to describe the state of the system at any time. The number of state variables required is the same as the order of the system. The state variables define a location in **state space**, a vector space of the same dimension as the order of the system. As a system changes state with time it follows a **trajectory** through state space.

# State Variables

Example



It is possible to specify the state of this system by two state variables, the capacitor voltage  $v_C(t)$  and the inductor current  $i_L(t)$ . The forcing function  $i_{in}(t)$  and the initial state of the system determine how the system will move through state space and the state variables describe its position in state space as it follows that trajectory.

# State Variables

The state-space description of a system has a standard form, the **system** equations and the **output** equations. Each system equation has on its left side the derivative of a state variable and on the right side a linear combination of state variables and excitations. For this example the state equations are

$$\begin{aligned}i'_L(t) &= (1/L)v_C(t) \\v'_C(t) &= -(1/C)i_L(t) - (G/C)v_C(t) + (1/C)i_{in}(t)\end{aligned}$$

The output equations express the responses of the system as linear combinations of the state variables and the excitations. In this case if we choose  $v_{out}(t)$  and  $i_R(t)$  as the responses the output equations are

$$\begin{aligned}v_{out}(t) &= v_C(t) \\i_R(t) &= G v_C(t) \quad , \quad (G = 1/R)\end{aligned}$$

# State Variables

The system and output equations can be expressed in matrix form as

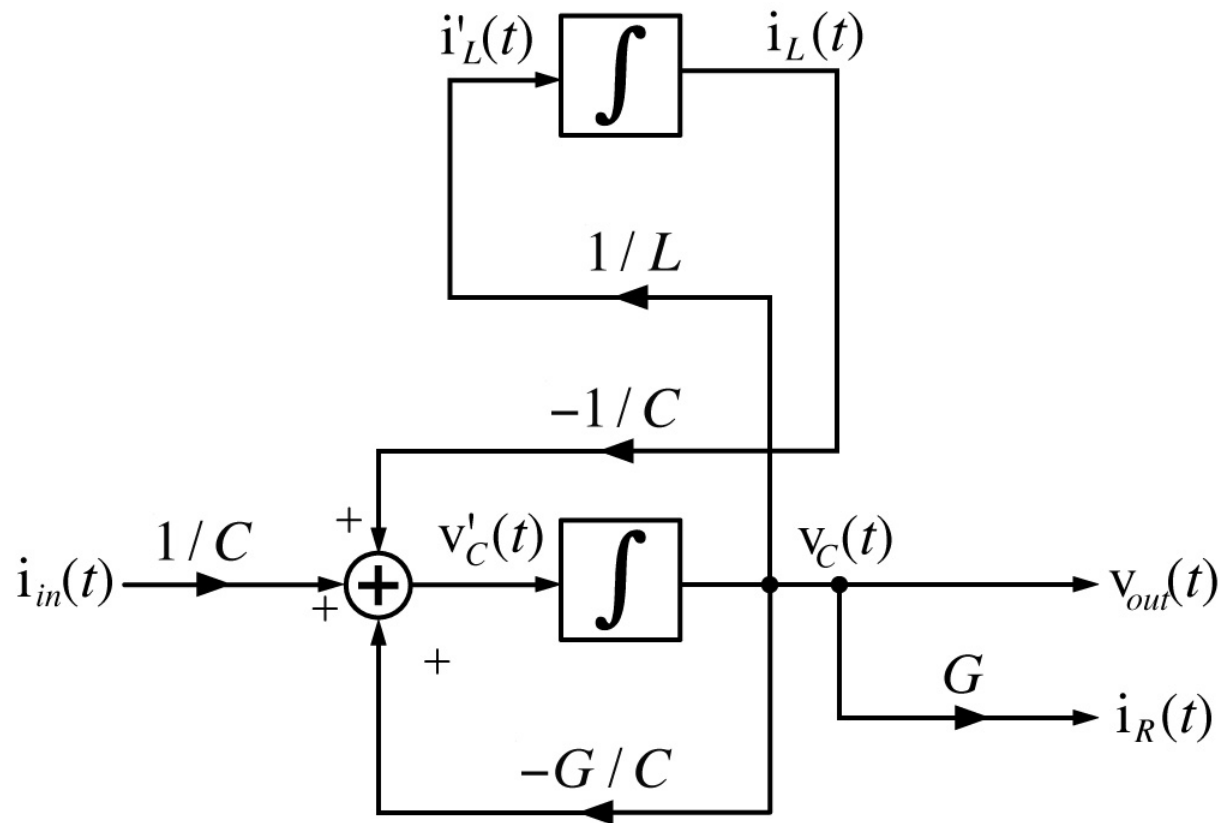
$$\begin{bmatrix} \dot{i}_L(t) \\ \dot{v}_C(t) \end{bmatrix} = \begin{bmatrix} 0 & 1/L \\ -1/C & -G/C \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/C \end{bmatrix} i_{in}(t)$$

and

$$\begin{bmatrix} v_{out}(t) \\ i_R(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & G \end{bmatrix} \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} i_{in}(t)$$

