Linear CW Modulation

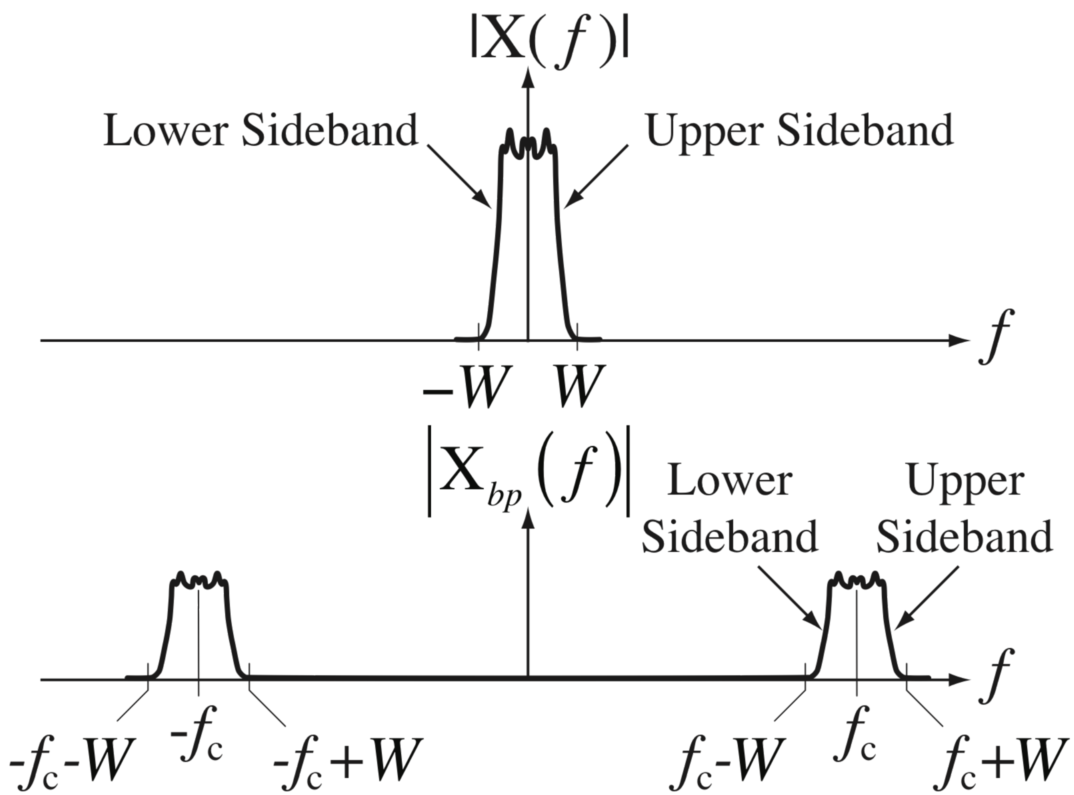
Bandpass Signals and Systems

Let $\mathrm{x}(t)$ represent a **message** to be sent from one location to another. Let $\mathrm{x}(t)$ have negligible spectral content for $|f| > W$. Let $|\mathrm{x}(t)| \leq 1$. Then $\langle \mathrm{x}^2(t) \rangle \leq 1$, meaning its average signal power cannot exceed one. Then $\mathrm{x}(t)$ will represent a typical message signal. Sometimes, when the analysis of a general signal $\mathrm{x}(t)$ becomes difficult we will let $\mathrm{x}(t) = A_m \cos(\omega_m t)$ with $A_m \leq 1$ and $f_m < W$. This could be one sinusoid of a spectrum of sinusoids that make up the real message signal.

Bandpass Signals and Systems

Let $\mathrm{x}(t)$ modulate a cosine to form a **bandpass** signal

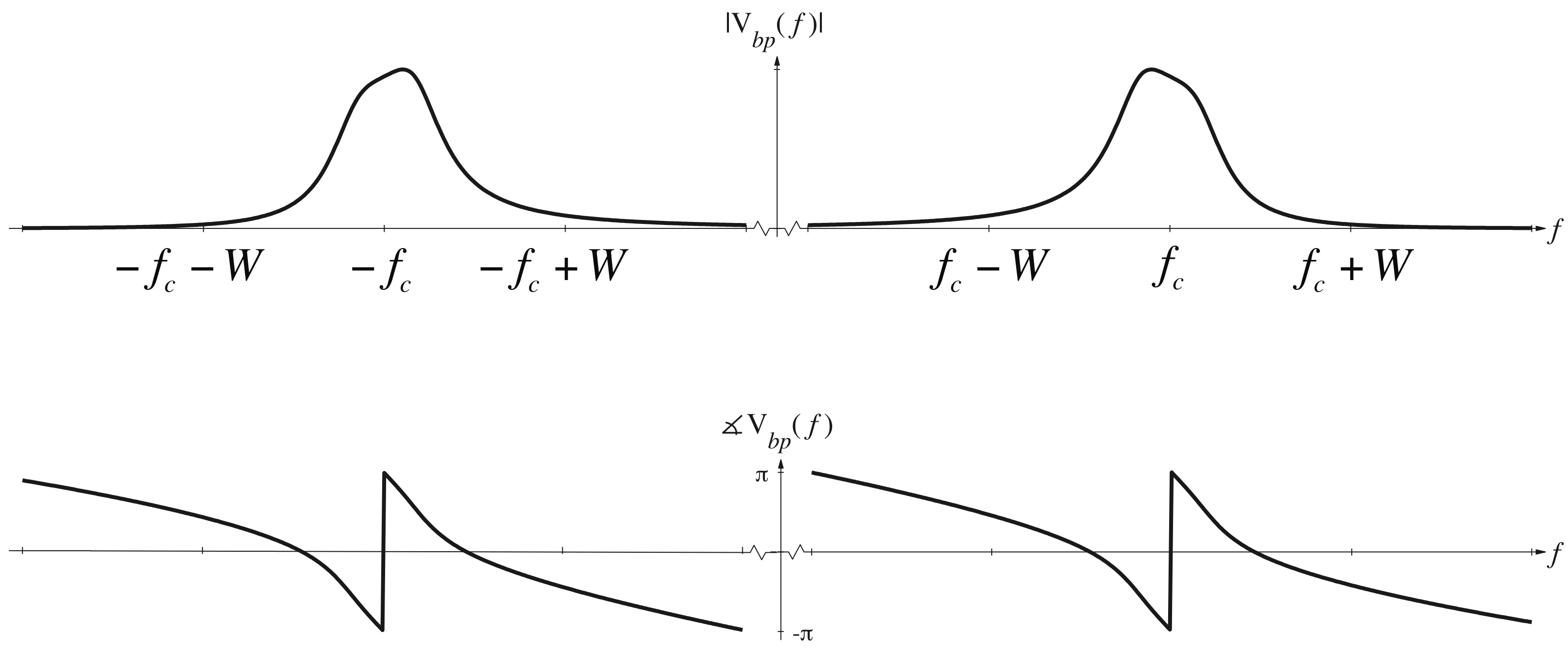
$$\mathrm{x}_{bp}(t) = \mathrm{x}(t)\cos(2\pi f_c t) \xleftrightarrow{\mathscr{F}} \mathrm{X}_{bp}(f) = (1/2)\left[\mathrm{X}(f - f_c) + \mathrm{X}(f + f_c)\right].$$



Bandpass Signals and Systems

Consider an energy signal $v_{bp}(t)$ whose Fourier transform $V_{bp}(f)$ has a bandpass characteristic, which means

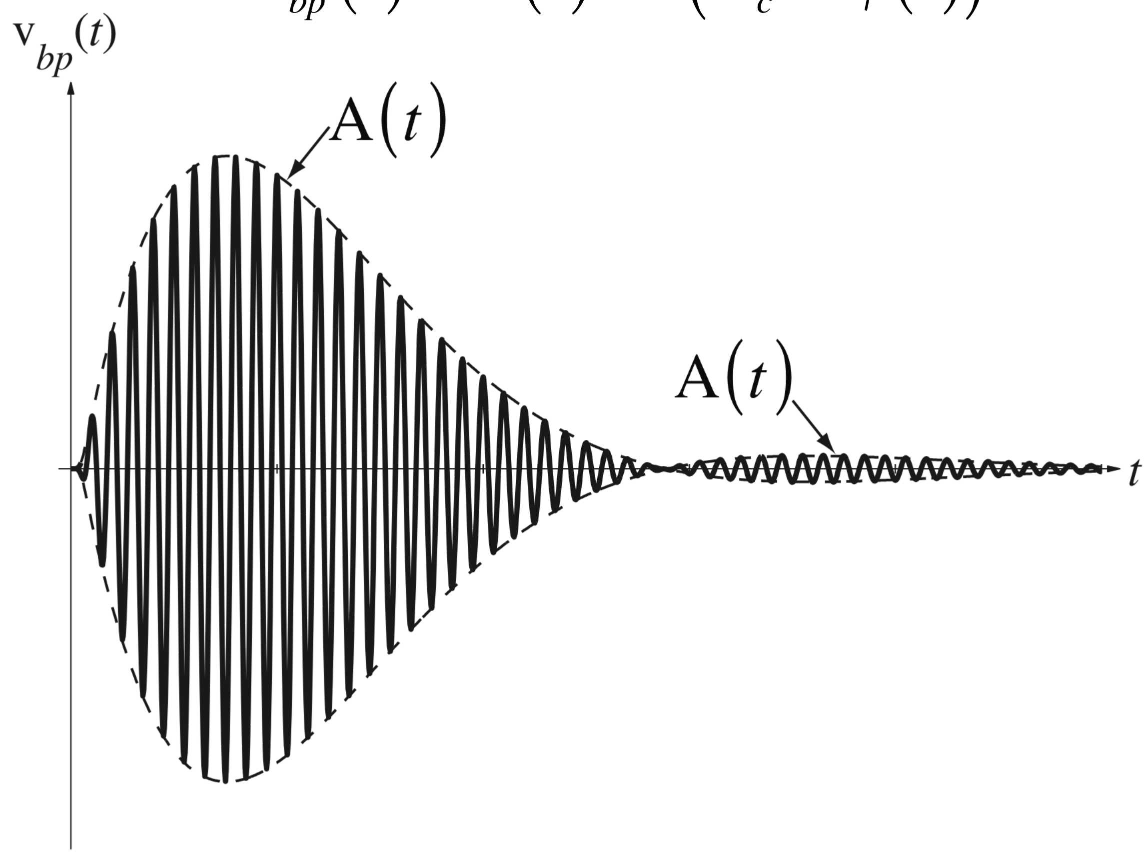
$$V_{bp}(f) \cong 0 \ , \ |f| < f_c - W \ \text{and} \ |f| > f_c + W$$



Bandpass Signals and Systems

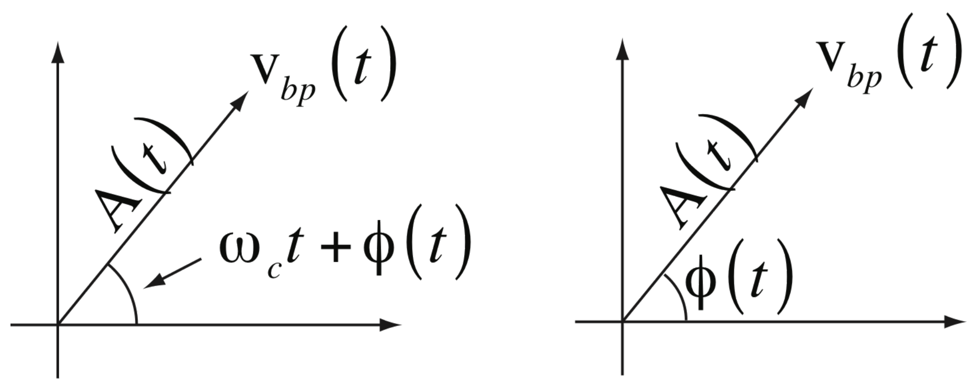
The time-domain signal $\mathrm{v}_{bp}(t)$ will have the form of a sinusoid with a slowly-changing **envelope** $\mathrm{A}(t)$ and **phase shift** $\phi(t)$.

$$\mathrm{v}_{bp}(t) = \mathrm{A}(t)\cos\left(\omega_c t + \phi(t)\right)$$



Bandpass Signals and Systems

The bandpass signal $v_{bp}(t)$ is characterized by its amplitude $A(t)$, frequency f_c or ω_c and phase shift $\phi(t)$. We can represent the signal in a vector diagram in which $A(t)$ is the length of the vector and $\omega_c t + \phi(t)$ is its angle. Since $\omega_c t$ represents a rotation at a constant angular velocity, we can suppress it and characterize the signal by $A(t)$ and $\phi(t)$ only, always realizing that the $\omega_c t$ term is there and can be re-introduced if needed.



