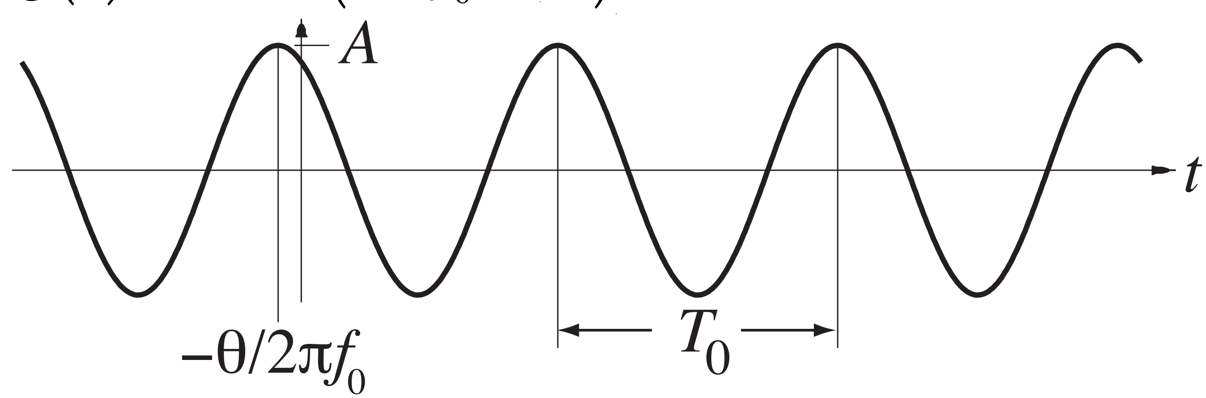
Signals and Systems Review

Continuous-Time Sinusoids

$$g(t) = A\cos(2\pi t / T_0 + \theta) = A\cos(2\pi f_0 t + \theta) = A\cos(\omega_0 t + \theta)$$

- A ↑ Amplitude
- T_0 ↑ Period (s)
- θ ↑ Phase Shift (radians)
- f_0 ↑ Cyclic Frequency (Hz)
- ω_0 ↑ Radian Frequency (radians/s) $(\omega_0 = 2\pi f_0)$

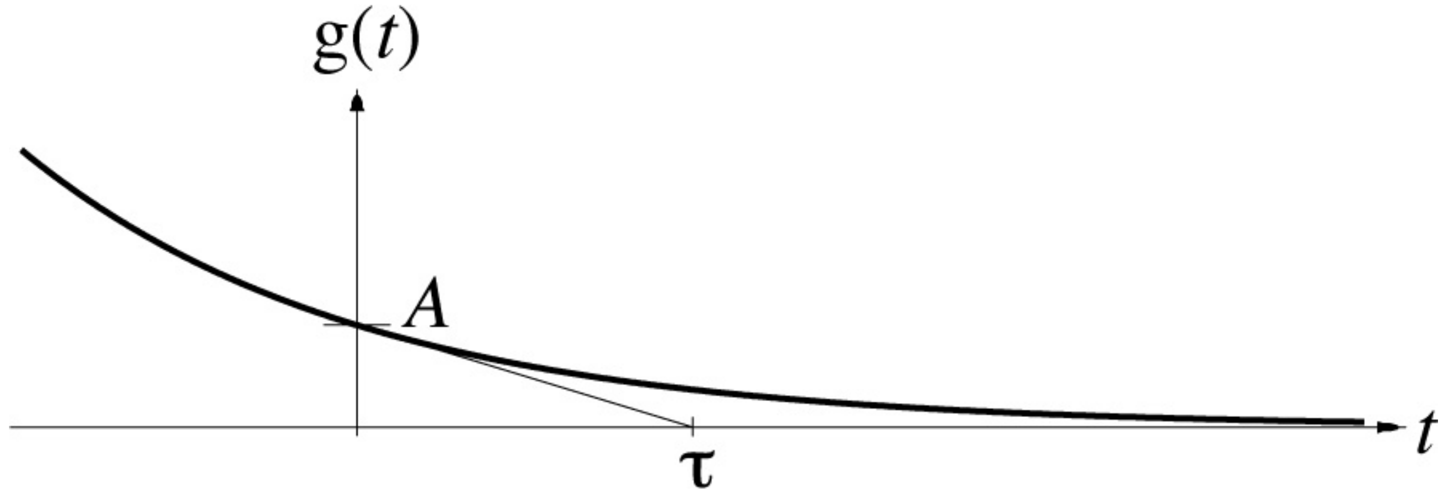
$$g(t) = A\cos(2\pi f_0 t + \theta)$$



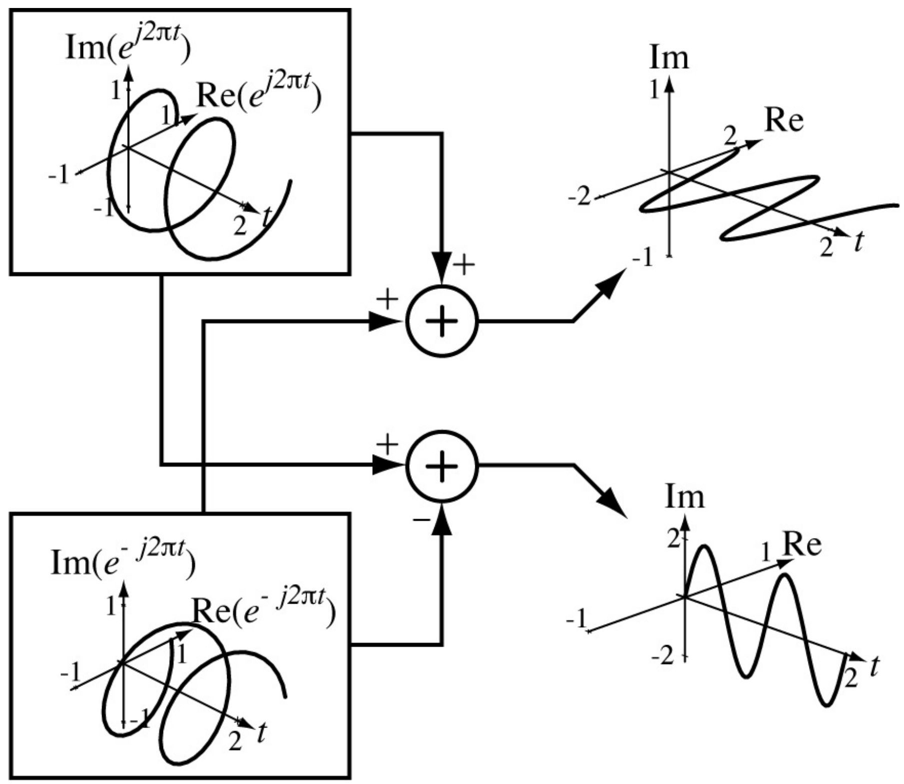
Continuous-Time Exponentials

$$\mathrm{g}(t) = Ae^{-t/\tau}$$

$\uparrow$ Amplitude $\uparrow$ Time Constant (s)



Complex Sinusoids

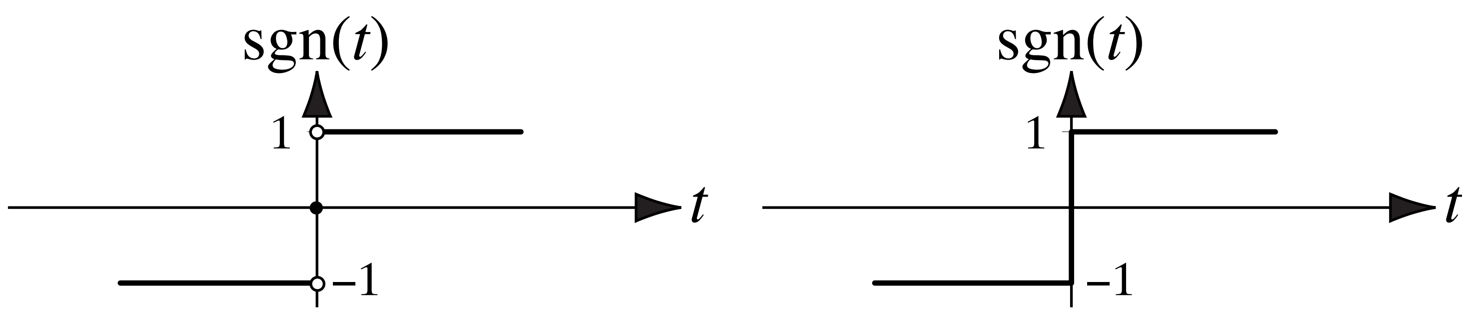


Euler's Identity: $e^{jx} = \cos(x) + j\sin(x)$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \quad , \quad \sin(x) = \frac{e^{jx} - e^{-jx}}{j2}$$

The Signum Function

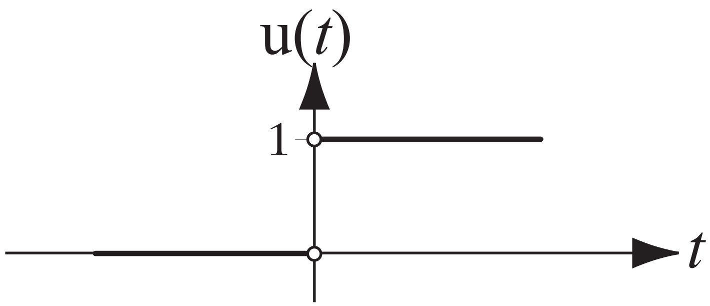
$$\operatorname{sgn}(t) = \begin{cases} 1 & , t > 0 \\ 0 & , t = 0 \\ -1 & , t < 0 \end{cases}$$

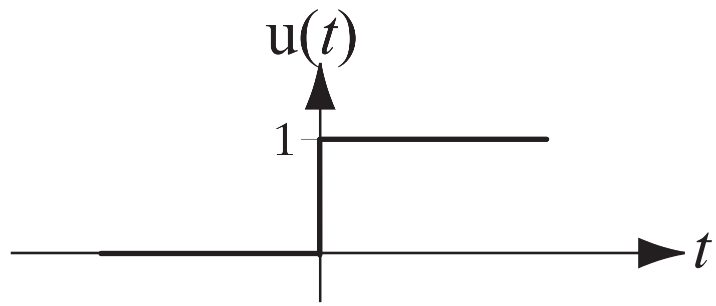


The signum function, in a sense, returns an indication of the sign of its argument.

The Unit Step Function

$$\mathrm{u}(t) = \begin{cases} 1 & , \ t > 0 \\ 0 & , \ t < 0 \\ \text{Undefined} & , \ t = 0 \\ \text{(but finite)} & \end{cases} \quad , \quad \mathrm{u}(t) = \frac{1}{2}\left[\mathrm{sgn}(t) + 1\right] \quad , \ t \neq 0$$

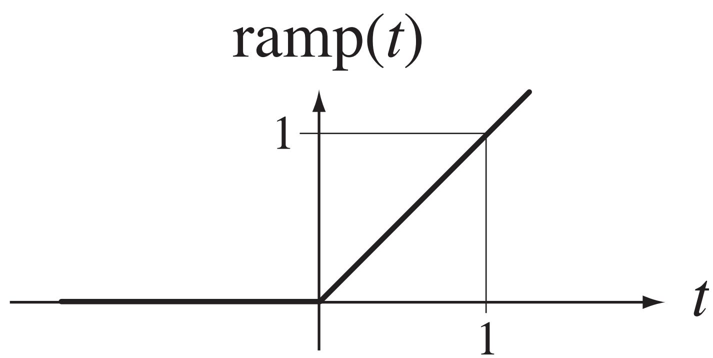




The product signal $\mathrm{g}(t)\mathrm{u}(t)$ can be thought of as the signal $\mathrm{g}(t)$ "turned on" at time $t = 0$.

The Unit Ramp Function

$$\mathrm{ramp}(t) = \begin{cases} t & , \ t > 0 \\ 0 & , \ t \le 0 \end{cases} = \int_{-\infty}^{t} \mathrm{u}(\lambda)\, d\lambda = t\,\mathrm{u}(t)$$



The Impulse

The continuous-time unit impulse is implicitly defined by

$$\mathrm{g}(0) = \int_{-\infty}^{\infty} \delta(t)\,\mathrm{g}(t)\,dt$$

The unit step is the integral of the unit impulse and the unit impulse is the generalized derivative of the unit step.

Properties of the Impulse

The Sampling Property

$$\int_{-\infty}^{\infty} g(t)\delta(t-t_0)\,dt = g(t_0)$$

The sampling property "extracts" the value of a function at a point. (In Ziemer and Tranter this is called the "sifting" property.)

The Scaling Property

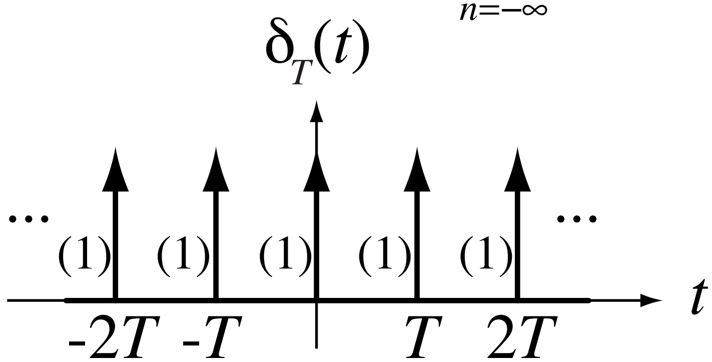
$$\delta(a(t-t_0)) = \frac{1}{|a|}\delta(t-t_0)$$

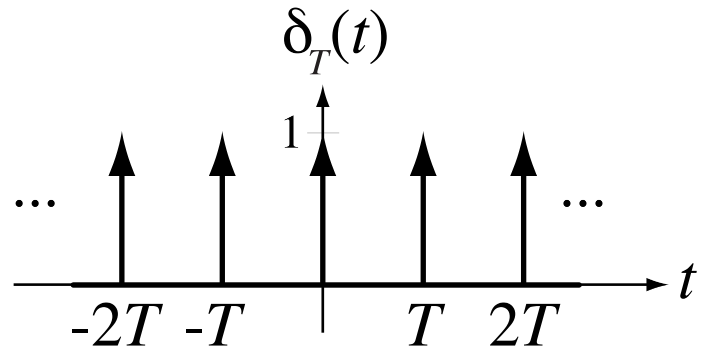
This property illustrates that the impulse is different from ordinary mathematical functions.

