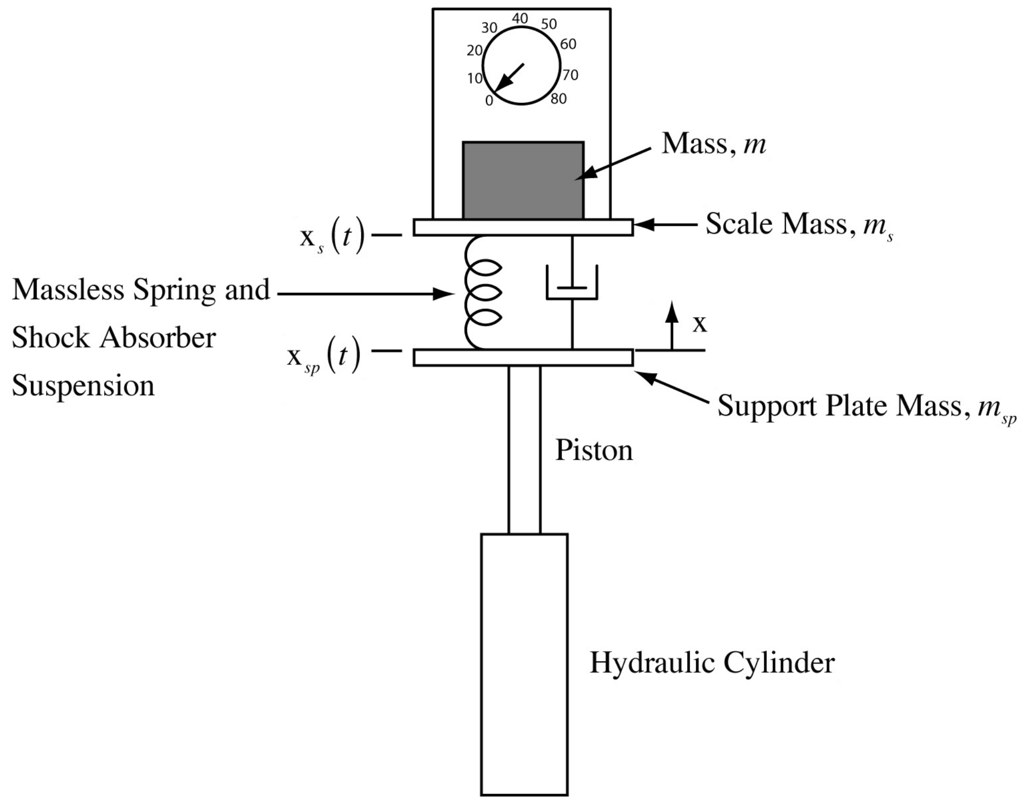
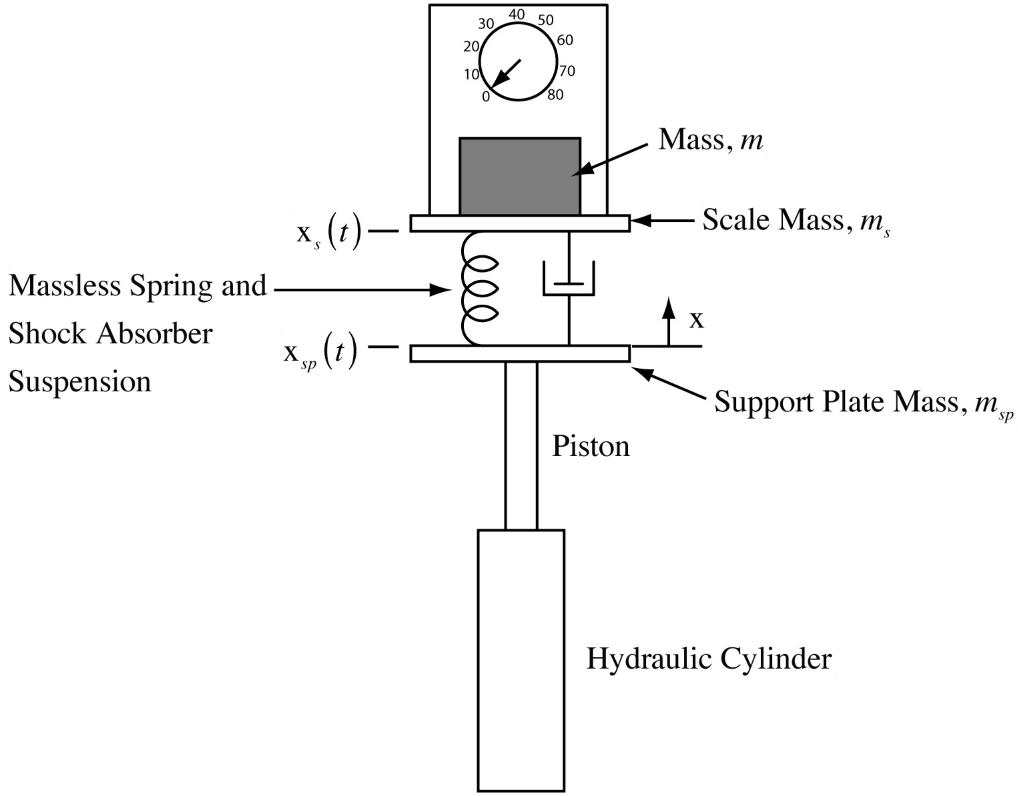