Bandpass Signals and Systems

$$v_{bp}(t) = A(t)\cos(\omega_c t + \phi(t))$$

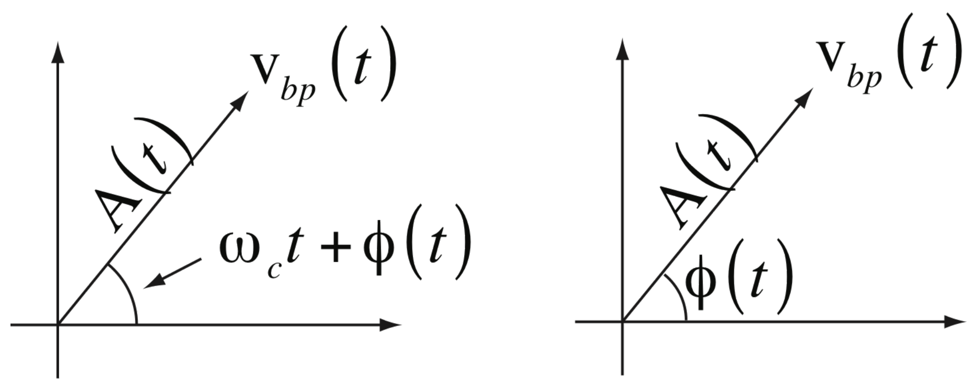
Using the trigonometric identity $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$

$$v_{bp}(t) = A(t)\cos(\phi(t))\cos(\omega_c t) - A(t)\sin(\phi(t))\sin(\omega_c t)$$

Therefore, the bandpass signal can also be characterized by its **in-phase** component $v_i(t) \triangleq A(t)\cos(\phi(t))$ and its **quadrature** component $v_q(t) \triangleq A(t)\sin(\phi(t))$. Then

$$v_{bp}(t) = v_i(t)\cos(\omega_c t) - v_q(t)\sin(\omega_c t)$$

$$v_{bp}(t) = v_i(t)\cos(\omega_c t) + v_q(t)\cos(\omega_c t + 90°)$$



Bandpass Signals and Systems

Fourier transforming

$$\mathrm{v}_{bp}(t) = \mathrm{v}_i(t)\cos(\omega_c t) - \mathrm{v}_q(t)\sin(\omega_c t)$$

yields

$$\mathrm{V}_{bp}(f) = (1/2)\left[\mathrm{V}_i(f - f_c) + \mathrm{V}_i(f + f_c)\right] - (j/2)\left[\mathrm{V}_q(f + f_c) - \mathrm{V}_q(f - f_c)\right]$$

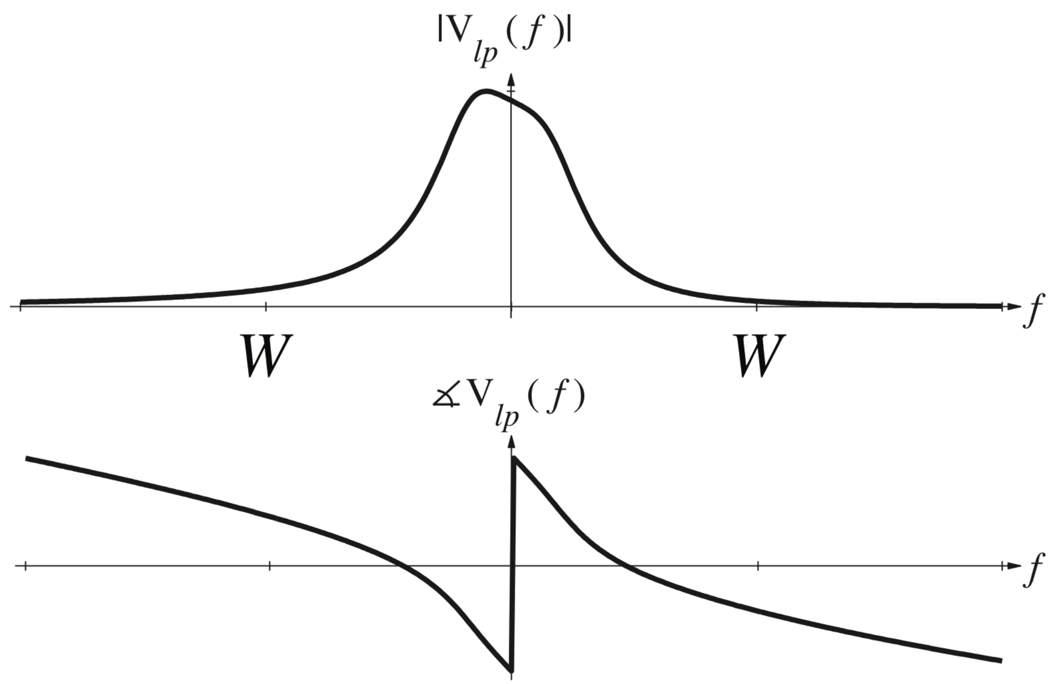
We can see from this result that, for $\mathrm{v}_{bp}(t)$ to be bandpass,

$$\mathrm{V}_i(f) = \mathrm{V}_q(f) = 0 \quad , \quad |f| > W$$

Bandpass Signals and Systems

Now we can conceive $V_{bp}(f)$ as consisting of two lowpass spectra that have been shifted by $\pm f_c$ and, in the case of $V_q(f)$, quadrature phase shifted. Then we can define a **lowpass equivalent spectrum**

$$V_{lp}(f) \triangleq (1/2)\left[V_i(f) + jV_q(f)\right] = V_{bp}(f + f_c)\,u(f + f_c)$$



Bandpass Signals and Systems

Notice that $V_{lp}(f)$ does not have Hermitian symmetry, therefore the inverse transform of $V_{lp}(f)$ is not a real-valued function of time. It is instead

$$v_{lp}(t) = (1/2)\left[v_i(t) + j\,v_q(t)\right]$$

which can also be written as

$$v_{lp}(t) = (1/2)A(t)\left[\cos(\phi(t)) + j\sin(\phi(t))\right] = (1/2)A(t)e^{j\phi(t)}$$

Then the relation between $v_{bp}(t)$ and $v_{lp}(t)$ can be derived as follows

$$v_{bp}(t) = A(t)\cos(\omega_c t + \phi(t)) = \mathrm{Re}\left(A(t)e^{j(\omega_c t + \phi(t))}\right)$$

$$v_{bp}(t) = 2\,\mathrm{Re}\left((1/2)A(t)e^{j\omega_c t}e^{j\phi(t)}\right) = 2\,\mathrm{Re}\left(v_{lp}(t)e^{j\omega_c t}\right)$$

Bandpass Signals and Systems

Fourier transforming $\mathrm{v}_{bp}(t) = 2\,\mathrm{Re}\left(\mathrm{v}_{lp}(t)e^{j\omega_c t}\right)$ we get

$$\mathrm{V}_{bp}(f) = \mathscr{F}\left(2\,\mathrm{Re}\left((1/2)\left[\mathrm{v}_i(t) + j\,\mathrm{v}_q(t)\right]e^{j\omega_c t}\right)\right)$$

$$\mathrm{V}_{bp}(f) = \mathscr{F}\left(\mathrm{v}_i(t)\cos(\omega_c t) - \mathrm{v}_q(t)\sin(\omega_c t)\right)$$

$$\mathrm{V}_{bp}(f) = (1/2)\left[\mathrm{V}_i(f - f_c) + \mathrm{V}_i(f + f_c)\right]$$
$$-(j/2)\left[\mathrm{V}_q(f + f_c) - \mathrm{V}_q(f - f_c)\right]$$

$$\mathrm{V}_{bp}(f) = (1/2)\left[\mathrm{V}_i(f - f_c) + j\mathrm{V}_q(f - f_c)\right]$$
$$+(1/2)\left[\mathrm{V}_i(f + f_c) - j\mathrm{V}_q(f + f_c)\right]$$

$$\mathrm{V}_{bp}(f) = \mathrm{V}_{lp}(f - f_c) + \mathrm{V}_{lp}^{*}(f + f_c)$$

Bandpass Signals and Systems

The response of a bandpass system to an excitation can be found using $\mathrm{Y}_{bp}(f) = \mathrm{H}_{bp}(f)\mathrm{X}_{bp}(f)$. But it can also be found using the lowpass equivalent spectrum of $\mathrm{x}_{bp}(t)$

$$\mathrm{X}_{lp}(f) = \mathrm{X}_{bp}(f + f_c)\mathrm{u}(f + f_c)$$

and the **lowpass equivalent transfer function** (frequency response)

$$\mathrm{H}_{lp}(f) = \mathrm{H}_{bp}(f + f_c)\mathrm{u}(f + f_c)$$

to form the lowpass equivalent spectrum of $\mathrm{y}_{bp}(t)$

$$\mathrm{Y}_{lp}(f) = \mathrm{H}_{lp}(f)\mathrm{X}_{lp}(f).$$

Then $\mathrm{y}_{lp}(t) = \mathscr{F}^{-1}\left(\mathrm{Y}_{lp}(f)\right)$ and $\mathrm{y}_{bp}(t) = 2\,\mathrm{Re}\left(\mathrm{y}_{lp}(t)e^{j\omega_c t}\right)$.

Bandpass Signals and Systems

$$y_{lp}(t) = \mathscr{F}^{-1}\left(Y_{lp}(f)\right) \text{ and } y_{bp}(t) = 2\,\mathrm{Re}\left(y_{lp}(t)\,e^{j\omega_c t}\right).$$

$$y_i(t) = 2\,\mathrm{Re}\left(y_{lp}(t)\right) \quad , \quad y_q(t) = 2\,\mathrm{Im}\left(y_{lp}(t)\right)$$

$$A_y(t) = 2\left|y_{lp}(t)\right| \quad , \quad \phi_y(t) = \measuredangle\, y_{lp}(t)$$

Bandpass Signals and Systems

Example 4.1-1 in Carlson and Crilly: Let $\mathrm{H}_{bp}(f) = Ke^{j\theta(f)}$, $f_l < |f| < f_h$.

Then $\mathrm{H}_{lp}(f) = Ke^{j\theta(f+f_c)}\,\mathrm{u}(f+f_c)$, $f_l < |f+f_c| < f_h$.

If $\theta(f+f_c)$ is relatively slowly varying with f, we can approximate it by the first two terms in its Taylor series expansion about the point $f = 0$.

$$\theta(f+f_c) \cong \theta(f_c) + \frac{(f+f_c)-(0+f_c)}{1!}\left[\frac{d\theta(f+f_c)}{df}\right]_{f=0}$$

$$\theta(f+f_c) \cong \theta(f_c) - f\left[\frac{d\theta(f+f_c)}{df}\right]_{f=0} = \theta(f_c) - f\left[\frac{d\theta(f)}{df}\right]_{f=f_c}$$

$$\theta(f+f_c) \cong -2\pi(t_0 f_c + t_1 f)$$

where $-2\pi t_0 f_c = \theta(f_c) \Rightarrow t_0 = -\dfrac{\theta(f_c)}{2\pi f_c}$

and $-2\pi t_1 f = -f\left[\dfrac{d\theta(f)}{df}\right]_{f=f_c} \Rightarrow t_1 = \dfrac{1}{2\pi}\left[\dfrac{d\theta(f)}{df}\right]_{f=f_c}$

Bandpass Signals and Systems

$$\theta(f+f_c) \cong -2\pi(t_0 f_c + t_1 f)$$

Let $\mathrm{x}_{bp}(t) = \mathrm{A_x}(t)\cos(\omega_c t)$,(implying that $\phi(t)=0$). Then

$$\mathrm{x}_{lp}(t) = (1/2)\mathrm{A_x}(t)\left[\cos(\phi(t)) + j\sin(\phi(t))\right] \text{ or } (1/2)\mathrm{A_x}(t)$$

because $\phi(t)=0$. Then, using $\mathrm{Y}_{lp}(f) = \mathrm{H}_{lp}(f)\mathrm{X}_{lp}(f)$,

$$\mathrm{Y}_{lp}(f) = Ke^{j\theta(f+f_c)}\,\mathrm{u}(f+f_c)\mathrm{X}_{lp}(f) = Ke^{-j2\pi(t_0 f_c + t_1 f)}\,\mathrm{u}(f+f_c)\mathrm{X}_{lp}(f)$$

$\mathrm{Y}_{lp}(f) = Ke^{-j\omega_c t_0}\left[\mathrm{X}_{lp}(f)e^{-j2\pi f t_1}\right]$, (the $\mathrm{u}(f+f_c)$ term can be omitted because $\mathrm{X}_{lp}(f)$ is bandlimited to the range $-f_c \ll -W < f < W \ll f_c$).