# State Variables

A block diagram description of the  $RLC$  circuit can be drawn directly from the system and output equations.



# State Variables

The vector of state variables will be designated by  $\mathbf{q}(t)$  and the matrix that multiplies  $\mathbf{q}(t)$  in the system equations is designated  $\mathbf{A}$ . The vector of excitations will be designated  $\mathbf{x}(t)$  and the matrix that multiplies  $\mathbf{x}(t)$  is designated  $\mathbf{B}$ . The matrix that multiplies  $\mathbf{q}(t)$  in the output equations is designated  $\mathbf{C}$  and the matrix that multiplies  $\mathbf{x}(t)$  in the output equations is designated  $\mathbf{D}$ . The vector of responses is designated  $\mathbf{y}(t)$ . So the standard form of the system equations is  $\mathbf{q}'(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}\mathbf{x}(t)$  and the standard form of the output equations is  $\mathbf{y}(t) = \mathbf{C}\mathbf{q}(t) + \mathbf{D}\mathbf{x}(t)$ .



# State Variables

For the *RLC* circuit example

$$\mathbf{x}(t) = [i_{in}(t)] \quad , \quad \mathbf{q}(t) = \begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} \quad , \quad \mathbf{y}(t) = \begin{bmatrix} v_{out}(t) \\ i_R(t) \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1/L \\ -1/C & -G/C \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1/C \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \\ 0 & G \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

# State Variables

Techniques exist for solving the system and output equations in the time domain, but the solution is generally easier using the Laplace transform. Laplace transforming the system equation we get

$$s\mathbf{Q}(s) - \mathbf{q}(0^-) = \mathbf{A}\mathbf{Q}(s) + \mathbf{B}\mathbf{X}(s)$$

or

$$[s\mathbf{I} - \mathbf{A}]\mathbf{Q}(s) = \mathbf{B}\mathbf{X}(s) + \mathbf{q}(0^-)$$

Solving for  $\mathbf{Q}(s)$

$$\mathbf{Q}(s) = [s\mathbf{I} - \mathbf{A}]^{-1} [\mathbf{B}\mathbf{X}(s) + \mathbf{q}(0^-)]$$

The matrix  $[s\mathbf{I} - \mathbf{A}]^{-1}$  is conventionally designated as  $\Phi(s)$ .

# State Variables

$\mathbf{Q}(s)$  can now be expressed in the form

$$\mathbf{Q}(s) = \Phi(s) \left[ \mathbf{B}\mathbf{X}(s) + \mathbf{q}(0^-) \right] = \underbrace{\Phi(s)\mathbf{B}\mathbf{X}(s)}_{\text{zero-state response}} + \underbrace{\Phi(s)\mathbf{q}(0^-)}_{\text{zero-input response}}$$

To find the time-domain solution for the state variables we inverse Laplace transform to get

$$\mathbf{q}(t) = \underbrace{\phi(t) * \mathbf{B}\mathbf{x}(t)}_{\text{zero-state response}} + \underbrace{\phi(t)\mathbf{q}(0^-)}_{\text{zero-input response}}$$

and  $\phi(t)$  is called the **state transition matrix** because it describes how the system transitions from one state to the next.

