Signal Transmission and Filtering

Response of LTI Systems

LTI means **linear and time - invariant**.

Linear means that if we multiply the excitation of the system by a constant we multiply the response by the same constant
It also means that the response to multiple excitations can be found by finding the responses to the individual excitations and adding them.

Response of LTI Systems

Time invariant means that if we excite the system and get a response at one time and then excite the system at a different time, the response is the same except shifted by the difference in time. In other words, the way the system responds to an excitation does not change with time. Hence the term "time invariant".

Response of LTI Systems

An LTI system is completely characterized by its impulse response $\mathrm{h}(t)$. The response $\mathrm{y}(t)$ of an LTI system to an excitation $\mathrm{x}(t)$ is the convolution of $\mathrm{x}(t)$ with $\mathrm{h}(t)$.

$$\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t) = \int_{-\infty}^{\infty} \mathrm{x}(\lambda)\mathrm{h}(t-\lambda)\,d\lambda$$

Response of LTI Systems

The Fourier transform of the impulse response is **frequency response**. (The book calls it **transfer function**.)

$$\mathrm{H}(f) = \int_{-\infty}^{\infty} \mathrm{h}(t) e^{-j2\pi ft}\, dt$$

For any physically realizable system $\mathrm{H}(f)$ has **Hermitian symmetry**. That is, $\mathrm{H}(f) = \mathrm{H}^*(-f)$ and that fact can be used to show that its magnitude is an even function and its phase can be expressed as an odd function.

$$|\mathrm{H}(f)| = |\mathrm{H}(-f)| \text{ and } \measuredangle \mathrm{H}(f) = -\measuredangle \mathrm{H}(-f)$$

(In the book, $\arg \mathrm{H}(f) = -\arg\, \mathrm{H}(-f)$.)

Response of LTI Systems

If the excitation $\mathrm{x}(t)$ is a **phasor** or **complex sinusoid** of frequency f_0, of the form

$$\mathrm{x}(t) = A_{\mathrm{x}} e^{j\phi_{\mathrm{x}}} e^{j2\pi f_0 t}$$

then the response $\mathrm{y}(t)$ is of the form

$$\mathrm{y}(t) = \mathrm{H}(f_0)\mathrm{x}(t) = \mathrm{H}(f_0) A_{\mathrm{x}} e^{j\phi_{\mathrm{x}}} e^{j2\pi f_0 t} .$$

The response can also be written in the form

$\mathrm{y}(t) = A_{\mathrm{y}} e^{j\phi_{\mathrm{y}}} e^{j2\pi f_0 t}$ where $A_{\mathrm{y}} = |\mathrm{H}(f_0)| A_{\mathrm{x}}$ and $\phi_{\mathrm{y}} = \phi_{\mathrm{x}} + \measuredangle \mathrm{H}(f_0)$.

Applying this to real sinusoids, if $\mathrm{x}(t) = A_{\mathrm{x}} \cos(2\pi f_0 t + \phi_{\mathrm{x}})$ then $\mathrm{y}(t) = A_{\mathrm{y}} \cos(2\pi f_0 t + \phi_{\mathrm{y}})$.

Response of LTI Systems

If the Fourier transform of the excitation $\mathrm{x}(t)$ is $\mathrm{X}(f)$ and the Fourier transform of the response $\mathrm{y}(t)$ is $\mathrm{Y}(f)$, then $\mathrm{Y}(f) = \mathrm{H}(f)\mathrm{X}(f)$ and $|\mathrm{Y}(f)| = |\mathrm{H}(f)||\mathrm{X}(f)|$ and $\measuredangle \mathrm{Y}(f) = \measuredangle \mathrm{H}(f) + \measuredangle \mathrm{X}(f)$.

If $\mathrm{x}(t)$ is an energy signal (finite signal energy) then, from Parseval's theorem $E_{\mathrm{x}} = \int_{-\infty}^{\infty} |\mathrm{X}(f)|^2 \, df$ and $E_{\mathrm{y}} = \int_{-\infty}^{\infty} |\mathrm{Y}(f)|^2 \, df = \int_{-\infty}^{\infty} |\mathrm{H}(f)|^2 |\mathrm{X}(f)|^2 \, df$

Signal Distortion in Transmission

Distortion means changing the shape of a signal. Two changes to a signal are not considered distortion, multiplying it by a constant and shifting it in time. The impulse response of an LTI system that does not distort is of the general form $\mathrm{h}(t) = K\delta(t - t_d)$. where K and t_d are constants. The corresponding frequency response of such a system is $\mathrm{H}(f) = Ke^{-j2\pi f t_d}$.

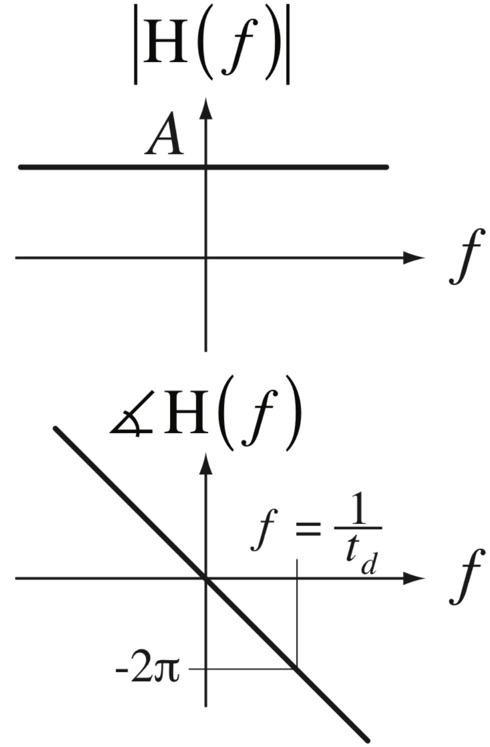
$|\mathrm{H}(f)| = K$ and $\measuredangle\mathrm{H}(f) = -2\pi f t_d$. If $|\mathrm{H}(f)| \neq K$ the system has **amplitude distortion**. If $\measuredangle\mathrm{H}(f) \neq -2\pi f t_d$ the system has **delay** or **phase distortion**. Both of these types of distortion are classified as **linear distortions**.

Signal Distortion in Transmission

If $\measuredangle \mathrm{H}(f) = -2\pi f t_d$, then $t_d = -\dfrac{\measuredangle \mathrm{H}(f)}{2\pi f}$ and t_d is a constant

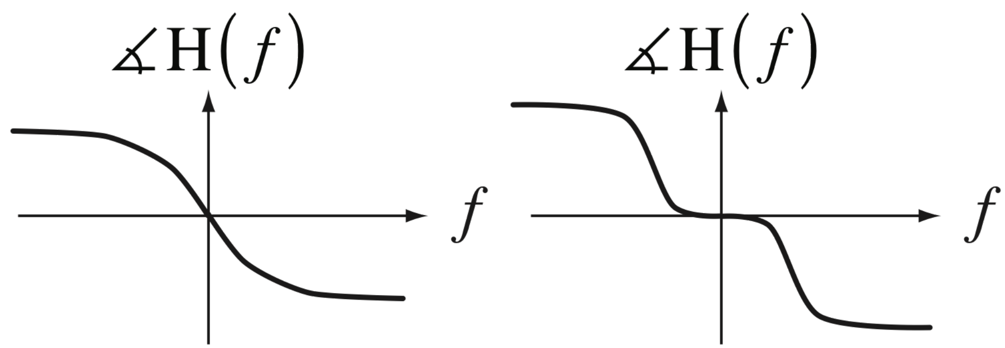
<u>if</u> $\measuredangle \mathrm{H}(f) = -Kf$ (K a constant). If t_d is not a constant, phase distortion results.

Frequency response of a distortionless LTI system $\rightarrow$



Signal Distortion in Transmission

Most real systems do not have simple delay. They have phases that are not linear functions of frequency.