Let $x_{sp}(0) = x_{sp0}$ and $x_s(0) = x_{s0}$ when the system is not moving. Let the spring constant be k_s (N/m) and the shock absorber constant be k_a $(\mathrm{N \cdot s/m})$. Let the unstretched length of the spring be Δx_u (m). Then the length of the spring when the system is not moving is $x_{s0} - x_{sp0} = \Delta x_u - (m + m_s) g / k_s$ where g is the gravitational constant 9.8 $\mathrm{m/s^2}$.



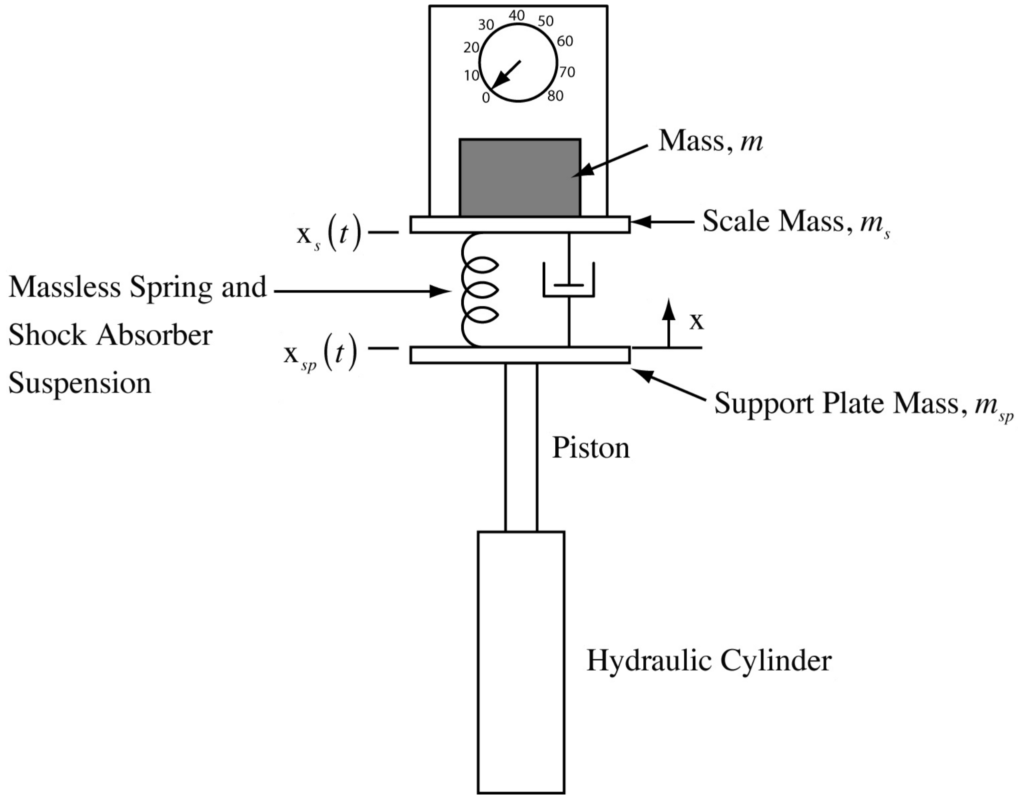
When the system is not moving the equilibrium of forces on the support plate is

$$f - m_{sp} g + k_s \left(x_{s0} - x_{sp0} - \Delta x_u \right) = 0$$

where f is the upward force of the piston. The equilibrium of forces on the scale and mass is

$$-\left(m_s + m \right) g - k_s \left(x_{s0} - x_{sp0} - \Delta x_u \right) = 0$$

Adding equations we get $f = \left(m_s + m + m_{sp} \right) g$ which says that the force of the piston supports all the weight.



