Linear Modulation

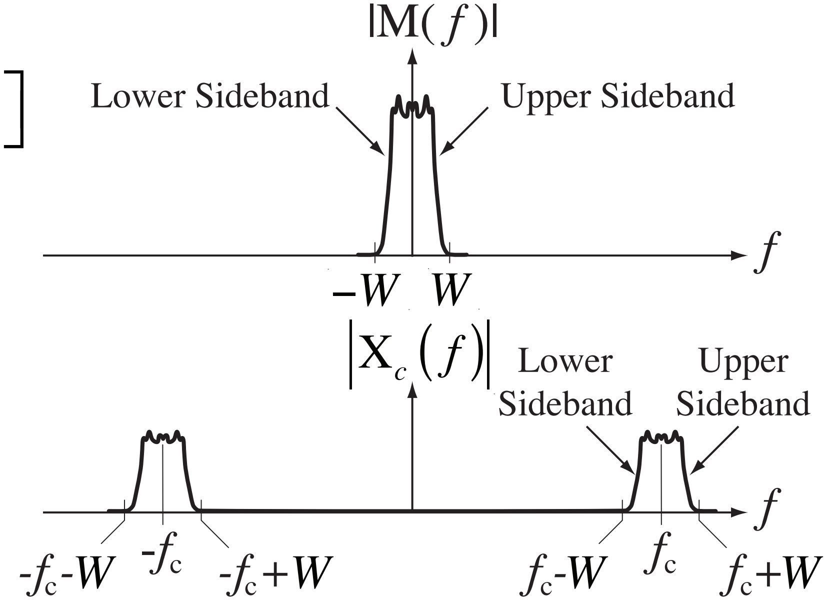
Double-Sideband Suppressed-Carrier Modulation

The simplest type of linear modulation to describe mathematically is **Double – sideband suppressed – carrier modulation** (**DSB**). It is done by directly multiplying a carrier by a message signal of bandwidth W.

$$x_c(t) = A_c\, m(t)\cos(2\pi f_c t)$$

Its Fourier transform is

$$X_c(f) = (A_c/2)\left[M(f - f_c) + M(f + f_c)\right]$$

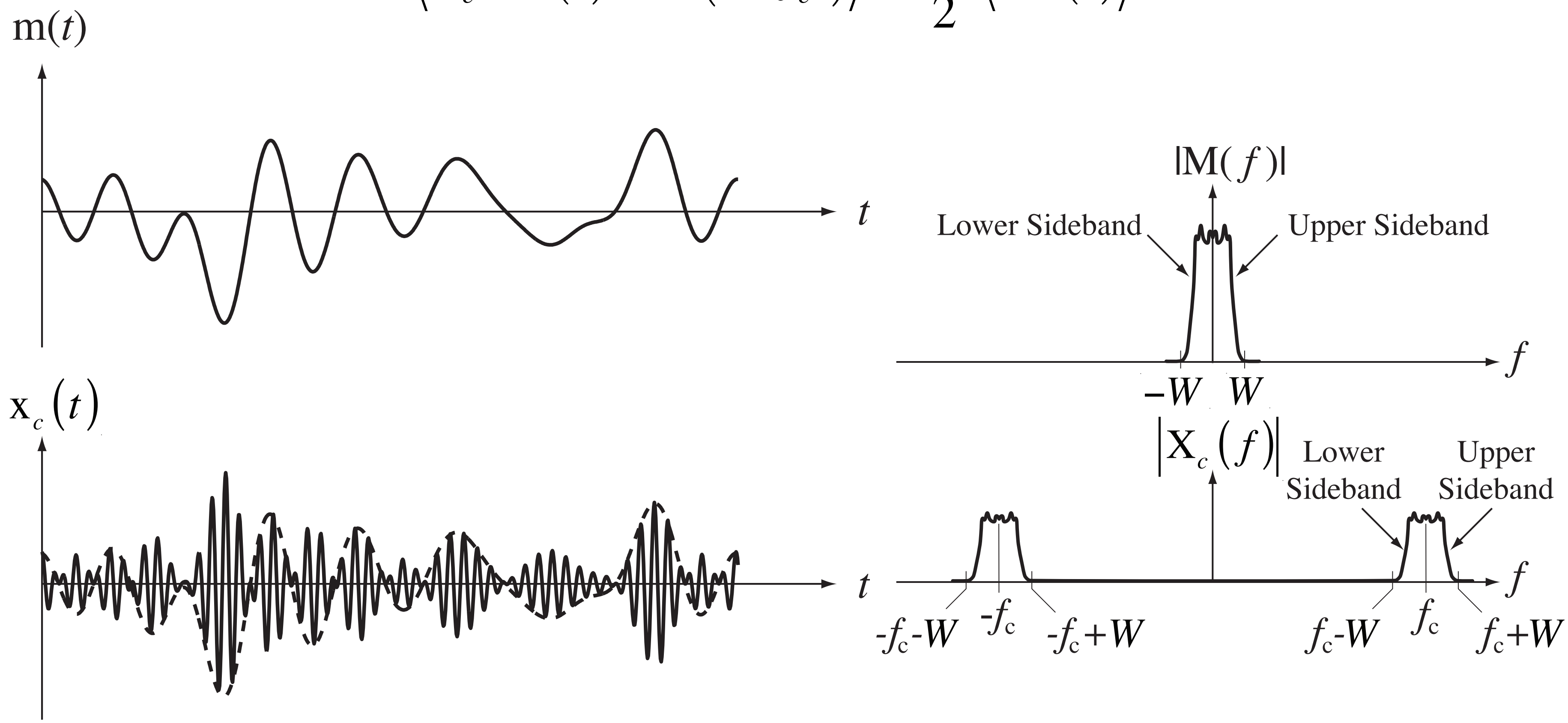


Double-Sideband Suppressed-Carrier Modulation

The average signal power of the DSB signal is

$$\left\langle A_c^2\,\mathrm{m}^2(t)\cos^2(2\pi f_c t)\right\rangle = \frac{A_c^2}{2}\left\langle \mathrm{m}^2(t)\left[1+\cos(4\pi f_c t)\right]\right\rangle$$

$$\left\langle A_c^2\,\mathrm{m}^2(t)\cos^2(2\pi f_c t)\right\rangle = \frac{A_c^2}{2}\left\langle \mathrm{m}^2(t)\right\rangle$$



Double-Sideband Suppressed-Carrier Modulation

DSB signals can be demodulated with a **synchronous demodulator**.

The received signal $x_r(t)$ is multiplied by a local oscillator $2\cos(2\pi f_c t)$ to form

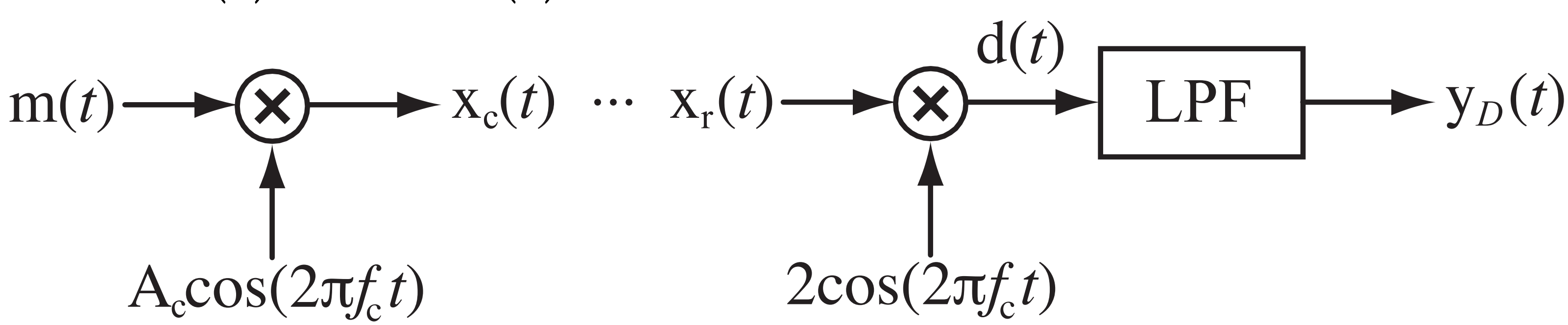
$$d(t) = x_r(t) \times 2\cos(2\pi f_c t) = K\,x_c(t) \times 2\cos(2\pi f_c t)$$