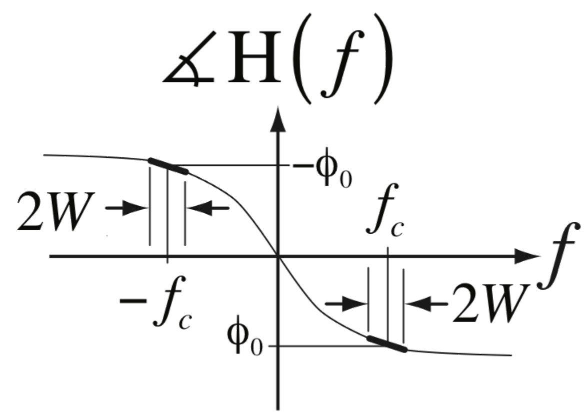


Signal Distortion in Transmission

For a bandpass signal with a small bandwidth W compared to its center frequency f_c, we can model the frequency response phase variation as approximately linear over the frequency ranges $f_c - W < |f| < f_c + W$, and the frequency response magnitude as approximately constant, of the form

$$\mathrm{H}(f) \cong A e^{-j2\pi f t_g} \begin{cases} e^{j\phi_0} & , \quad f_c - W < f < f_c + W \\ e^{-j\phi_0} & , \quad -f_c - W < f < -f_c + W \end{cases}$$

where $\phi_0 = \measuredangle \mathrm{H}(f_c)$.



Signal Distortion in Transmission

If we now let the bandpass signal be

$$\mathrm{x}(t) = \mathrm{x}_1(t)\cos(2\pi f_c t) + \mathrm{x}_2(t)\sin(2\pi f_c t)$$

Its Fourier transform is

$$\mathrm{X}(f) = \begin{Bmatrix} \mathrm{X}_1(f) * (1/2)\left[\delta(f - f_c) + \delta(f + f_c)\right] \\ + \mathrm{X}_2(f) * (j/2)\left[\delta(f + f_c) - \delta(f - f_c)\right] \end{Bmatrix}$$

$$\mathrm{X}(f) = (1/2)\left\{\left[\mathrm{X}_1(f - f_c) + \mathrm{X}_1(f + f_c)\right] + j\left[\mathrm{X}_2(f + f_c) - \mathrm{X}_2(f - f_c)\right]\right\}$$

The frequency response is modeled by

$$\mathrm{H}(f) \cong Ae^{-j2\pi f t_g} \begin{cases} e^{j\phi_0} & , \quad f_c - W < f < f_c + W \\ e^{-j\phi_0} & , \quad -f_c - W < f < -f_c + W \end{cases}$$

then the Fourier transform of the response $\mathrm{y}(t)$ is

$$\mathrm{Y}(f) \cong \mathrm{H}(f)\mathrm{X}(f) = (A/2)\begin{Bmatrix} \mathrm{X}_1(f - f_c)e^{-j(2\pi f t_g - \phi_0)} + \mathrm{X}_1(f + f_c)e^{-j(2\pi f t_g + \phi_0)} \\ + j\,\mathrm{X}_2(f + f_c)e^{-j(2\pi f t_g + \phi_0)} - j\,\mathrm{X}_2(f - f_c)e^{-j(2\pi f t_g - \phi_0)} \end{Bmatrix}$$

Signal Distortion in Transmission

$$\mathrm{Y}(f) \cong \mathrm{H}(f)\mathrm{X}(f) = (A/2)\left\{ \begin{array}{l} \mathrm{X}_1(f - f_c)e^{-j(2\pi f t_g - \phi_0)} + \mathrm{X}_1(f + f_c)e^{-j(2\pi f t_g + \phi_0)} \\ + j\mathrm{X}_2(f + f_c)e^{-j(2\pi f t_g + \phi_0)} - j\mathrm{X}_2(f - f_c)e^{-j(2\pi f t_g - \phi_0)} \end{array} \right\}$$

Inverse Fourier transforming, using the time and frequency shifting properties,

$$\mathrm{y}(t) \cong (A/2)\left\{ \begin{array}{l} e^{j\phi_0}\,\mathrm{x}_1(t - t_g)e^{j2\pi f_c t} + e^{-j\phi_0}\,\mathrm{x}_1(t - t_g)e^{-j2\pi f_c t} \\ + je^{-j\phi_0}\,\mathrm{x}_2(t - t_g)e^{-j2\pi f_c t} - je^{j\phi_0}\,\mathrm{x}_2(t - t_g)e^{j2\pi f_c t} \end{array} \right\}$$

$$\mathrm{y}(t) \cong (A/2)\left\{ \begin{array}{l} \mathrm{x}_1(t - t_g)\left[e^{j(2\pi f_c t + \phi_0)} + e^{-j(2\pi f_c t + \phi_0)}\right] \\ + \mathrm{x}_2(t - t_g)\,j\left[e^{-j(2\pi f_c t + \phi_0)} - e^{j(2\pi f_c t + \phi_0)}\right] \end{array} \right\}$$

$$\mathrm{y}(t) \cong A\left\{ \mathrm{x}_1(t - t_g)\cos(2\pi f_c t + \phi_0) + \mathrm{x}_2(t - t_g)\sin(2\pi f_c t + \phi_0) \right\}$$

$$\mathrm{y}(t) \cong A\left\{ \mathrm{x}_1(t - t_g)\cos\left(2\pi f_c (t - t_d)\right) + \mathrm{x}_2(t - t_g)\sin\left(2\pi f_c (t - t_d)\right) \right\}$$

where $t_d = -\dfrac{\phi_0}{2\pi f_c} = -\dfrac{\measuredangle \mathrm{H}(f_c)}{2\pi f_c}$ is known as the **phase** or **carrier delay**.

Signal Distortion in Transmission

From the approximate form of the system frequency response

$$\mathrm{H}(f) \cong A e^{-j2\pi f t_{\mathrm{g}}} \begin{cases} e^{j\phi_0} & , \quad f_c - W < f < f_c + W \\ e^{-j\phi_0} & , \quad -f_c - W < f < -f_c + W \end{cases}$$

we get

$$\measuredangle \mathrm{H}(f) \cong \begin{cases} -2\pi f t_{\mathrm{g}} + \phi_0 & , \quad f_c - W < f < f_c + W \\ -2\pi f t_{\mathrm{g}} - \phi_0 & , \quad -f_c - W < f < -f_c + W \end{cases}$$

If we differentiate both sides w.r.t. f we get

$$\frac{d}{df}\left(\measuredangle \mathrm{H}(f)\right) \cong -2\pi t_{\mathrm{g}} \quad , \qquad f_c - W < |f| < f_c + W$$