Inverse Fourier transforming,

$$\mathrm{y}_{lp}(t) = Ke^{-j\omega_c t_0}\,\mathrm{x}_{lp}(t-t_1) = (1/2)Ke^{-j\omega_c t_0}\,\mathrm{A_x}(t-t_1).$$

Then, using $\mathrm{y}_{bp}(t) = 2\,\mathrm{Re}\left(\mathrm{y}_{lp}(t)e^{j\omega_c t}\right)$,

$$\mathrm{y}_{bp}(t) = K\,\mathrm{A_x}(t-t_1)\cos(\omega_c(t-t_0)).$$

Bandpass Signals and Systems

The result from the previous slide

$$\mathrm{y}_{lp}(t) = K\,\mathrm{A}_{\mathrm{x}}(t - t_1)\cos(\omega_c(t - t_0))$$

indicates that t_0 is the **carrier delay** and t_1 is the **envelope or group delay**. In this example, the original signal $\mathrm{x}_{bp}(t)$ has experienced no delay distortion, at least within the limits of the approximation $\theta(f + f_c) \cong -2\pi(t_0 f_c + t_1 f)$.

Bandpass Signals and Systems

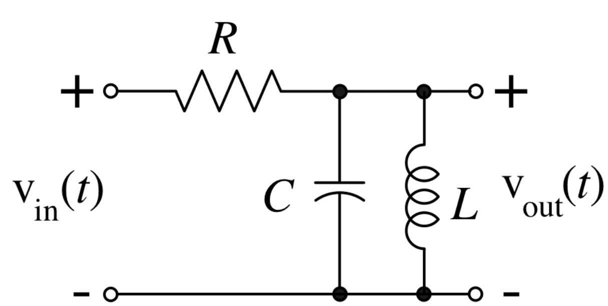
The simplest and most commonly used bandpass system is the parallel resonant *RLC* circuit below. Its frequency response is

$$\mathrm{H}(f) = \frac{\mathrm{Z}_{LC}(f)}{\mathrm{Z}_{LC}(f) + R} \quad \text{where } \mathrm{Z}_{LC}(f) = \frac{j2\pi fL / j2\pi fC}{j2\pi fL + 1/j2\pi fC} = \frac{j2\pi fL}{1 - (2\pi f)^2 LC}$$

$$\text{and } \mathrm{Z}_{LC}(f) = \frac{j2\pi fL}{1 - (f/f_0)^2} \text{ where } f_0 = \frac{1}{2\pi\sqrt{LC}}$$

$$\mathrm{H}(f) = \frac{\dfrac{j2\pi fL}{1-(f/f_0)^2}}{\dfrac{j2\pi fL}{1-(f/f_0)^2} + R} = \frac{1}{1 + jQ\left(\dfrac{f}{f_0} - \dfrac{f_0}{f}\right)} \quad \text{where } Q = R\sqrt{\frac{C}{L}}$$

f_0 is the **resonant cyclic frequency** and Q is the **quality factor**.



Bandpass Signals and Systems

$$\mathrm{H}(f) = \frac{1}{1 + jQ\left(\frac{f}{f_0} - \frac{f_0}{f}\right)}$$

The maximum response occurs when $f = f_0$ and $\mathrm{H}(f_0) = 1$. The -3 dB bandwidth is defined by the frequencies at which $\left|\mathrm{H}(f)\right|^2 = 1/2$.

$$\left|\mathrm{H}(f)\right|^2 = \frac{1}{1 + jQ\left(\frac{f}{f_0} - \frac{f_0}{f}\right)} \times \frac{1}{1 - jQ\left(\frac{f}{f_0} - \frac{f_0}{f}\right)} = \frac{1}{1 + Q^2\left(\frac{f}{f_0} - \frac{f_0}{f}\right)^2} = \frac{1}{2}$$

Bandpass Signals and Systems

$$Q^2\left(\frac{f}{f_0}-\frac{f_0}{f}\right)^2=1 \;\Rightarrow\; \left(f/f_0\right)^2-2-\frac{1}{Q^2}+\left(f_0/f\right)^2=0$$

$$\left(\frac{f^2}{f_0^2}\right)^2-\left(2+\frac{1}{Q^2}\right)\left(\frac{f^2}{f_0^2}\right)+1=0\Rightarrow\frac{f^2}{f_0^2}=\frac{\left(2+1/Q^2\right)\pm\sqrt{\left(2+1/Q^2\right)^2-4}}{2}$$

$$\frac{f^2}{f_0^2}=1+\frac{1}{2Q^2}\pm\sqrt{\frac{4/Q^2+1/Q^4}{4}}=1+\frac{1}{2Q^2}\pm\sqrt{\frac{4Q^2+1}{4Q^4}}=1+\frac{1\pm\sqrt{4Q^2+1}}{2Q^2}$$

For large Q, $4Q^2>>1$ and $2Q>>1$ and $\frac{f^2}{f_0^2}\cong 1\pm\frac{1}{Q}\Rightarrow f^2\cong f_0^2\left(1\pm 1/Q\right)$

$\therefore f\cong \pm f_0\sqrt{1\pm 1/Q}$. Again, for large Q, $\sqrt{1\pm 1/Q}\cong 1\pm 1/2Q$ and $f\cong \pm f_0\left(1\pm 1/2Q\right)$

So the 3 dB bandwidth B is $B\cong f_0/Q$ if Q is large. As a practical matter the Q of this type of tuned circuit is between 10 and 100. Also, as a practical matter, the **fractional bandwidth** B/f_0 should be in the range $0.01<B/f_0<0.1$ to avoid some design problems. Therefore large bandwidths require high center frequencies.

Bandpass Signals and Systems

There are many definitions of "bandwidth".

Absolute bandwidth

The band of frequencies outside of which there is absolutely no signal energy. This only applies to ideal situations in which we have signals that are unlimited in time and filters that are ideal.

Null - to - Null Bandwidth

The spacing between zero crossings of a filter or the spectrum of a signal.

-3 dB Bandwidth

The frequency range between frequencies at which a signal's power is down 3 dB from its maximum (1/2 power points) or at which a filter's power gain is down 3 dB from its maximum.

There are many other definitions for various purposes.

Double-Sideband Amplitude Modulation

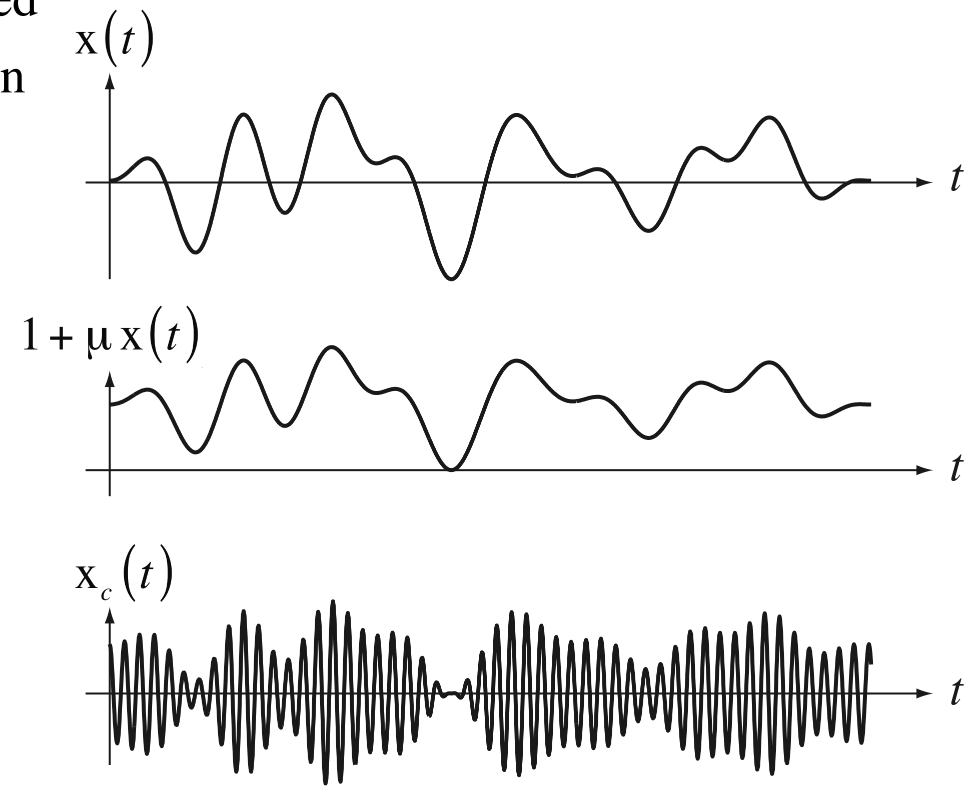
The most common type of amplitude modulation used in practice is **standard amplitude modulation** (**AM**) (also known as Double-Sideband Transmitted Carrier Modulation or DSBTC). In this type of modulation the envelope of the modulated carrier has the shape of the message signal. The modulated carrier is

$$x_c(t) = A_c\left[1 + \mu\, x(t)\right]\cos(\omega_c t)$$

where A_c is the amplitude of the unmodulated carrier and μ is the **modulation index**. Then

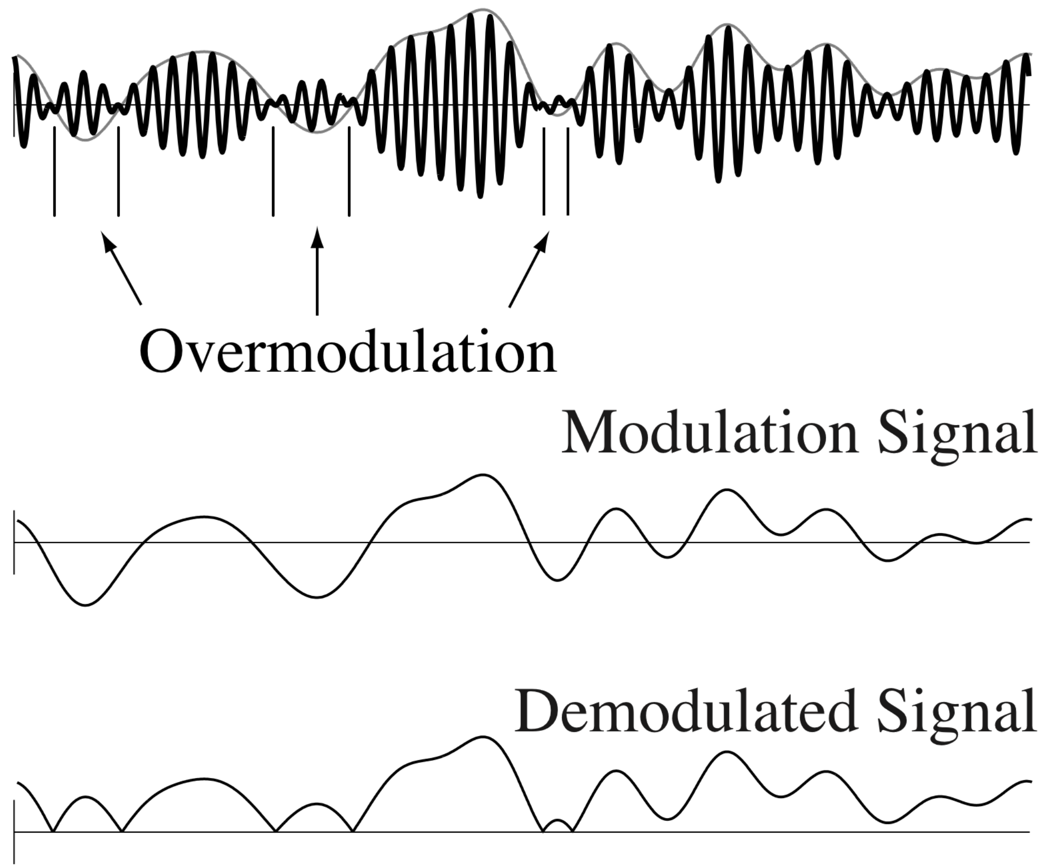
$$A(t) = A_c\left[1 + \mu\, x(t)\right]$$

and $\phi(t) = 0$. Therefore $x_{ci}(t) = A(t)$ and $x_{cq}(t) = 0$.