$$d(t) = KA_c\,m(t)\cos(2\pi f_c t) \times 2\cos(2\pi f_c t)$$

$$d(t) = KA_c\,m(t)\left[1 + \cos(4\pi f_c t)\right]$$

Then $d(t)$ is lowpass filtered to produce the demodulated output signal

$$y_D(t) = KA_c\,m(t)$$



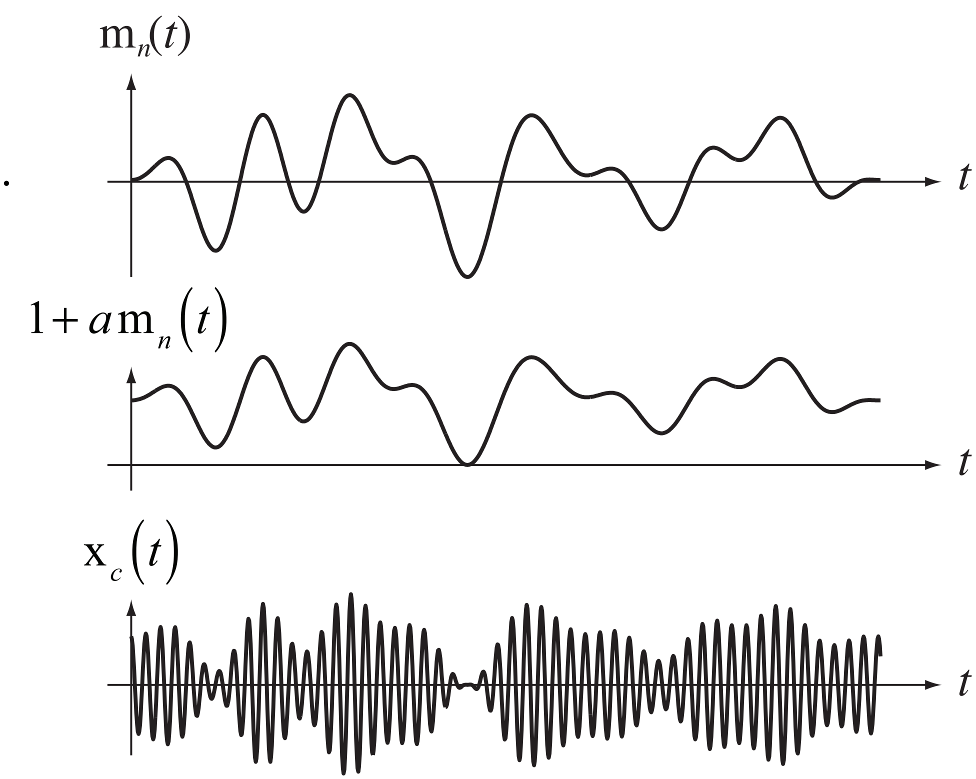
Double-Sideband Amplitude Modulation

The most common type of amplitude modulation used in practice is **standard amplitude modulation** (**AM**) (also known as Double-Sideband Transmitted Carrier Modulation or DSBTC). In this type of modulation, the envelope of the modulated carrier has the shape of the message signal. The modulated carrier is

$$x_c(t) = A_c\left[1 + a\,m_n(t)\right]\cos(2\pi f_c t)$$

where A_c is the amplitude of the unmodulated carrier, a is the **modulation index** and $m_n(t)$ is a scaled version of the message signal $m(t)$.

$$m_n(t) = \frac{m(t)}{\left|m(t)\right|_{\max}}$$

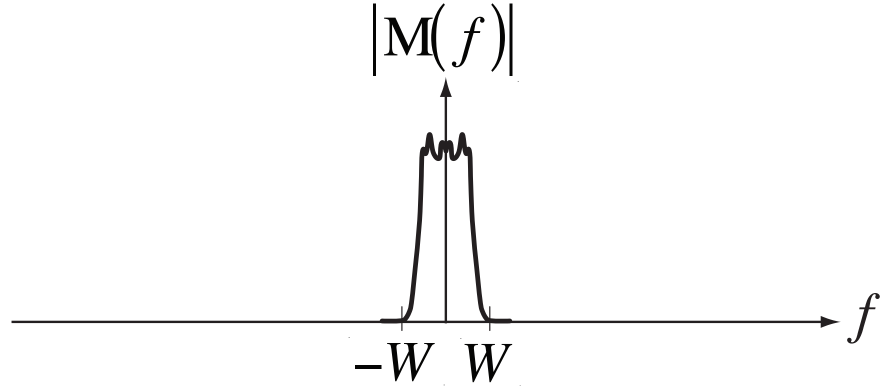


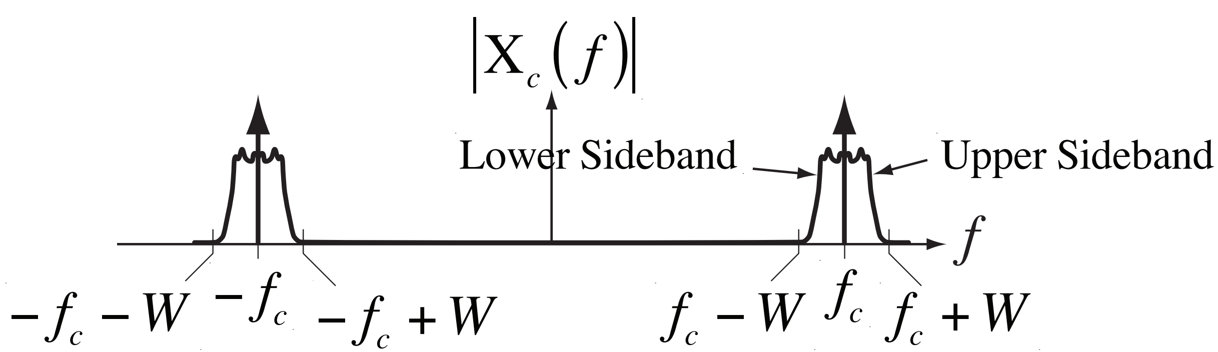
Double-Sideband Amplitude Modulation

The Fourier transform of the AM signal $x_c(t) = A_c\left[1 + a\,m_n(t)\right]\cos(2\pi f_c t)$ is

$$X_c(f) = A_c\left[\delta(f) + a\,M_n(f)\right] * (1/2)\left[\delta(f - f_c) + \delta(f + f_c)\right]$$

$$X_c(f) = \frac{A_c}{2}\left\{\delta(f - f_c) + \delta(f + f_c) + a\left[M_n(f - f_c) + M_n(f + f_c)\right]\right\}$$





Double-Sideband Amplitude Modulation

The average signal power of an AM signal is

$$\langle \mathrm{x}_c(t)\rangle = \left\langle A_c^2\left[1+a\,\mathrm{m}_n(t)\right]^2\cos^2\left(2\pi f_{ct}\right)\right\rangle$$

$$\langle \mathrm{x}_c(t)\rangle = \frac{A_c^2}{2}\left\langle\left[1+2a\,\mathrm{m}_n(t)+a^2\,\mathrm{m}_n^2(t)\right]\left[1+\cos\left(4\pi f_c t\right)\right]\right\rangle$$

$$\langle \mathrm{x}_c(t)\rangle = \frac{A_c^2}{2}\left\langle 1+2a\,\mathrm{m}_n(t)+a^2\,\mathrm{m}_n^2(t)\right\rangle = \frac{A_c^2}{2}\left[1+\langle 2a\,\mathrm{m}_n(t)\rangle+\langle a^2\,\mathrm{m}_n^2(t)\rangle\right]$$

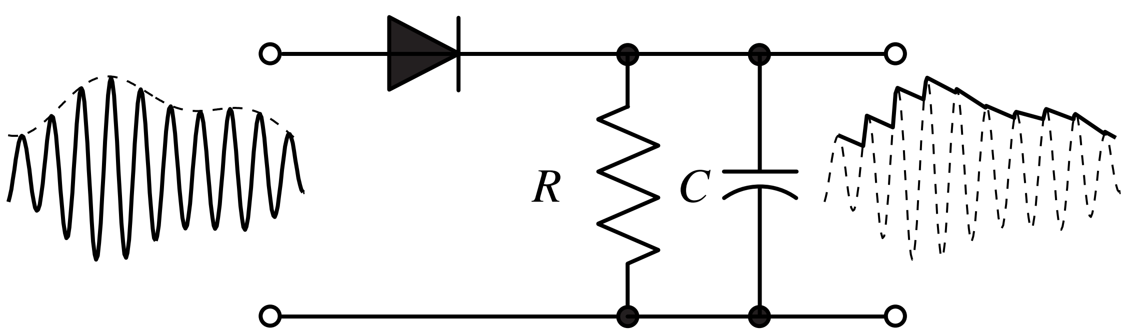
If $\langle \mathrm{m}_n(t)\rangle = 0$ (a very common case), $\langle \mathrm{x}_c(t)\rangle = \frac{A_c^2}{2}\left[1+a^2\langle \mathrm{m}_n^2(t)\rangle\right]$