# State Variables

We can now finish the *RLC* circuit example. To make the example concrete let  $i(t) = Au(t)$ , let the initial conditions be

$$\mathbf{q}(0^-) = \begin{bmatrix} i_L(0^-) \\ v_C(0^-) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and let } R = 1/3, C = 1, L = 1.$$

Then

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -1/L \\ 1/C & s + G/C \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s + G/C & -1/C \\ 1/L & s \end{bmatrix}^T}{s^2 + (G/C)s + 1/LC}$$

# State Variables

$$\Phi(s) = \frac{\begin{bmatrix} s + G/C & 1/L \\ -1/C & s \end{bmatrix}}{s^2 + (G/C)s + 1/LC} \text{ and } \mathbf{Q}(s) = \Phi(s) [\mathbf{B}\mathbf{X}(s) + \mathbf{q}(0^-)]$$

Multiplying matrices, simplifying and substituting in numerical

component values we get.  $\mathbf{Q}(s) = \begin{bmatrix} \frac{1}{s(s^2 + 3s + 1)} + \frac{1}{s^2 + 3s + 1} \\ \frac{1}{s^2 + 3s + 1} + \frac{s}{s^2 + 3s + 1} \end{bmatrix}.$

# State Variables

Expanding in partial fractions and combining like denominators

$$\mathbf{Q}(s) = \begin{bmatrix} \frac{1}{s} - \frac{0.277}{s + 2.62} - \frac{0.723}{s + 0.382} \\ \frac{0.723}{s + 2.62} + \frac{0.277}{s + 0.382} \end{bmatrix}$$

Inverse Laplace transforming

$$\mathbf{q}(t) = \begin{bmatrix} 1 - 0.277e^{-2.62t} - 0.723e^{-0.382t} \\ 0.723e^{-0.382t} + 0.277e^{-2.62t} \end{bmatrix} \mathbf{u}(t)$$

Now that we have this solution we can immediately solve for the responses also

$$\mathbf{y}(t) = \begin{bmatrix} 0 & 1 \\ 0 & G \end{bmatrix} \mathbf{q} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 - 0.277e^{-2.62t} - 0.723e^{-0.382t} \\ 0.723e^{-0.382t} + 0.277e^{-2.62t} \end{bmatrix} \mathbf{u}(t)$$
$$\mathbf{y}(t) = \begin{bmatrix} 0.723e^{-0.382t} + 0.277e^{-2.62t} \\ 2.169e^{-0.382t} + 0.831e^{-2.62t} \end{bmatrix} \mathbf{u}(t)$$

# Transfer Functions

From the system equation

$$s\mathbf{Q}(s) - \mathbf{q}(0^-) = \mathbf{A}\mathbf{Q}(s) + \mathbf{B}\mathbf{X}(s)$$

we can find the transfer function of the system. Transfer functions defined only for zero-state responses. Therefore  $\mathbf{q}(0^-) = 0$  and

$$\mathbf{Q}(s) = [s\mathbf{I} - \mathbf{A}]^{-1} \mathbf{B}\mathbf{X}(s) = \Phi(s)\mathbf{B}\mathbf{X}(s)$$

and

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{Q}(s) + \mathbf{D}\mathbf{X}(s) = \mathbf{C}\Phi(s)\mathbf{B}\mathbf{X}(s) + \mathbf{D}\mathbf{X}(s) = \underbrace{[\mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D}]}_{\mathbf{H}(s)}\mathbf{X}(s)$$

Therefore  $\mathbf{H}(s) = \mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D} = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1} \mathbf{B} + \mathbf{D}$ . This is a **vector** transfer function, valid for multiple-input-multiple-output systems.

# Alternate State Variables

The choice of state variables is not unique. The variables chosen as state variables must be independent and the number of state variables must be the same as the order of the system. If an alternate set of state variables is substituted for an original set the  $\mathbf{A}$  matrix changes but the determinant  $|s\mathbf{I} - \mathbf{A}|$  does not change. This determinant says something fundamental about the system and is not dependent on the choice of state variables.



# Transformations of State Variables

Any set of state variables can be transformed into another set through a linear transformation. Let an original state-variable vector be  $\mathbf{q}_1(t)$  and let a new choice of state-variable vector be  $\mathbf{q}_2(t) = \mathbf{T}\mathbf{q}_1(t)$ . Then

$$\mathbf{q}'_2(t) = \mathbf{T}\mathbf{q}'_1(t) = \mathbf{T}(\mathbf{A}_1\mathbf{q}_1(t) + \mathbf{B}_1\mathbf{x}(t)) = \mathbf{T}\mathbf{A}_1\mathbf{q}_1(t) + \mathbf{T}\mathbf{B}_1\mathbf{x}(t)$$