Double-Sideband Amplitude Modulation

The envelope $A(t)$ is defined as being non-negative. So if μ is too large a problem called **overmodulation** occurs as illustrated below. Simple detection of the envelope causes distortion of the original message.

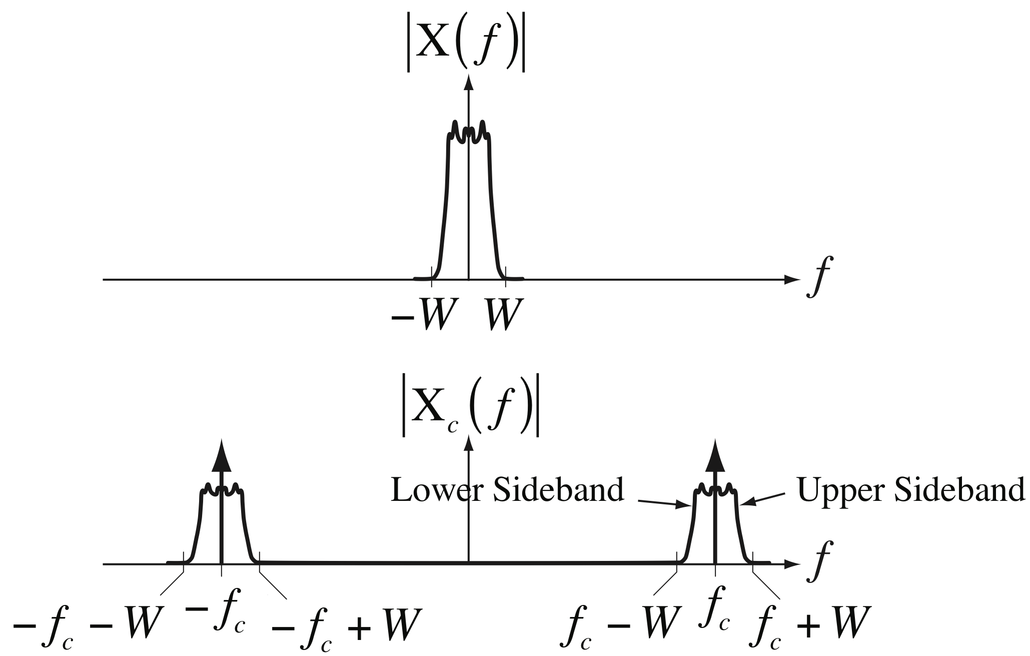


Double-Sideband Amplitude Modulation

The Fourier transform of the AM signal $x_c(t) = A_c[1+\mu x(t)]\cos(\omega_c t)$ is

$$X_c(f) = A_c[\delta(f)+\mu X(f)] * (1/2)[\delta(f-f_c)+\delta(f+f_c)]$$

$$X_c(f) = \frac{A_c}{2}\{\delta(f-f_c)+\delta(f+f_c)+\mu[X(f-f_c)+X(f+f_c)]\}$$



Double-Sideband Amplitude Modulation

The transmission bandwidth required to transmit a message with bandwidth W is $B_T = 2W$. The average transmitted power is $S_T \triangleq \left\langle \mathrm{x}_c^2(t) \right\rangle$. Using $\mathrm{x}_c(t) = A_c\left[1+\mu\,\mathrm{x}(t)\right]\cos(\omega_c t)$

$$S_T = \left\langle A_c^2\left[1+\mu\,\mathrm{x}(t)\right]^2 \cos^2(\omega_c t)\right\rangle$$

$$S_T = A_c^2\left\langle\left[1+2\mu\,\mathrm{x}(t)+\mu^2\,\mathrm{x}^2(t)\right](1/2)\left[1+\cos(2\omega_c t)\right]\right\rangle$$

$$S_T = \frac{A_c^2}{2}\left\langle\left[1+2\mu\,\mathrm{x}(t)+\mu^2\,\mathrm{x}^2(t)\right]+\left[1+2\mu\,\mathrm{x}(t)+\mu^2\,\mathrm{x}^2(t)\right]\cos(2\omega_c t)\right\rangle$$

$$S_T = \frac{A_c^2}{2}\left\langle 1+2\mu\,\mathrm{x}(t)+\mu^2\,\mathrm{x}^2(t)+\right\rangle + \underbrace{\frac{A_c^2}{2}\left\langle\left[1+2\mu\,\mathrm{x}(t)+\mu^2\,\mathrm{x}^2(t)\right]\cos(2\omega_c t)\right\rangle}_{=0 \text{ if } f_c >> W}$$

$$S_T = \frac{A_c^2}{2}\left\langle 1+2\mu\,\mathrm{x}(t)+\mu^2\,\mathrm{x}^2(t)\right\rangle$$

Double-Sideband Amplitude Modulation

$$S_T = \frac{A_c^2}{2}\left\langle 1 + 2\mu\,\mathrm{x}(t) + \mu^2\,\mathrm{x}^2(t)\right\rangle$$

If $\left\langle \mathrm{x}(t)\right\rangle = 0$ (which is typical of most messages) then $S_T = \frac{A_c^2}{2}\left(1 + \mu^2 S_{\mathrm{x}}\right)$ where S_{x} is the average signal power of the message. Then $S_T = P_c + 2P_{sb}$ where $P_c = A_c^2/2$ is the average signal power in the unmodulated carrier and

$$P_{sb} = \frac{1}{2}\times\frac{A_c^2}{2}\mu^2 S_{\mathrm{x}} = (1/2)\mu^2 S_{\mathrm{x}} P_c$$

is the average signal power in each sideband. To avoid overmodulation $\left|\mu\,\mathrm{x}(t)\right| \le 1$. Then $\mu^2 S_{\mathrm{x}} \le 1$ and $P_{sb} \le (1/2)P_c$ and

$$P_c = S_T - 2P_{sb} \ge S_T/2 \text{ and } P_{sb} \le S_T/4$$

So at least half of the total transmitted power is in the carrier and conveys no information. The efficiency can be improved by using other modulation techniques but with an attendant complication in receiver design.

Double-Sideband Suppressed-Carrier Modulation

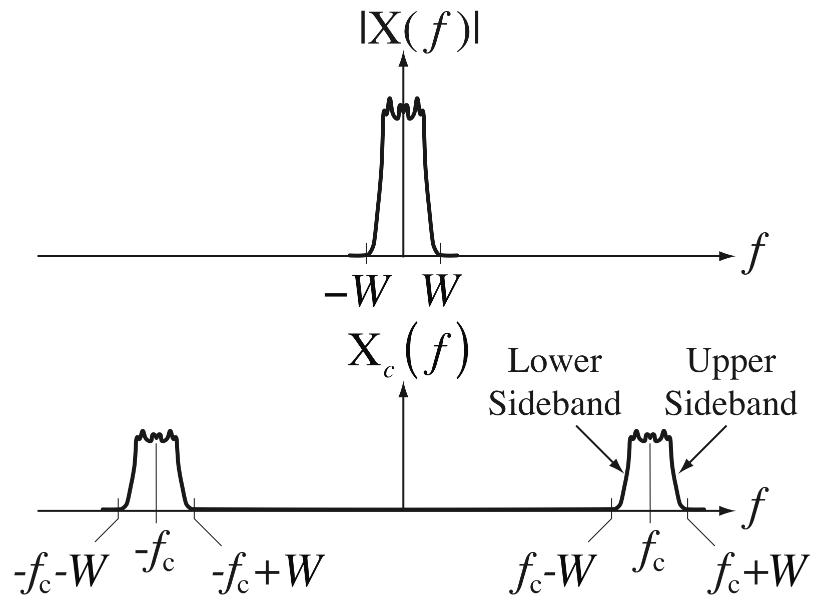
If, instead of transmitting $\mathrm{x}_c(t) = A_c[1 + \mu\,\mathrm{x}(t)]\cos(\omega_c t)$ we remove the "1" and set $\mu = 1$, we get

$$\mathrm{x}_c(t) = A_c\,\mathrm{x}(t)\cos(\omega_c t).$$

This type of modulation is called **double-sideband suppressed-carrier modulation** (**DSB** for short), (also sometimes called DSBSC or DSSC).

Its Fourier transform is

$$\mathrm{X}_c(f) = (A_c/2)\left[\mathrm{X}(f - f_c) + \mathrm{X}(f + f_c)\right]$$

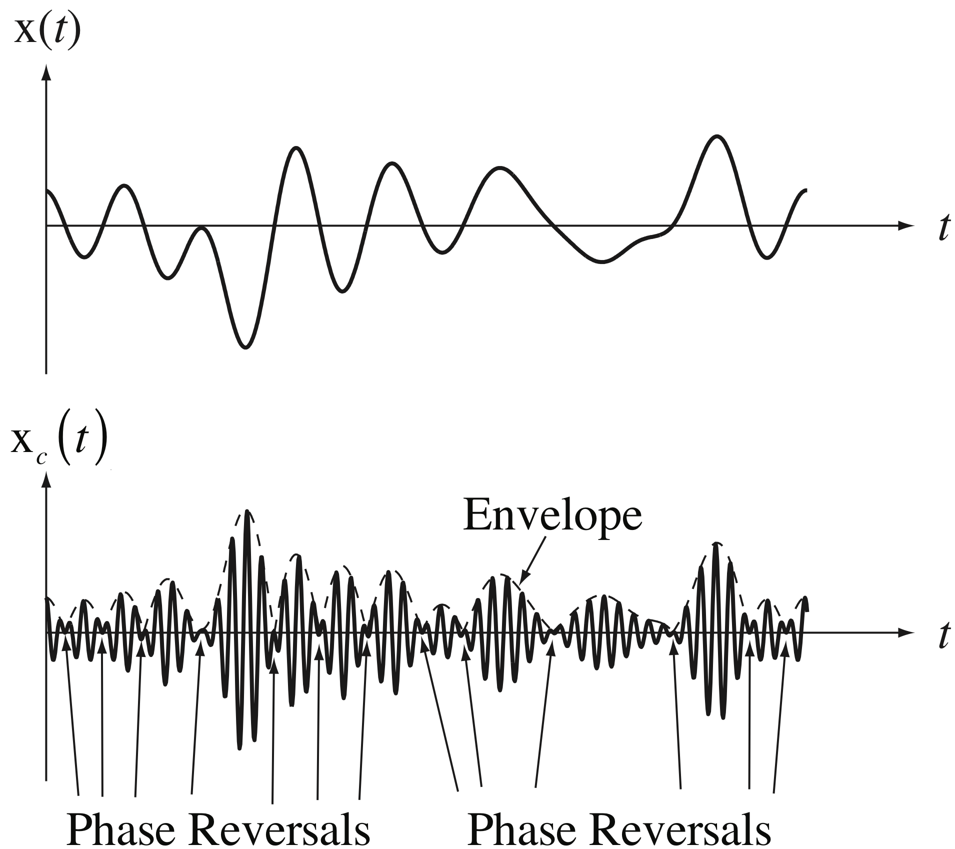


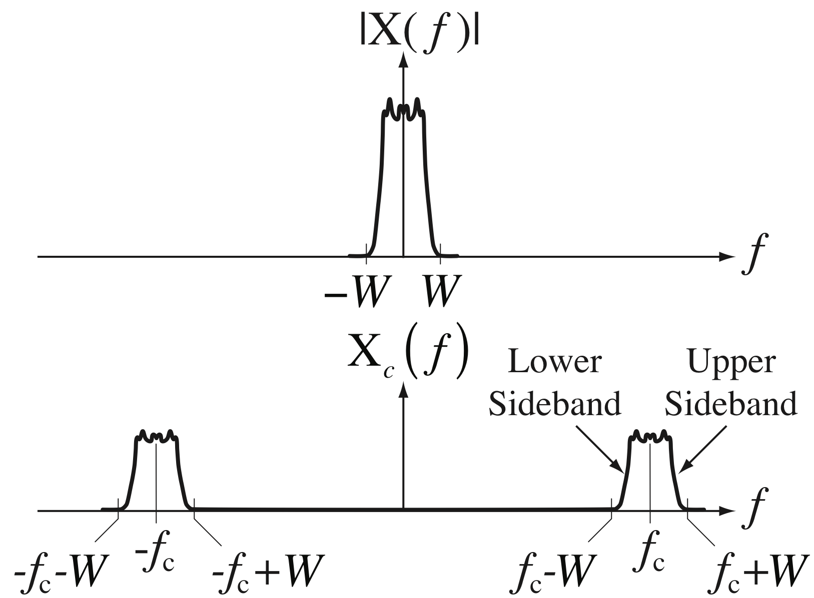
Double-Sideband Suppressed-Carrier Modulation

In DSB modulation, the impulses at the carrier frequency are gone and all the transmitted power is in the sidebands. $S_T = 2P_{sb} = \left(A_c^2/2\right)S_x$

The envelope is defined as non-negative so, in this case, it is $\mathrm{A}(t) = A_c\left|\mathrm{x}(t)\right|$.