The two parts of the signal power are $\frac{A_c^2}{2}$, which is the power in the carrier and $\frac{A_c^2 a^2}{2}\langle \mathrm{m}_n^2(t)\rangle$, which is the power in the sidebands. The efficiency of the modulation in defined as the ratio of the power in the sidebands to the total power.

$$Eff = \frac{\frac{A_c^2 a^2}{2}\langle \mathrm{m}_n^2(t)\rangle}{\frac{A_c^2}{2}+\frac{A_c^2 a^2}{2}\langle \mathrm{m}_n^2(t)\rangle} = \frac{a^2\langle \mathrm{m}_n^2(t)\rangle}{1+a^2\langle \mathrm{m}_n^2(t)\rangle}$$

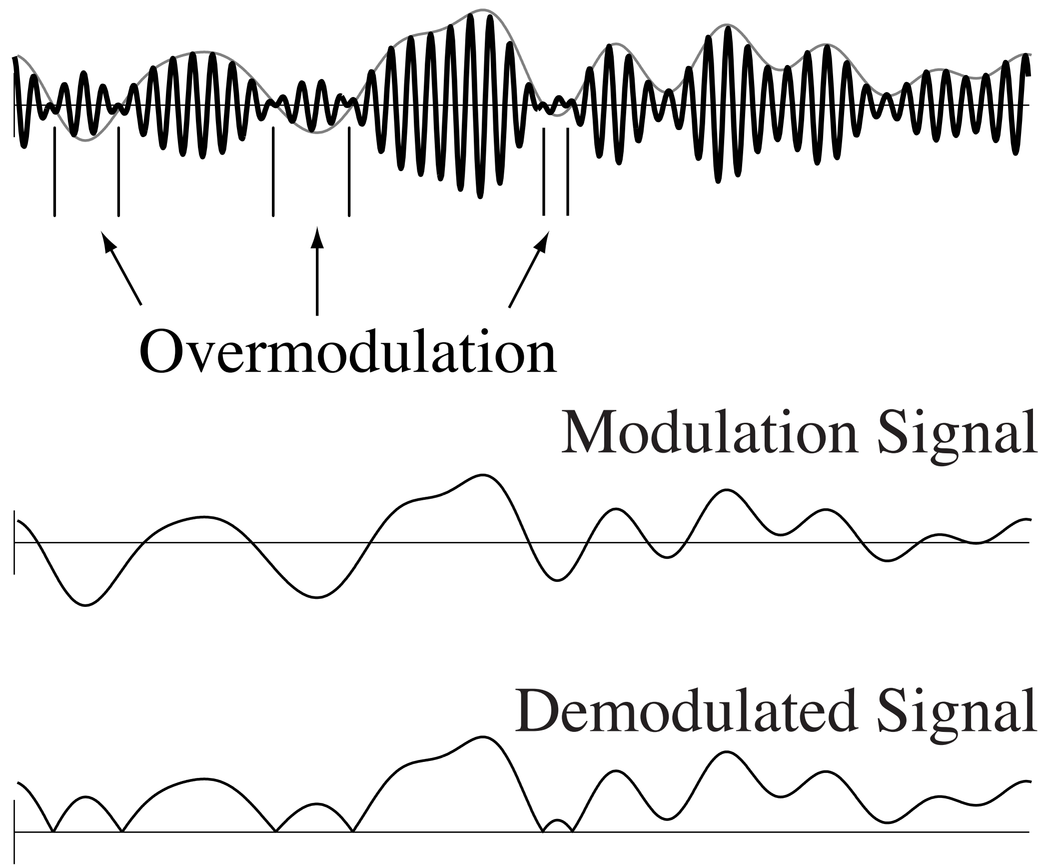
Envelope Detection

An AM signal carries the message (plus a constant) directly in its envelope. An **envelope detector** is a simple electrical circuit designed to extract the envelope from the AM signal. On each positive half-cycle of the carrier, the diode is forward biased for a short time during which the voltage across the *RC* parallel combination follows the AM signal, charging the capacitor. As the AM signal descends from its peak, its voltage falls below the capacitor voltage and the diode is reverse biased and is effectively an open circuit until the next positive half-cycle of the AM signal. While the diode is reverse biased the voltage across the *RC* parallel combination decays exponentially. This action causes the output voltage to be an approximate replica of the envelope of the AM signal. This type of detection does not require a synchronized local oscillator and is therefore an **asynchronous** detection method.



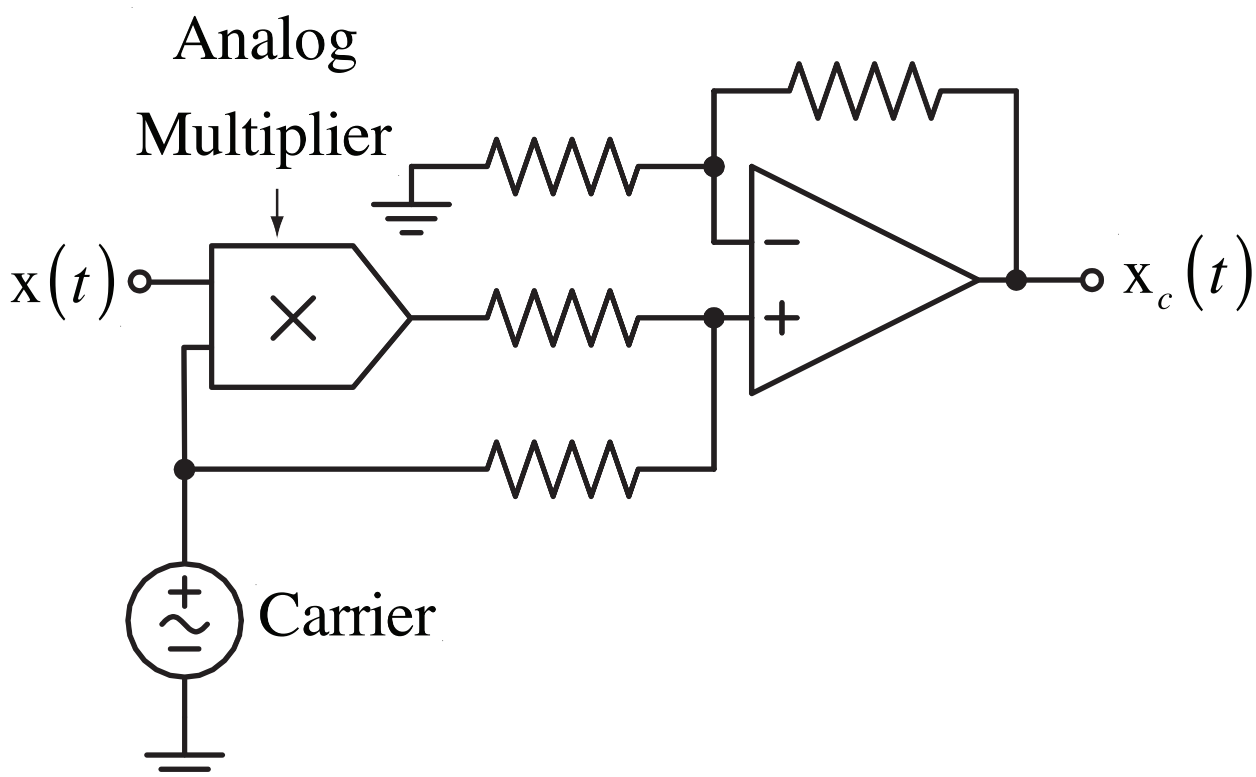
Double-Sideband Amplitude Modulation

The envelope is defined as being non-negative. So if a is too large a problem called **overmodulation** occurs as illustrated below. Simple detection of the envelope causes distortion of the original message.