The Unit Periodic Impulse

The unit periodic impulse is defined by

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad , \; n \text{ an integer}$$



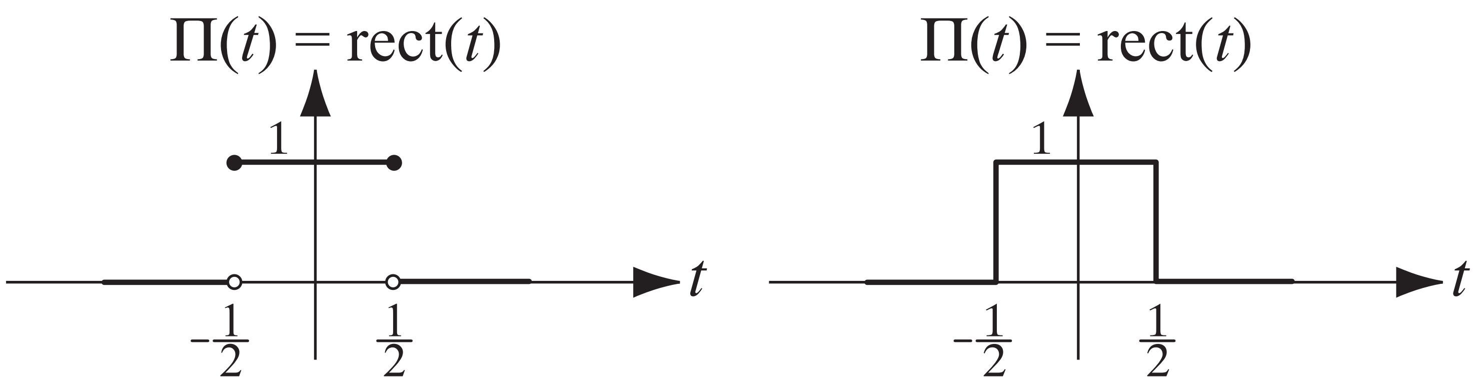


The periodic impulse is a sum of infinitely many impulses uniformly-spaced apart by T.

$$\delta_T\left(a(t - t_0)\right) = \frac{1}{|a|}\delta_{T/a}(t - t_0) \quad , \; n \text{ an integer}$$

The Unit Rectangle Function

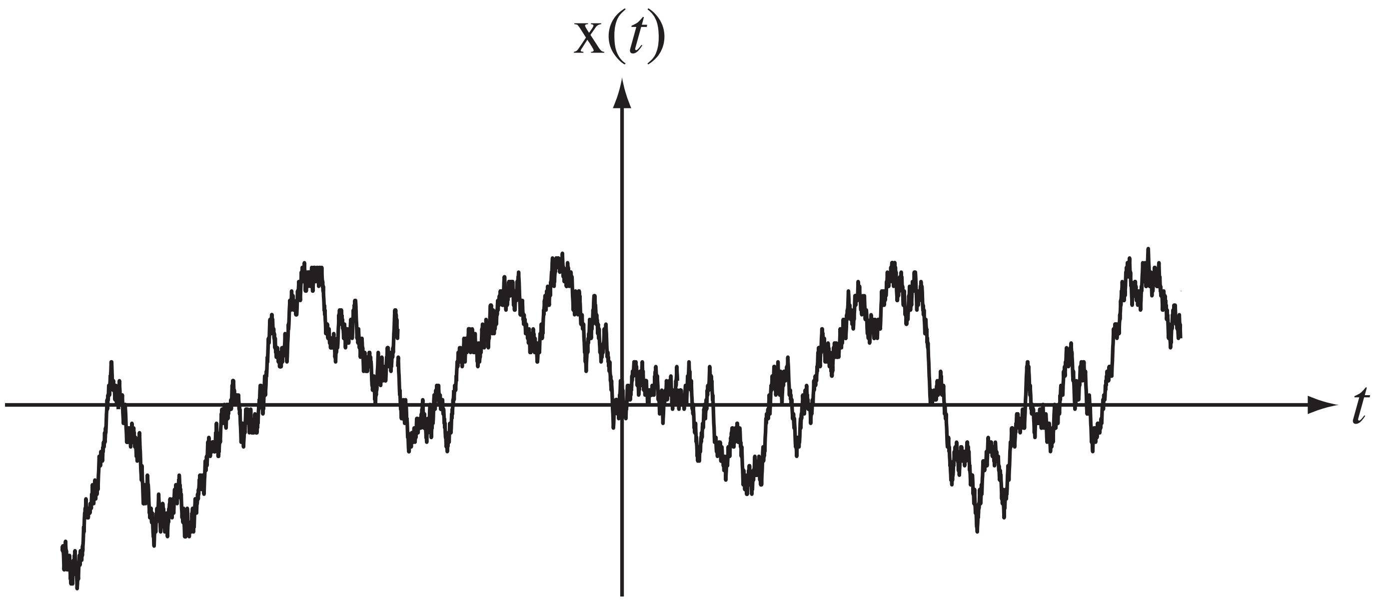
$$\Pi(t) = \text{rect}(t) = \begin{Bmatrix} 1 & , & |t| \le 1/2 \\ 0 & , & |t| > 1/2 \end{Bmatrix} = \text{u}(t+1/2) - \text{u}(t-1/2) \quad , \quad |t| \ne \frac{1}{2}$$



The product signal $\text{g}(t)\text{rect}(t)$ can be thought of as the signal $\text{g}(t)$ "turned on" at time $t = -1/2$ and "turned back off" at time $t = +1/2$.

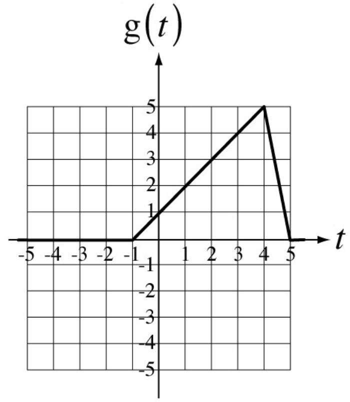
Random Signals

The sinusoid, exponential, signum, unit step, unit ramp, and unit rectangle are all **deterministic** signals. The term deterministic means that their values are specified at all times. Signals that are not deterministic are **random**. The exact values of random signals are unpredictable although their general behavior may be known to some degree.

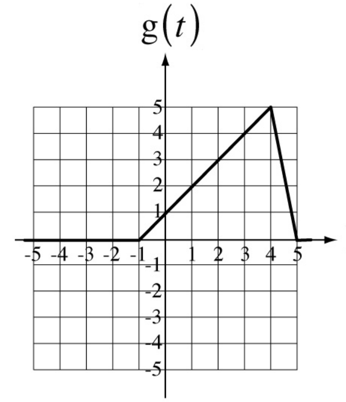


Shifting and Scaling Functions

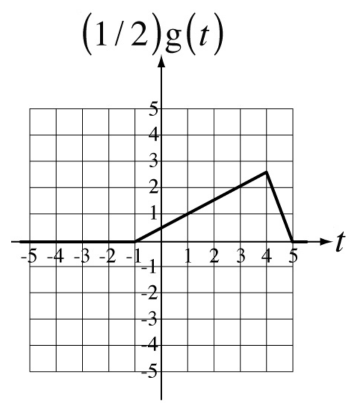
Amplitude Scaling, $g(t) \to A\,g(t)$



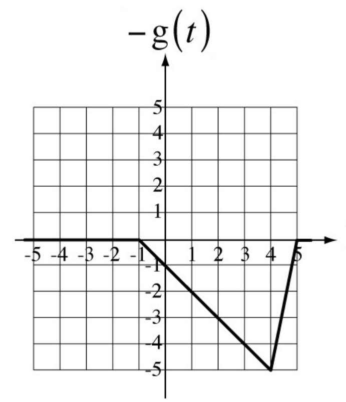
t	−5	−4	−3	−2	−1	0	1	2	3	4	5
$g(t)$	0	0	0	0	0	1	2	3	4	5	0



t	−5	−4	−3	−2	−1	0	1	2	3	4	5
$g(t)$	0	0	0	0	0	1	2	3	4	5	0



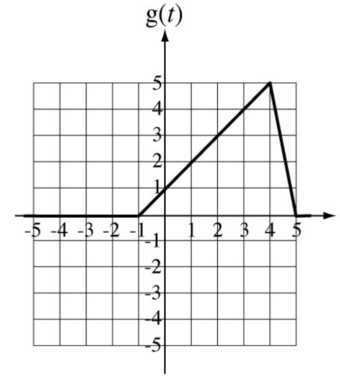
t	−5	−4	−3	−2	−1	0	1	2	3	4	5
$(1/2)g(t)$	0	0	0	0	0	1/2	1	3/2	2	5/2	0



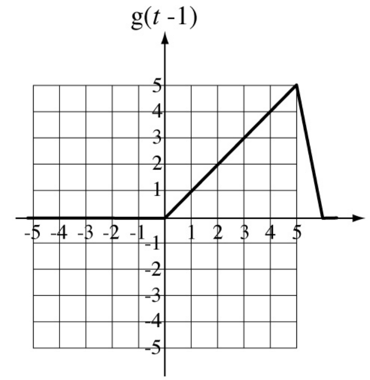
t	−5	−4	−3	−2	−1	0	1	2	3	4	5
$-g(t)$	0	0	0	0	0	−1	−2	−3	−4	−5	0

Shifting and Scaling Functions

Time shifting, $t \rightarrow t - t_0$



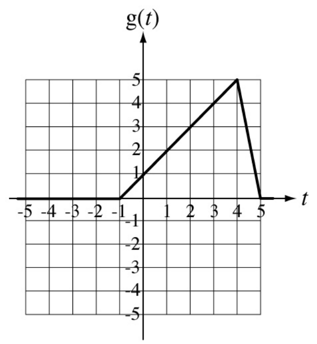
t	-5	-4	-3	-2	-1	0	1	2	3	4	5
$g(t)$	0	0	0	0	0	1	2	3	4	5	0



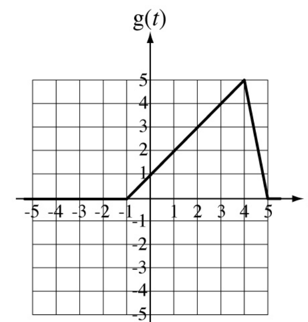
t	-5	-4	-3	-2	-1	0	1	2	3	4	5
$t-1$	-6	-5	-4	-3	-2	-1	0	1	2	3	4
$g(t-1)$	0	0	0	0	0	0	1	2	3	4	5

Shifting and Scaling Functions

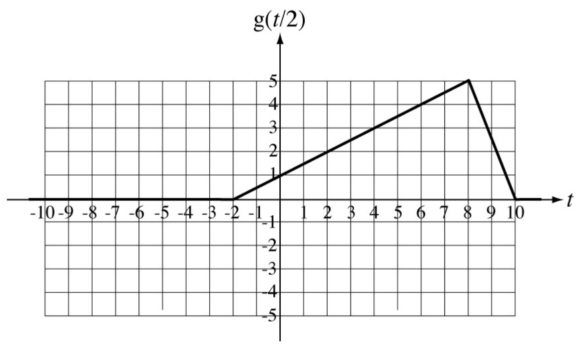
Time scaling, $t \rightarrow t/a$



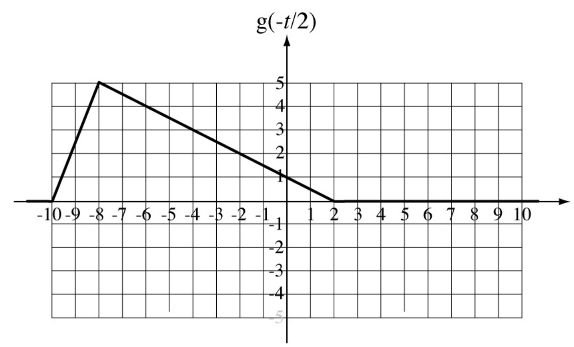
t	-5	-4	-3	-2	-1	0	1	2	3	4	5
$g(t)$	0	0	0	0	0	1	2	3	4	5	0