The message cannot be recovered by a simple envelope detector.





Double-Sideband Amplitude Modulation

In the special case in which $\mathrm{x}(t) = A_m \cos(\omega_m t)$, the DSB-modulated carrier becomes $\mathrm{x}_c(t) = A_c A_m \cos(\omega_m t)\cos(\omega_c t)$ which, using a trigonometric identity, can be written as

$$\mathrm{x}_c(t) = \frac{A_c A_m}{2}\left[\cos(\omega_m - \omega_c)t + \cos(\omega_m + \omega_c)t\right]$$

which is the sum of two sinusoids, one at the sum frequency and one at the difference frequency. If we use the same message signal in an AM system we get

$$\mathrm{x}_c(t) = A_c\left[1 + \mu A_m \cos(\omega_m t)\right]\cos(\omega_c t)$$

$$\mathrm{x}_c(t) = A_c \cos(\omega_c t) + \mu A_c A_m \cos(\omega_m t)\cos(\omega_c t)$$

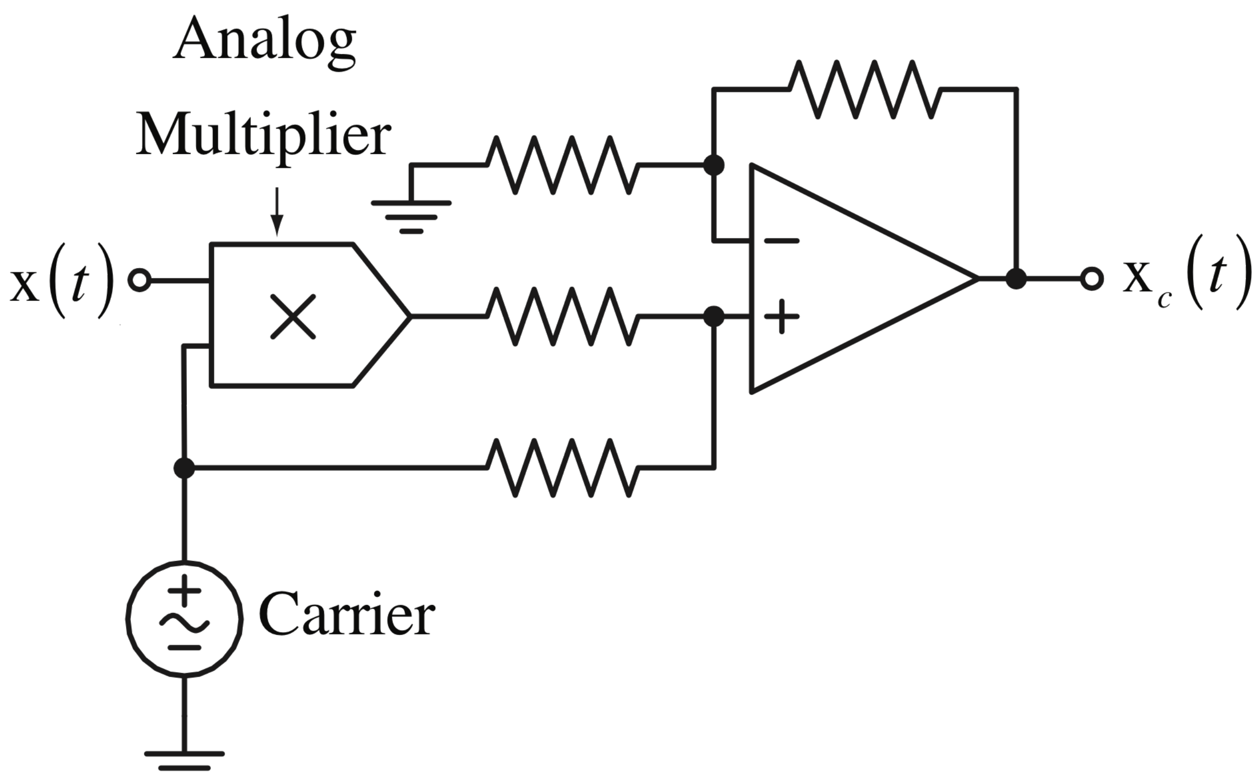
$$\mathrm{x}_c(t) = A_c \cos(\omega_c t) + \frac{\mu A_c A_m}{2}\left[\cos(\omega_m - \omega_c)t + \cos(\omega_m + \omega_c)t\right]$$

which is three sinusoids, one at the carrier frequency and one each at the sum and difference frequencies.

Modulation with a sinusoid is referred to as **tone modulation**.

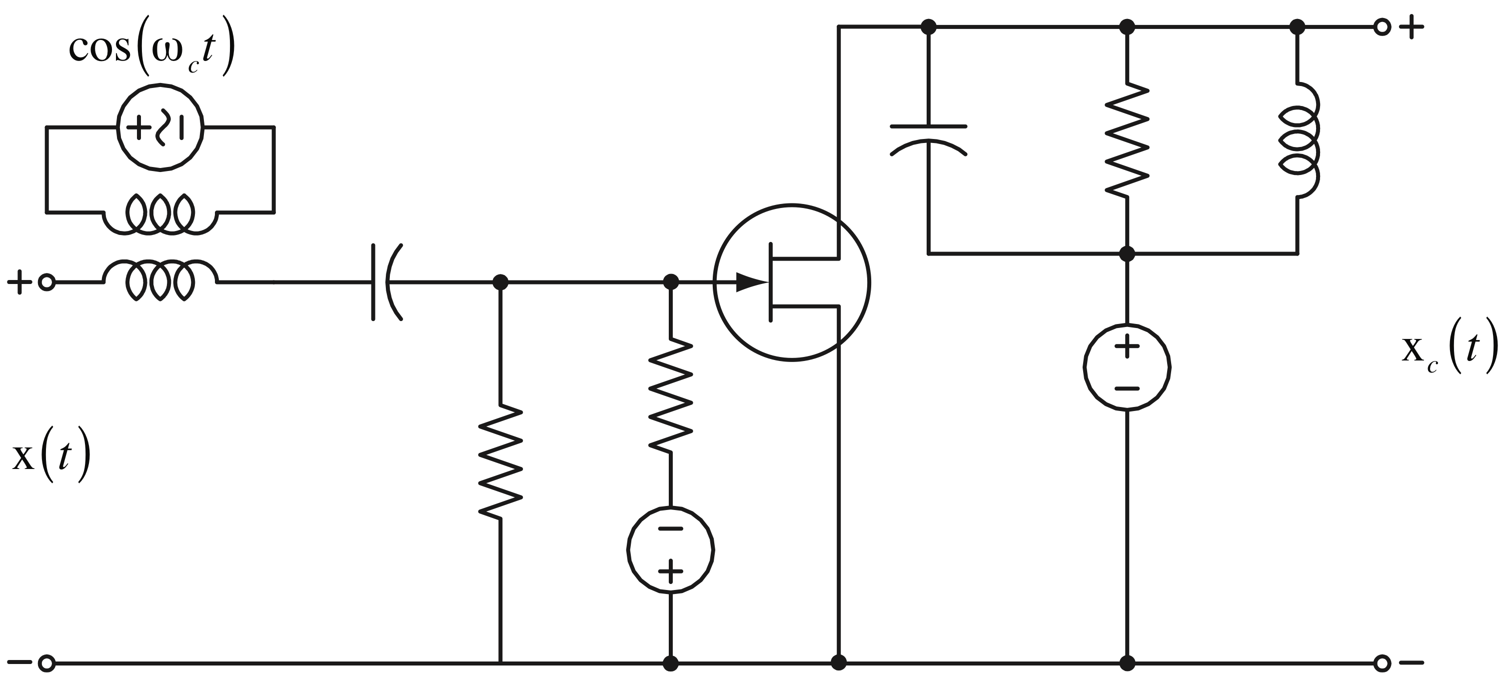
Modulators and Transmitters

The electronic hardware to implement AM or DSB modulation can take any of several forms. The most direct and obvious form is the **product modulator** illustrated below for AM modulation.



Modulators and Transmitters

Another way of obtaining the product of two signals is to use a **square-law modulator**. This type of circuit takes advantage of the inherent non-linearity of a solid-state device. In the example below the device is a field-effect transistor (FET).



Modulators and Transmitters

If the FET has a transfer characteristic $\mathrm{v}_{out} = a_1\,\mathrm{v}_{in} + a_2\,\mathrm{v}_{in}^2$ and if $\mathrm{v}_{in}(t) = \mathrm{x}(t) + \cos(\omega_c t)$, then

$$\mathrm{v}_{out}(t) = a_1\left[\mathrm{x}(t) + \cos(\omega_c t)\right] + a_2\left[\mathrm{x}(t) + \cos(\omega_c t)\right]^2$$

$$\mathrm{v}_{out}(t) = a_1\,\mathrm{x}(t) + a_1\cos(\omega_c t) + a_2\left[\mathrm{x}^2(t) + \cos^2(\omega_c t) + 2\,\mathrm{x}(t)\cos(\omega_c t)\right]$$

$$\mathrm{v}_{out}(t) = a_1\,\mathrm{x}(t) + a_1\cos(\omega_c t) + a_2\,\mathrm{x}^2(t) + a_2\cos^2(\omega_c t) + 2a_2\,\mathrm{x}(t)\cos(\omega_c t)$$

$$\mathrm{v}_{out}(t) = a_1\,\mathrm{x}(t) + a_2\,\mathrm{x}^2(t) + a_2\cos^2(\omega_c t) + a_1\left[1 + 2(a_2/a_1)\mathrm{x}(t)\right]\cos(\omega_c t)$$

$$\mathrm{v}_{out}(t) = a_1\,\mathrm{x}(t) + a_2\,\mathrm{x}^2(t) + a_2\cos^2(\omega_c t) + \underbrace{A_c\left[1 + \mu\,\mathrm{x}(t)\right]\cos(\omega_c t)}_{\text{Desired AM Wave}}$$

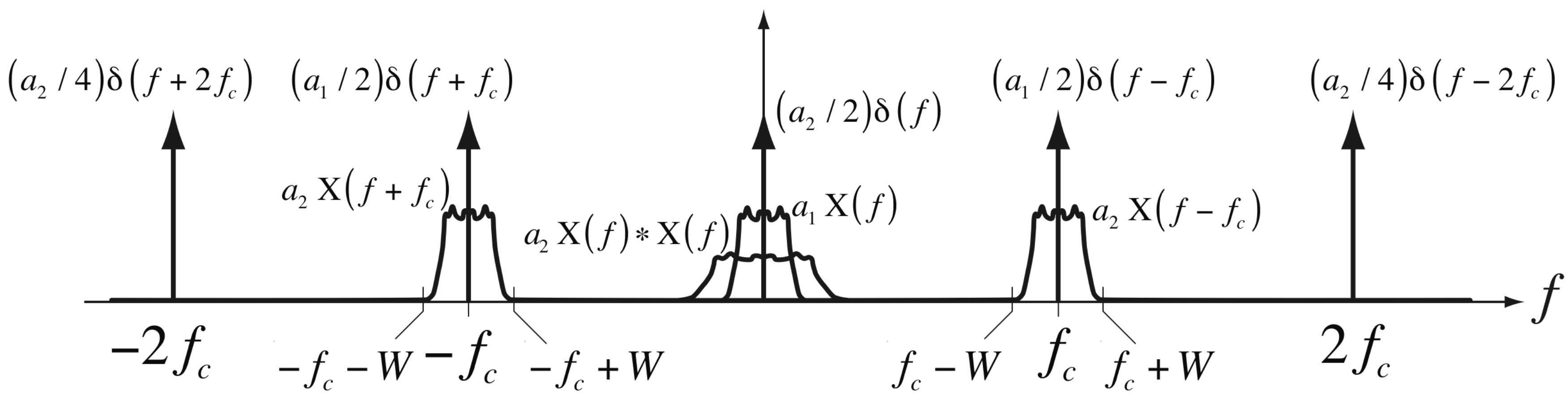
where $A_c = a_1$ and $\mu = 2(a_2/a_1)$.