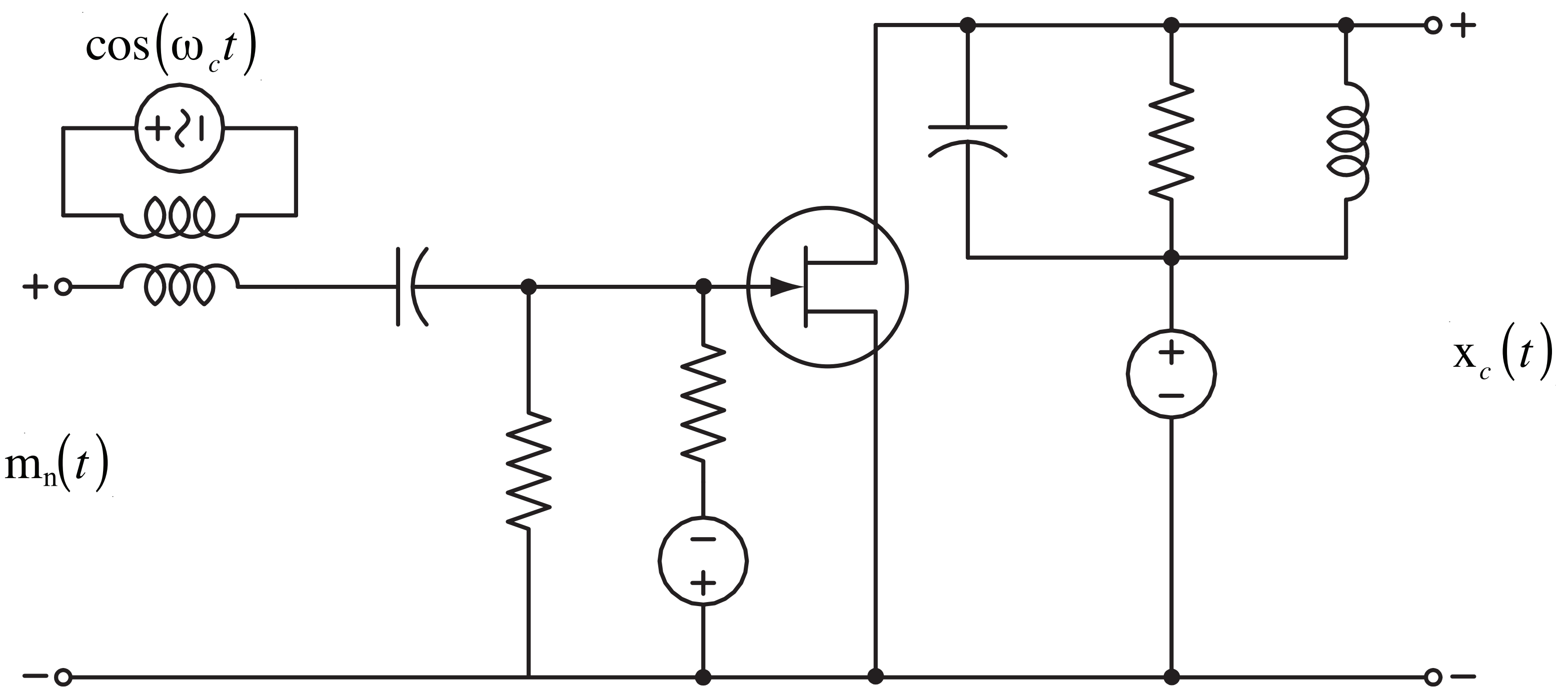
Modulators and Transmitters

The electronic hardware to implement AM or DSB modulation can take any of several forms. The most direct and obvious form is the **product modulator** illustrated below for AM modulation.



Modulators and Transmitters

Another way of obtaining the product of two signals is to use a **square-law modulator**. This type of circuit takes advantage of the inherent non-linearity of a solid-state device. In the example below the device is a field-effect transistor (FET).



Modulators and Transmitters

If the FET has a transfer characteristic $\mathrm{v}_{out} = b_1\,\mathrm{v}_{in} + b_2\,\mathrm{v}_{in}^2$ and if $\mathrm{v}_{in}(t) = \mathrm{m}_n(t) + \cos(2\pi f_c t)$, then

$$\mathrm{v}_{out}(t) = b_1\left[\mathrm{m}_n(t) + \cos(2\pi f_c t)\right] + b_2\left[\mathrm{m}_n(t) + \cos(2\pi f_c t)\right]^2$$

$$\mathrm{v}_{out}(t) = b_1\,\mathrm{m}_n(t) + b_1\cos(2\pi f_c t) + b_2\left[\mathrm{m}_n^2(t) + \cos^2(2\pi f_c t) + 2\,\mathrm{m}_n(t)\cos(2\pi f_c t)\right]$$

$$\mathrm{v}_{out}(t) = b_1\,\mathrm{m}_n(t) + b_1\cos(2\pi f_c t) + b_2\,\mathrm{m}_n^2(t) + b_2\cos^2(2\pi f_c t) + 2b_2\,\mathrm{m}_n(t)\cos(2\pi f_c t)$$

$$\mathrm{v}_{out}(t) = b_1\,\mathrm{m}_n(t) + b_2\,\mathrm{m}_n^2(t) + b_2\cos^2(2\pi f_c t) + b_1\left[1 + 2(b_2/b_1)\mathrm{m}_n(t)\right]\cos(2\pi f_c t)$$

$$\mathrm{v}_{out}(t) = b_1\,\mathrm{m}_n(t) + b_2\,\mathrm{m}_n^2(t) + b_2\cos^2(2\pi f_c t) + \underbrace{A_c\left[1 + a\,\mathrm{m}_n(t)\right]\cos(2\pi f_c t)}_{\text{Desired AM Wave}}$$

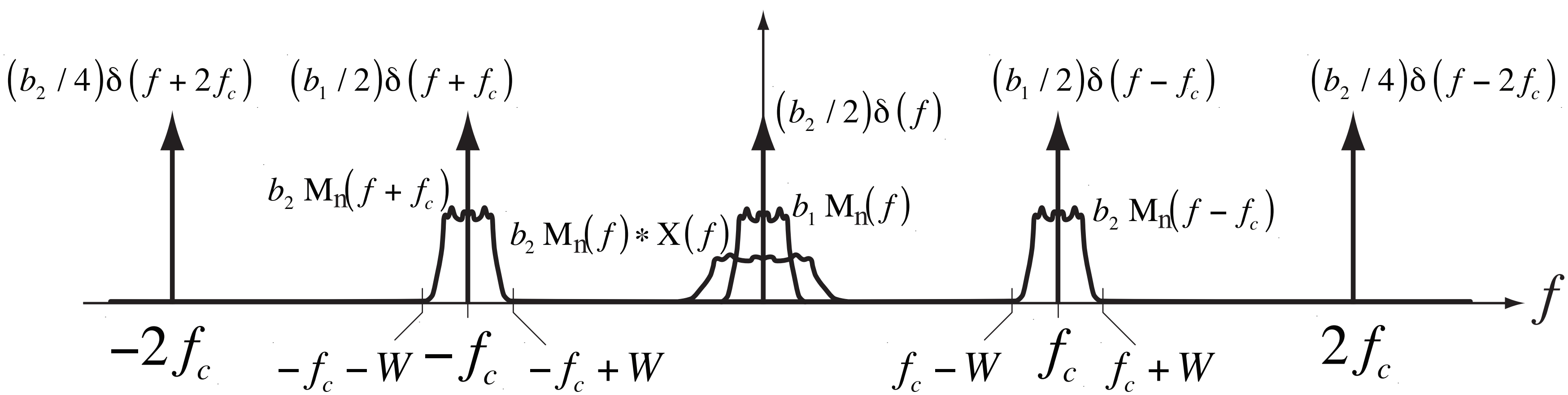
where $A_c = b_1$ and $a = 2(b_2/b_1)$.

Modulators and Transmitters

$$\mathrm{v}_{out}(t) = b_1\,\mathrm{m}_n(t) + b_2\,\mathrm{m}_n^2(t) + b_2\cos^2(2\pi f_c t) + b_1\left[1 + 2(b_2/b_1)\mathrm{m}_n(t)\right]\cos(2\pi f_c t)$$