t	-5	-4	-3	-2	-1	0	1	2	3	4	5
$g(t)$	0	0	0	0	0	1	2	3	4	5	0

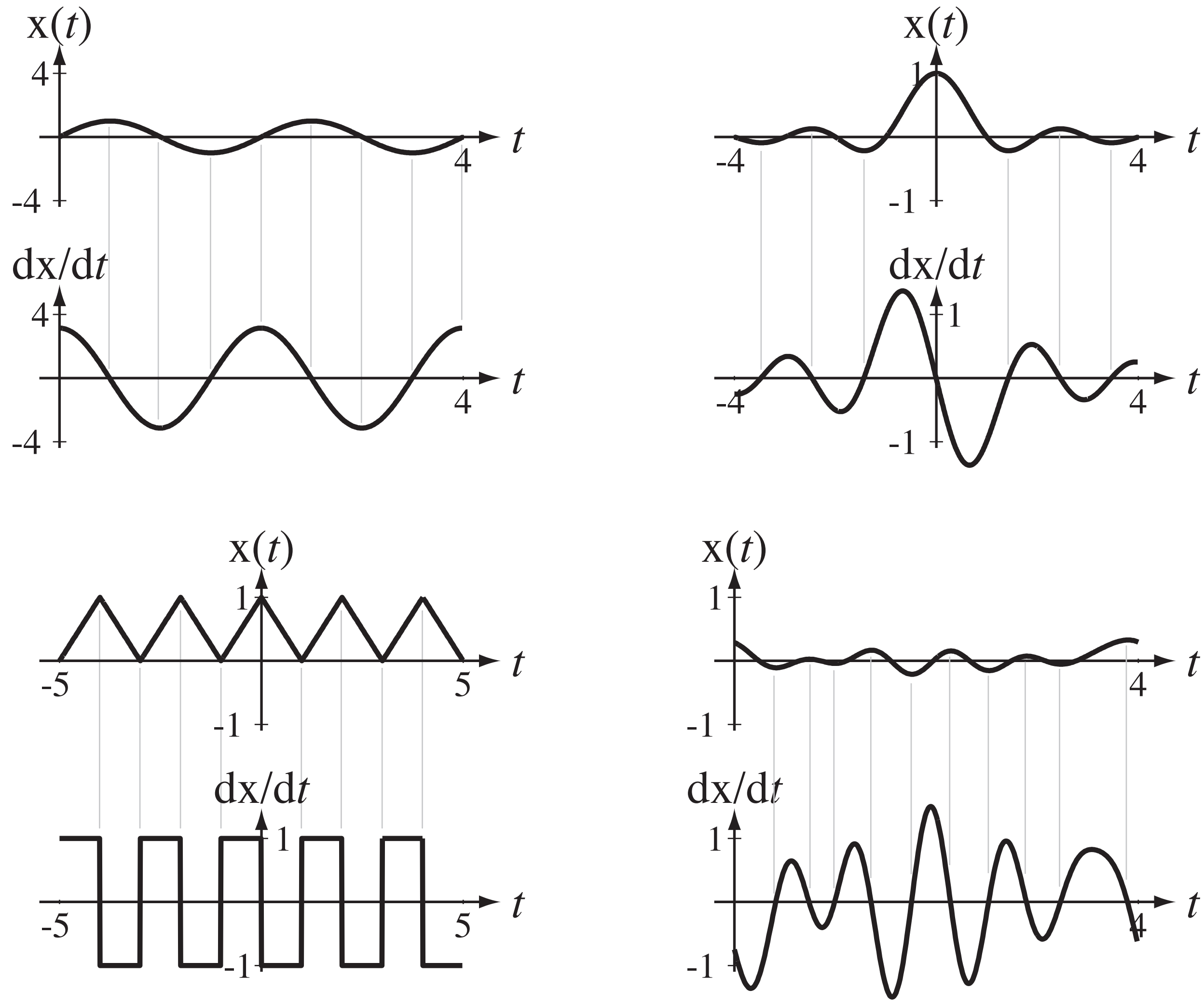


t	-10	-8	-6	-4	-2	0	2	4	6	8	10
$t/2$	-5	-4	-3	-2	-1	0	1	2	3	4	5
$g(t/2)$	0	0	0	0	0	1	2	3	4	5	0

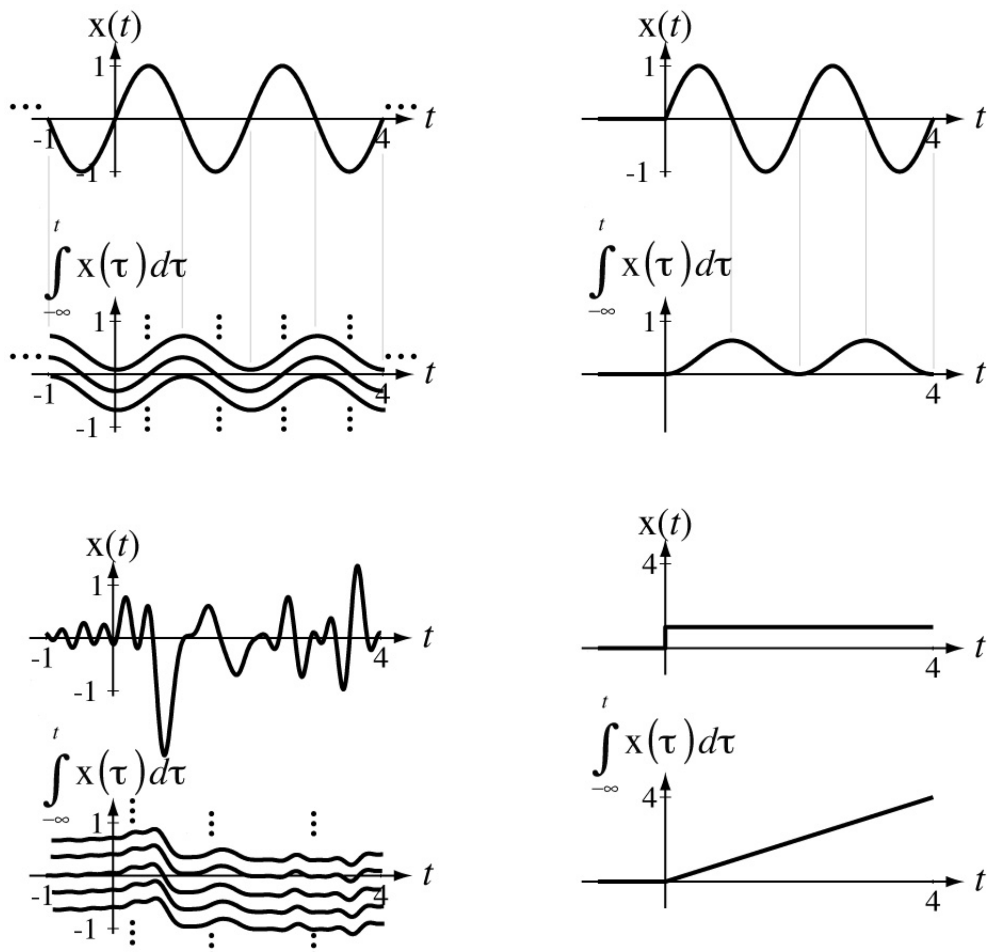


t	-10	-8	-6	-4	-2	0	2	4	6	8	10
$-t/2$	5	4	3	2	1	0	-1	-2	-3	-4	-5
$g(-t/2)$	0	5	4	3	2	1	0	0	0	0	0

Differentiation



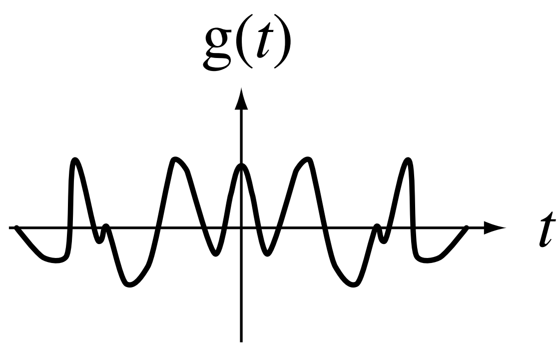
Integration



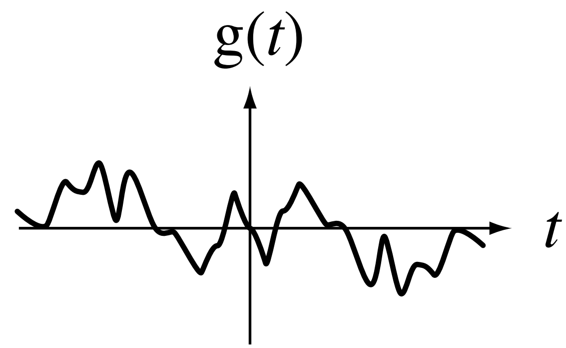
Even and Odd Signals

Even Functions	Odd Functions
$\mathrm{g}(t) = \mathrm{g}(-t)$	$\mathrm{g}(t) = -\mathrm{g}(-t)$

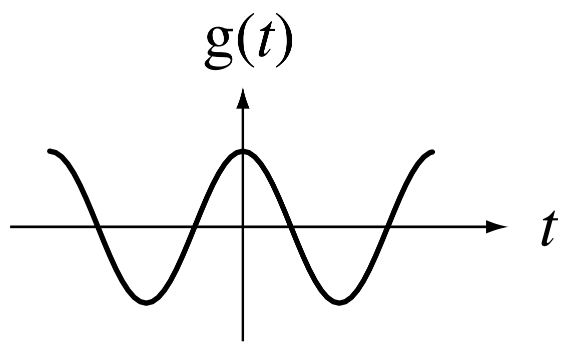
Even Function



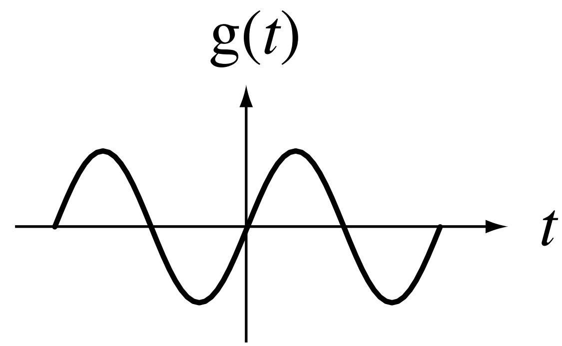
Odd Function



Even Function



Odd Function



Even and Odd Parts of Functions

The **even part** of a function is $g_e(t) = \frac{g(t) + g(-t)}{2}$.

The **odd part** of a function is $g_o(t) = \frac{g(t) - g(-t)}{2}$.

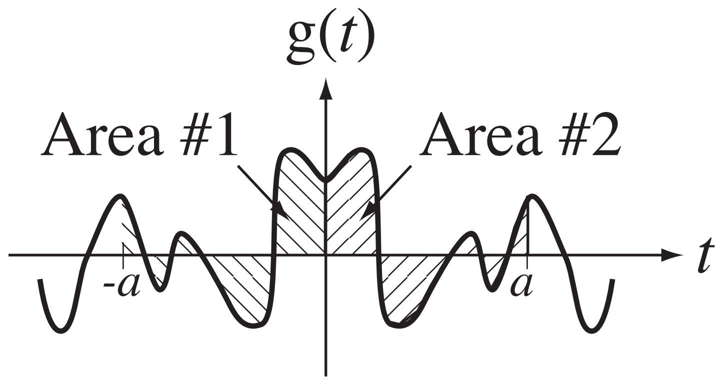
A function whose even part is zero is odd and a function whose odd part is zero is even.

The derivative of an even function is odd and the derivative of an odd function is even.

The integral of an even function is an odd function, plus a constant, and the integral of an odd function is even.

Integrals of Even and Odd Functions

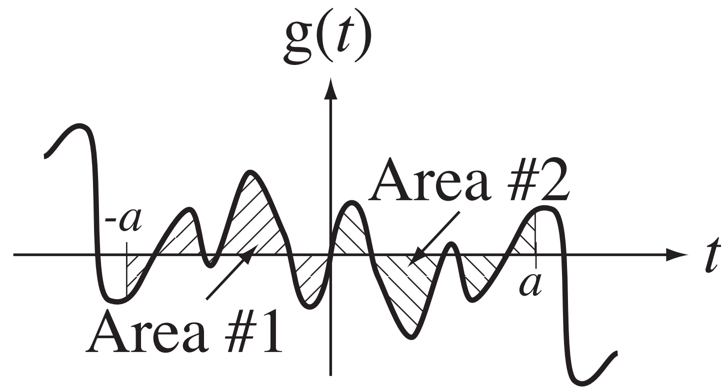
Even Function



Area #1 = Area #2

$$\int_{-a}^{a} g(t)\,dt = 2\int_{0}^{a} g(t)\,dt$$

Odd Function



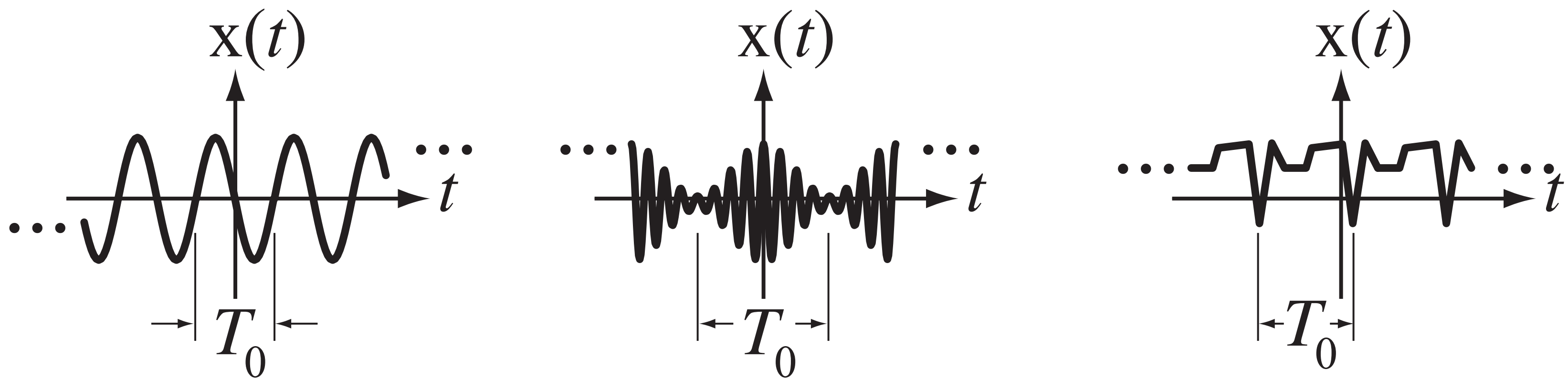
Area #1 = - Area #2

$$\int_{-a}^{a} g(t)\,dt = 0$$

The integral of an odd function, over limits that are symmetrical about zero, is zero.

Periodic Signals

If a function $\mathrm{g}(t)$ is **periodic**, $\mathrm{g}(t)=\mathrm{g}(t+nT)$ where n is any integer and T is a **period** of the function. The minimum positive value of T for which $\mathrm{g}(t)=\mathrm{g}(t+T)$ is called the **fundamental period** T_0 of the function. The reciprocal of the fundamental period is the **fundamental frequency** $f_0 = 1/T_0$.



A function that is not periodic is **aperiodic**.