Modulators and Transmitters

$$\mathrm{v}_{out}(t) = a_1\,\mathrm{x}(t) + a_2\,\mathrm{x}^2(t) + a_2\cos^2(\omega_c t) + a_1\left[1 + 2(a_2/a_1)\mathrm{x}(t)\right]\cos(\omega_c t)$$

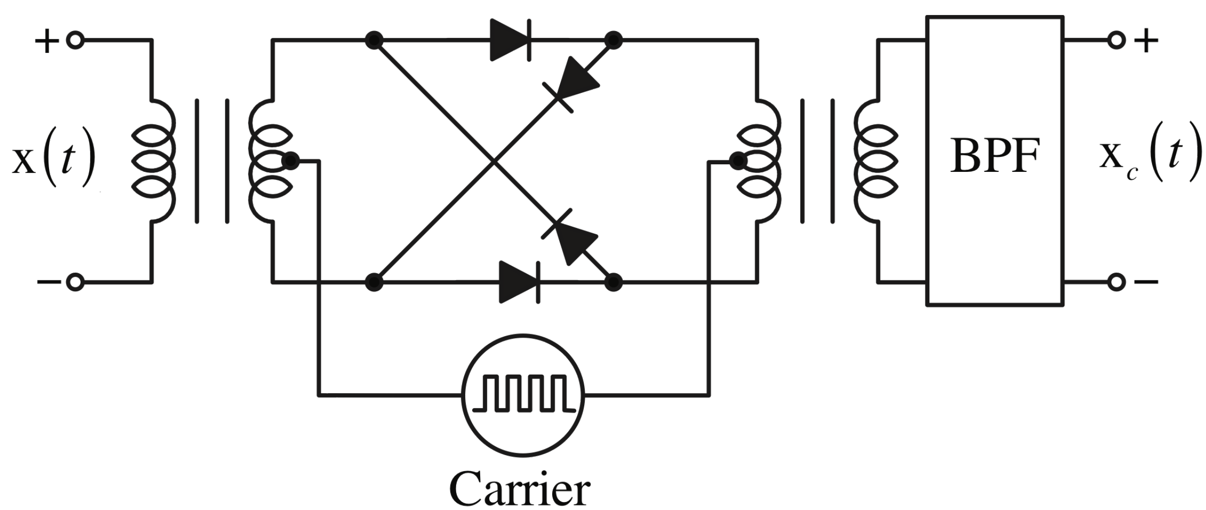
The Fourier transform of $\mathrm{v}_{out}(t)$ is

$$\mathrm{V}_{out}(f) = a_1\,\mathrm{X}(f) + a_2\,\mathrm{X}(f) * \mathrm{X}(f) + (a_2/2)\left\{\delta(f) + (1/2)\left[\delta(f - 2f_c) + \delta(f + 2f_c)\right]\right\}$$
$$+ (a_1/2)\left[\delta(f - f_c) + \delta(f + f_c)\right] + a_2\left[\mathrm{X}(f - f_c) + \mathrm{X}(f + f_c)\right]$$



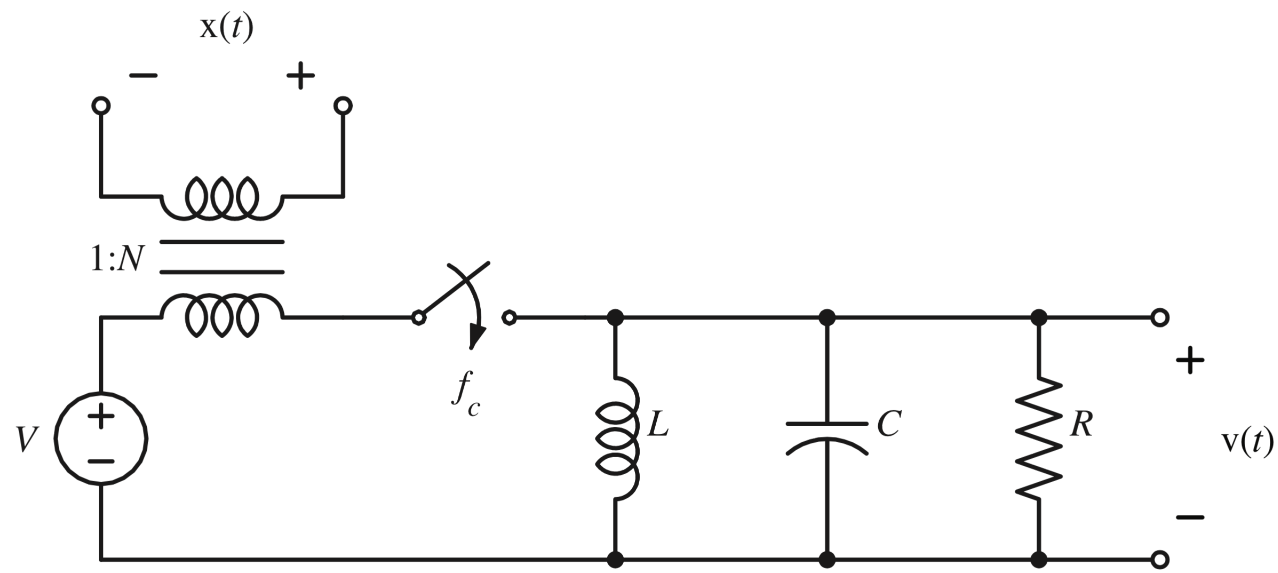
Modulators and Transmitters

The circuit below is a **ring modulator**. When the carrier signal is positive on the left side, the top and bottom diodes are forward biased and the inner diodes are reverse biased, effectively connecting the top of the left transformer secondary to the top of the right transformer primary and the bottoms also and $x_c(t) = x(t)$. When the carrier signal is positive on the right side, the diodes all switch their bias to the opposite state and the tops and bottoms of the transformers are now cross connected making $x_c(t) = -x(t)$. (The ring modulator circuit in the book is drawn wrong.)



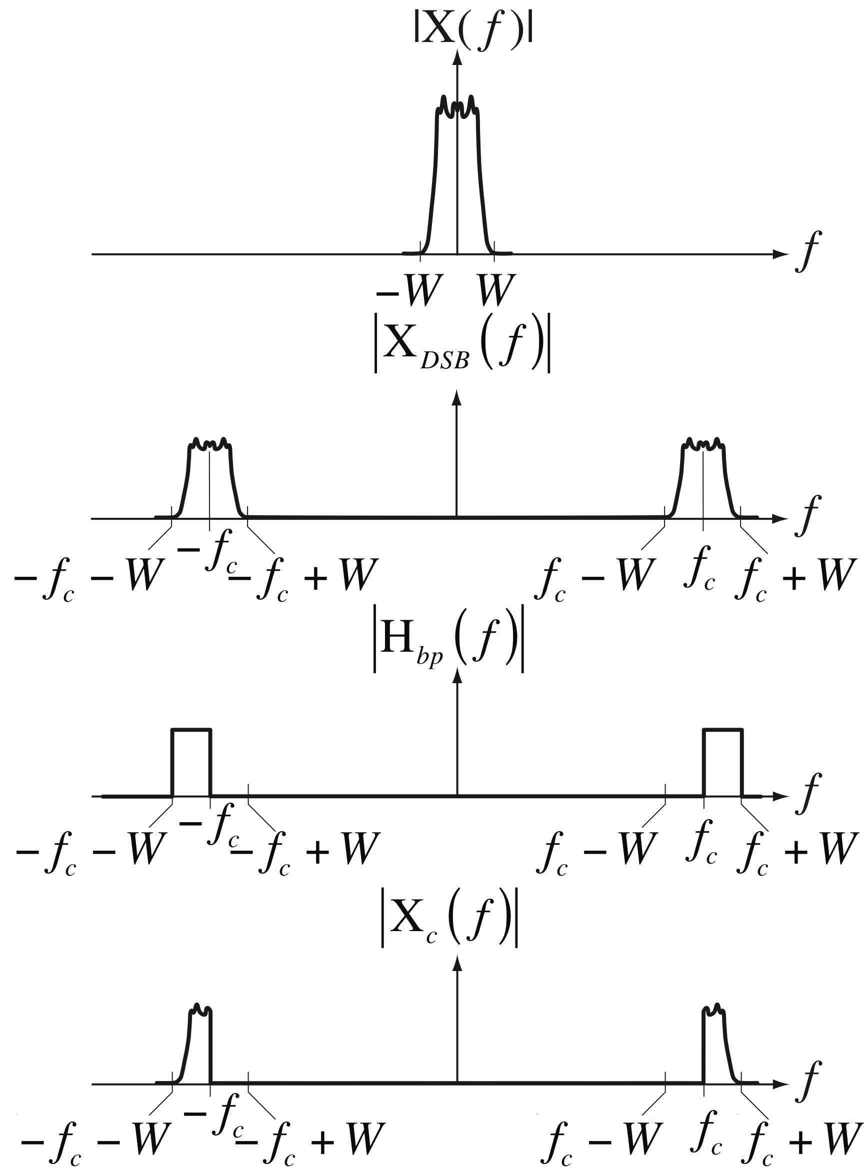
Modulators and Transmitters

The circuit below is a **switching modulator**. The switch closes briefly every $1/f_c$ seconds. The tank circuit (the parallel RLC circuit) on the right is tuned to resonate at f_c Hz. Every $1/f_c$ seconds the tank circuit is hit with a pulse of energy and "rings" at its resonant frequency. Then at the end of one cycle of ringing it is hit again the same way. If the driving voltage is of constant amplitude the output signal is effectively a sinusoid. The driving voltage is a constant plus the message signal. It changes slowly (compared with the resonant frequency) so the overall effect is to AM modulate the sinusoid with the message signal.



Suppressed-Sideband Amplitude Modulation

In DSB modulation the upper and lower sidebands are related through Hermitian symmetry and, therefore, they both contain *all* of the message information.
So it should be possible to transmit all the message information using only one of the two sidebands. This can be done by **suppressing** one of the sidebands and transmitting only the other sideband. This type of modulation is called **single-sideband suppressed-carrier** (**SSB**) modulation.
SSB reduces the bandwidth requirement by a factor of two thus using frequency space more efficiently.