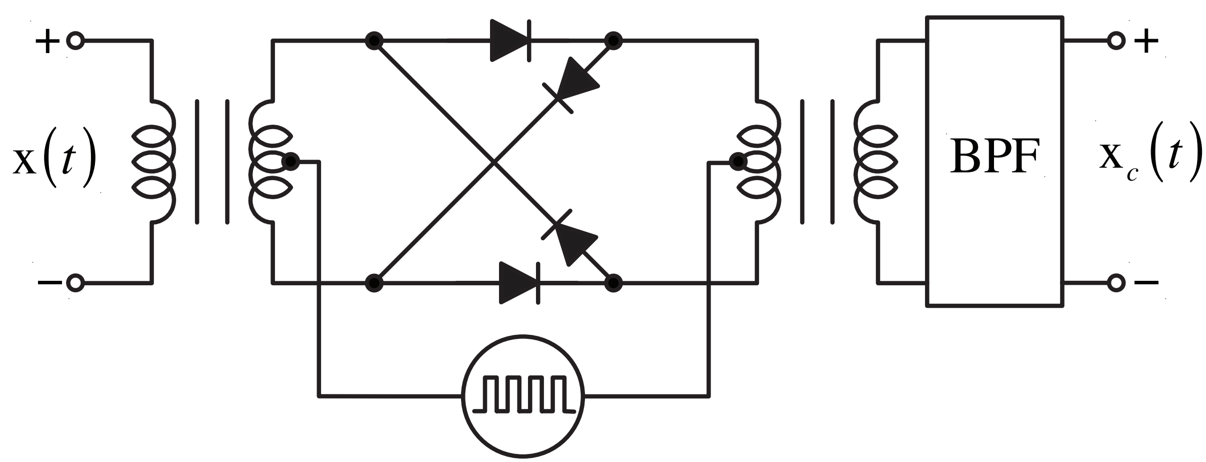
The Fourier transform of $\mathrm{v}_{out}(t)$ is

$$\mathrm{V}_{out}(f) = b_1\,\mathrm{M}_n(f) + b_2\,\mathrm{M}_n(f) * \mathrm{M}_n(f) + (b_2/2)\left\{\delta(f) + (1/2)\left[\delta(f - 2f_c) + \delta(f + 2f_c)\right]\right\}$$
$$+(b_1/2)\left[\delta(f - f_c) + \delta(f + f_c)\right] + b_2\left[\mathrm{M}_n(f - f_c) + \mathrm{M}_n(f + f_c)\right]$$



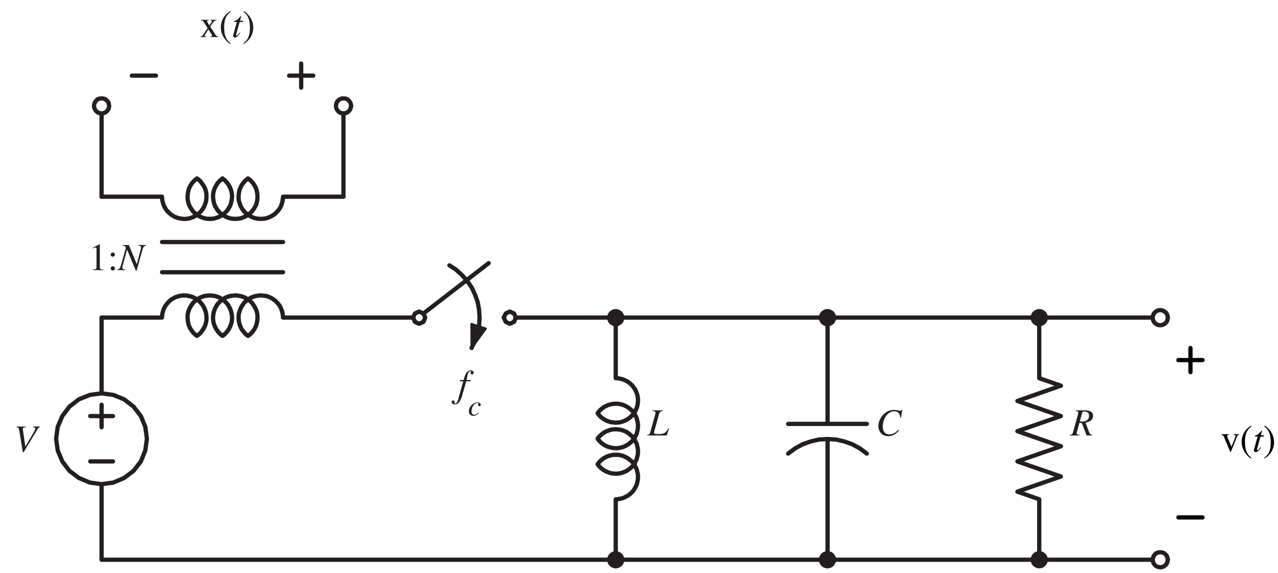
Modulators and Transmitters

The circuit below is a **ring modulator**. When the carrier signal is positive on the left side, the top and bottom diodes are forward biased and the inner diodes are reverse biased, effectively connecting the top of the left transformer secondary to the top of the right transformer primary and the bottoms also and $x_c(t) = x(t)$. When the carrier signal is positive on the right side, the diodes all switch their bias to the opposite state and the tops and bottoms of the transformers are now cross connected making $x_c(t) = -x(t)$.



Modulators and Transmitters

The circuit below is a **switching modulator**. The switch closes briefly every $1/f_c$ seconds. The tank circuit (the parallel RLC circuit) on the right is tuned to resonate at f_c Hz. Every $1/f_c$ seconds the tank circuit is hit with a pulse of energy and "rings" at its resonant frequency. Then at the end of one cycle of ringing it is hit again the same way. If the driving voltage is of constant amplitude the output signal is effectively a sinusoid. The driving voltage is a constant plus the message signal. It changes slowly (compared with the resonant frequency) so the overall effect is to AM modulate the sinusoid with the message signal.



Suppressed-Sideband Amplitude Modulation

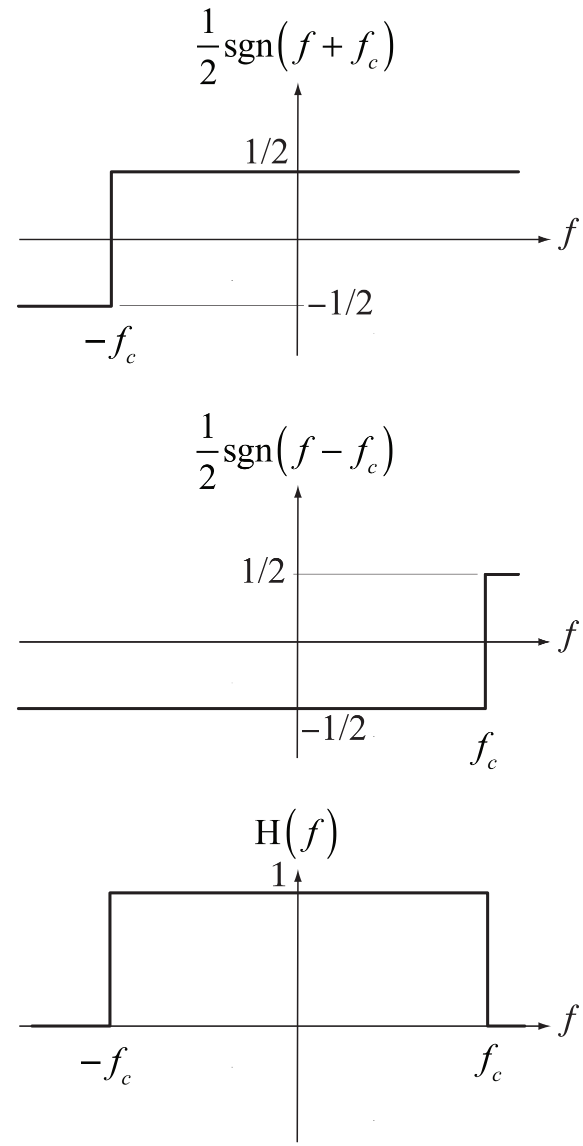
In DSB modulation the upper and lower sidebands are related through Hermitian symmetry and, therefore, they both contain *all* of the message information. So it should be possible to transmit all the message information using only one of the two sidebands. This can be done by **suppressing** one of the sidebands and transmitting only the other sideband. This type of modulation is called **single-sideband suppressed-carrier** (**SSB**) modulation. SSB reduces the bandwidth requirement bya factor of two thus using frequency space more efficiently.

Suppressed-Sideband Amplitude Modulation

We will look at two methods for creating an SSB signal. The first method is sideband filtering. A lower sideband signal can be produced by passing the DSB signal through a filter whose frequency response is

$$\mathrm{H}(f)=\frac{1}{2}\left[\operatorname{sgn}(f+f_c)-\operatorname{sgn}(f-f_c)\right],$$

an ideal lowpass filter.