Also  $\mathbf{q}_1(t) = \mathbf{T}^{-1}\mathbf{q}_2(t)$ . Therefore

$$\mathbf{q}'_2(t) = \mathbf{T}\mathbf{A}_1\mathbf{T}^{-1}\mathbf{q}_2(t) + \mathbf{T}\mathbf{B}_1\mathbf{x}(t) = \mathbf{A}_2\mathbf{q}_2(t) + \mathbf{B}_2\mathbf{x}(t)$$

and  $\mathbf{A}_2 = \mathbf{T}\mathbf{A}_1\mathbf{T}^{-1}$  and  $\mathbf{B}_2 = \mathbf{T}\mathbf{B}_1$ . Also

$$\mathbf{y}(t) = \mathbf{C}_1\mathbf{q}_1(t) + \mathbf{D}_1\mathbf{x}(t) = \mathbf{C}_1\mathbf{T}^{-1}\mathbf{q}_2(t) + \mathbf{D}_1\mathbf{x}(t) = \mathbf{C}_2\mathbf{q}_2(t) + \mathbf{D}_2\mathbf{x}(t)$$

and  $\mathbf{C}_2 = \mathbf{C}_1\mathbf{T}^{-1}$  and  $\mathbf{D}_2 = \mathbf{D}_1$ .

# Diagonalization

If the eigenvalues of a system are distinct (no repetitions) it is possible to choose the state variables in such a way that the system matrix  $\mathbf{A}$  is diagonal of the form

$$\mathbf{A} = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{NN} \end{bmatrix}$$

Then  $|s\mathbf{I} - \mathbf{A}| = (s - a_{11})(s - a_{22}) \cdots (s - a_{NN})$  and the elements on the diagonal are the eigenvalues implying

$$\mathbf{A} = \Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix}$$

# Diagonalization

If we have a system matrix  $\mathbf{A}$  and we want to make a diagonal system matrix  $\Lambda$  using a transformation matrix  $\mathbf{T}$ , then  $\Lambda = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$ . Postmultiplying both sides by  $\mathbf{T}$ ,  $\Lambda\mathbf{T} = \mathbf{T}\mathbf{A}$ . Since  $\Lambda$  and  $\mathbf{A}$  are known, this equation can be solved for  $\mathbf{T}$ . If we find a  $\mathbf{T}$  that solves this equation and multiply it by  $K$  to form  $\mathbf{T}_2 = K\mathbf{T}$  we can say  $\Lambda\mathbf{T}_2 = \Lambda K\mathbf{T} = K\Lambda\mathbf{T}$  and  $\Lambda\mathbf{T}_2 = K\mathbf{T}\mathbf{A} = \mathbf{T}_2\mathbf{A}$  or simply  $\Lambda\mathbf{T}_2 = \mathbf{T}_2\mathbf{A}$ . This shows that the solution is not unique.

# Diagonalization

Once we have found a  $T$  to diagonalize a system matrix we can then write

$$\begin{bmatrix} q'_1(t) \\ q'_2(t) \\ \vdots \\ q'_N(t) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \\ \vdots \\ q_N(t) \end{bmatrix} + \mathbf{B}\mathbf{x}(t)$$

This is a set of  $N$  uncoupled differential equations which could be solved one at a time. So diagonalization converts  $N$  coupled differential equations into  $N$  uncoupled or independent differential equations which can be solved one at a time.

# Diagonalization

Example

A system has  $\mathbf{A}_1 = \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$  and  $\mathbf{B}_1 = \begin{bmatrix} 4 & 0 \\ -2 & 1 \end{bmatrix}$ . Find a matrix  $\mathbf{T}$  that diagonalizes the  $\mathbf{A}$  matrix and the new state variables.

The eigenvalues are the solutions of  $|s\mathbf{I} - \mathbf{A}| = 0$  or

$$\begin{vmatrix} s-2 & 1 \\ 3 & s-4 \end{vmatrix} = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 5. \text{ To find } \mathbf{T} \text{ solve}$$

$$\Lambda \mathbf{T} = \mathbf{T} \mathbf{A}_1 \Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix}$$

# Diagonalization

Example

$$\text{From } \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 4 \end{bmatrix} \text{ we get}$$

$$t_{11} = 2t_{11} - 3t_{12}, \quad t_{12} = -t_{11} + 4t_{12}$$

$$5t_{21} = 2t_{21} - 3t_{22}, \quad 5t_{22} = -t_{21} + 4t_{22}$$

and the top two equations both simplify to  $-t_{11} + 3t_{12} = 0$  so they are not linearly independent. The same is true of the bottom two.