The differential equations account for the dynamics through f = ma.

$$f(t) - m_{sp} g + k_s \left(x_s(t) - x_{sp}(t) - \Delta x_u \right) + k_a \left(x_s'(t) - x_{sp}'(t) \right) = m_{sp}\, x_{sp}''(t)$$

$$-(m_s + m) g - k_s \left(x_s(t) - x_{sp}(t) - \Delta x_u \right) - k_a \left(x_s'(t) - x_{sp}'(t) \right) = (m_s + m) x_s''(t)$$

Laplace transforming the differential equations and applying initial conditions of no movement,

$$\mathrm{F}(s) - m_{sp} g / s + k_s \left(\mathrm{X}_s(s) - \mathrm{X}_{sp}(s) - \Delta \mathrm{x}_\mathrm{u} / s \right) + k_a \left(s\,\mathrm{X}_s(s) - \mathrm{x}_{s0} - s\,\mathrm{X}_{sp}(s) + \mathrm{x}_{sp0} \right)$$

$$= m_{sp} \left[s^2\,\mathrm{X}_{sp}(s) - s\,\mathrm{x}_{sp0} \right]$$

$$-(m_s + m) g / s - k_s \left(\mathrm{X}_s(s) - \mathrm{X}_{sp}(s) - \Delta \mathrm{x}_\mathrm{u} / s \right) - k_a \left(s\,\mathrm{X}_s(t) - \mathrm{x}_{s0} - s\,\mathrm{X}_{sp}(t) + \mathrm{x}_{sp0} \right)$$

$$= (m_s + m) \left[s^2\,\mathrm{X}_s(s) - s\,\mathrm{x}_{s0} \right]$$

$$s\mathrm{F}(s) - m_{sp} g + s k_s\,\mathrm{X}_s(s) - s k_s\,\mathrm{X}_{sp}(s) - k_s \Delta \mathrm{x}_\mathrm{u} + s^2 k_a\,\mathrm{X}_s(s) - s k_a\,\mathrm{x}_{s0} - s^2 k_a\,\mathrm{X}_{sp}(s) + s k_a\,\mathrm{x}_{sp0}$$

$$= s^3 m_{sp}\,\mathrm{X}_{sp}(s) - s^2 m_{sp}\,\mathrm{x}_{sp0}$$

$$-(m_s + m) g - s k_s\,\mathrm{X}_s(s) + s k_s\,\mathrm{X}_{sp}(s) + k_s \Delta \mathrm{x}_\mathrm{u} - s^2 k_a\,\mathrm{X}_s(t) + s k_a\,\mathrm{x}_{s0} + s^2 k_a\,\mathrm{X}_{sp}(t) - s k_a\,\mathrm{x}_{sp0}$$

$$= s^3 (m_s + m) \mathrm{X}_s(s) - s^2 (m_s + m) \mathrm{x}_{s0}$$

$$\begin{bmatrix} s\left(m_{sp} s^2 + s k_a + k_s\right) & -s\left(s k_a + k_s\right) \\ -s\left(s k_a + k_s\right) & s\left[(m_s + m) s^2 + s k_a + k_s\right] \end{bmatrix} \begin{bmatrix} \mathrm{X}_{sp}(s) \\ \mathrm{X}_s(s) \end{bmatrix}$$

$$= \begin{bmatrix} s\mathrm{F}(s) - m_{sp} g - k_s \Delta \mathrm{x}_\mathrm{u} - s k_a \left(\mathrm{x}_{s0} - \mathrm{x}_{sp0} \right) + s^2 m_{sp}\,\mathrm{x}_{sp0} \\ -(m_s + m) g + k_s \Delta \mathrm{x}_\mathrm{u} + s k_a \left(\mathrm{x}_{s0} - \mathrm{x}_{sp0} \right) + s^2 (m_s + m) \mathrm{x}_{s0} \end{bmatrix}$$

Let $\mathrm{f}(t)=\left[\left(m+m_s+m_{sp}\right)g+1\right]\mathrm{u}(t)$. Then $\mathrm{F}(s)=\left[\left(m+m_s+m_{sp}\right)g+1\right]/s$. Also let

$\mathrm{x}_{sp0}=0\text{ m}$, $\mathrm{x}_{s0}=0.2\text{ m}$, $m=20\text{ kg}$, $m_s=5\text{ kg}$, $m_{sp}=5\text{ kg}$