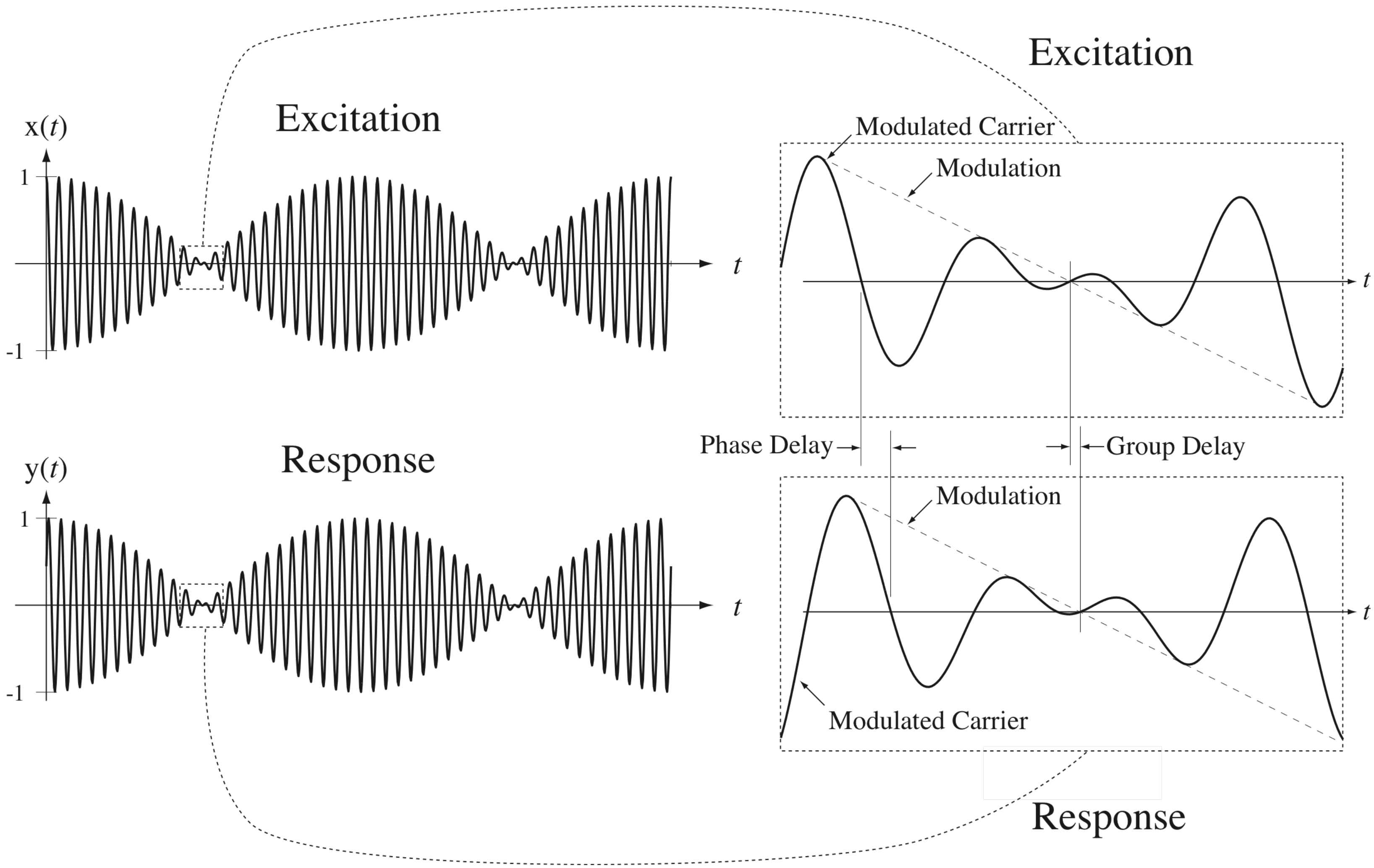
or

$$t_{\mathrm{g}} \cong -\frac{1}{2\pi}\frac{d}{df}\left(\measuredangle \mathrm{H}(f)\right) \ , \qquad f_c - W < |f| < f_c + W$$

t_{g} is known as the **group delay**.

Signal Distortion in Transmission

Phase and Group Delay

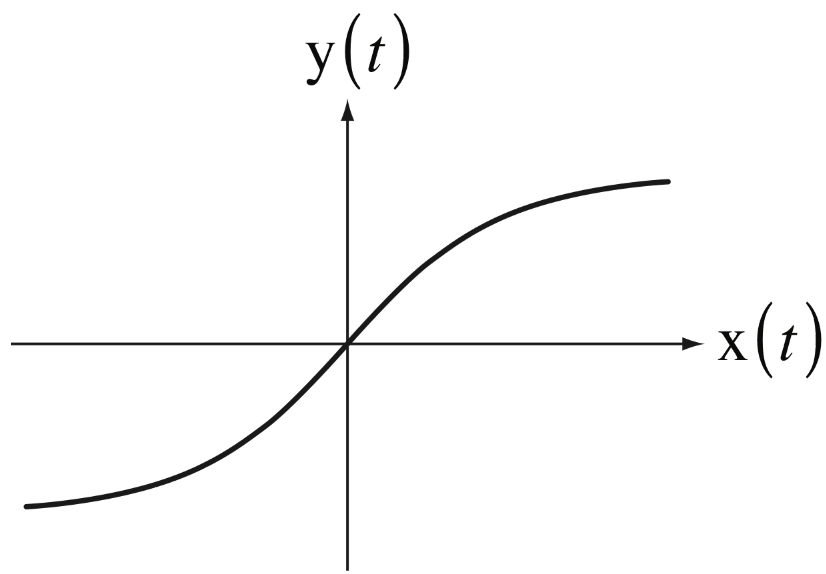


Signal Distortion in Transmission

Linear distortion can be corrected (theoretically) by an **equalization** network. If the communication channel's frequency response is $\mathrm{H}_C(f)$ and it is followed by an equalization network with frequency response $\mathrm{H}_{eq}(f)$ then the overall frequency response is $\mathrm{H}(f) = \mathrm{H}_C(f)\mathrm{H}_{eq}(f)$ and the overall frequency response will be distortionless if $\mathrm{H}(f) = \mathrm{H}_C(f)\mathrm{H}_{eq}(f) = Ke^{-j\omega t_d}$. Therefore, the frequency response of the equalization network should be $\mathrm{H}_{eq}(f) = \dfrac{Ke^{-j\omega t_d}}{\mathrm{H}_C(f)}$. It is very rare in practice that this can be done exactly but in many cases an excellent approximation can be made that greatly reduces linear distortion.

Signal Distortion in Transmission

Communication systems can also have nonlinear distortion caused by elements in the system that are **statically nonlinear**. In that case the excitation and response are related through a **transfer characteristic** of the form $\mathrm{y}(t)=\mathrm{T}(\mathrm{x}(t))$. For example, some amplifiers experience a "soft" saturation in which the ratio of the response to the excitation decreases with an increase in the excitation level.



Signal Distortion in Transmission

The transfer characteristic is usually not a simple known function but can often be closely approximated by a polynomial curve fit of the form $\mathrm{y}(t) = a_1\,\mathrm{x}(t) + a_2\,\mathrm{x}^2(t) + a_3\,\mathrm{x}^3(t) + \cdots$. The Fourier transform of $\mathrm{y}(t)$ is

$$\mathrm{Y}(f) = a_1\,\mathrm{X}(f) + a_2\,\mathrm{X}(f) * \mathrm{X}(f) + a_3\,\mathrm{X}(f) * \mathrm{X}(f) * \mathrm{X}(f) + \cdots$$

In a linear system if the excitation is bandlimited, the response has the same band limits. The response cannot contain frequencies not present in the excitation. But in a nonlinear system of this type if $\mathrm{X}(f)$ contains a range of frequencies, $\mathrm{X}(f) * \mathrm{X}(f)$ contains a greater range of frequencies and $\mathrm{X}(f) * \mathrm{X}(f) * \mathrm{X}(f)$ contains a still greater range of frequencies.

Signal Distortion in Transmission

If $\mathrm{X}(f) * \mathrm{X}(f)$ contains frequencies that are all outside the range of $\mathrm{X}(f)$ then a filter can be used to eliminate them. But often $\mathrm{X}(f) * \mathrm{X}(f)$ contains frequencies both inside and outside that range, and those inside the range cannot be filtered out without affecting the spectrum of $\mathrm{X}(f)$. As a simple example of the kind of nonlinear distortion that can occur let $\mathrm{x}(t) = A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t)$ and let $\mathrm{y}(t) = \mathrm{x}^2(t)$. Then

$$\begin{aligned}
\mathrm{y}(t) &= \left[A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t)\right]^2 \\
&= A_1^2 \cos^2(\omega_1 t) + A_2^2 \cos^2(\omega_2 t) + 2 A_1 A_2 \cos(\omega_1 t)\cos(\omega_2 t) \\
&= \left(A_1^2 / 2\right)\left[1 + \cos(2\omega_1 t)\right] + \left(A_2^2 / 2\right)\left[1 + \cos(2\omega_2 t)\right] \\
&\quad + A_1 A_2 \left[\cos\left((\omega_1 - \omega_2)t\right) + \cos\left((\omega_1 + \omega_2)t\right)\right]
\end{aligned}$$

Signal Distortion in Transmission

$$\mathrm{y}(t) = \left(A_1^2/2\right)\left[1+\cos\left(2\omega_1 t\right)\right] + \left(A_2^2/2\right)\left[1+\cos\left(2\omega_2 t\right)\right]$$
$$+ A_1 A_2 \left[\cos\left(\left(\omega_1 - \omega_2\right)t\right) + \cos\left(\left(\omega_1 + \omega_2\right)t\right)\right]$$

$\mathrm{y}(t)$ contains frequencies $2\omega_1$, $2\omega_2$, $\omega_1 - \omega_2$ and $\omega_1 + \omega_2$. The frequencies $\omega_1 - \omega_2$ and $\omega_1 + \omega_2$ are called **intermodulation distortion products**. When the excitation contains more frequencies (which it usually does) and the nonlinearity is of higher order (which it often is), many more intermodulation distortion products occur. All systems have nonlinearities and intermodulation disortion will occur. But, by careful design, it can often be reduced to a negligible level.

Transmission Loss and Decibels

Communication systems affect the power of a signal. If the signal power at the input is P_{in} and the signal power at the output is P_{out}, the **power gain** g of the system is $g = P_{out} / P_{in}$. It is very common to express this gain in **decibels**. A decibel is one-tenth of a **bel**, a unit named in honor of Alexander Graham Bell. The system gain g expressed in decibels would be $g_{dB} = 10\log_{10}\left(P_{out} / P_{in}\right)$.

g	0.1	1	10	100	1000	10,000	100,000
g_{dB}	−10	0	10	20	30	40	50

Because gains expressed in dB are logarithmic, they compress the range of numbers. If two systems are cascaded, the overall power gain is the product of the two individual power gains $g = g_1 g_2$. The overall power gain expressed in dB is the sum of the two power gains expressed in dB , $g_{dB} = g_{1,dB} + g_{2,dB}$.