Suppressed-Sideband Amplitude Modulation

The Fourier transform of the DSB signal is

$$\mathrm{X}_{DSB}(f)=\frac{1}{2}A_c\,\mathrm{M}(f+f_c)+\frac{1}{2}A_c\,\mathrm{M}(f-f_c)$$

so the output of the lowpass filter is

$$\mathrm{X}_c(f)=\mathrm{H}(f)\mathrm{X}_{DSB}(f)$$

$$\mathrm{X}_c(f)=\frac{1}{4}A_c\begin{bmatrix}\mathrm{M}(f+f_c)\operatorname{sgn}(f+f_c)+\mathrm{M}(f-f_c)\operatorname{sgn}(f+f_c)\\ -\mathrm{M}(f+f_c)\operatorname{sgn}(f-f_c)-\mathrm{M}(f-f_c)\operatorname{sgn}(f-f_c)\end{bmatrix}$$

This can be rewritten in this form

$$\mathrm{X}_c(f)=\frac{1}{4}A_c\begin{bmatrix}\mathrm{M}(f+f_c)+\mathrm{M}(f-f_c)\\ +\mathrm{M}(f+f_c)\operatorname{sgn}(f+f_c)-\mathrm{M}(f-f_c)\operatorname{sgn}(f-f_c)\end{bmatrix}$$

Suppressed-Sideband Amplitude Modulation

We now inverse Fourier transform

$$\mathrm{X}_c(f) = \frac{1}{4} A_c \begin{bmatrix} \mathrm{M}(f+f_c) + \mathrm{M}(f-f_c) \\ +\mathrm{M}(f+f_c)\operatorname{sgn}(f+f_c) - \mathrm{M}(f-f_c)\operatorname{sgn}(f-f_c) \end{bmatrix}$$

using $\hat{\mathrm{m}}(t) \xleftrightarrow{\mathscr{F}} -j\operatorname{sgn}(f)\mathrm{M}(f)$ and $\mathrm{m}(t)e^{j2\pi f_c t} \xleftrightarrow{\mathscr{F}} \mathrm{M}(f-f_c)$

to yield

$$\mathrm{x}_c(t) = \frac{1}{4} A_c \left[\mathrm{m}(t)e^{-j2\pi f_c t} + \mathrm{m}(t)e^{+j2\pi f_c t} + j\hat{\mathrm{m}}(t)e^{-j2\pi f_c t} - j\hat{\mathrm{m}}(t)e^{j2\pi f_c t} \right]$$

$$\mathrm{x}_c(t) = \frac{1}{2} A_c \left[\mathrm{m}(t)\cos(2\pi f_c t) + \hat{\mathrm{m}}(t)\sin(2\pi f_c t) \right]$$

For upper sideband SSB the corresponding result is

$$\mathrm{x}_c(t) = \frac{1}{2} A_c \left[\mathrm{m}(t)\cos(2\pi f_c t) - \hat{\mathrm{m}}(t)\sin(2\pi f_c t) \right]$$

Suppressed-Sideband Amplitude Modulation

The generation of an SSB signal, as presented so far, requires an ideal filter (one with vertical sides and a flat top). Ideal filters don't exist. Real filters can have transition regions that are steep but not vertical and real filters can never have a perfectly flat top. So if we use a real filter we can either

1. Set the filter transition region inside the sideband to be retained and lose some of the sideband information

or

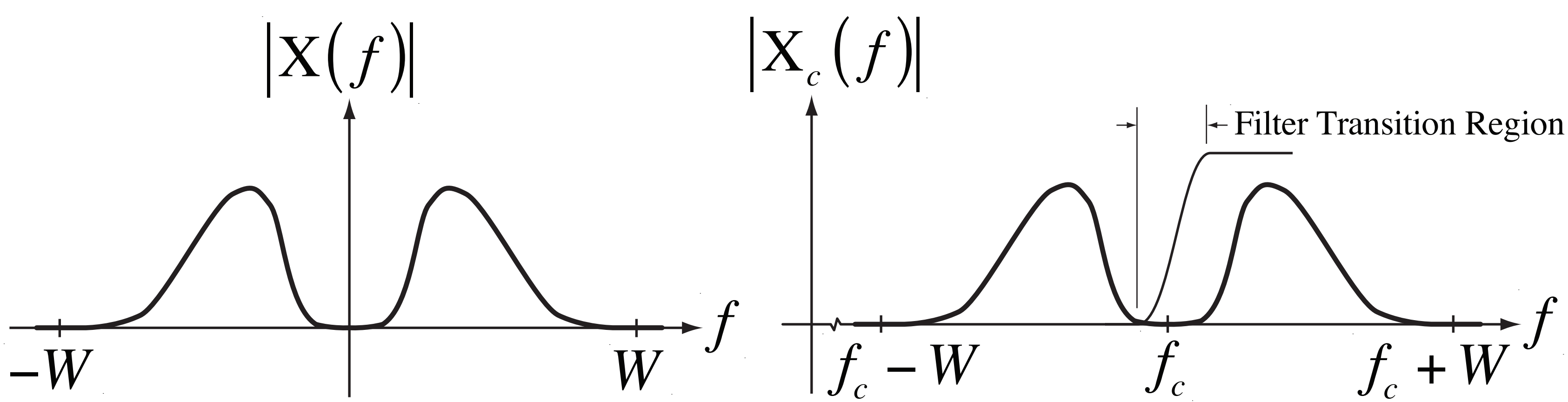
2. Set the filter transition region inside the sideband to be removed retain some of the unwanted sideband

or

3. some combination of 1 and 2.

Suppressed-Sideband Amplitude Modulation

Fortunately, for many practical messages, the spectral content at very low frequencies is very small. This gives the designer of an SSB system a little room to maneuver. The transition region of the filter used to eliminate the unwanted sideband can be placed in the region around the carrier frequency where the DSB signal has very little signal power.



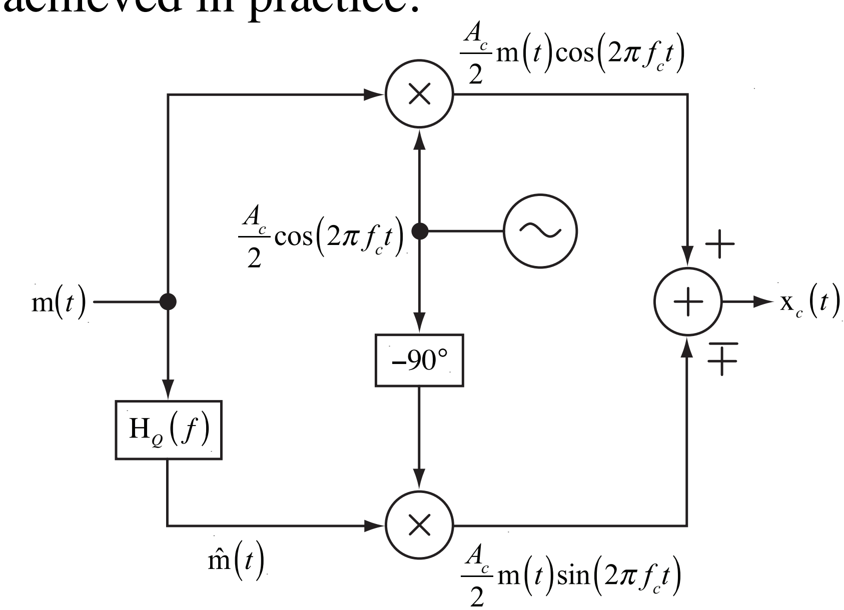
Suppressed-Sideband Amplitude Modulation

Another method for generating SSB is suggested by the relationship

$$x_c(t) = (A_c/2)\,m(t)\cos(2\pi f_c t) \mp (A_c/2)\,\hat{m}(t)\sin(2\pi f_c t).$$

This is written as though SSB consists of the sum of two DSB signals, with carriers that are in quadrature (phase shifted by 90°) and modulated by $x(t)$ and $\hat{x}(t)$. That could be accomplished (theoretically) by the system below where $H_Q(f)$ is a **quadrature phase shifter**.

Unfortunately a quadrature phase shifter is an idealization that can never quite be achieved in practice.