Signal Energy and Power

The signal energy of a signal $\mathrm{x}(t)$ is

$$E_{\mathrm{x}} = \int_{-\infty}^{\infty} \left|\mathrm{x}(t)\right|^2 dt$$

Signal Energy and Power

Some signals have infinite signal energy. In that case it is more convenient to deal with average signal power. The average signal power of a signal $\mathrm{x}(t)$ is

$$P_{\mathrm{x}} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \left|\mathrm{x}(t)\right|^2 dt$$

For a periodic signal $\mathrm{x}(t)$ the average signal power is

$$P_{\mathrm{x}} = \frac{1}{T} \int_{T} \left|\mathrm{x}(t)\right|^2 dt$$

where T is any period of the signal.

Signal Energy and Power

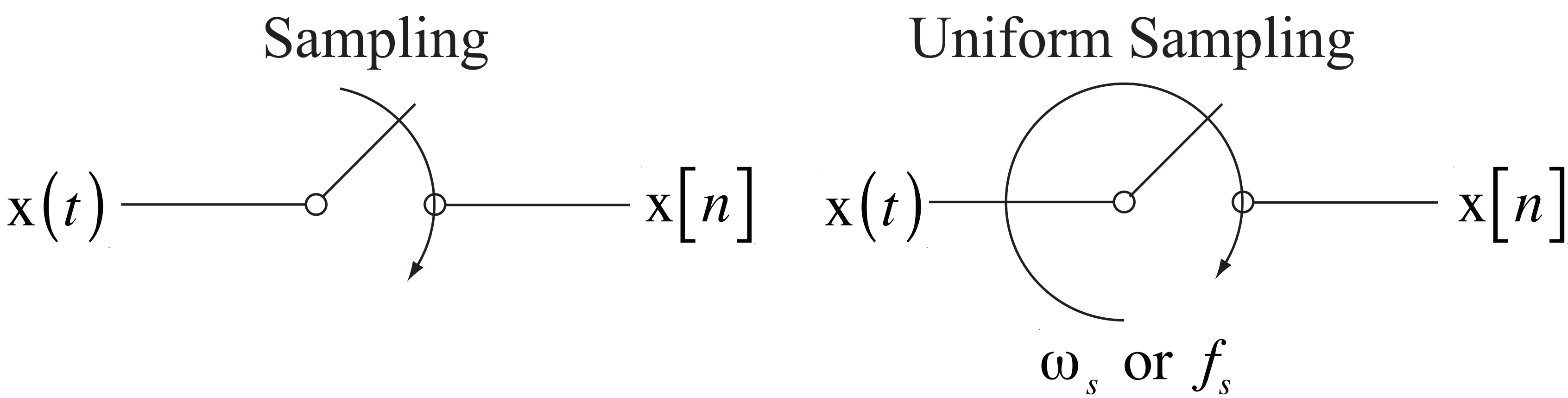
A signal with finite signal energy is called an **energy signal**.

A signal with infinite signal energy and finite average signal power is called a **power signal**.

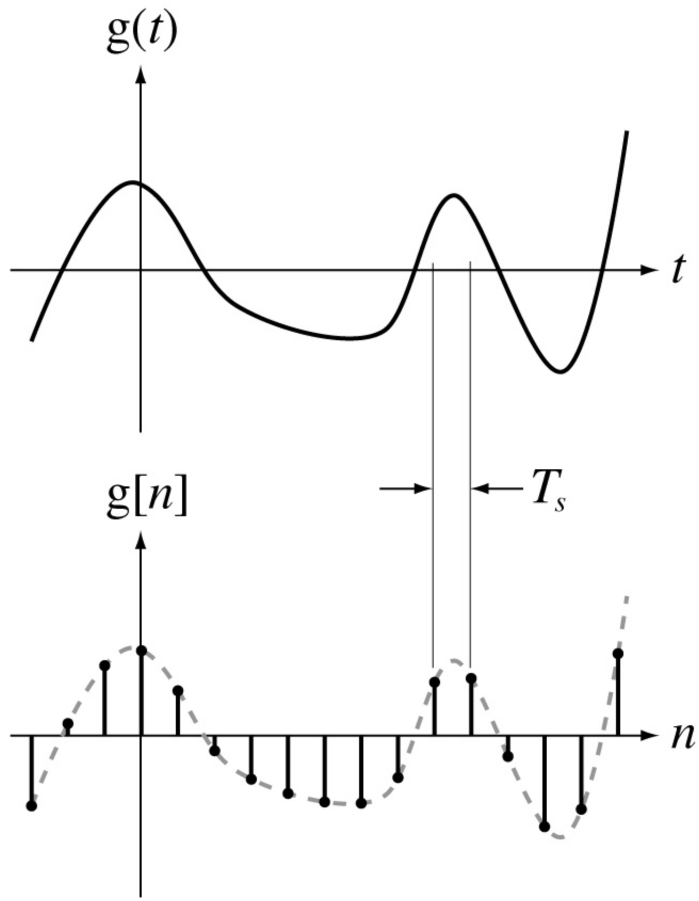
Sampling and Discrete Time

Sampling is the acquisition of the values of a continuous-time signal at discrete points in time. $x(t)$ is a continuous-time signal, $x[n]$ is a discrete-time signal.

$$x[n] = x(nT_s) \text{ where } T_s \text{ is the time between samples}$$



Sampling and Discrete Time

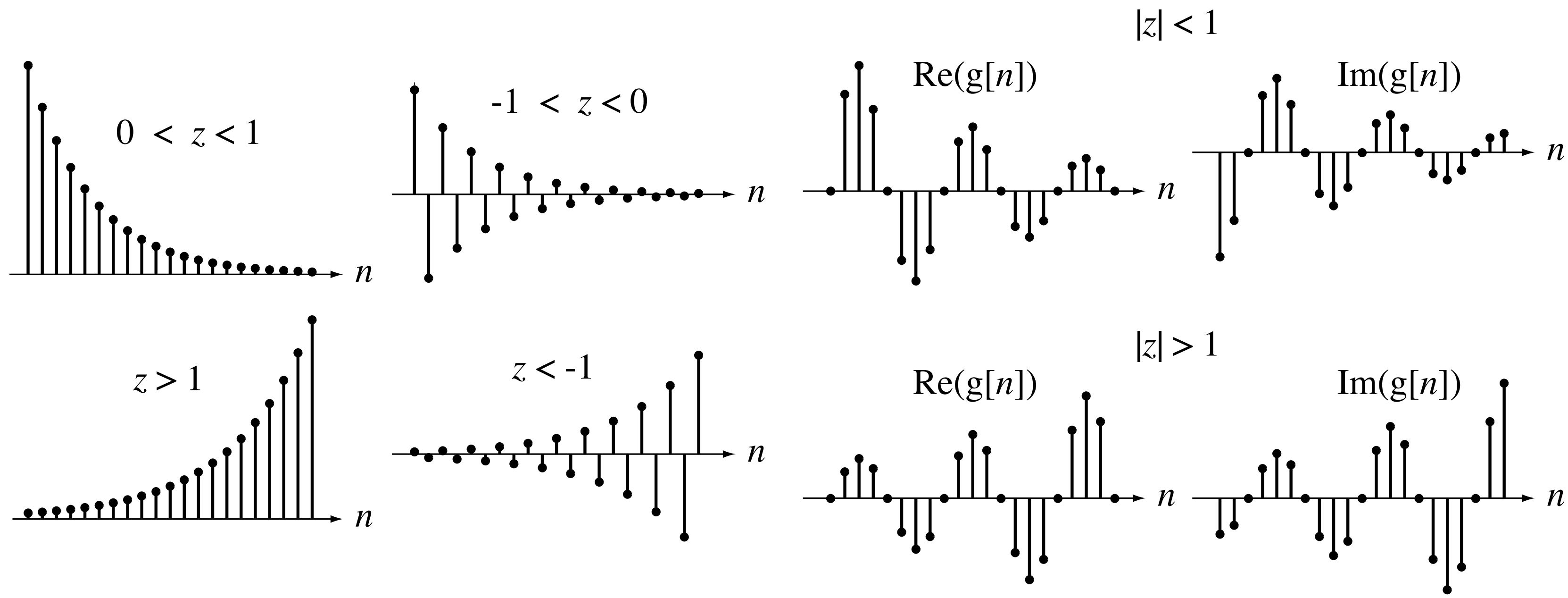


Exponentials

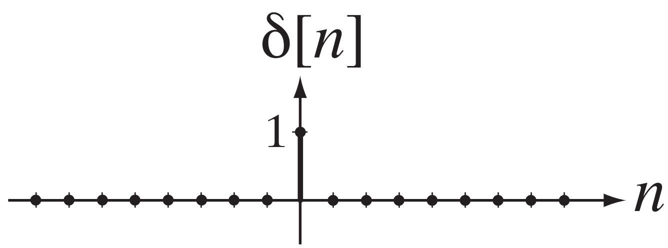
The form of the exponential is

$$\underbrace{\mathrm{x}[n] = A\alpha^n}_{\text{Preferred}} \quad \text{or} \quad \mathrm{x}[n] = Ae^{\beta n} \quad \text{where } \alpha = e^{\beta}$$

Real α | Complex α



The Unit Impulse Function



$$\delta[n] = \begin{cases} 1 \ , & n = 0 \\ 0 \ , & n \neq 0 \end{cases}$$

The discrete-time unit impulse (also known as the "**Kronecker delta function**") is a function in the ordinary sense (in contrast with the continuous-time unit impulse). It has a sampling property,

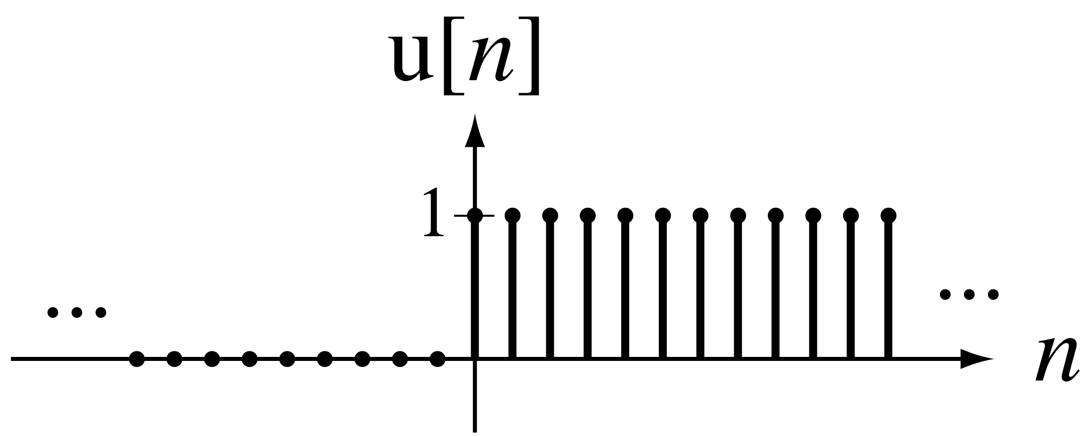
$$\sum_{n=-\infty}^{\infty} A\delta[n - n_0]\mathrm{x}[n] = A\,\mathrm{x}[n_0]$$

but no scaling property. That is,

$\delta[n] = \delta[an]$ for any non-zero, finite integer a.

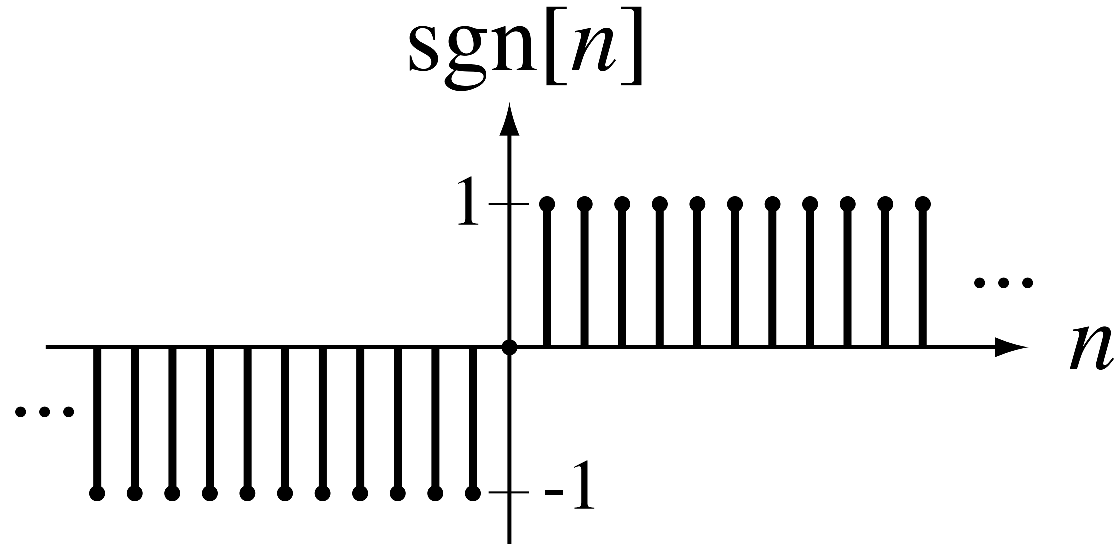
The Unit Sequence Function

$$\mathrm{u}[n] = \begin{cases} 1 \ , & n \geq 0 \\ 0 \ , & n < 0 \end{cases}$$



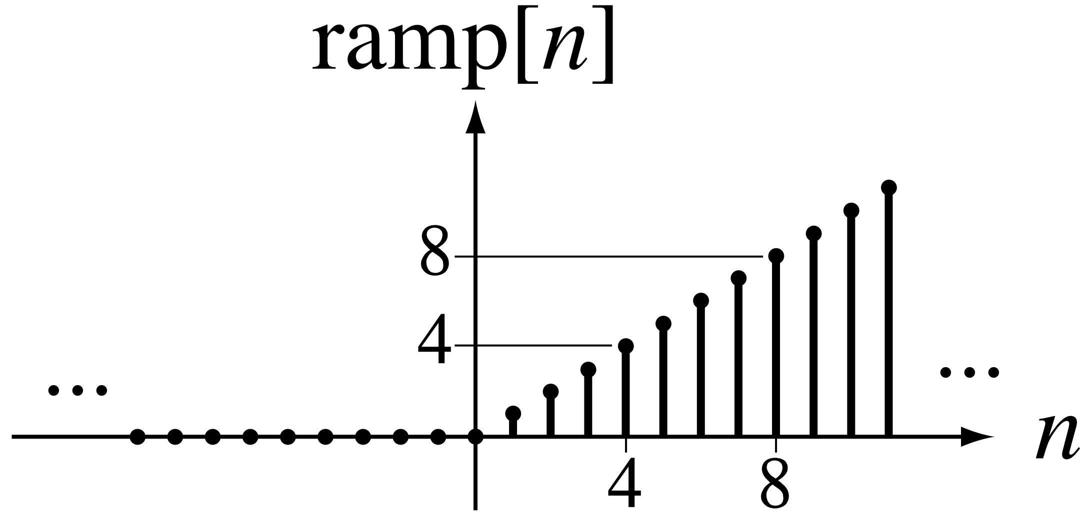
The Signum Function

$$\operatorname{sgn}[n] = \begin{cases} 1 & , \; n > 0 \\ 0 & , \; n = 0 \\ -1 & , \; n < 0 \end{cases} = 2\,\mathrm{u}[n] - \delta[n] - 1$$