Transmission Loss and Decibels

The decibel was defined based on a power ratio, but it is often used to indicate the power of a single signal. Two common types of power indication of this type are **dBW** and **dBm**. dBW is the power of a signal with reference to one watt. That is, a one watt signal would have a power expressed in dBW of 0 dBW. dBm is the power of a signal with reference to one milliwatt. A 20 mW signal would have a power expressed in dBm of 13.0103 dBm. Signal power gain as a function of frequency is the square of the magnitude of frequency response $\left|\mathrm{H}(f)\right|^2$. Frequency response magnitude is often expressed in dB also. $\left|\mathrm{H}(f)\right|_{dB} = 10\log_{10}\left(\left|\mathrm{H}(f)\right|^2\right) = 20\log_{10}\left(\left|\mathrm{H}(f)\right|\right)$.

Transmission Loss and Decibels

A communication system generally consists of components that amplify a signal and components that attenuate a signal. Any cable, optical or copper, attenuates the signal as it propagates. Also there are noise processes in all cables and amplifiers that generate random noise. If the power level gets too low, the signal power becomes comparable to the noise power and the fidelity of analog signals is degraded too far or the detection probability for digital signals becomes too low. So, before that signal level is reached, we must boost the signal power back up to transmit it further. Amplifiers used for this purpose are called **repeaters**.

Transmission Loss and Decibels

On a signal cable of 100's or 1000's of kilometers many repeaters will be needed. How many are needed depends on the **attenuation** per kilometer of the cable and the power gains of the repeaters. Attenuation will be symbolized by $L = 1/g = P_{in}/P_{out}$ or $L_{dB} = -g_{dB} = 10\log 10\left(P_{in}/P_{out}\right)$, (L for "loss".) For optical and copper cables the attenuation is typically exponential and $P_{out} = 10^{-\alpha l/10} P_{in}$ where l is the length of the cable and α is the **attenuation coefficient** in dB/unit length. Then $L = 10^{\alpha l/10}$ and $L_{dB} = \alpha l$.

Filters and Filtering

An **ideal** bandpass filter has the frequency response

$$\mathrm{H}(f) = \begin{cases} Ke^{-j\omega t_d} & , f_l \le |f| \le f_h \\ 0 & , \text{ otherwise} \end{cases}$$

where f_l is the lower cutoff frequency and f_h is the upper cutoff frequency and K and t_d are constants. The filter's bandwidth is $B = f_h - f_l$. An ideal lowpass filter has the same frequency response but with $f_l = 0$ and $B = f_h$. An ideal highpass filter has the same frequency response but with $f_h \to \infty$ and $B \to \infty$. These filters are called ideal because they cannot actually be built. They cannot be built because they are non-causal. But they are useful fictions for introducing in a simplified way some of the concepts of communication systems.

Filters and Filtering

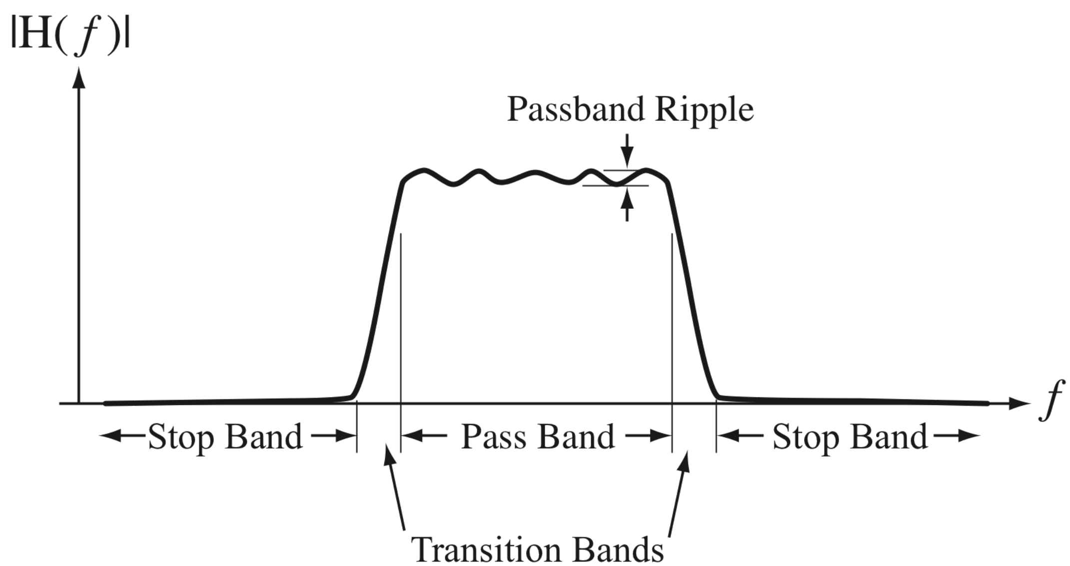
Strictly speaking a signal cannot be both bandlimited and timelimited. But many signals are almost bandlimited and timelimited. That is, many signals have very little signal energy outside a defined bandwidth and, at the same time, very little signal energy outside a defined time range. A good example of this is a Gaussian pulse

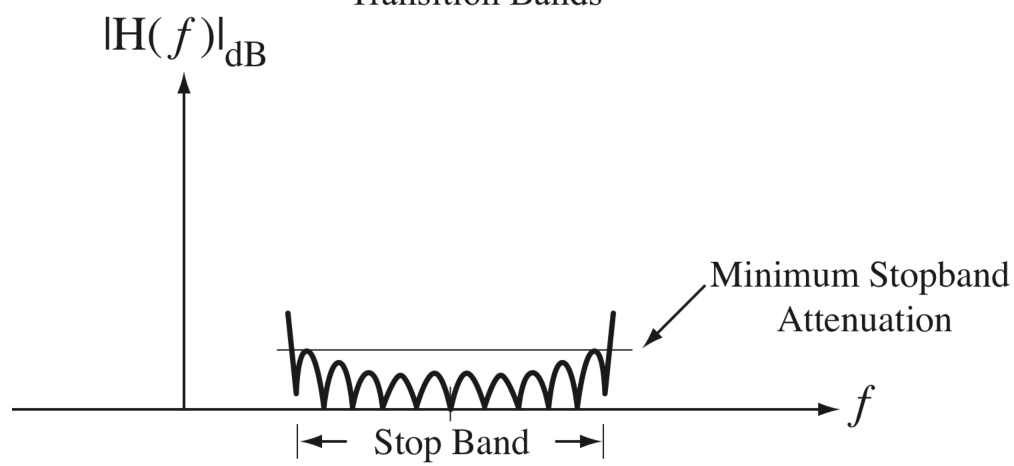
$$\mathrm{x}(t) = e^{-\pi t^2} \xleftrightarrow{\mathcal{F}} \mathrm{X}(f) = e^{-\pi f^2}$$

Strictly speaking, this signal is not bandlimited or timelimited. The total signal energy of this signal is $1/\sqrt{2}$. 99% of its energy lies in the time range $-0.74 < t < 0.74$ and in the frequency range $-0.74 < f < 0.74$. So in many practical calculations this signal could be considered both bandlimited and timelimited with very little error.

Filters and Filtering

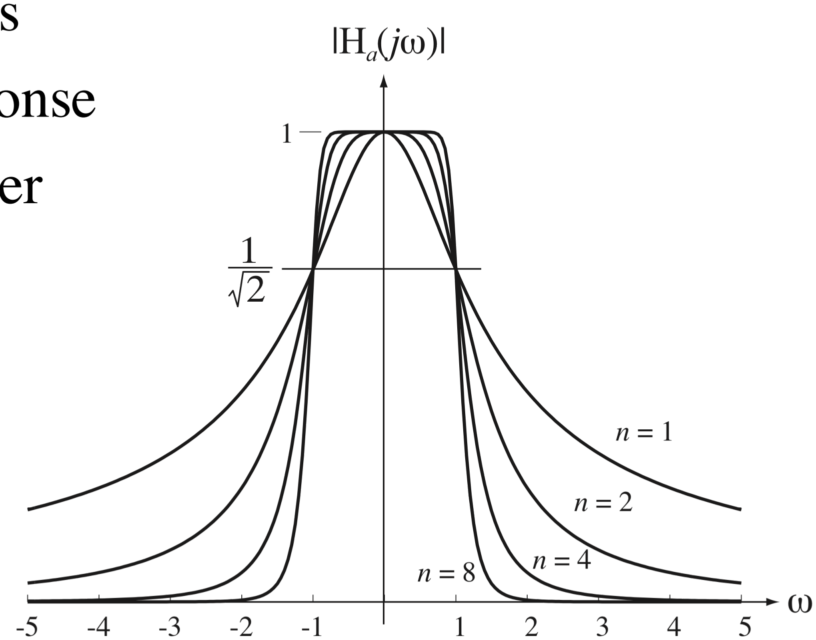
Real filters cannot have constant amplitude response and linear phase response in their passbands like ideal filters.