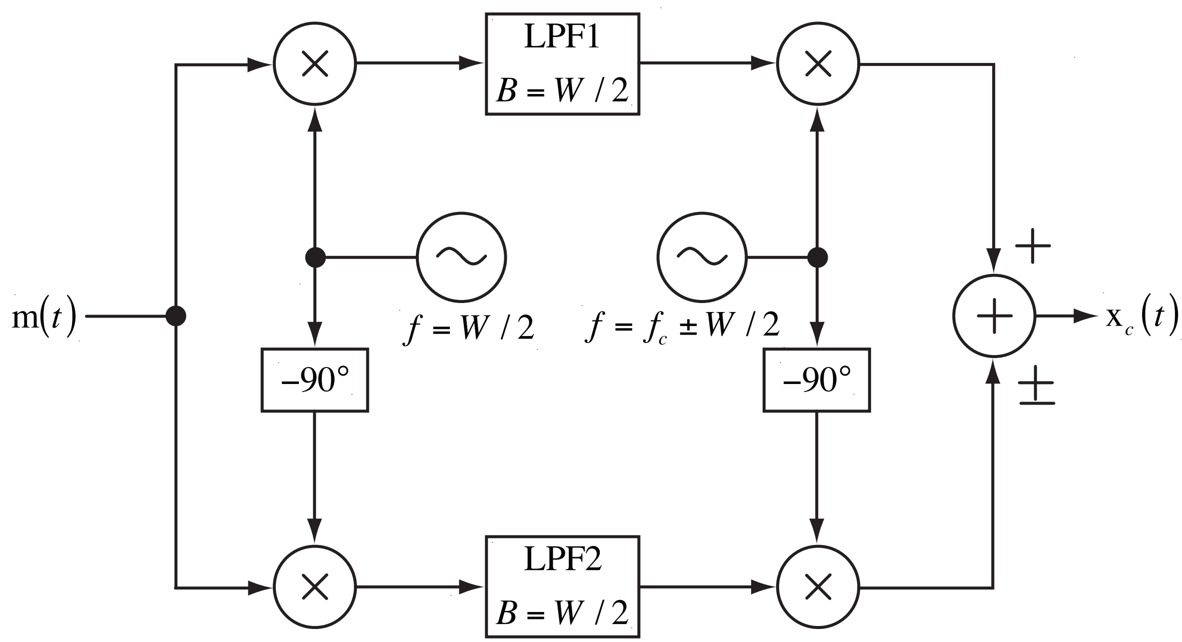
Suppressed-Sideband Amplitude Modulation

A third, more practical, method is Weaver's SSB modulator, diagrammed below.

Let $x(t) = \cos(2\pi f_m t)$ with $0 < f_m < W$ (tone modulation). Then $x_c(t) = v_1(t) \pm v_2(t)$ where $v_1(t)$ is the signal from the upper part of the loop and $v_2(t)$ is the signal from the lower part. The input signal to LPF1 is

$$\cos(2\pi f_m t)\cos(2\pi W t/2) = (1/2)\left[\cos\left(2\pi\left(f_m - W/2\right)t\right) + \cos\left(2\pi\left(f_m + W/2\right)t\right)\right]$$

Since the filter cuts off at $W/2$ its output signal is $(1/2)\left[\cos\left(2\pi\left(f_m - W/2\right)t\right)\right]$.



Suppressed-Sideband Amplitude Modulation

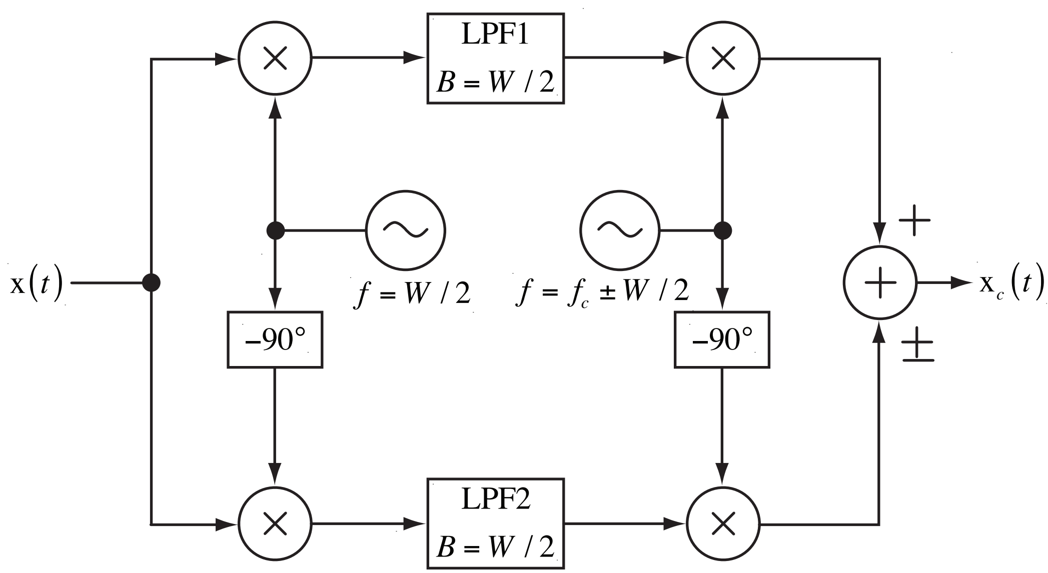
The LPF1 output signal is multiplied by $\cos\left(2\pi\left(f_c \pm W/2\right)t\right)$. Therefore,

$$\mathrm{v}_1(t) = (1/4)\left[\cos\left(2\pi\left(f_c \pm W/2 + f_m - W/2\right)t\right) + \cos\left(2\pi\left(f_c \pm W/2 - f_m + W/2\right)t\right)\right]$$

The input signal to the LPF2 is $\cos\left(2\pi f_m t\right)\sin\left(2\pi W t/2\right)$ and (using similar reasoning)

$$\mathrm{v}_2(t) = (1/4)\left[\cos\left(2\pi\left(f_c \pm W/2 + f_m - W/2\right)t\right) - \cos\left(2\pi\left(f_c \pm W/2 - f_m + W/2\right)t\right)\right].$$

Taking the upper signs, $\mathrm{x}_c(t) = (1/2)\cos\left(2\pi\left(f_c + f_m\right)t\right)$. This is a USB signal. If we instead take the lower signs we get $\mathrm{x}_c(t) = (1/2)\cos\left(2\pi\left(f_c - f_m\right)t\right)$, an LSB signal.



Frequency Conversion and Demodulation

So far we have described several ways of modulating a carrier with a message signal. Now we turn to **demodulation**, the recovery of the message from the modulated carrier. An essential process in most demodulation methods is **frequency conversion**. Consider a DSB signal of the form $\mathrm{x}(t)\cos(2\pi f_1 t)$. If we multiply it by $\cos(2\pi f_2 t)$ we get

$$\mathrm{x}(t)\cos(2\pi f_1 t)\cos(2\pi f_2 t) = (1/2)\left[\cos(2\pi(f_1 - f_2)t) + \cos(2\pi(f_1 + f_2)t)\right]$$

The DSB spectrum has been shifted in frequency up and down by f_2 resulting in **sum** and **difference** spectral components. Devices that do this operation are called **frequency converters** or **mixers** and the operation is called **heterodyning** or **mixing**. The "hetero" prefix refers to two things that are different, in this case f_1 and f_2. If we make f_1 and f_2 the same, heterodyning becomes a special case called **homodyning** in which the prefix "homo" refers to two things that are the same. The most common current uses of the prefixes "hetero" and "homo" in everyday speech are in the words "heterosexual" and "homosexual" where they have the same general significance.

Frequency Conversion and Demodulation

A basic process in many demodulation systems is **synchronous detection**. In synchronous detection the received signal is multiplied by the signal from a **local oscillator** that is at the same frequency as the carrier of the received signal and in phase with that carrier (as received). Let the received signal be represented by

$$\mathrm{x}_c(t) = \left[K_c + K_\mu \,\mathrm{x}(t)\right]\cos(\omega_c t) - K_\mu \,\mathrm{x}_q(t)\sin(\omega_c t)$$

If $K_c = 0$, we have a suppressed carrier. If $\mathrm{x}_q(t) = 0$, we have double sideband. This form can represent the types of modulation we have seen so far. So the demodulation process begins with the product

$$\mathrm{x}_c(t) A_{LO}\cos(\omega_c t) = \left\{\left[K_c + K_\mu \,\mathrm{x}(t)\right]\cos(\omega_c t) - K_\mu \,\mathrm{x}_q(t)\sin(\omega_c t)\right\} A_{LO}\cos(\omega_c t)$$

$$\mathrm{x}_c(t) A_{LO}\cos(\omega_c t) = A_{LO}\left\{K_c\cos^2(\omega_c t) + K_\mu \,\mathrm{x}(t)\cos^2(\omega_c t) - K_\mu \,\mathrm{x}_q(t)\cos(\omega_c t)\sin(\omega_c t)\right\}$$

$$\mathrm{x}_c(t) A_{LO}\cos(\omega_c t) = (A_{LO}/2)\left\{\begin{array}{l} K_c + K_\mu \,\mathrm{x}(t) + K_c\cos(2\omega_c t) + K_\mu \,\mathrm{x}(t)\cos(2\omega_c t) \\ -K_\mu \,\mathrm{x}_q(t)\sin(2\omega_c t) \end{array}\right\}$$

Frequency Conversion and Demodulation

From the previous slide,

$$\mathrm{x}_c(t)A_{LO}\cos(\omega_c t)=(A_{LO}/2)\left\{\begin{matrix}K_c+K_\mu\,\mathrm{x}(t)+K_c\cos(2\omega_c t)+K_\mu\,\mathrm{x}(t)\cos(2\omega_c t)\\ -K_\mu\,\mathrm{x}_q(t)\sin(2\omega_c t)\end{matrix}\right\}$$

We can filter out the double-frequency components of this signal leaving $\mathrm{y}_D(t)=K_D\left[K_c+K_\mu\,\mathrm{x}(t)\right]$, where K_D is a constant accounting for any gain or attenuation in the multiplication/filtering process. Since the local oscillator and the carrier frequency are the same, this is a case of homodyning. We can, if desired, also filter out the K_DK_c constant component of the signal with a blocking capacitor or other suitable DC blocking filter. This process seems simple enough until we come to the problem of how to generate a local oscillator that is locked in both frequency and phase to the carrier of the incoming signal. If the incoming signal is DSB, the carrier has been suppressed. So locking to it is non-trivial. Sometimes a small **pilot carrier** is added to the transmitted signal to facilitate detection of and locking on to the carrier.