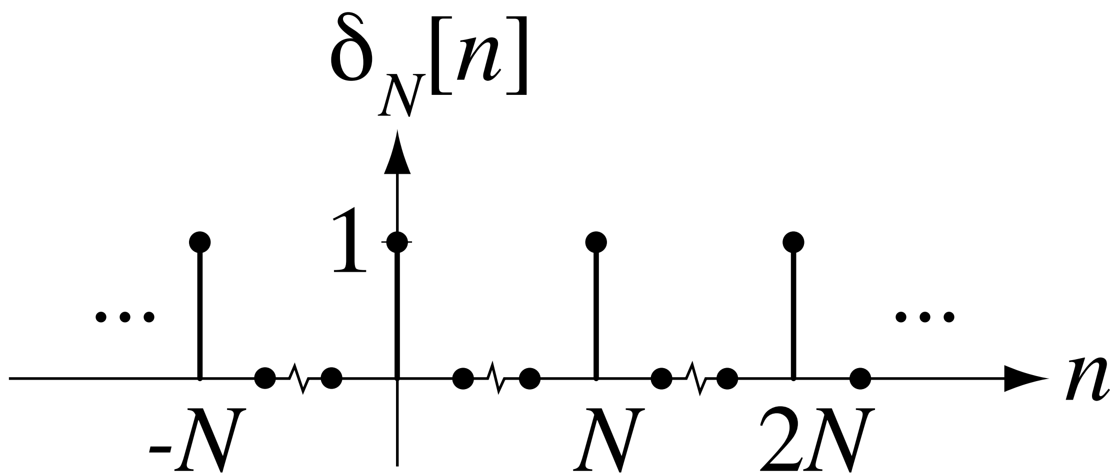
The Unit Ramp Function

$$\text{ramp}[n] = \begin{cases} n & , \; n \geq 0 \\ 0 & , \; n < 0 \end{cases} = n\,\text{u}[n] = \sum_{m=-\infty}^{n} \text{u}[m-1]$$



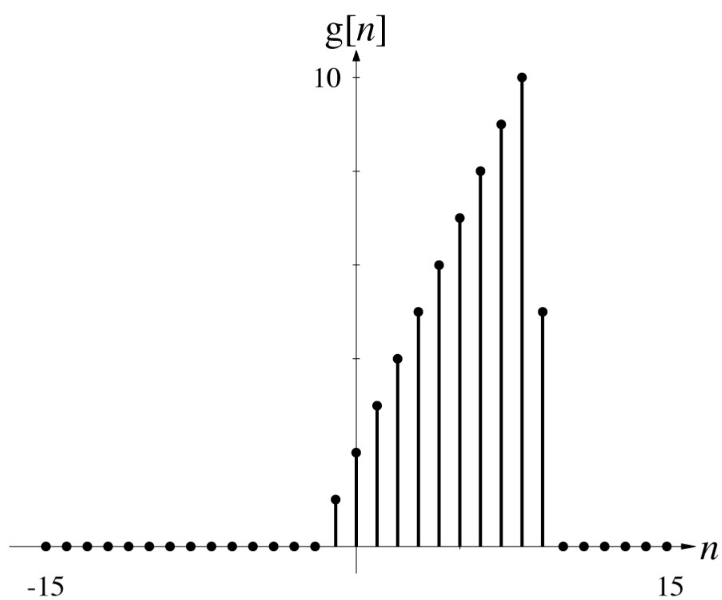
The Periodic Impulse Function

$$\delta_N[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN]$$

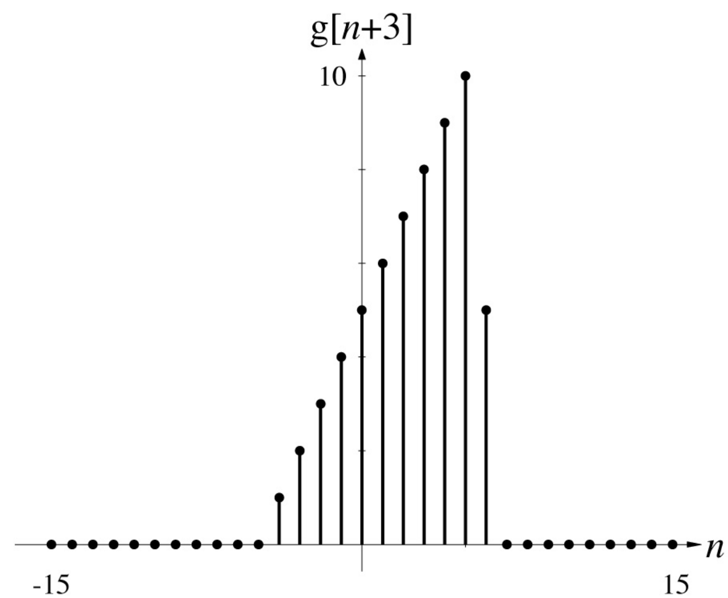


Scaling and Shifting Functions

Time shifting $n \to n + n_0 \, , n_0$ an integer



n	−1	0	1	2	3	4	5	6	7	8	9	10
$g[n]$	1	2	3	4	5	6	7	8	9	10	5	0



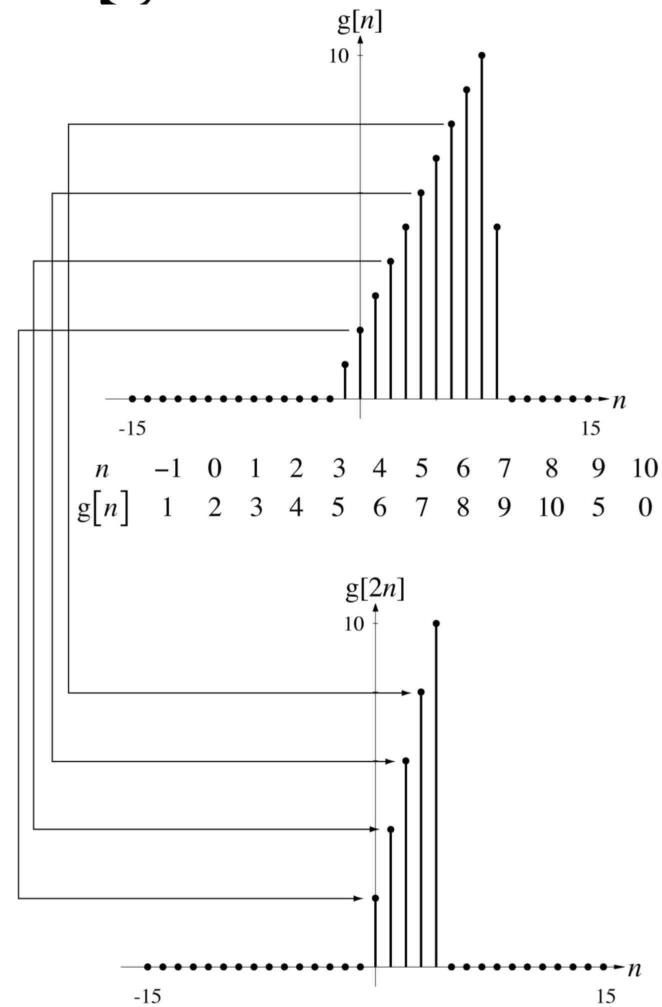
n	−4	−3	−2	−1	0	1	2	3	4	5	6	7
$n+3$	−1	0	1	2	3	4	5	6	7	8	9	10
$g[n+3]$	1	2	3	4	5	6	7	8	9	10	5	0

Scaling and Shifting Functions

Time compression

$$n \to Kn$$

K an integer > 1



n	−1	0	1	2	3	4	5	6	7	8	9	10
$g[n]$	1	2	3	4	5	6	7	8	9	10	5	0

n	0	1	2	3	4
$2n$	0	2	4	6	8
$g[2n]$	2	4	6	8	10

Scaling and Shifting Functions

Time expansion $n \to n/K\,,\, K > 1$

For all n such that n/K is an integer, $\mathrm{g}[n/K]$ is defined.

For all n such that n/K is not an integer, $\mathrm{g}[n/K]$ is <u>not defined</u>.

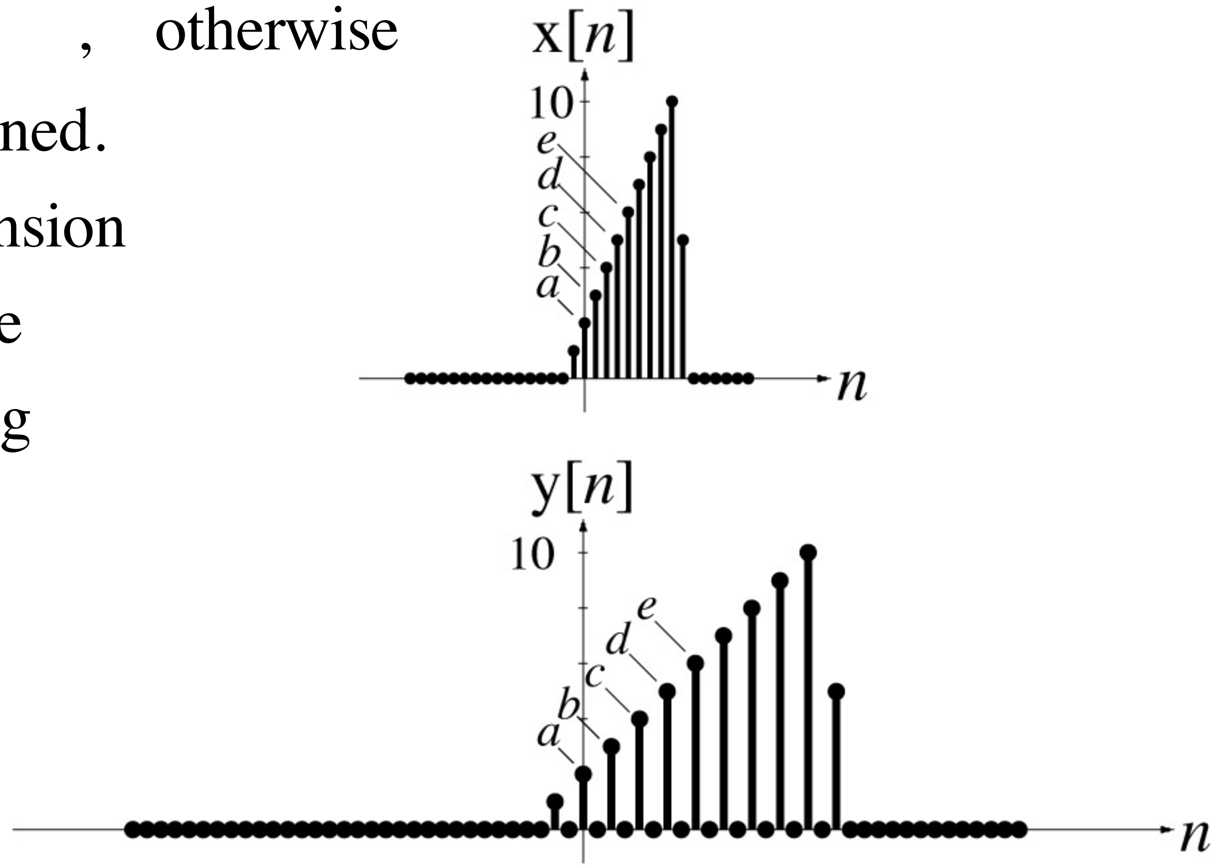
Scaling and Shifting Functions

There is a form of time expansion that is useful. Let

$$\mathrm{y}[n] = \begin{cases} \mathrm{x}[n/m] & , \quad n/m \text{ an integer} \\ 0 & , \quad \text{otherwise} \end{cases}$$