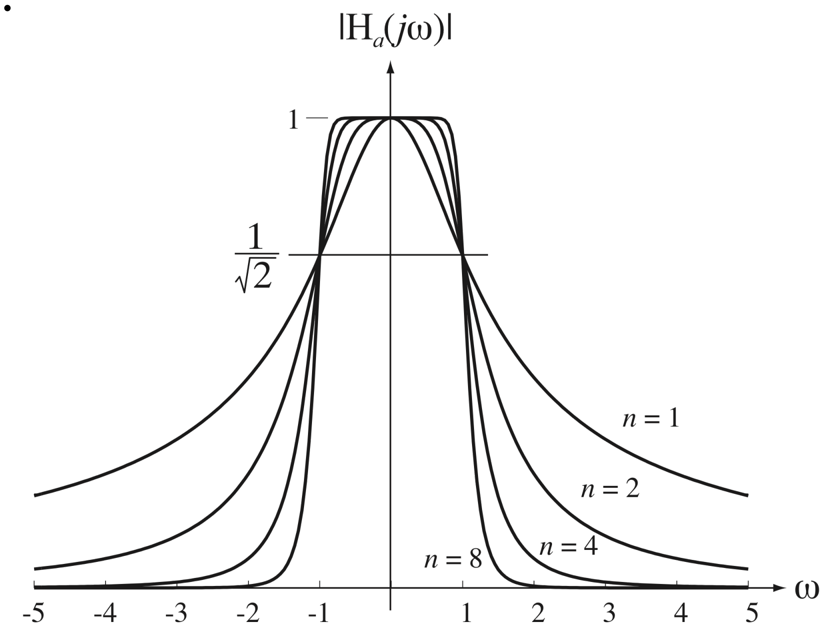
Filters and Filtering

There are many types of standardized filters. One very common and useful one is the **Butterworth** filter. The frequency response of a lowpass Butterworth filter is of the form $\left|\mathrm{H}(f)\right| = \dfrac{1}{\sqrt{1+\left(f/B\right)^{2n}}}$ where n is the **order** of the filter. As the order is increased, its magnitude response approaches that of an ideal filter, constant in the passband and zero outside the passband. (Below is illustrated the magnitude frequency response of a normalized lowpass Butterworth filter with a corner frequency of 1 radian/s.)



Filters and Filtering

The Butterworth filter is said to be **maximally flat** in its passband. It is given this description because the first n derivatives of its magnitude frequency response are all zero at $f = 0$ (for a lowpass filter). The passband of a lowpass Butterworth filter is defined as the frequency at which its magnitude frequency response is reduced from its maximum by a factor of $1/\sqrt{2}$. This is also known as its **half-power** bandwidth because, at this frequency the power gain of the filter is half its maximum value.



Filters and Filtering

The step response of a filter is

$$\mathrm{h}_{-1}(t) = \int_{-\infty}^{\infty} \mathrm{h}(\lambda)\mathrm{u}(t-\lambda)\,d\lambda = \int_{-\infty}^{t} \mathrm{h}(\lambda)\,d\lambda$$

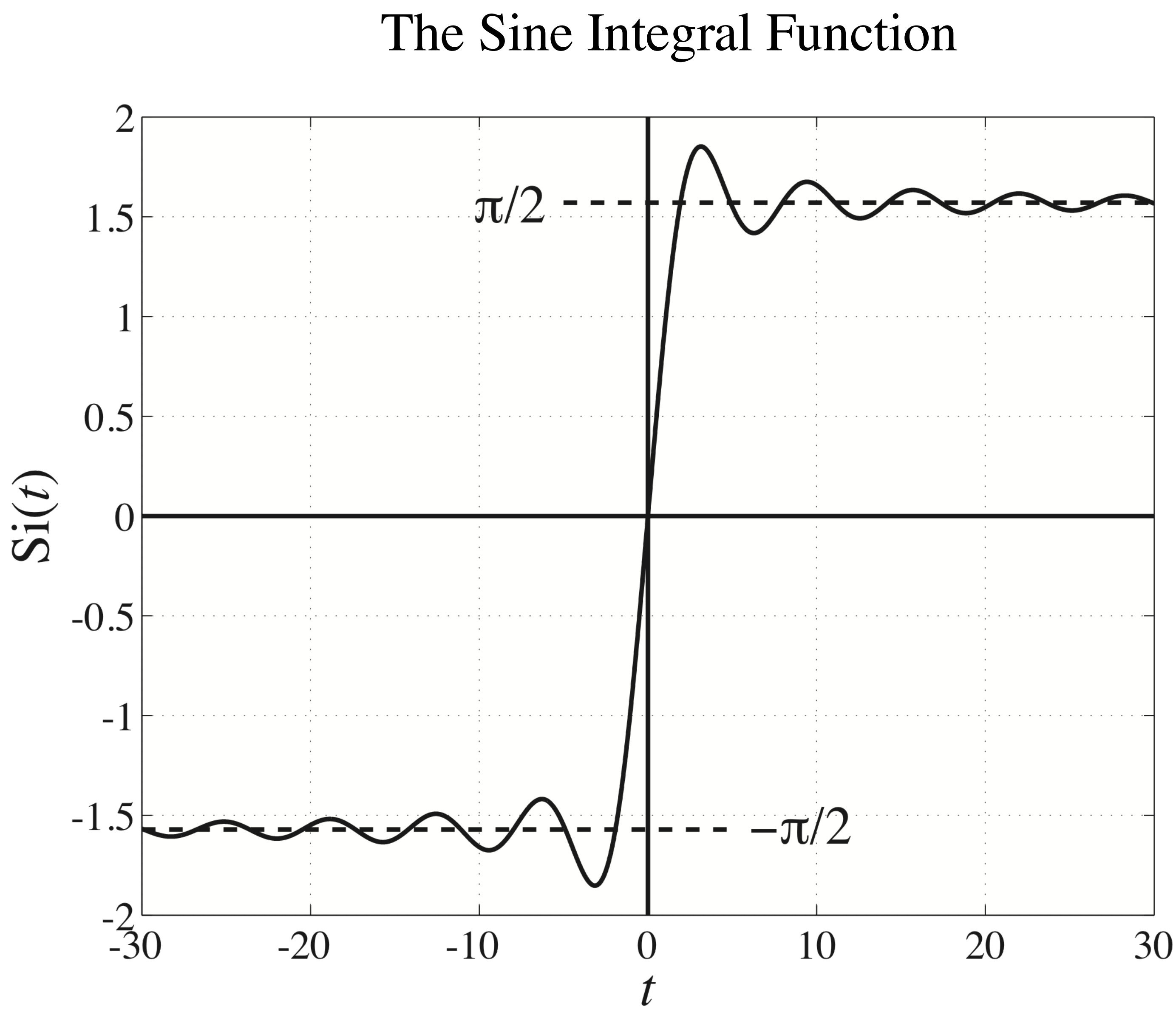
($\mathrm{g}(t)$ in the book). That is, the step response is the integral of the impulse response. The impulse response of a unity-gain ideal lowpass filter with no delay is $\mathrm{h}(t) = 2B\mathrm{sinc}(2Bt)$ where B is its bandwidth. Its step response is therefore

$$\mathrm{h}_{-1}(t) = \int_{-\infty}^{t} 2B\mathrm{sinc}(2B\lambda)\,d\lambda = 2B\left[\int_{-\infty}^{0} \mathrm{sinc}(2B\lambda)\,d\lambda + \int_{0}^{t} \mathrm{sinc}(2B\lambda)\,d\lambda\right]$$

This result can be further simplified by using the definition of the **sine integral function**

$$\mathrm{Si}(\theta) \triangleq \int_{0}^{\theta} \frac{\sin(\alpha)}{\alpha}\,d\alpha = \pi \int_{0}^{\theta/\pi} \mathrm{sinc}(\lambda)\,d\lambda$$

Filters and Filtering



Filters and Filtering

$$\mathrm{h}_{-1}(t) = 2B\left[\int_{-\infty}^{0} \mathrm{sinc}(2B\lambda)\,d\lambda + \int_{0}^{t} \mathrm{sinc}(2B\lambda)\,d\lambda\right]$$

Let $2B\lambda = \alpha$. Then $\mathrm{h}_{-1}(t) = \int_{-\infty}^{0} \mathrm{sinc}(\alpha)\,d\alpha + \int_{0}^{2Bt} \mathrm{sinc}(\alpha)\,d\alpha$.