Suppressed-Sideband Amplitude Modulation

We can analyze SSB modulation using the equivalent lowpass method. Let $\mathrm{x}_{bp}(t)$ be the output signal from DSB modulation and let $\mathrm{x}_c(t)$ be the SSB signal. Then $\mathrm{x}_{bp}(t) = A_c\,\mathrm{x}(t)\cos(\omega_c t)$. Using some results from the derivation of the lowpass equivalent method

$$\mathrm{v}_{lp}(t) = (1/2)\left[\mathrm{v}_i(t) + j\,\mathrm{v}_q(t)\right]$$

and

$$\mathrm{v}_i(t) \triangleq \mathrm{A}(t)\cos(\phi(t)) \text{ and } \mathrm{v}_q(t) \triangleq \mathrm{A}(t)\sin(\phi(t))$$

we get

$$\mathrm{x}_{lp}(t) = (1/2)\left[\mathrm{x}_i(t) + j\,\mathrm{x}_q(t)\right] = (1/2)\left[\mathrm{A}(t)\cos(\phi(t)) + j\,\mathrm{A}(t)\sin(\phi(t))\right]$$

In this case $\mathrm{x}_{lp}(t) = (1/2)A_c\left|\mathrm{x}(t)\right|e^{j\phi(t)}$, $\phi(t) = \begin{cases} 0 & , \ \mathrm{x}(t) > 0 \\ \pm 180^\circ & , \ \mathrm{x}(t) < 0 \end{cases}$

and $\mathrm{X}_{lp}(f) = (1/2)A_c\,\mathrm{X}(f)$

Suppressed-Sideband Amplitude Modulation

The bandpass filter transfer function is

$$\mathrm{H}_{bp}(f) = \begin{cases} 1 \ , \ f_c - W < |f| < f_c \\ 0 \ , \ \text{otherwise} \end{cases} \text{ for LSSB}$$

or

$$\mathrm{H}_{bp}(f) = \begin{cases} 1 \ , \ f_c < |f| < f_c + W \\ 0 \ , \ \text{otherwise} \end{cases} \text{ for USSB}$$

The equivalent lowpass transfer function is

$$\mathrm{H}_{lp}(f) = \mathrm{H}_{bp}(f + f_c)\mathrm{u}(f + f_c) = \begin{cases} \mathrm{u}(f + W) - \mathrm{u}(f) \ \text{ for LSSB} \\ \mathrm{u}(f) - \mathrm{u}(f - W) \ \text{ for USSB} \end{cases}$$

They can both be expressed in the form $\mathrm{H}_{lp}(f) = \begin{cases} (1/2)\left[1 \pm \mathrm{sgn}(f)\right] \ , \ |f| < W \\ 0 \ , \ \text{otherwise} \end{cases}$

where the plus sign is taken for USSB and the minus sign is taken for LSSB.

Suppressed-Sideband Amplitude Modulation

The equivalent lowpass response of the system is

$$\mathrm{Y}_{lp}(f) = \mathrm{H}_{lp}(f)\,\mathrm{X}_{lp}(f) = \begin{cases} (1/2)\left[1 \pm \mathrm{sgn}(f)\right] & , \ |f| < W \\ 0 & , \ \text{otherwise} \end{cases} (1/2) A_c\, \mathrm{X}(f)$$

$$\mathrm{Y}_{lp}(f) = (A_c/4)\left[\mathrm{X}(f) \pm \mathrm{sgn}(f)\,\mathrm{X}(f)\right]$$

(The requirement $|f| < W$ is satisfied by the fact that $\mathrm{X}(f)$ is bandlimited to W.)

Now, using the Hilbert transform relationship $\hat{\mathrm{x}}(t) \xleftrightarrow{\mathscr{F}} -j\,\mathrm{sgn}(f)\,\mathrm{X}(f)$

$$\mathrm{y}_{lp}(t) = (A_c/4)\left[\mathrm{x}(t) \pm j\,\hat{\mathrm{x}}(t)\right]$$

Then, transforming from lowpass to bandpass,

$$\mathrm{x}_c(t) = \mathrm{y}_{bp}(t) = 2\,\mathrm{Re}\left[\mathrm{y}_{lp}(t)\,e^{j\omega_c t}\right] = (A_c/2)\left[\mathrm{x}(t)\cos(\omega_c t) \mp \hat{\mathrm{x}}(t)\sin(\omega_c t)\right]$$

The in-phase and quadrature components are

$$\mathrm{x}_{ci}(t) = (A_c/2)\,\mathrm{x}(t) \text{ and } \mathrm{x}_{cq}(t) = \pm (A_c/2)\,\hat{\mathrm{x}}(t)$$

and the envelope is $\mathrm{A}(t) = (A_c/2)\sqrt{\mathrm{x}^2(t) + \hat{\mathrm{x}}^2(t)}$

Suppressed-Sideband Amplitude Modulation

The generation of an SSB signal, as presented so far, requires an ideal filter (one with vertical sides and a flat top). Ideal filters don't exist. Real filters can have transition regions that are steep but not vertical and real filters can never have a perfectly flat top. So if we use a real filter we can either

1. Set the filter transition region inside the sideband to be retained and lose some of the sideband information

or

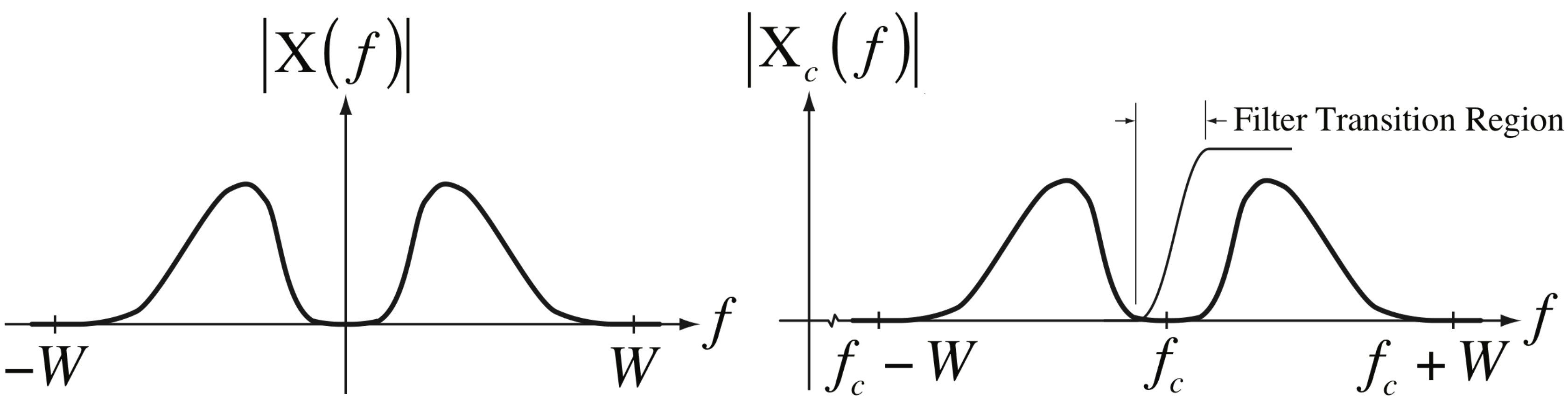
2. Set the filter transition region inside the sideband to be removed retain some of the unwanted sideband

or

3. some combination of 1 and 2.

Suppressed-Sideband Amplitude Modulation

Fortunately, for many practical messages, the spectral content at very low frequencies is very small. This gives the designer of an SSB system a little room to maneuver. The transition region of the filter used to eliminate the unwanted sideband can be placed in the region around the carrier frequency where the DSB signal has very little signal power.

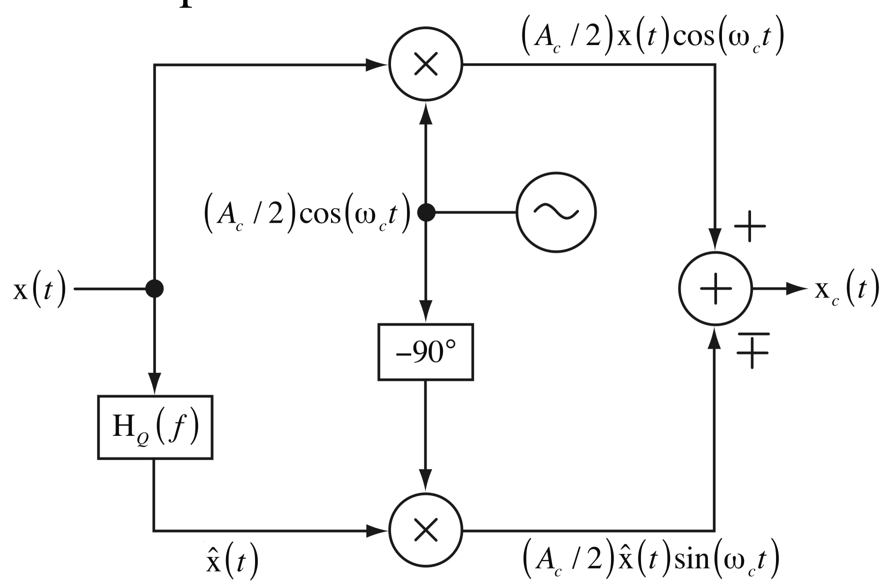


Suppressed-Sideband Amplitude Modulation

Another method for generating SSB is suggested by the relationship

$$x_c(t) = (A_c/2)x(t)\cos(\omega_c t) \mp (A_c/2)\hat{x}(t)\cos(\omega_c t - 90°).$$

This is written as though SSB consists of the sum of two DSB signals, with carriers that are in quadrature (phase shifted by 90°) and modulated by $x(t)$ and $\hat{x}(t)$. That could be accomplished (theoretically) by the system below where $H_Q(f)$ is a **quadrature phase shifter**. Unfortunately a quadrature phase shifter is an idealization that can never quite be achieved in practice.



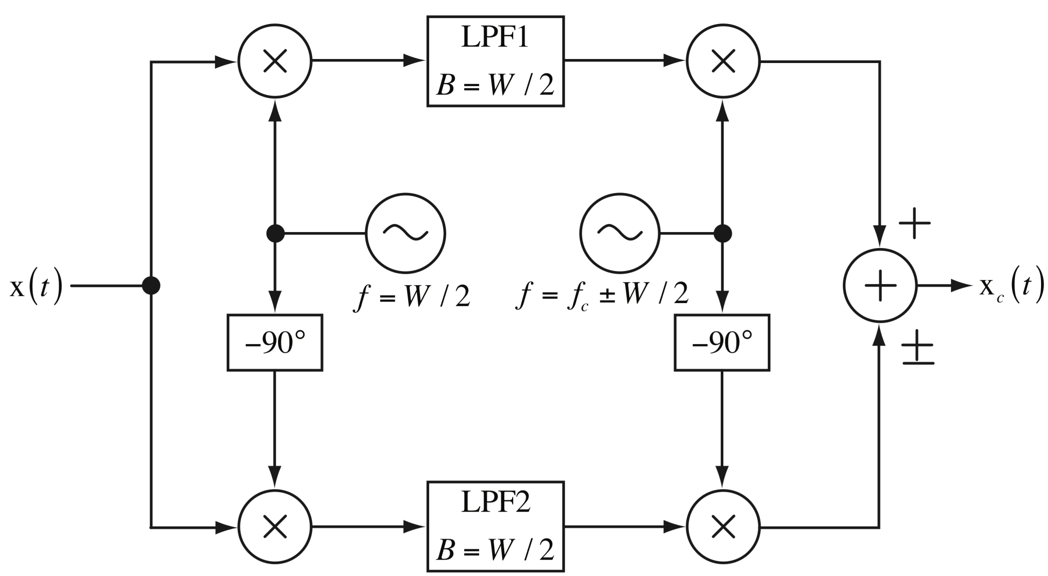
Suppressed-Sideband Amplitude Modulation

A third, more practical, method is Weaver's SSB modulator, diagrammed below.

Let $\mathrm{x}(t) = \cos(2\pi f_m t)$ with $0 < f_m < W$ (tone modulation). Then $\mathrm{x}_c(t) = \mathrm{v}_1(t) \pm \mathrm{v}_2(t)$ where $\mathrm{v}_1(t)$ is the signal from the upper part of the loop and $\mathrm{v}_2(t)$ is the signal from the lower part. The input signal to LPF1 is

$$\cos(2\pi f_m t)\cos(2\pi W t/2) = (1/2)\left[\cos\left(2\pi\left(f_m - W/2\right)t\right) + \cos\left(2\pi\left(f_m + W/2\right)t\right)\right]$$

Since the filter cuts off at $W/2$ its output signal is $(1/2)\left[\cos\left(2\pi\left(f_m - W/2\right)t\right)\right]$.