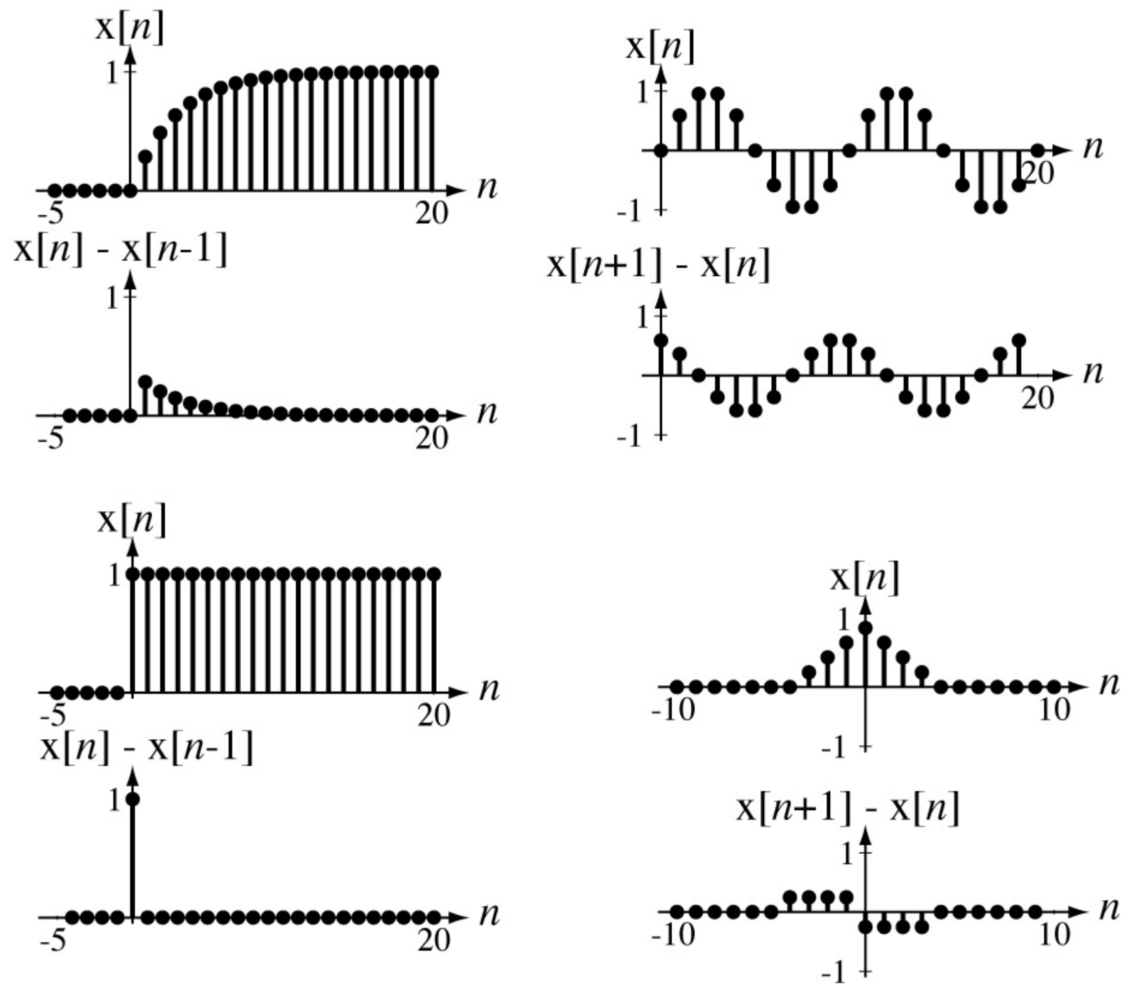
All values of y are defined.

This type of time expansion is actually used in some digital signal processing operations.



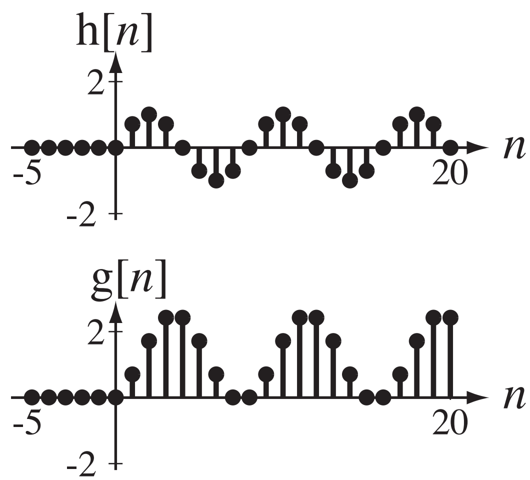
Differencing

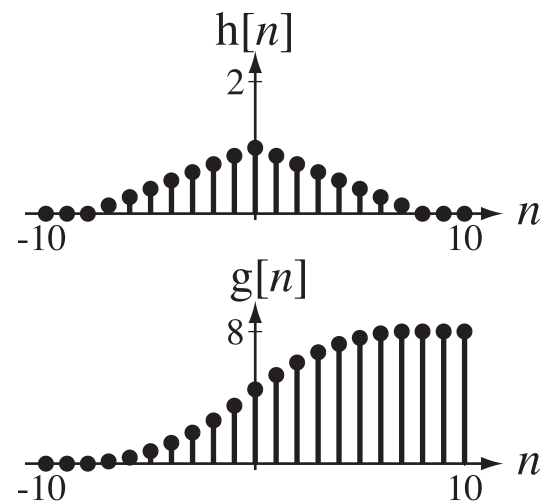
Backward Differences **Forward Differences**



Accumulation

$$g[n] = \sum_{m=-\infty}^{n} h[m]$$



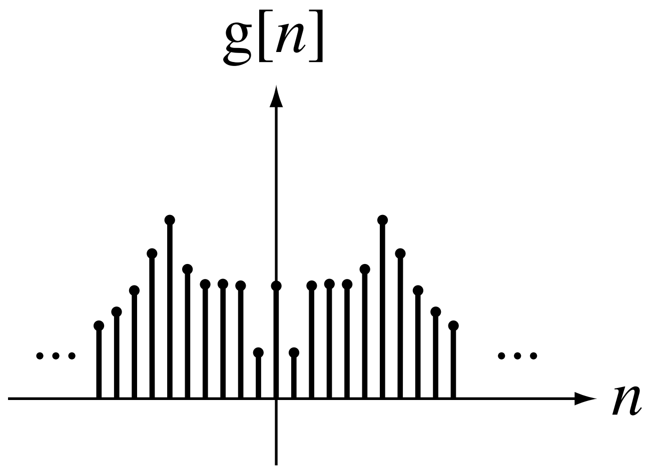


Even and Odd Signals

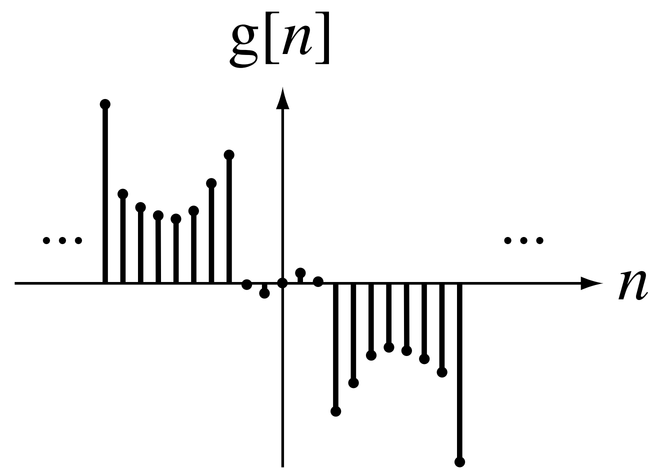
$$g[n] = g[-n]$$

$$g[n] = -g[-n]$$

Even Function



Odd Function

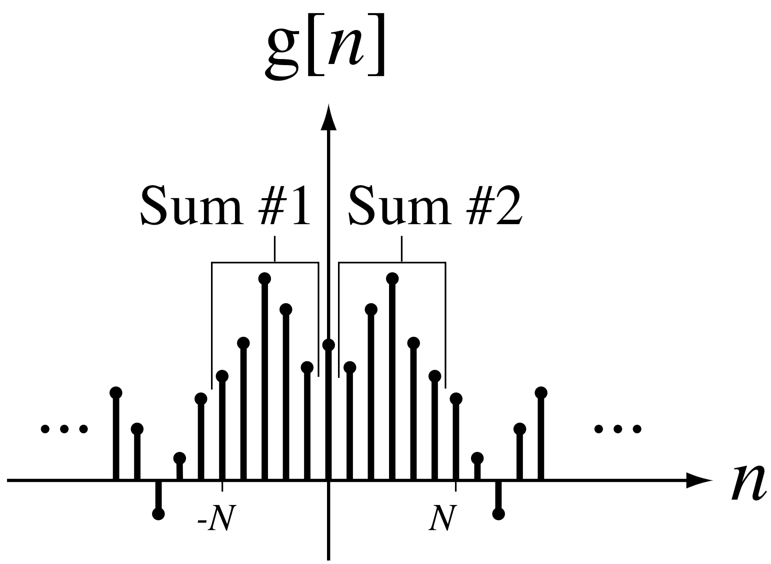


$$g_e[n] = \frac{g[n] + g[-n]}{2}$$

$$g_o[n] = \frac{g[n] - g[-n]}{2}$$

Symmetric Finite Summation

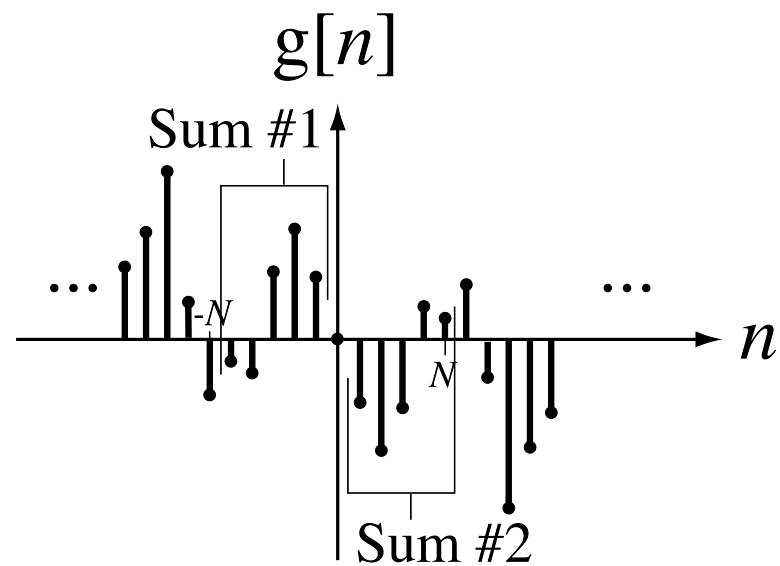
Even Function



Sum #1 = Sum #2

$$\sum_{n=-N}^{N} g[n] = g[0] + 2\sum_{n=1}^{N} g[n]$$

Odd Function



Sum #1 = - Sum #2

$$\sum_{n=-N}^{N} g[n] = 0$$

Periodic Functions

A **periodic** function is one that is invariant to the change of variable $n \to n + mN$ where N is a **period** of the function and m is any integer.

The minimum positive integer value of N for which $\mathrm{g}[n] = \mathrm{g}[n+N]$ is called the **fundamental period** N_0.

Signal Energy and Power

The signal energy of a signal $\mathrm{x}[n]$ is

$$E_{\mathrm{x}} = \sum_{n=-\infty}^{\infty} \left|\mathrm{x}[n]\right|^2$$

Signal Energy and Power

Some signals have infinite signal energy. In that case It is usually more convenient to deal with average signal power. The average signal power of a signal $\mathrm{x}[n]$ is

$$P_{\mathrm{x}} = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N-1} \left|\mathrm{x}[n]\right|^2$$

For a periodic signal $\mathrm{x}[n]$ the average signal power is

$$P_{\mathrm{x}} = \frac{1}{N} \sum_{n=\langle N\rangle} \left|\mathrm{x}[n]\right|^2$$

(The notation $\sum_{n=\langle N\rangle}$ means the sum over any set of consecutive n's exactly N in length.)

Signal Energy and Power

A signal with finite signal energy is called an **energy signal**.

A signal with infinite signal energy and finite average signal power is called a **power signal**.

Linearity and LTI Systems

- If a system is both homogeneous and additive it is **linear**.
- If a system is both linear and time-invariant it is called an **LTI** system
- Some systems that are non-linear can be accurately approximated for analytical purposes by a linear system for small excitations

Response of LTI Systems

An LTI system is completely characterized by its impulse response $\mathrm{h}(t)$. The response $\mathrm{y}(t)$ of an LTI system to an excitation $\mathrm{x}(t)$ is the convolution of $\mathrm{x}(t)$ with $\mathrm{h}(t)$.

$$\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t) = \int_{-\infty}^{\infty} \mathrm{x}(\lambda)\mathrm{h}(t-\lambda)\,d\lambda$$

Convolution Integral Properties

$$\mathrm{x}(t) * A\delta(t - t_0) = A\,\mathrm{x}(t - t_0)$$

If $\mathrm{g}(t) = \mathrm{g}_0(t) * \delta(t)$ then $\mathrm{g}(t - t_0) = \mathrm{g}_0(t - t_0) * \delta(t) = \mathrm{g}_0(t) * \delta(t - t_0)$

If $\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t)$ then $\mathrm{y}'(t) = \mathrm{x}'(t) * \mathrm{h}(t) = \mathrm{x}(t) * \mathrm{h}'(t)$

and $\mathrm{y}(at) = |a|\,\mathrm{x}(at) * \mathrm{h}(at)$

Commutativity

$$\mathrm{x}(t) * \mathrm{y}(t) = \mathrm{y}(t) * \mathrm{x}(t)$$

Associativity

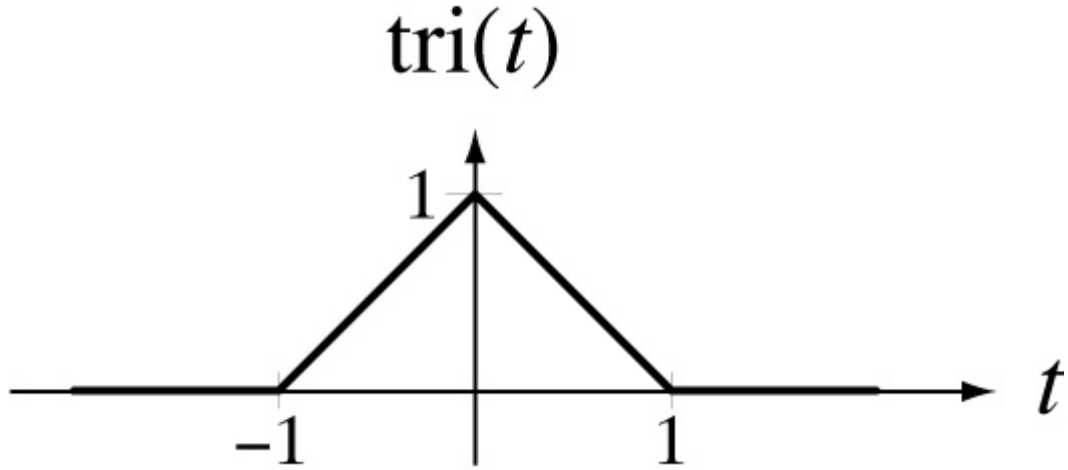
$$\left[\mathrm{x}(t) * \mathrm{y}(t)\right] * \mathrm{z}(t) = \mathrm{x}(t) * \left[\mathrm{y}(t) * \mathrm{z}(t)\right]$$

Distributivity

$$\left[\mathrm{x}(t) + \mathrm{y}(t)\right] * \mathrm{z}(t) = \mathrm{x}(t) * \mathrm{z}(t) + \mathrm{y}(t) * \mathrm{z}(t)$$

The Unit Triangle Function

$$\text{tri}(t) = \begin{cases} 1-|t| & , |t| < 1 \\ 0 & , |t| \geq 1 \end{cases}$$



The unit triangle, is the convolution of a unit rectangle with Itself.