$k_s=1000\text{ N/m}$, $k_a=50\text{ N}\cdot\text{s/m}$

Then $\Delta\mathrm{x}_{\mathrm{u}}=\mathrm{x}_{s0}-\mathrm{x}_{sp0}+\left(m+m_s\right)g/k_s=0.445\text{ m}$

$$\begin{bmatrix} 5s\left(s^2+10s+200\right) & -50s\left(s+20\right) \\ -50s\left(s+20\right) & 25s\left(s^2+2s+40\right) \end{bmatrix}\begin{bmatrix} \mathrm{X}_{sp}(s) \\ \mathrm{X}_s(s) \end{bmatrix}=\begin{bmatrix} -10\left(s+19.9\right) \\ 5\left(s^2+2s+40\right) \end{bmatrix}$$

$$\Delta=5s\left(s^2+10s+200\right)\times 25s\left(s^2+2s+40\right)-\left[50s\left(s+20\right)\right]^2$$

$$\Delta=125s^2\left(s^4+12s^3+260s^2+800s+8000\right)-2500s^2\left(s^2+40s+400\right)$$

$$\Delta=125s^4\left(s^2+12s+240\right)$$

$$\mathrm{X}_s(s)=\frac{5s\left(s^2+10s+200\right)\times 5\left(s^2+2s+40\right)-10\left(s+19.9\right)50s\left(s+20\right)}{125s^4\left(s^2+12s+240\right)}$$

$$\mathrm{X}_s(s)=\frac{\left(s^2+10s+200\right)\left(s^2+2s+40\right)-20\left(s+19.9\right)\left(s+20\right)}{5s^3\left(s^2+12s+240\right)}$$

$$\mathrm{X}_s(s)=\frac{s^4+12s^3+240s^2+2s+40}{5s^3\left(s^2+12s+240\right)}=\frac{0.19978}{s}+\frac{0.0333333}{s^3}+0.0002222\frac{s-102}{s^2+12s+240}$$

$$\mathrm{X}_s(s) = \frac{0.19978}{s} + \frac{0.0333333}{s^3} + 0.0002222\frac{s-102}{s^2+12s+240}$$

$$\mathrm{X}_s(s) = \frac{0.19978}{s} + \frac{0.0333333}{s^3} + 0.0002222\frac{s-102}{s^2+12s+240}$$

$$\mathrm{x}_s(t) = 0.19978\,\mathrm{u}(t) + 0.0166667t^2\,\mathrm{u}(t) + 0.0016948e^{-6t}\cos(14.2829t + 1.4393)\mathrm{u}(t)$$

$$\mathrm{x}_s(0^+) = 0.2 \quad \text{Check.}$$

$$\mathrm{x}_s''(t) = \mathscr{L}^{-1}\left[s^2\,\mathrm{X}_s(s) - s\,\mathrm{x}_s(0^-) - \underbrace{\left(\frac{d}{dt}\mathrm{x}_s(t)\right)_{t=0^-}}_{=0}\right]$$