This means that we can arbitrarily choose two elements of  $\mathbf{T}$  and then solve for the other two. Let  $t_{11} = a$  and  $t_{21} = b$ . Then

$$\begin{matrix} t_{12} = a/3 \\ t_{22} = -b \end{matrix} \Rightarrow \mathbf{T} = \begin{bmatrix} a & a/3 \\ b & -b \end{bmatrix}$$

# Diagonalization

Example

$\mathbf{T} = \begin{bmatrix} a & a/3 \\ b & -b \end{bmatrix}$ . We could choose  $t_{11} = a = 3$  and  $t_{21} = b = 1$ . Then

$\mathbf{T} = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}$ . The state-variable vector corresponding to the new

diagonalized A is  $\mathbf{q}_2 = \mathbf{T}\mathbf{q}_1 = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix}\mathbf{q}_1$  and the new system equation

is  $\mathbf{q}'_2(t) = \mathbf{T}\mathbf{A}_1\mathbf{T}^{-1}\mathbf{q}_2(t) + \mathbf{T}\mathbf{B}_1\mathbf{x}(t) = \mathbf{A}_2\mathbf{q}_2(t) + \mathbf{B}_2\mathbf{x}(t)$  or

$$\mathbf{q}'_2(t) = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\mathbf{q}_2(t) + \begin{bmatrix} 10 & 1 \\ 6 & -1 \end{bmatrix}\mathbf{x}(t).$$

# Discrete Time



# State Variables

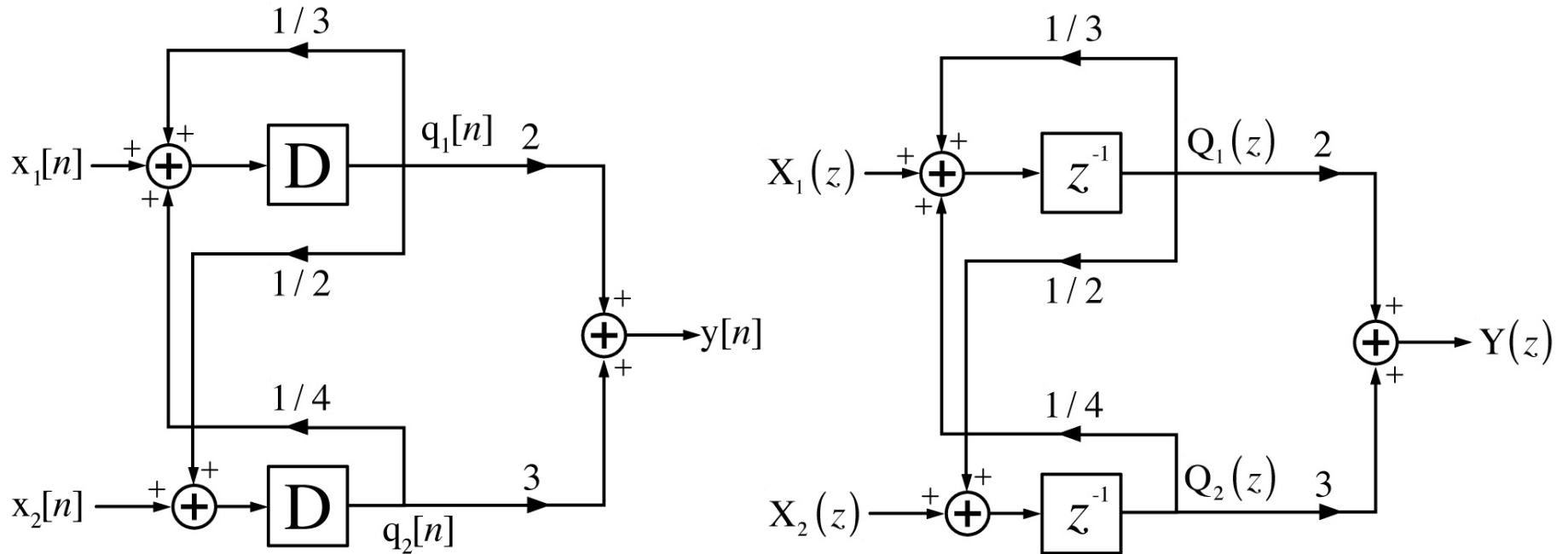
In continuous-time-system, state-space realization, the derivatives of the state variables are set equal to a linear combination of the state variables and the excitations. In discrete-time system state-space realization, the next state-variable values are equated to a linear combination of the present state-variable values and the present excitations.

$$\mathbf{q}[n + 1] = \mathbf{A}\mathbf{q}[n] + \mathbf{B}\mathbf{x}[n]$$

$$\mathbf{y}[n] = \mathbf{C}\mathbf{q}[n] + \mathbf{D}\mathbf{x}[n]$$

The state variables are the responses of the delay blocks.