Systems Described by Differential Equations

The transfer function:

$$\mathrm{H}(s) = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_2 s^2 + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_2 s^2 + a_1 s + a_0}$$

This type of function is called a **rational function** because it is a ratio of polynomials in s. The transfer function encapsulates all the system characteristics and is of great importance in signal and system analysis.

Response of LTI Systems

If the excitation $\mathrm{x}(t)$ is a **phasor** or **complex sinusoid** of frequency f_0, of the form

$$\mathrm{x}(t) = A_{\mathrm{x}} e^{j\phi_{\mathrm{x}}} e^{j2\pi f_0 t}$$

then the response $\mathrm{y}(t)$ is of the form

$$\mathrm{y}(t) = \mathrm{H}(f_0)\mathrm{x}(t) = \mathrm{H}(f_0) A_{\mathrm{x}} e^{j\phi_{\mathrm{x}}} e^{j2\pi f_0 t}.$$

The response can also be written in the form

$\mathrm{y}(t) = A_{\mathrm{y}} e^{j\phi_{\mathrm{y}}} e^{j2\pi f_0 t}$ where $A_{\mathrm{y}} = |\mathrm{H}(f_0)| A_{\mathrm{x}}$ and $\phi_{\mathrm{y}} = \phi_{\mathrm{x}} + \measuredangle \mathrm{H}(f_0)$.

Applying this to real sinusoids, if $\mathrm{x}(t) = A_{\mathrm{x}} \cos(2\pi f_0 t + \phi_{\mathrm{x}})$ then $\mathrm{y}(t) = A_{\mathrm{y}} \cos(2\pi f_0 t + \phi_{\mathrm{y}})$.

The Convolution Sum

The response $\mathrm{y}[n]$ of an LTI system with impulse response $\mathrm{h}[n]$ to an arbitrary excitation $\mathrm{x}[n]$ is

$$\mathrm{y}[n] = \sum_{m=-\infty}^{\infty} \mathrm{x}[m]\mathrm{h}[n-m]$$

Convolution Sum Properties

$$x[n]*A\delta[n-n_0]=A\,x[n-n_0]$$

Let $y[n]=x[n]*h[n]$ then

$$y[n-n_0]=x[n]*h[n-n_0]=x[n-n_0]*h[n]$$

$$y[n]-y[n-1]=x[n]*\left(h[n]-h[n-1]\right)=\left(x[n]-x[n-1]\right)*h[n]$$

and the sum of the impulse strengths in y is the product of the sum of the impulse strengths in x and the sum of the impulse strengths in h.

Systems Described by Difference Equations

The transfer function is

$$\mathrm{H}(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

or, alternately,

$$\mathrm{H}(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = z^{N-M} \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_{M-1} z + b_M}{a_0 z^N + a_1 z^{N-1} + \cdots + a_{N-1} z + a_N}$$

The transfer function can be written directly from the system difference equation and vice versa. $\mathrm{H}(e^{j\Omega})$ is the system's **frequency response**. It is the transfer function $\mathrm{H}(z)$ with z replaced by $e^{j\Omega}$.

Continuous-Time Fourier Series Definition

$$\mathrm{x}(t) = \sum_{n=-\infty}^{\infty} X_n e^{j2\pi nt/T} \quad \text{and} \quad X_n = \frac{1}{T}\int_{t_0}^{t_0+T} \mathrm{x}(t) e^{-j2\pi nt/T}\, dt.$$

The signal and its harmonic function form a **Fourier series pair** $\mathrm{x}(t) \xleftrightarrow[T]{\mathcal{FS}} X_n$ where T is the representation time and, therefore, the fundamental period of the continuous-time Fourier series (CTFS) representation of $\mathrm{x}(t)$. If T is also a period of $\mathrm{x}(t)$, the CTFS representation of $\mathrm{x}(t)$ is valid for all time. This is, by far, the most common use of the CTFS in engineering applications. If T is not a period of $\mathrm{x}(t)$, the CTFS representation is generally valid only in the interval $t_0 \le t < t_0 + T$.

CTFS of a Real Function

It can be shown that the continuous-time Fourier series (CTFS) harmonic function of any real-valued function $\mathrm{x}(t)$ has the property that $X_n = X_{-n}^*$.

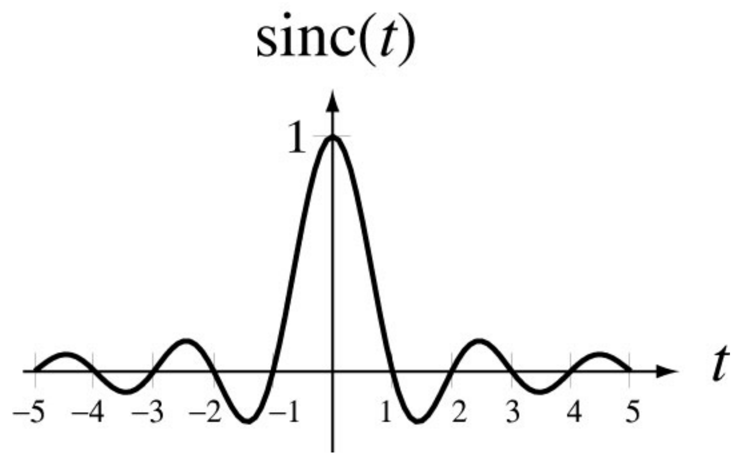
One implication of this fact is that, for real-valued functions, the magnitudes of their harmonic functions are even functions and their phases can be expressed as odd functions of harmonic number k.

The Sinc Function

Let $\mathrm{x}(t) = A\,\mathrm{rect}(t/w) * \delta_{T_0}(t)$, $w < T_0$. Then

$$\mathrm{x}(t) = A\,\mathrm{rect}(t/w) * \delta_{T_0}(t) \xleftrightarrow[T_0]{\mathcal{FS}} X_n = A\frac{\sin(\pi n w / T_0)}{\pi n}$$

The mathematical form $\frac{\sin(\pi x)}{\pi x}$ arises frequently enough to be given its own name, "sinc". That is $\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$.



The Uniqueness Property

If we find a Fourier series representation of a signal, it is unique. That is, no other or alternate Fourier series representation exists.

Example: Let $x(t) = 3\cos(8\pi t - \pi/4) + 4\sin(4\pi t)$

Using trigonometric identities, this can be rewritten as

$$x(t) = 3\left[\cos(8\pi t)\cos(\pi/4) - \sin(8\pi t)\sin(\pi/4)\right] + 4\sin(4\pi t)$$

$$x(t) = \frac{3\sqrt{2}}{2}\left[\cos(8\pi t) - \sin(8\pi t)\right] + 4\sin(4\pi t)$$

The Uniqueness Property

$$\mathrm{x}(t)=\frac{3\sqrt{2}}{2}\left[\frac{e^{j8\pi t}+e^{-j8\pi t}}{2}-\frac{e^{j8\pi t}-e^{-j8\pi t}}{j2}\right]+4\frac{e^{j4\pi t}-e^{-j4\pi t}}{j2}$$

$$\mathrm{x}(t)=\frac{3\sqrt{2}}{4}\left[(1+j)e^{j8\pi t}+(1-j)e^{-j8\pi t}\right]-j2\left(e^{j4\pi t}-e^{-j4\pi t}\right)$$

$$\mathrm{x}(t)=\frac{3\sqrt{2}}{4}(1+j)e^{j8\pi t}+\frac{3\sqrt{2}}{4}(1-j)e^{-j8\pi t}-j2e^{j4\pi t}+j2e^{-j4\pi t}$$

This is THE (complex)CTFS representation of $\mathrm{x}(t)$ in which

$$\mathrm{x}(t)=\sum_{n=-\infty}^{\infty}X_n e^{j2\pi n f_0 t}\ ,\ f_0=2\ ,X_{-2}=\frac{3\sqrt{2}}{4}(1-j)\ ,X_{-1}=j2\ ,X_1=-j2\ ,$$

$X_2=\dfrac{3\sqrt{2}}{4}(1+j)$ and all other CTFS coefficients are zero.

Some Common CTFS Pairs

$$1 \xleftrightarrow[T]{\mathcal{FS}} \delta[n] \text{ , } T \text{ arbitrary}$$

$$\delta_{T_0}(t) \xleftrightarrow[mT_0]{\mathcal{FS}} \begin{cases} (1/T_0) & \text{, } n/m \text{ an integer} \\ 0 & \text{, otherwise} \end{cases}$$

$$e^{j2\pi qt/T_0} \xleftrightarrow[mT_0]{\mathcal{FS}} \delta[n-mq]$$

$$\sin(2\pi qt/T_0) \xleftrightarrow[mT_0]{\mathcal{FS}} (j/2)(\delta[n+mq]-\delta[n-mq])$$

$$\cos(2\pi qt/T_0) \xleftrightarrow[mT_0]{\mathcal{FS}} (1/2)(\delta[n-mq]+\delta[n+mq])$$

$$\text{rect}(t/w) * \delta_{T_0}(t) \xleftrightarrow[mT_0]{\mathcal{FS}} (w/T_0)\text{sinc}(wn/mT_0)\delta_m[n]$$

$$\text{tri}(t/w) * \delta_{T_0}(t) \xleftrightarrow[mT_0]{\mathcal{FS}} (w/T_0)\text{sinc}^2(wn/mT_0)\delta_m[n]$$

$$(m \text{ an integer})$$

Definition of the CTFT

Forward — f form — **Inverse**

$$\mathrm{X}(f)=\mathcal{F}\left(\mathrm{x}(t)\right)=\int_{-\infty}^{\infty}\mathrm{x}(t)e^{-j2\pi ft}\,dt \qquad \mathrm{x}(t)=\mathcal{F}^{-1}\left(\mathrm{X}(f)\right)=\int_{-\infty}^{\infty}\mathrm{X}(f)e^{+j2\pi ft}\,df$$

Forward — ω form — **Inverse**

$$\mathrm{X}(j\omega)=\mathcal{F}\left(\mathrm{x}(t)\right)=\int_{-\infty}^{\infty}\mathrm{x}(t)e^{-j\omega t}\,dt \qquad \mathrm{x}(t)=\mathcal{F}^{-1}\left(\mathrm{X}(j\omega)\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{X}(j\omega)e^{+j\omega t}\,d\omega$$

Commonly-used notation:

$$\mathrm{x}(t)\xleftrightarrow{\mathcal{F}}\mathrm{X}(f) \quad \text{or} \quad \mathrm{x}(t)\xleftrightarrow{\mathcal{F}}\mathrm{X}(j\omega)$$

Some CTFT Pairs

$$\delta(t) \xleftrightarrow{\mathcal{F}} 1$$

$$e^{-\alpha t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{F}} 1/(j\omega+\alpha) \;,\; \alpha > 0 \qquad -e^{-\alpha t}\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{F}} 1/(j\omega+\alpha) \;,\; \alpha < 0$$

$$te^{-\alpha t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{F}} 1/(j\omega+\alpha)^2 \;,\; \alpha > 0 \qquad -te^{-\alpha t}\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{F}} 1/(j\omega+\alpha)^2 \;,\; \alpha < 0$$

$$t^n e^{-\alpha t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{F}} \frac{n!}{(j\omega+\alpha)^{n+1}} \;,\; \alpha > 0 \qquad -t^n e^{-\alpha t}\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{F}} \frac{n!}{(j\omega+\alpha)^{n+1}} \;,\; \alpha < 0$$

$$e^{-\alpha t}\sin(\omega_0 t)\mathrm{u}(t) \xleftrightarrow{\mathcal{F}} \frac{\omega_0}{(j\omega+\alpha)^2+\omega_0^2} \;,\; \alpha > 0 \qquad -e^{-\alpha t}\sin(\omega_0 t)\mathrm{u}(-t) \xleftrightarrow{\mathcal{F}} \frac{\omega_0}{(j\omega+\alpha)^2+\omega_0^2} \;,\; \alpha < 0$$

$$e^{-\alpha t}\cos(\omega_0 t)\mathrm{u}(t) \xleftrightarrow{\mathcal{F}} \frac{j\omega+\alpha}{(j\omega+\alpha)^2+\omega_0^2} \;,\; \alpha > 0 \qquad -e^{-\alpha t}\cos(\omega_0 t)\mathrm{u}(-t) \xleftrightarrow{\mathcal{F}} \frac{j\omega+\alpha}{(j\omega+\alpha)^2+\omega_0^2} \;,\; \alpha < 0$$

$$e^{-\alpha|t|} \xleftrightarrow{\mathcal{F}} \frac{2\alpha}{\omega^2+\alpha^2} \;,\; \alpha > 0$$