Using the fact that sinc is an even function, $\int_{-\infty}^{0} \mathrm{sinc}(\alpha)\,d\alpha = \int_{0}^{\infty} \mathrm{sinc}(\alpha)\,d\alpha$.

Then, using $\mathrm{Si}(\theta) = \pi \int_{0}^{\theta/\pi} \mathrm{sinc}(\alpha)\,d\alpha$ and $\mathrm{Si}(\infty) = \pi / 2$, we get

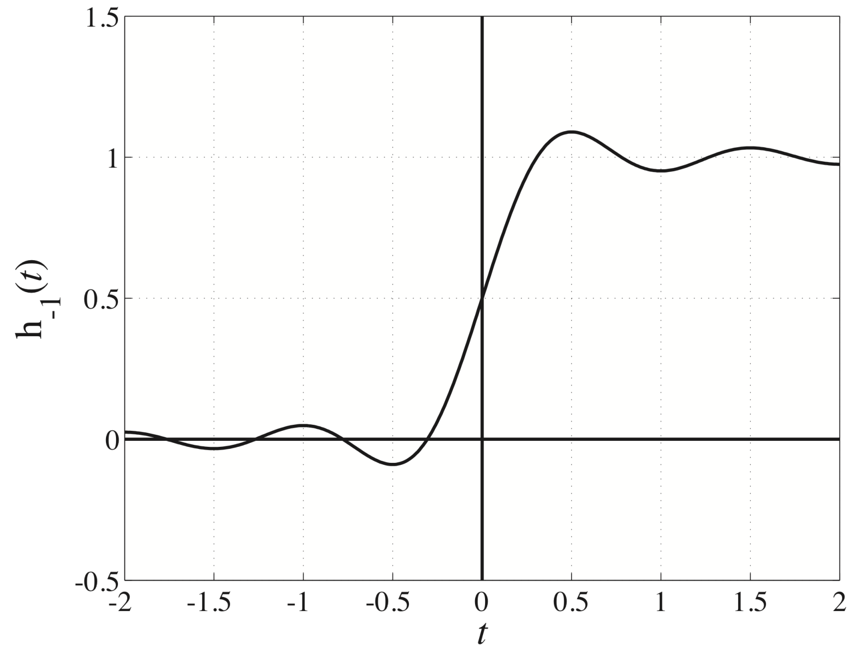
$$\mathrm{h}_{-1}(t) = \frac{\mathrm{Si}(\infty)}{\pi} + \frac{1}{\pi}\mathrm{Si}(2\pi Bt) = \frac{1}{2} + \frac{1}{\pi}\mathrm{Si}(2\pi Bt)$$

Filters and Filtering

$$h_{-1}(t) = \frac{1}{2} + \frac{1}{\pi}\mathrm{Si}(2\pi Bt)$$

This step response has **precursors**, **overshoot**, and **oscillations** (**ringing**). **Risetime** is defined as the time required to move from 10% of the final value to 90% of the final value. For this ideal lowpass filter the rise time is $0.44/B$. The rise time for a single-pole, lowpass filter is $0.35/B$.

Step response of an Ideal Lowpass Filter with $B = 1$ →

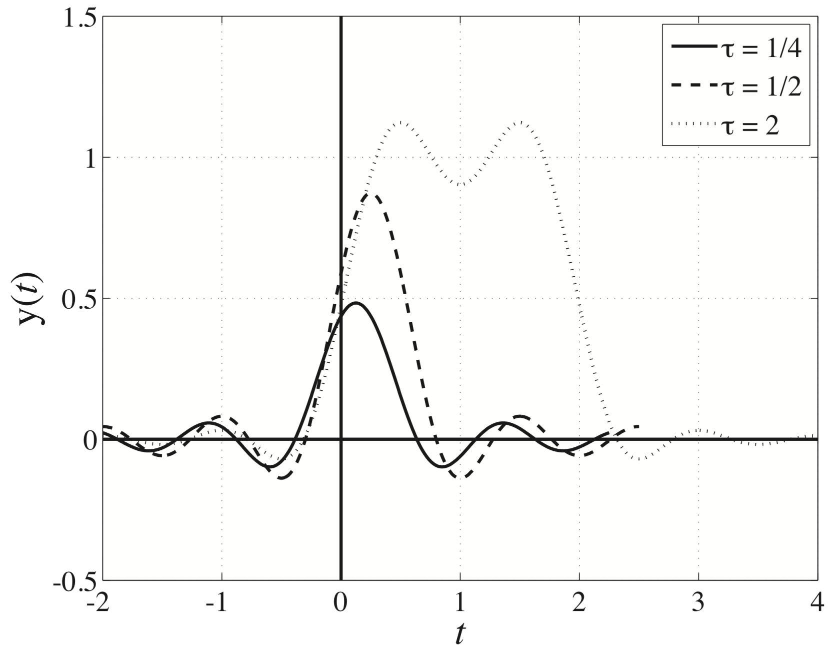


Filters and Filtering

The response of an ideal lowpass filter to a rectangular pulse of width τ is

$$y(t) = h_{-1}(t) - h_{-1}(t-\tau) = \frac{1}{\pi}\left[\mathrm{Si}(2\pi Bt) - \mathrm{Si}(2\pi B(t-\tau))\right].$$

From the graph (in which $B = 1$) we see that, to reproduce the rectangular pulse shape, even very crudely, requires a bandwidth much greater than $1/\tau$. If we have a pulse train with pulse widths τ and spaces between pulses also τ and we want to simply detect whether or not a pulse is present at some time, we will need at least $B \geq 1/2\tau$. If the bandwidth is any lower the overlap between pulses makes them very hard to resolve.



Quadrature Filters and Hilbert Transforms

A **quadrature filter** is an allpass network that shifts the phase of positive frequency components by $-90°$ and negative frequency components by $+90°$. Its frequency response is therefore

$$\mathrm{H}_Q(f) = \begin{cases} -j & , f > 0 \\ j & , f < 0 \end{cases} = -j\,\mathrm{sgn}(f).$$

Its magnitude is one at all frequencies, therefore an even function of f and its phase is an odd function of f. The inverse Fourier transform of $\mathrm{H}_Q(f)$ is the impulse response $\mathrm{h}_Q(t) = 1/\pi t$. The **Hilbert transform** $\hat{\mathrm{x}}(t)$ of a signal $\mathrm{x}(t)$ is defined as the response of a quadrature filter to $\mathrm{x}(t)$. That is $\hat{\mathrm{x}}(t) = \mathrm{x}(t) * \mathrm{h}_Q(t) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{x}(\lambda)}{t-\lambda}\,d\lambda.$

$$\mathscr{F}\left(\hat{\mathrm{x}}(t)\right) = -j\,\mathrm{sgn}(f)\,\mathrm{X}(f)$$

Quadrature Filters and Hilbert Transforms

The impulse response of a quadrature filter $\mathrm{h}_Q(t) = 1/\pi t$ is non-causal. That means it is physically unrealizable. Some important properties of the Hilbert transform are

1. The Fourier transforms of a signal and its Hilbert transform have the same magnitude. Therefore the signal and its Hilbert transform have the same signal energy.
2. If $\hat{\mathrm{x}}(t)$ is the Hilbert transform of $\mathrm{x}(t)$ then $-\mathrm{x}(t)$ is the Hilbert transform of $\hat{\mathrm{x}}(t)$.
3. A signal $\mathrm{x}(t)$ and its Hilbert transform are orthogonal on the entire real line. That means for energy signals $\int_{-\infty}^{\infty} \mathrm{x}(t)\hat{\mathrm{x}}(t)\,dt = 0$ and for power signals $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{x}(t)\hat{\mathrm{x}}(t)\,dt = 0$.