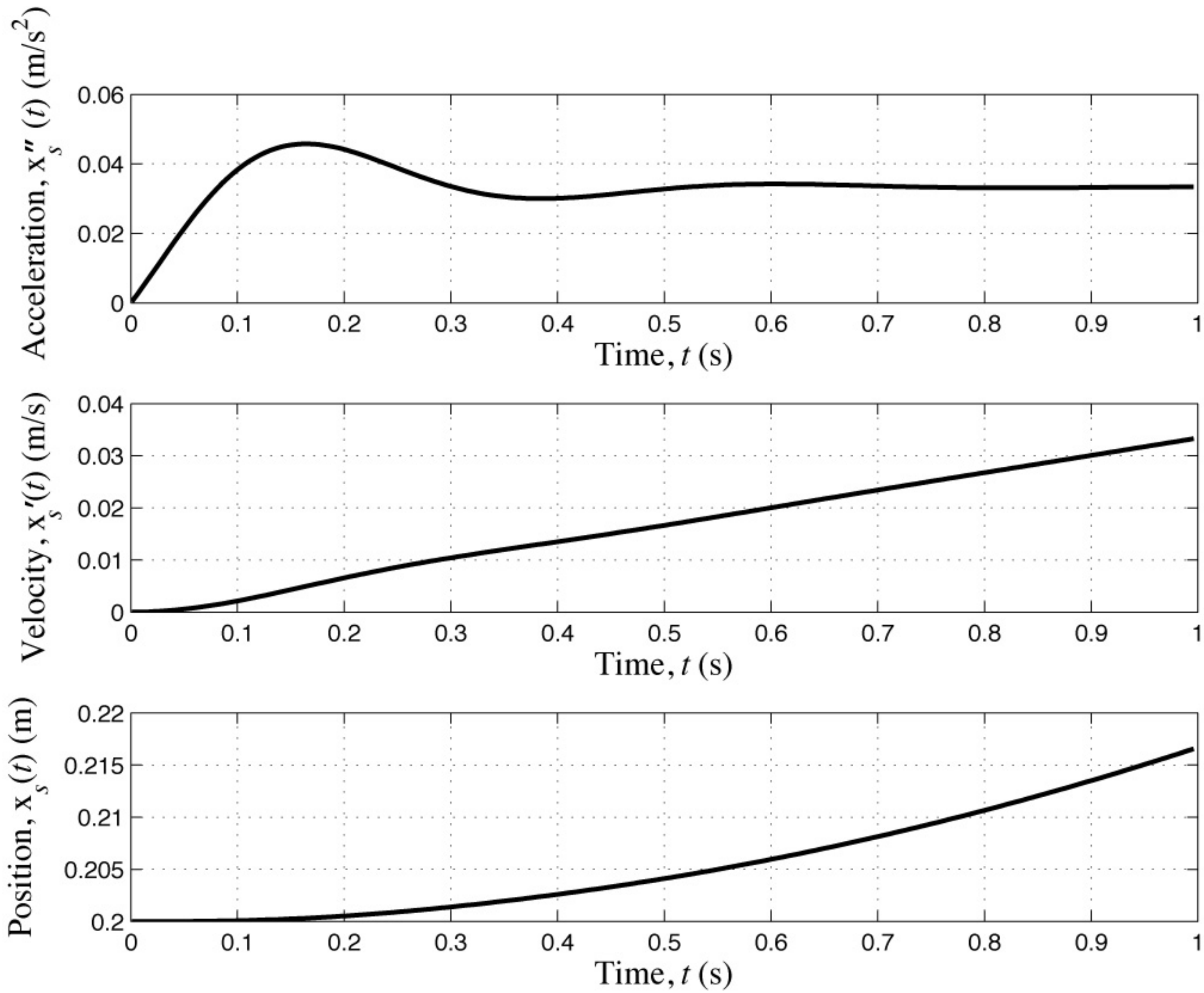
$$\mathrm{x}_s''(t) = \mathscr{L}^{-1}\left[s^2\,\frac{s^4+12s^3+240s^2+2s+40}{5s^3(s^2+12s+240)} - 0.2s\right]$$

$$\mathrm{x}_s''(t) = \mathscr{L}^{-1}\left[\frac{s^6+12s^5+240s^4+2s^3+40s^2-s^6-12s^5-240s^4}{5s^3(s^2+12s+240)}\right]$$

$$\mathrm{x}_s''(t) = \mathscr{L}^{-1}\left[\frac{2s+40}{5s(s^2+12s+240)}\right] = \mathscr{L}^{-1}\left[0.03333\left(\frac{1}{s} - \frac{s}{s^2+12s+240}\right)\right]$$

$$\mathrm{x}_s''(t) = \mathscr{L}^{-1}\left[0.0333333\left(\frac{1}{s} - \frac{s+6}{(s+6)^2+204} + \frac{6}{14.2829}\frac{14.2829}{(s+6)^2+204}\right)\right]$$

$$\mathrm{x}_s''(t) = 0.0333333\left\{1 - e^{-6t}\left[\cos(14.2829t) - 0.42\sin(14.2829t)\right]\right\}\mathrm{u}(t) \quad \mathrm{m/s^2}$$


Acceleration, $x_s''(t)$ (m/s^2)
0.06
0.04
0.02
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
Time, t (s)
Velocity, $x_s'(t)$ (m/s)
0.04
0.03
0.02
0.01
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
Time, t (s)
Position, $x_s(t)$ (m)
0.22
0.215
0.21
0.205
0.2
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
Time, t (s)

If the scale is used under standard conditions at sea level it can be calibrated to indicate the mass, in kg, of the item placed on it. The relationship is $m_i = mg / g = m$ where m_i is the indicated mass. When subjected to the force of the piston and accelerated upward the mass indication will be $m_i = m(g+a)/g = m(1+a/\mathrm{g})$ where a is the upward acceleration in $\mathrm{m/s^2}$. In this case the acceleration settles out at 0.033333 $\mathrm{m/s^2}$ and the indicated mass would be

$$m_i = m(1+0.033333/9.8) = 1.0034m$$

an error of about 0.34%.