Frequency Conversion and Demodulation

There are many techniques for generating a local oscillator that is phase-locked to the incoming carrier but there is always at least a little asynchronism. Let the local oscillator be $\cos(\omega_c t + \omega' t + \phi')$ where ω' accounts for frequency error and ϕ' accounts for phase error. Let the signal be DSB with tone modulation $\cos(\omega_m t)$. Then $\mathrm{x}_c(t) = K_\mu \cos(\omega_m t)\cos(\omega_c t)$ and, multiplying by the local oscillator, we get

$$K_\mu \cos(\omega_m t)\cos(\omega_c t)\cos(\omega_c t + \omega' t + \phi')$$

$$= (K_\mu / 2)\left[\cos(\omega_m t)\cos(\omega' t + \phi') + \cos(\omega_m t)\cos(2\omega_c t + \omega' t + \phi')\right]$$

After lowpass filtering (possibly with gain or attenuation) this becomes

$$\mathrm{y}_D(t) = K_D \cos(\omega_m t)\cos(\omega' t + \phi')$$

$$\mathrm{y}_D(t) = \begin{cases} (K_D / 2)\left[\cos((\omega_m - \omega')t)\right] + \cos((\omega_m + \omega')t) & , \ \phi' = 0 \\ K_D \cos(\omega_m t)\cos(\phi') & , \ \omega' = 0 \end{cases}$$

The frequency error causes a shift in the signal's frequency both up an down by f'. The phase error causes a loss in signal power. If $\phi' = 90°$, the detected signal is zero.

Frequency Conversion and Demodulation

Now let the signal be SSB with $\mathrm{x}_c(t) = \cos((\omega_c \pm \omega_m)t)$

Multiplying by the local oscillator, we get

$$\cos((\omega_c \pm \omega_m)t)\cos(\omega_c t + \omega' t + \phi')$$
$$= (1/2)\left[\cos(\omega' t + \phi' \mp \omega_m t) + \cos(2\omega_c t + \omega' t + \phi' \pm \omega_m t)\right]$$

After lowpass filtering (possibly with gain or attenuation) this becomes

$$\mathrm{y}_D(t) = K_D \cos(\omega' t + \phi' \mp \omega_m t) = K_D \cos(\pm\omega_m t - (\omega' t + \phi'))$$

Multiplying the cosine argument through by ± 1 we get

$$\mathrm{y}_D(t) = K_D \begin{cases} \cos((\omega_m \mp \omega')t) & , \ \phi' = 0 \\ \cos(\omega_m t \mp \phi') & , \ \omega' = 0 \end{cases}$$

The frequency error causes a shift in the frequency of the message either up or down by f'. The phase error causes a phase shift in the signal. For tone modulation this is not a problem. But for a more general signal, the different frequencies would all experience the same phase shift, therefore different time delays, causing delay distortion.

Bandpass Signals and Systems

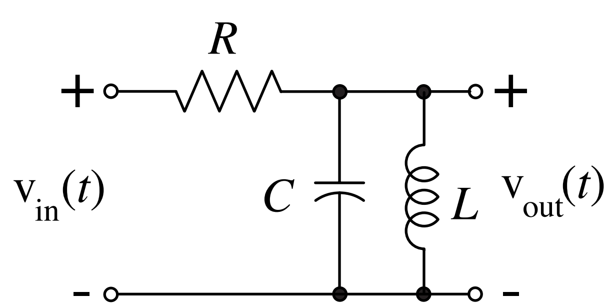
The simplest and most commonly used bandpass system is the parallel resonant *RLC* circuit below. Its frequency response is

$$\mathrm{H}(f)=\frac{\mathrm{Z}_{LC}(f)}{\mathrm{Z}_{LC}(f)+R} \text{ where } \mathrm{Z}_{LC}(f)=\frac{j2\pi fL / j2\pi fC}{j2\pi fL + 1/j2\pi fC}=\frac{j2\pi fL}{1-(2\pi f)^2 LC}$$

$$\text{and } \mathrm{Z}_{LC}(f)=\frac{j2\pi fL}{1-(f/f_0)^2} \text{ where } f_0=\frac{1}{2\pi\sqrt{LC}}$$

$$\mathrm{H}(f)=\frac{\dfrac{j2\pi fL}{1-(f/f_0)^2}}{\dfrac{j2\pi fL}{1-(f/f_0)^2}+R}=\frac{1}{1+jQ\left(\dfrac{f}{f_0}-\dfrac{f_0}{f}\right)} \quad \text{where } Q=R\sqrt{\frac{C}{L}}$$

f_0 is the **resonant cyclic frequency** and Q is the **quality factor**.



Bandpass Signals and Systems

$$\mathrm{H}(f) = \frac{1}{1 + jQ\left(\frac{f}{f_0} - \frac{f_0}{f}\right)}$$

The maximum response occurs when $f = f_0$ and $\mathrm{H}(f_0) = 1$. The -3 dB bandwidth is defined by the frequencies at which $|\mathrm{H}(f)|^2 = 1/2$.

$$|\mathrm{H}(f)|^2 = \frac{1}{1 + jQ\left(\frac{f}{f_0} - \frac{f_0}{f}\right)} \times \frac{1}{1 - jQ\left(\frac{f}{f_0} - \frac{f_0}{f}\right)} = \frac{1}{1 + Q^2\left(\frac{f}{f_0} - \frac{f_0}{f}\right)^2} = \frac{1}{2}$$

Bandpass Signals and Systems

$$Q^2\left(\frac{f}{f_0}-\frac{f_0}{f}\right)^2=1 \;\Rightarrow\; \left(f/f_0\right)^2-2-\frac{1}{Q^2}+\left(f_0/f\right)^2=0$$

$$\left(\frac{f^2}{f_0^2}\right)^2-\left(2+\frac{1}{Q^2}\right)\left(\frac{f^2}{f_0^2}\right)+1=0 \Rightarrow \frac{f^2}{f_0^2}=\frac{\left(2+1/Q^2\right)\pm\sqrt{\left(2+1/Q^2\right)^2-4}}{2}$$

$$\frac{f^2}{f_0^2}=1+\frac{1}{2Q^2}\pm\sqrt{\frac{4/Q^2+1/Q^4}{4}}=1+\frac{1}{2Q^2}\pm\sqrt{\frac{4Q^2+1}{4Q^4}}=1+\frac{1\pm\sqrt{4Q^2+1}}{2Q^2}$$

For large Q, $4Q^2 >> 1$ and $2Q >> 1$ and $\dfrac{f^2}{f_0^2}\cong 1\pm\dfrac{1}{Q}\Rightarrow f^2\cong f_0^2\left(1\pm 1/Q\right)$

$\therefore f\cong \pm f_0\sqrt{1\pm 1/Q}$. Again, for large Q, $\sqrt{1\pm 1/Q}\cong 1\pm 1/2Q$ and $f\cong \pm f_0\left(1\pm 1/2Q\right)$
So the 3 dB bandwidth B is $B\cong f_0/Q$ if Q is large. As a practical matter the Q of this type of tuned circuit is between 10 and 100. Also, as a practical matter, the **fractional bandwidth** B/f_0 should be in the range $0.01 < B/f_0 < 0.1$ to avoid some design problems. Therefore large bandwidths require high center frequencies.

Bandpass Signals and Systems

There are many definitions of "bandwidth".

Absolute bandwidth

The band of frequencies outside of which there is absolutely no signal energy. This only applies to ideal situations in which we have signals that are unlimited in time and filters that are ideal.

Null - to - Null Bandwidth

The spacing between zero crossings of a filter or the spectrum of a signal.

-3 dB Bandwidth

The frequency range between frequencies at which a signal's power is down 3 dB from its maximum (1/2 power points) or at which a filter's power gain is down 3 dB from its maximum.

There are many other definitions for various purposes.