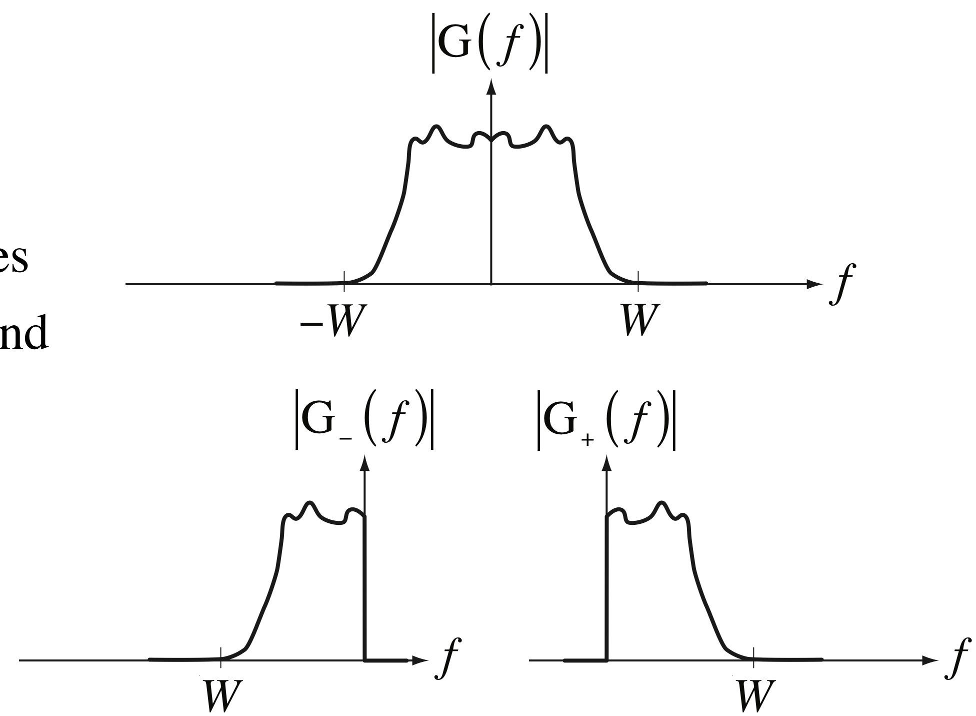
Quadrature Filters and Hilbert Transforms

The Hilbert transform will appear later in the exploration of single-sideband modulation. Let $\mathrm{g}(t)$ be a real signal and let $\mathrm{g}_+(t) = (1/2)\left[\mathrm{g}(t) + j\hat{\mathrm{g}}(t)\right]$ and let $\mathrm{g}_-(t) = (1/2)\left[\mathrm{g}(t) - j\hat{\mathrm{g}}(t)\right]$. Now look at their Fourier transforms.

$$\mathrm{G}_+(f) = (1/2)\left[\mathrm{G}(f) + j\left(-j\,\mathrm{sgn}(f)\right)\mathrm{G}(f)\right] = (1/2)\mathrm{G}(f)\left[1 + \mathrm{sgn}(f)\right] = \mathrm{G}(f)\mathrm{u}(f)$$

$$\mathrm{G}_-(f) = (1/2)\left[\mathrm{G}(f) - j\left(-j\,\mathrm{sgn}(f)\right)\mathrm{G}(f)\right] = (1/2)\mathrm{G}(f)\left[1 - \mathrm{sgn}(f)\right] = \mathrm{G}(f)\mathrm{u}(-f)$$

So $\mathrm{G}_+(f)$ is the positive-frequency half of $\mathrm{G}(f)$ and $\mathrm{G}_-(f)$ is the negative-frequency half of $\mathrm{G}(f)$. This separation of the spectrum of a signal into two halves will be useful in describing single-sideband modulation later.



Quadrature Filters and Hilbert Transforms

$\underline{\mathrm{g}(t)}$	$\underline{\hat{\mathrm{g}}(t)}$
$a_1\,\mathrm{g}_1(t)+a_2\,\mathrm{g}_2(t);\ a_1,a_2\in\mathbb{C}$	$a_1\,\hat{\mathrm{g}}_1(t)+a_2\,\hat{\mathrm{g}}_2(t)$
$\mathrm{h}(t-t_0)$	$\hat{\mathrm{h}}(t-t_0)$
$\mathrm{h}(at);a\neq 0$	$\mathrm{sgn}(a)\hat{\mathrm{h}}(at)$
$\frac{d}{dt}(\mathrm{h}(t))$	$\frac{d}{dt}(\hat{\mathrm{h}}(t))$
$\delta(t)$	$\frac{1}{\pi t}$
e^{jt}	$-je^{jt}$
e^{-jt}	je^{-jt}
$\cos(t)$	$\sin(t)$
$\mathrm{rect}(t)$	$\frac{1}{\pi}\ln\left\lvert\frac{2t+1}{2t-1}\right\rvert$
$\mathrm{sinc}(t)$	$(\pi t/2)\mathrm{sinc}^2(t/2)=\sin(\pi t/2)\mathrm{sinc}(t/2)$
$\frac{1}{1+t^2}$	$\frac{t}{1+t^2}$

Distribution Functions

The distribution function of a random variable X is the probability that it is less than or equal to some value, as a function of that value.

$$\mathrm{F}_X(x) = \mathrm{P}[X \le x]$$

Since the distribution function is a probability it must satisfy the requirements for a probability.

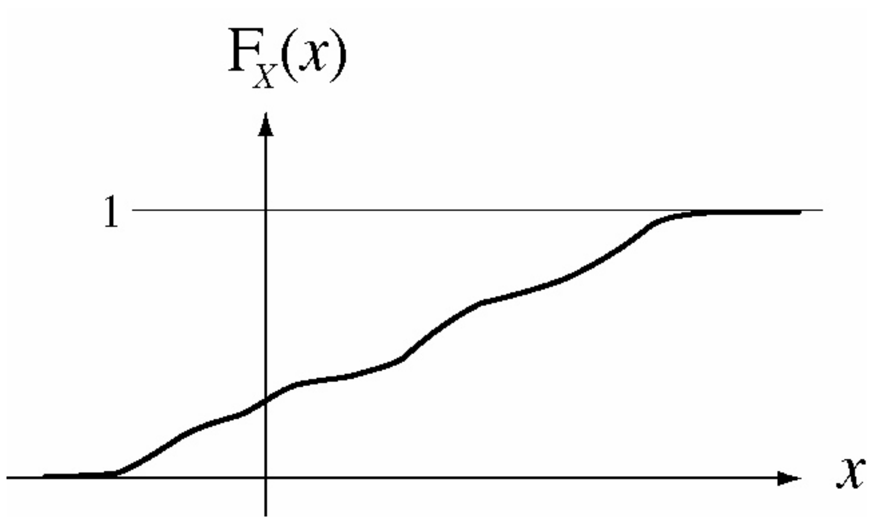
$$0 \le \mathrm{F}_X(x) \le 1 \quad , \quad -\infty < x < \infty$$

$$\mathrm{P}[x_1 < X \le x_2] = \mathrm{F}_X(x_2) - \mathrm{F}_X(x_1)$$

$\mathrm{F}_X(x)$ is a monotonic function and its derivative is never negative.

Distribution Functions

A possible distribution function for a continuous random variable.



Probability Density

The derivative of the distribution function is the **probability density function** (**PDF**)

$$\mathrm{p}_X(x) \equiv \frac{d}{dx}\left(\mathrm{F}_X(x)\right)$$

Probability density can also be defined by

$$\mathrm{p}_X(x)\,dx = \mathrm{P}\left[x < X \le x + dx\right]$$

Properties

$$\mathrm{p}_X(x) \ge 0 \quad , \quad -\infty < x < +\infty \qquad\qquad \int_{-\infty}^{\infty} \mathrm{p}_X(x)\,dx = 1$$

$$\mathrm{F}_X(x) = \int_{-\infty}^{x} \mathrm{p}_X(\lambda)\,d\lambda \qquad\qquad \mathrm{P}\left[x_1 < X \le x_2\right] = \int_{x_1}^{x_2} \mathrm{p}_X(x)\,dx$$

Expectation and Moments

The first moment of a random variable is its expected value

$\mathrm{E}(X) = \int_{-\infty}^{\infty} x\,\mathrm{p}_X(x)\,dx$. The second moment of a random variable is its mean-squared value (which is the mean of its square, not the square of its mean).

$$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2\,\mathrm{p}_X(x)\,dx$$

Expectation and Moments

A **central moment** of a random variable is the moment of that random variable after its expected value is subtracted.

$$\mathrm{E}\left(\left[X-\mathrm{E}(X)\right]^n\right)=\int_{-\infty}^{\infty}\left[x-\mathrm{E}(X)\right]^n \mathrm{p}_X(x)\,dx$$