Suppressed-Sideband Amplitude Modulation

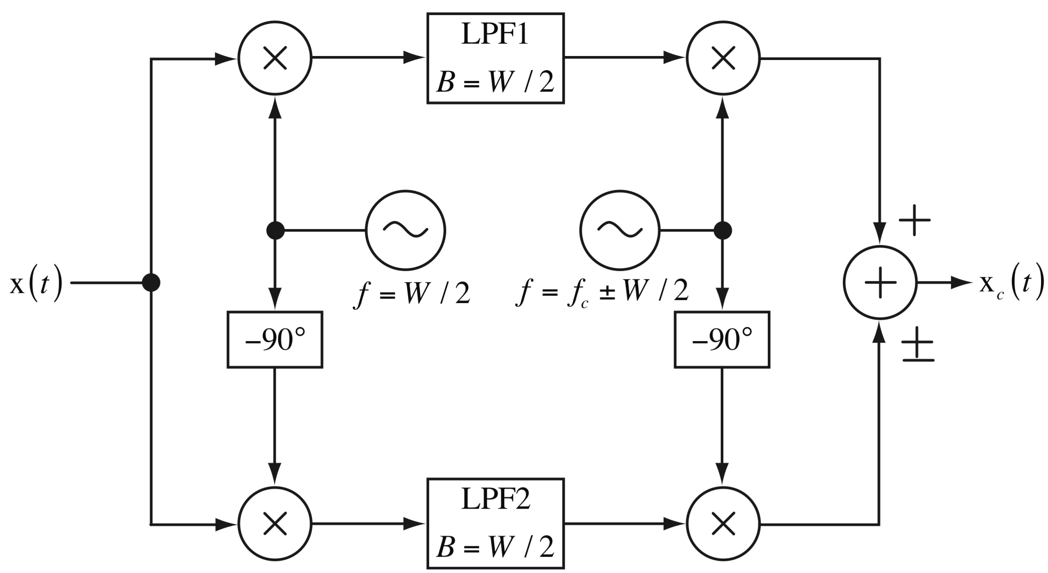
The LPF1 output signal is multiplied by $\cos\left(2\pi\left(f_c \pm W/2\right)t\right)$. Therefore,

$$\mathrm{v}_1(t) = (1/4)\left[\cos\left(2\pi\left(f_c \pm W/2 + f_m - W/2\right)t\right) + \cos\left(2\pi\left(f_c \pm W/2 - f_m + W/2\right)t\right)\right]$$

The input signal to the LPF2 is $\cos\left(2\pi f_m t\right)\sin\left(2\pi W t/2\right)$ and (using similar reasoning)

$$\mathrm{v}_2(t) = (1/4)\left[\cos\left(2\pi\left(f_c \pm W/2 + f_m - W/2\right)t\right) - \cos\left(2\pi\left(f_c \pm W/2 - f_m + W/2\right)t\right)\right].$$

Taking the upper signs, $\mathrm{x}_c(t) = (1/2)\cos\left(\left(\omega_c + \omega_m\right)t\right)$. This is a USB signal. If we instead take the lower signs we get $\mathrm{x}_c(t) = (1/2)\cos\left(\left(\omega_c - \omega_m\right)t\right)$, an LSB signal.



Vestigial-Sideband Amplitude Modulation

Vestigial sideband modulation (**VSB**) is a compromise between DSB, which has good fidelity for messages with significant low-frequency content, and SSB which does not, but uses only half the bandwidth. In VSB the transmitted signal is predominately one of the two sidebands but with a "vestige" of the other sideband. Sometimes a carrier is added to VSB. This generally makes the detector easier to design.

Frequency Conversion and Demodulation

So far we have described several ways of modulating a carrier with a message signal. Now we turn to **demodulation**, the recovery of the message from the modulated carrier. An essential process in most demodulation methods is **frequency conversion**. Consider a DSB signal of the form $\mathrm{x}(t)\cos(\omega_1 t)$.

If we multiply it by $\cos(\omega_2 t)$ we get

$$\mathrm{x}(t)\cos(\omega_1 t)\cos(\omega_2 t)=(1/2)\left[\cos((\omega_1-\omega_2)t)+\cos((\omega_1+\omega_2)t)\right]$$

The DSB spectrum has been shifted in frequency up and down by ω_2 resulting in **sum** and **difference** spectral components. Devices that do this operation are called **frequency converters** or **mixers** and the operation is called **heterodyning** or **mixing**. The "hetero" prefix refers to two things that are different, in this case ω_1 and ω_2. If we make ω_1 and ω_2 the same, heterodyning becomes a special case called **homodyning** in which the prefix "homo" refers to two things that are the same. The most common current uses of the prefixes "hetero" and "homo" in everyday speech are in the words "heterosexual" and "homosexual" where they have the same general significance.

Frequency Conversion and Demodulation

A basic process in many demodulation systems is **synchronous detection**. In synchronous detection the received signal is multiplied by the signal from a **local oscillator** that is at the same frequency as the carrier of the received signal and in phase with that carrier (as received). Let the received signal be represented by

$$x_c(t) = \left[K_c + K_\mu x(t)\right]\cos(\omega_c t) - K_\mu x_q(t)\sin(\omega_c t)$$

If $K_c = 0$, we have a suppressed carrier. If $x_q(t) = 0$, we have double sideband. This form can represent the types of modulation we have seen so far. So the demodulation process begins with the product

$$x_c(t) A_{LO}\cos(\omega_c t) = \left\{\left[K_c + K_\mu x(t)\right]\cos(\omega_c t) - K_\mu x_q(t)\sin(\omega_c t)\right\} A_{LO}\cos(\omega_c t)$$

$$x_c(t) A_{LO}\cos(\omega_c t) = A_{LO}\left\{K_c \cos^2(\omega_c t) + K_\mu x(t)\cos^2(\omega_c t) - K_\mu x_q(t)\cos(\omega_c t)\sin(\omega_c t)\right\}$$

$$x_c(t) A_{LO}\cos(\omega_c t) = (A_{LO}/2)\left\{\begin{array}{l} K_c + K_\mu x(t) + K_c\cos(2\omega_c t) + K_\mu x(t)\cos(2\omega_c t) \\ -K_\mu x_q(t)\sin(2\omega_c t) \end{array}\right\}$$

Frequency Conversion and Demodulation

From the previous slide,

$$\mathrm{x}_c(t)A_{LO}\cos(\omega_c t) = (A_{LO}/2)\begin{Bmatrix} K_c + K_\mu \mathrm{x}(t) + K_c\cos(2\omega_c t) + K_\mu \mathrm{x}(t)\cos(2\omega_c t) \\ -K_\mu \mathrm{x}_q(t)\sin(2\omega_c t) \end{Bmatrix}$$

We can filter out the double-frequency components of this signal leaving $\mathrm{y}_D(t) = K_D\left[K_c + K_\mu \mathrm{x}(t)\right]$, where K_D is a constant accounting for any gain or attenuation in the multiplication/filtering process. Since the local oscillator and the carrier frequency are the same, this is a case of homodyning. We can, if desired, also filter out the $K_D K_c$ constant component of the signal with a blocking capacitor or other suitable DC blocking filter. This process seems simple enough until we come to the problem of how to generate a local oscillator that is locked in both frequency and phase to the carrier of the incoming signal. If the incoming signal is DSB, the carrier has been suppressed. So locking to it is non-trivial. Sometimes a small **pilot carrier** is added to the transmitted signal to facilitate detection of and locking on to the carrier.

Frequency Conversion and Demodulation

There are many techniques for generating a local oscillator that is phase-locked to the incoming carrier but there is always at least a little asynchronism. Let the local oscillator be $\cos(\omega_c t + \omega' t + \phi')$ where ω' accounts for frequency error and ϕ' accounts for phase error. Let the signal be DSB with tone modulation $\cos(\omega_m t)$. Then $x_c(t) = K_\mu \cos(\omega_m t)\cos(\omega_c t)$ and, multiplying by the local oscillator, we get

$$K_\mu \cos(\omega_m t)\cos(\omega_c t)\cos(\omega_c t + \omega' t + \phi')$$

$$= (K_\mu / 2)\left[\cos(\omega_m t)\cos(\omega' t + \phi') + \cos(\omega_m t)\cos(2\omega_c t + \omega' t + \phi')\right]$$

After lowpass filtering (possibly with gain or attenuation) this becomes

$$y_D(t) = K_D \cos(\omega_m t)\cos(\omega' t + \phi')$$

$$y_D(t) = \begin{cases} (K_D / 2)\left[\cos((\omega_m - \omega')t)\right] + \cos((\omega_m + \omega')t) & , \ \phi' = 0 \\ K_D \cos(\omega_m t)\cos(\phi') & , \ \omega' = 0 \end{cases}$$

The frequency error causes a shift in the signal's frequency both up an down by f'. The phase error causes a loss in signal power. If $\phi' = 90°$, the detected signal is zero.

Frequency Conversion and Demodulation

Now let the signal be SSB with $x_c(t) = \cos((\omega_c \pm \omega_m)t)$

Multiplying by the local oscillator, we get

$$\cos((\omega_c \pm \omega_m)t)\cos(\omega_c t + \omega' t + \phi')$$

$$= (1/2)\left[\cos(\omega' t + \phi' \mp \omega_m t) + \cos(2\omega_c t + \omega' t + \phi' \pm \omega_m t)\right]$$

After lowpass filtering (possibly with gain or attenuation) this becomes

$$y_D(t) = K_D \cos(\omega' t + \phi' \mp \omega_m t) = K_D \cos(\pm\omega_m t - (\omega' t + \phi'))$$

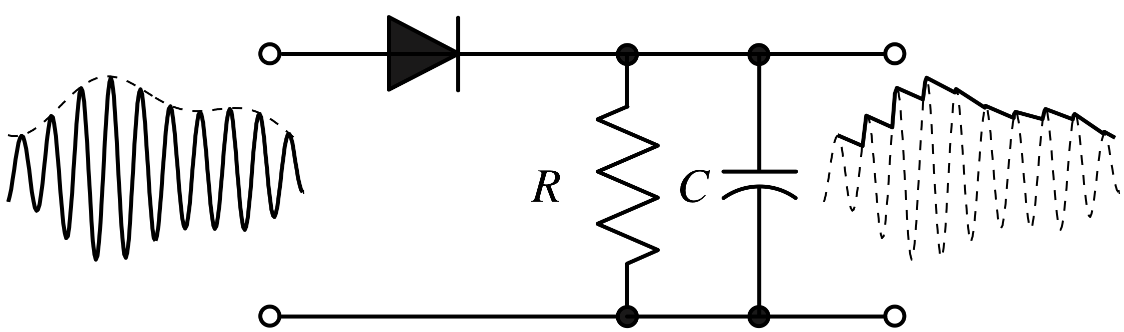
Multiplying the cosine argument through by ± 1 we get

$$y_D(t) = K_D \begin{cases} \cos((\omega_m \mp \omega')t) & , \ \phi' = 0 \\ \cos(\omega_m t \mp \phi') & , \ \omega' = 0 \end{cases}$$

The frequency error causes a shift in the frequency of the message either up or down by f'. The phase error causes a phase shift in the signal. For tone modulation this is not a problem. But for a more general signal, the different frequencies would all experience the same phase shift, therefore different time delays, causing delay distortion.

Envelope Detection

An AM signal carries the message (plus a constant) directly in its envelope. An **envelope detector** is a simple electrical circuit designed to extract the envelope from the AM signal. On each positive half-cycle of the carrier, the diode is forward biased for a short time during which the voltage across the *RC* parallel combination follows the AM signal, charging the capacitor. As the AM signal descends from its peak, its voltage falls below the capacitor voltage and the diode is reverse biased and is effectively an open circuit until the next positive half-cycle of the AM signal. While the diode is reverse biased the voltage across the *RC* parallel combination decays exponentially. This action causes the output voltage to be an approximate replica of the envelope of the AM signal. This type of detection does not require a synchronized local oscillator and is therefore an **asynchronous** detection method.



Envelope Detection

The output voltage of the envelope detector is not a perfect replica of the message. The added constant can be removed by a blocking capacitor. If necessary, the small ripple caused by the charge-discharge cycling of the capacitor voltage can be filtered out with a lowpass filter. If the carrier frequency is much larger than the bandwidth of the message, the reproduction of the message is good enough for most applications like voice or music. The envelope detector cannot be used directly for DSB or SSB. But if a local oscillator is used to inject a carrier into the DSB or SSB signal then the envelope detector can be used. But the problem of synchronizing the local oscillator to the carrier is still there just as it is in synchronous detection.