# State Variables



$$\mathbf{q}[n] = \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix}, \mathbf{A} = \begin{bmatrix} 1/3 & 1/4 \\ 1/2 & 0 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathbf{x}[n] = \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix}$$

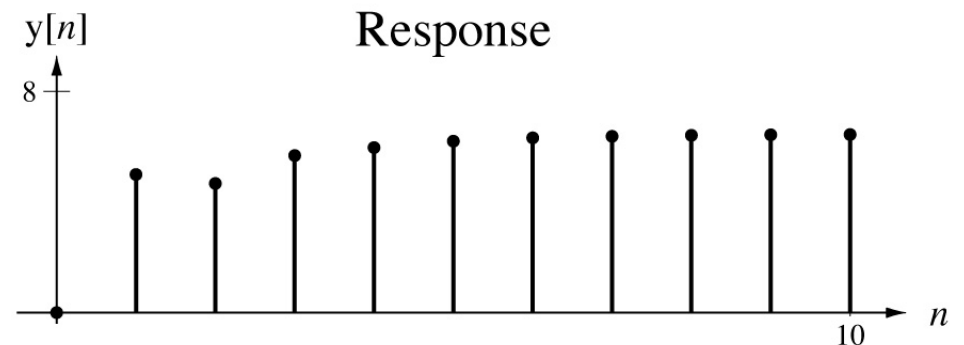
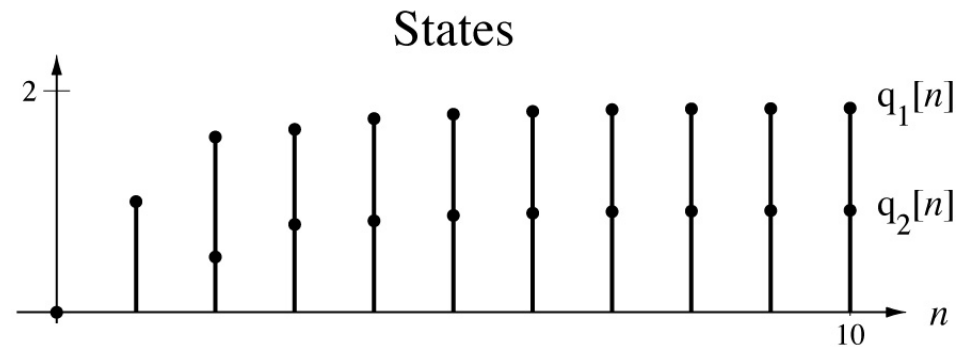
$$\mathbf{y}[n] = [y[n]], \mathbf{C} = [2 \quad 3], \mathbf{D} = [0 \quad 0]$$

# State Variables

One way to solve the state equations is by recursion. Let  $\mathbf{x}[n] = \begin{bmatrix} u[n] \\ \delta[n] \end{bmatrix}$

and let the system be initially at rest ( $\mathbf{q}[0] = [0]$ ). Then

$n$	$q_1[n]$	$q_2[n]$	$y[n]$
0	0	0	0
1	1	1	5
2	1.5833	0.5	4.667
3	1.6528	0.7917	5.681
$\vdots$	$\vdots$	$\vdots$	$\vdots$



# State Variables

The process of recursion is

$$\mathbf{q}[1] = \mathbf{A}\mathbf{q}[0] + \mathbf{B}\mathbf{x}[0]$$

$$\mathbf{q}[2] = \mathbf{A}\mathbf{q}[1] + \mathbf{B}\mathbf{x}[1] = \mathbf{A}^2\mathbf{q}[0] + \mathbf{A}\mathbf{B}\mathbf{x}[0] + \mathbf{B}\mathbf{x}[1]$$

$$\mathbf{q}[3] = \mathbf{A}\mathbf{q}[2] + \mathbf{B}\mathbf{x}[2] = \mathbf{A}^3\mathbf{q}[0] + \mathbf{A}^2\mathbf{B}\mathbf{x}[0] + \mathbf{A}\mathbf{B}\mathbf{x}[1] + \mathbf{B}\mathbf{x}[2]$$