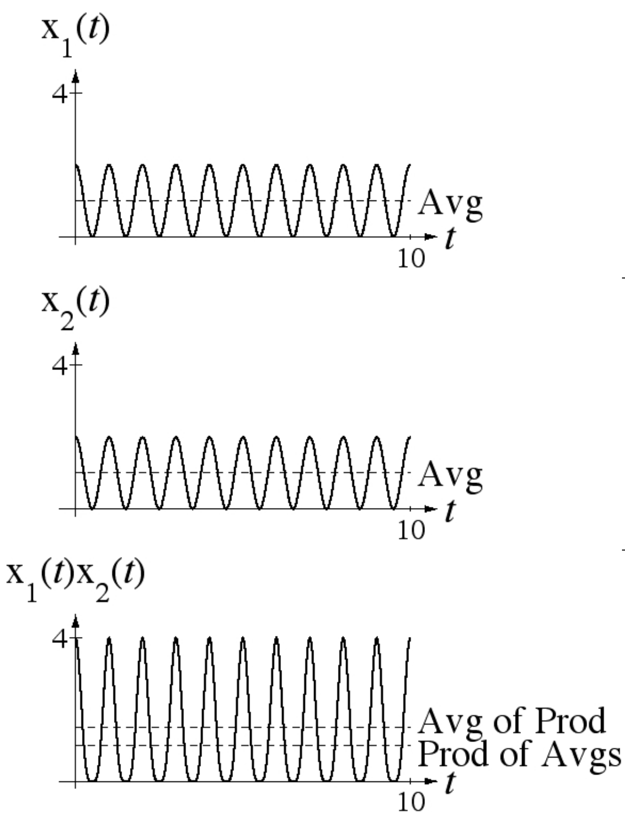
The first central moment is always zero. The second central moment (for real-valued random variables) is the **variance**,

$$\sigma_X^2=\mathrm{E}\left(\left[X-\mathrm{E}(X)\right]^2\right)=\int_{-\infty}^{\infty}\left[x-\mathrm{E}(X)\right]^2 \mathrm{p}_X(x)\,dx$$

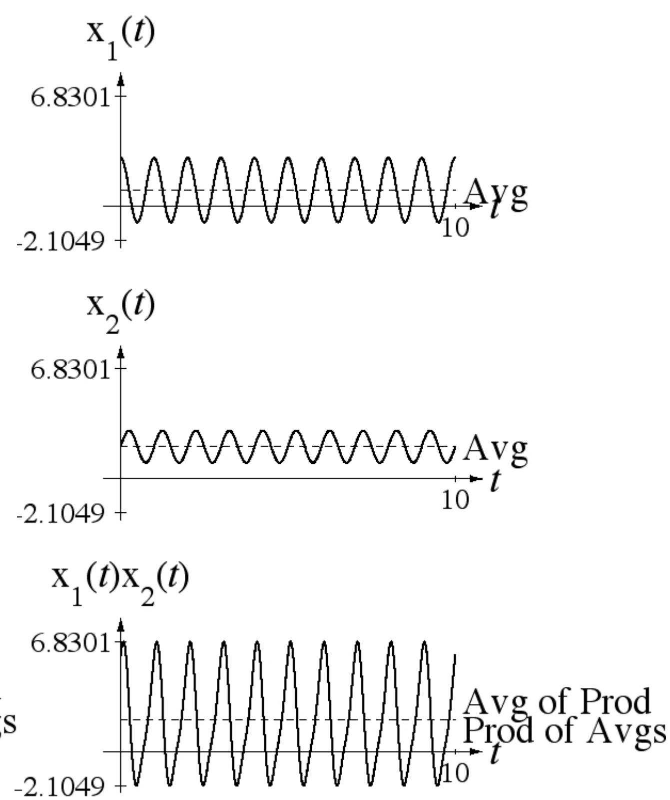
The positive square root of the variance is the **standard deviation**.

Correlation

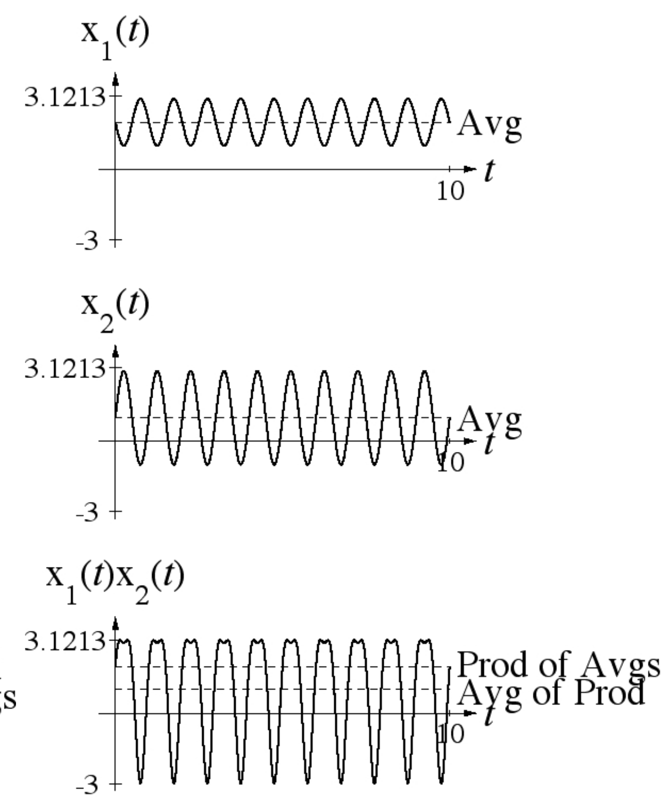
Positively Correlated Sinusoids with Non-Zero Mean



Uncorrelated Sinusoids with Non-Zero Mean



Negatively Correlated Sinusoids with Non-Zero Mean



Correlation

Let $\mathrm{v}(t)$ be a power signal, not necessarily real-valued or periodic, but with a well-defined average signal power

$$P_{\mathrm{v}} \triangleq \left\langle \left|\mathrm{v}(t)\right|^2 \right\rangle = \left\langle \mathrm{v}(t)\mathrm{v}^*(t) \right\rangle \geq 0$$

where $\langle \cdot \rangle$ means "time average of" and mathematically means

$$\left\langle \mathrm{z}(t) \right\rangle = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathrm{z}(t)\,dt.$$

Time averaging has the properties $\left\langle \mathrm{z}^*(t) \right\rangle = \left\langle \mathrm{z}(t) \right\rangle^*$, $\left\langle \mathrm{z}(t - t_d) \right\rangle = \left\langle \mathrm{z}(t) \right\rangle$ for any t_d and $\left\langle a_1 \mathrm{z}_1(t) + a_2 \mathrm{z}_2(t) \right\rangle = a_1 \left\langle \mathrm{z}_1(t) \right\rangle + a_2 \left\langle \mathrm{z}_2(t) \right\rangle$. If $\mathrm{v}(t)$ and $\mathrm{w}(t)$ are power signals, $\left\langle \mathrm{v}(t)\mathrm{w}^*(t) \right\rangle$ is the **scalar product** of $\mathrm{v}(t)$ and $\mathrm{w}(t)$. The scalar product is a measure of the similarity between two signals.

Correlation

Let $\mathrm{z}(t) = \mathrm{v}(t) - a\mathrm{w}(t)$ with a real. Then the average power of $\mathrm{z}(t)$ is

$$P_{\mathrm{z}} = \langle \mathrm{z}(t)\mathrm{z}^*(t) \rangle = \langle [\mathrm{v}(t) - a\mathrm{w}(t)][\mathrm{v}^*(t) - a^*\mathrm{w}^*(t)] \rangle .$$

Expanding,

$$P_{\mathrm{z}} = \langle \mathrm{v}(t)\mathrm{v}^*(t) - a\mathrm{w}(t)\mathrm{v}^*(t) - \mathrm{v}(t)a^*\mathrm{w}^*(t) + a^2\,\mathrm{w}(t)\mathrm{w}^*(t) \rangle$$

Using the fact that $a\mathrm{w}(t)\mathrm{v}^*(t)$ and $\mathrm{v}(t)a^*\mathrm{w}^*(t)$ are complex conjugates, and that the sum of a complex number and its complex conjugate is twice the real part of either one,

$$P_{\mathrm{z}} = P_{\mathrm{v}} + a^2 P_{\mathrm{w}} - 2a\,\mathrm{Re}\left[\langle \mathrm{v}(t)\mathrm{w}^*(t) \rangle\right] = P_{\mathrm{v}} + a^2 P_{\mathrm{w}} - 2aR_{vw}$$