⋮

$$\mathbf{q}[n] = \mathbf{A}^n\mathbf{q}[0] + \mathbf{A}^{n-1}\mathbf{B}\mathbf{x}[0] + \mathbf{A}^{n-2}\mathbf{B}\mathbf{x}[1] + \cdots + \mathbf{A}^1\mathbf{B}\mathbf{x}[n-2] + \mathbf{A}^0\mathbf{B}\mathbf{x}[n-1]$$

and

$$\mathbf{y}[1] = \mathbf{C}\mathbf{q}[1] + \mathbf{D}\mathbf{x}[1] = \mathbf{C}\mathbf{A}\mathbf{q}[0] + \mathbf{C}\mathbf{B}\mathbf{x}[0] + \mathbf{D}\mathbf{x}[1]$$

$$\mathbf{y}[2] = \mathbf{C}\mathbf{q}[2] + \mathbf{D}\mathbf{x}[2] = \mathbf{C}\mathbf{A}^2\mathbf{q}[0] + \mathbf{C}\mathbf{A}\mathbf{B}\mathbf{x}[0] + \mathbf{C}\mathbf{B}\mathbf{x}[1] + \mathbf{D}\mathbf{x}[2]$$

$$\mathbf{y}[3] = \mathbf{C}\mathbf{q}[3] + \mathbf{D}\mathbf{x}[3] = \mathbf{C}\mathbf{A}^3\mathbf{q}[0] + \mathbf{C}\mathbf{A}^2\mathbf{B}\mathbf{x}[0] + \mathbf{C}\mathbf{A}\mathbf{B}\mathbf{x}[1] + \mathbf{C}\mathbf{B}\mathbf{x}[2] + \mathbf{D}\mathbf{x}[3]$$

⋮

$$\mathbf{y}[n] = \mathbf{C}\mathbf{A}^n\mathbf{q}[0] + \mathbf{C}\mathbf{A}^{n-1}\mathbf{B}\mathbf{x}[0] + \mathbf{C}\mathbf{A}^{n-2}\mathbf{B}\mathbf{x}[1] + \cdots + \mathbf{C}\mathbf{A}^0\mathbf{B}\mathbf{x}[n-1] + \mathbf{D}\mathbf{x}[n]$$

# State Variables

$$\mathbf{q}[n] = \underbrace{\mathbf{A}^n \mathbf{q}[0]}_{\text{Zero-input response}} + \underbrace{\sum_{m=0}^{n-1} \mathbf{A}^{n-m-1} \mathbf{B} \mathbf{x}[m]}_{\text{Zero-State Response}}$$

$$\mathbf{q}[n] = \underbrace{\phi[n] \mathbf{q}[0]}_{\text{zero-input response}} + \underbrace{\phi[n-1] \mathbf{u}[n-1] * \mathbf{B} \mathbf{x}[n]}_{\text{zero-state response}}$$

where  $\phi[n] = \mathbf{A}^n$  is the state transition matrix. Similarly

$$\mathbf{y}[n] = \mathbf{C} \mathbf{A}^n \mathbf{q}[0] + \mathbf{C} \sum_{m=0}^{n-1} \mathbf{A}^{n-m-1} \mathbf{B} \mathbf{x}[m] + \mathbf{D} \mathbf{x}[n]$$

becomes

$$\mathbf{y}[n] = \mathbf{C} \phi[n] \mathbf{q}[0] + \mathbf{C} \phi[n-1] \mathbf{u}[n-1] * \mathbf{B} \mathbf{x}[n] + \mathbf{D} \mathbf{x}[n]$$

These are the discrete-time solutions of the system and output equations.

# State Variables

Using the  $z$  transform

$$z\mathbf{Q}(z) - z\mathbf{q}[0] = \mathbf{A}\mathbf{Q}(z) + \mathbf{B}\mathbf{X}(z)$$

$$\mathbf{Q}(z) = [z\mathbf{I} - \mathbf{A}]^{-1} [\mathbf{B}\mathbf{X}(z) + z\mathbf{q}[0]] = \underbrace{[z\mathbf{I} - \mathbf{A}]^{-1} \mathbf{B}\mathbf{X}(z)}_{\text{zero-state response}} + \underbrace{z[z\mathbf{I} - \mathbf{A}]^{-1} \mathbf{q}[0]}_{\text{zero-input response}}$$

where  $\phi[n] \xleftrightarrow{\mathcal{Z}} z[z\mathbf{I} - \mathbf{A}]^{-1} \Rightarrow \Phi(z) = z[z\mathbf{I} - \mathbf{A}]^{-1}$  (analogous to the continuous-time result  $\Phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1}$ ). With the same excitation and

initial conditions as before  $\mathbf{x}[n] = \begin{bmatrix} u[n] \\ \delta[n] \end{bmatrix}$  and  $\mathbf{q}[0] = [0]$ ,

$$\mathbf{Q}(z) = \begin{bmatrix} z - 1/3 & -1/4 \\ -1/2 & z \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ z - 1 \\ 1 \end{bmatrix}$$

# State Variables

$$\mathbf{Q}(z) = \left[ \begin{array}{c} \frac{z^2 + z/4 - 1/4}{(z-1)(z-0.5575)(z+0.2242)} \\ \frac{z^2 - 5z/6 + 1/3}{(z-1)(z-0.5575)(z+0.2242)} \end{array} \right] = \left[ \begin{array}{c} \frac{1.846}{z-1} - \frac{0.578}{z-0.5575} - \frac{0.268}{z+0.2242} \\ \frac{0.923}{z-1} - \frac{0.519}{z-0.5575} + \frac{0.596}{z+0.2242} \end{array} \right]$$

Inverse  $z$  transforming

$$\mathbf{q}[n] = \left[ \begin{array}{c} 1.846 - 0.578(0.5575)^{(n-1)} - 0.268(-0.2242)^{(n-1)} \\ 0.923 - 0.519(0.5575)^{(n-1)} + 0.596(-0.2242)^{(n-1)} \end{array} \right] \mathbf{u}[n-1]$$

After solving for the states the response can be found in one simple step.

$$\mathbf{y}[n] = \left[ 6.461 - 2.713(0.5575)^{(n-1)} + 1.252(-0.2242)^{(n-1)} \right] \mathbf{u}[n-1]$$

# Transfer Functions

The system equation is

$$z\mathbf{Q}(z) - z\mathbf{q}[0] = \mathbf{A}\mathbf{Q}(z) + \mathbf{B}\mathbf{X}(z)$$

The initial conditions must be zero to find a transfer function. Then, solving for  $\mathbf{Q}(z)$ ,

$$\mathbf{Q}(z) = [z\mathbf{I} - \mathbf{A}]^{-1} \mathbf{B}\mathbf{X}(z) = z^{-1}\Phi(z)\mathbf{B}\mathbf{X}(z).$$

The response is

$$\mathbf{Y}(z) = \mathbf{C}\mathbf{Q}(z) + \mathbf{D}\mathbf{X}(z) = z^{-1}\mathbf{C}\Phi(z)\mathbf{B}\mathbf{X}(z) + \mathbf{D}\mathbf{X}(z)$$

and the transfer function is then

$$\mathbf{H}(z) = z^{-1}\mathbf{C}\Phi(z)\mathbf{B} + \mathbf{D} = \mathbf{C}[z\mathbf{I} - \mathbf{A}]^{-1} \mathbf{B} + \mathbf{D}$$



# Transformations of State Variables

If  $\mathbf{q}_2[n] = \mathbf{T}\mathbf{q}_1[n]$  and  $\mathbf{q}_1[n+1] = \mathbf{A}_1\mathbf{q}_1[n] + \mathbf{B}_1\mathbf{x}[n]$   
then  $\mathbf{q}_2[n+1] = \mathbf{A}_2\mathbf{q}_2[n] + \mathbf{B}_2\mathbf{x}[n]$  where  $\mathbf{A}_2 = \mathbf{T}\mathbf{A}_1\mathbf{T}^{-1}$  and  
 $\mathbf{B}_2 = \mathbf{T}\mathbf{B}_1$ . Also  $\mathbf{y}[n] = \mathbf{C}_2\mathbf{q}_2[n] + \mathbf{D}_2\mathbf{x}[n]$  where  $\mathbf{C}_2 = \mathbf{C}_1\mathbf{T}^{-1}$   
and  $\mathbf{D}_2 = \mathbf{D}_1$ . These transformation relations are exactly the  
same as in continuous-time.

# Use of MATLAB

A system object described by a state-space model in MATLAB can be formed using

$$\text{sys} = \text{ss}(\text{A}, \text{B}, \text{C}, \text{D}, \text{Ts}) ;$$

where A, B, C and D are the A, B, C and D matrices and Ts is the time between samples. The function `ssdata` extracts state space matrices from a system object and the function `ss2ss` transforms one state-space model into another. (See the help files for more detail.)