More CTFT Pairs

$$\delta(t) \xleftrightarrow{\mathcal{F}} 1 \qquad 1 \xleftrightarrow{\mathcal{F}} \delta(f)$$

$$\text{sgn}(t) \xleftrightarrow{\mathcal{F}} 1/j\pi f \qquad \text{u}(t) \xleftrightarrow{\mathcal{F}} (1/2)\delta(f) + 1/j2\pi f$$

$$\text{rect}(t) \xleftrightarrow{\mathcal{F}} \text{sinc}(f) \qquad \text{sinc}(t) \xleftrightarrow{\mathcal{F}} \text{rect}(f)$$

$$\text{tri}(t) \xleftrightarrow{\mathcal{F}} \text{sinc}^2(f) \qquad \text{sinc}^2(t) \xleftrightarrow{\mathcal{F}} \text{tri}(f)$$

$$\delta_{T_0}(t) \xleftrightarrow{\mathcal{F}} f_0 \delta_{f_0}(f) \ , \ f_0 = 1/T_0 \qquad T_0 \delta_{T_0}(t) \xleftrightarrow{\mathcal{F}} \delta_{f_0}(f) \ , \ T_0 = 1/f_0$$

$$\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} (1/2)\left[\delta(f - f_0) + \delta(f + f_0)\right] \qquad \sin(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} (j/2)\left[\delta(f + f_0) - \delta(f - f_0)\right]$$

Numerical Computation of the CTFT

It can be shown that the **DFT** can be used to approximate samples from the CTFT. If the signal $\mathrm{x}(t)$ is a **causal energy signal** and N samples are taken from it over a finite time beginning at $t = 0$, at a rate f_s then the relationship between the CTFT of $\mathrm{x}(t)$ and the DFT of the samples taken from it is

$$\mathrm{X}(kf_s / N) \cong T_s e^{-j\pi k/N} \,\mathrm{sinc}(k / N)\,\mathrm{X}_{DFT}[k]$$

For those harmonic numbers k for which $k << N$

$$\mathrm{X}(kf_s / N) \cong T_s\,\mathrm{X}_{DFT}[k]$$

As the sampling rate and number of samples are increased, this approximation is improved.

The Discrete-Time Fourier Series

The discrete-time Fourier series (DTFS) is similar to the CTFS. A periodic discrete-time signal can be expressed as

$$\mathrm{x}[n] = \sum_{k=\langle N \rangle} \mathrm{c_x}[k] e^{j2\pi kn/N} \qquad \mathrm{c_x}[k] = \frac{1}{N} \sum_{n=n_0}^{n_0+N-1} \mathrm{x}[n] e^{-j2\pi kn/N}$$

where $\mathrm{c_x}[k]$ is the harmonic function, N is any period of $\mathrm{x}[n]$ and the notation, $\sum_{k=\langle N \rangle}$ means a summation over any range of consecutive k's exactly N in length.

The Discrete Fourier Transform

The discrete Fourier transform (DFT) is almost identical to the DTFS.

A periodic discrete-time signal can be expressed as

$$\mathrm{x}[n] = \frac{1}{N} \sum_{k=\langle N \rangle} \mathrm{X}[k] e^{j2\pi kn/N} \qquad \mathrm{X}[k] = \sum_{n=n_0}^{n_0+N-1} \mathrm{x}[n] e^{-j2\pi kn/N}$$

where $\mathrm{X}[k]$ is the DFT harmonic function and N is any period of $\mathrm{x}[n]$. The main difference between the DTFS and the DFT is the location of the $1/N$ term. So $\mathrm{X}[k] = N\,\mathrm{c}_{\mathrm{x}}[k]$.

The Discrete Fourier Transform

Because the DTFS and DFT are so similar, and because the DFT is so widely used in digital signal processing (DSP), we will concentrate on the DFT realizing we can always form the DTFS from $c_x[k] = X[k]/N$.

The Discrete Fourier Transform

Notice that in

$$\mathrm{x}[n]=\frac{1}{N}\sum_{k=\langle N\rangle}\mathrm{X}[k]e^{j2\pi kn/N}$$

the summation is over N values of k, a finite summation. This is because of the periodicity of the complex sinusoid, $e^{-j2\pi kn/N}$ in harmonic number k. If k is increased by any integer multiple of N the complex sinusoid does not change.

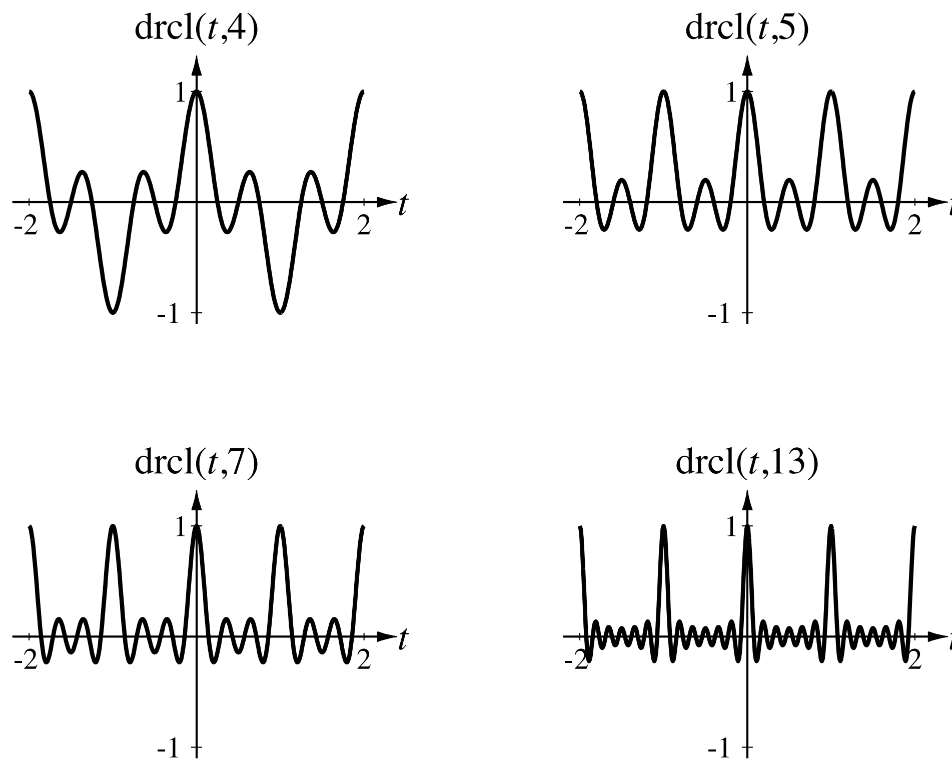
$$e^{-j2\pi kn/N}=e^{-j2\pi(k+mN)n/N}=e^{-j2\pi kn/N}\underbrace{e^{-j2\pi mn}}_{=1}\quad ,\ m \text{ an integer}$$

This occurs because discrete time n is always an integer.

The Dirichlet Function

The functional form $\dfrac{\sin(\pi N t)}{N\sin(\pi t)}$ appears often in discrete-time signal analysis and is given the special name **Dirichlet** function. That is

$$\text{drcl}(t,N) = \frac{\sin(\pi N t)}{N\sin(\pi t)}$$



Response of LTI Systems

If the Fourier transform of the excitation $\mathrm{x}(t)$ is $\mathrm{X}(f)$ and the Fourier transform of the response $\mathrm{y}(t)$ is $\mathrm{Y}(f)$, then

$\mathrm{Y}(f)=\mathrm{H}(f)\mathrm{X}(f)$ and $\left|\mathrm{Y}(f)\right|=\left|\mathrm{H}(f)\right|\left|\mathrm{X}(f)\right|$ and

$\measuredangle\mathrm{Y}(f)=\measuredangle\mathrm{H}(f)+\measuredangle\mathrm{X}(f)$.

If $\mathrm{x}(t)$ is an energy signal (finite signal energy) then, from Parseval's theorem $E_{\mathrm{x}}=\int_{-\infty}^{\infty}\left|\mathrm{X}(f)\right|^2 df$ and $E_{\mathrm{y}}=\int_{-\infty}^{\infty}\left|\mathrm{Y}(f)\right|^2 df=\int_{-\infty}^{\infty}\left|\mathrm{H}(f)\right|^2\left|\mathrm{X}(f)\right|^2 df$

Signal Distortion in Transmission

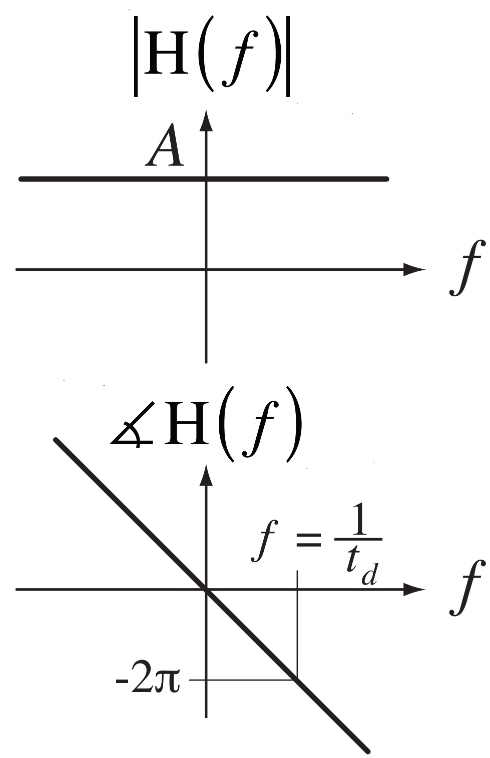
Distortion means changing the shape of a signal. Two changes to a signal are not considered distortion, multiplying it by a constant and shifting it in time. The impulse response of an LTI system that does not distort is of the general form $\mathrm{h}(t) = K\delta(t - t_d)$. where K and t_d are constants. The corresponding frequency response of such a system is $\mathrm{H}(f) = Ke^{-j2\pi f t_d}$.
$|\mathrm{H}(f)| = K$ and $\measuredangle \mathrm{H}(f) = -2\pi f t_d$. If $|\mathrm{H}(f)| \neq K$ the system has **amplitude distortion**. If $\measuredangle \mathrm{H}(f) \neq -2\pi f t_d$ the system has **delay** or **phase distortion**. Both of these types of distortion are classified as **linear distortions**.

Signal Distortion in Transmission

If $\measuredangle \mathrm{H}(f) = -2\pi f t_d$, then $t_d = -\dfrac{\measuredangle \mathrm{H}(f)}{2\pi f}$ and t_d is a constant

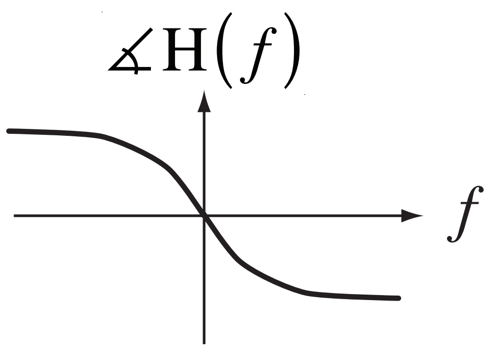
<u>if</u> $\measuredangle \mathrm{H}(f) = -Kf$ (K a constant). If t_d is not a constant, phase distortion results.

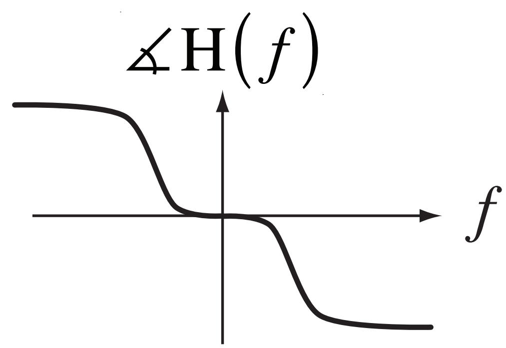
Frequency response of a distortionless LTI system →



Signal Distortion in Transmission

Most real systems do not have simple delay. They have phases that are not linear functions of frequency.



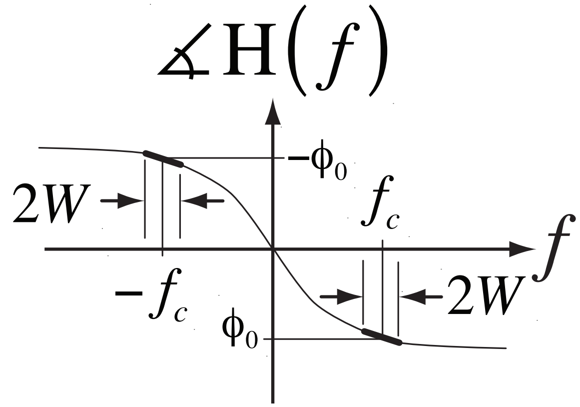


Signal Distortion in Transmission

For a bandpass signal with a small bandwidth W compared to its center frequency f_c, we can model the frequency response phase variation as approximately linear over the frequency ranges $f_c - W < |f| < f_c + W$, and the frequency response magnitude as approximately constant, of the form

$$\mathrm{H}(f) \cong A e^{-j2\pi f t_g} \begin{cases} e^{j\phi_0} & , \quad f_c - W < f < f_c + W \\ e^{-j\phi_0} & , \quad -f_c - W < f < -f_c + W \end{cases}$$

where $\phi_0 = \measuredangle \mathrm{H}(f_c)$.



Signal Distortion in Transmission

If we now let the bandpass signal be

$$x(t) = x_1(t)\cos(2\pi f_c t) + x_2(t)\sin(2\pi f_c t)$$

Its Fourier transform is

$$X(f) = \begin{Bmatrix} X_1(f) * (1/2)\left[\delta(f - f_c) + \delta(f + f_c)\right] \\ + X_2(f) * (j/2)\left[\delta(f + f_c) - \delta(f - f_c)\right] \end{Bmatrix}$$

$$X(f) = (1/2)\left\{\left[X_1(f - f_c) + X_1(f + f_c)\right] + j\left[X_2(f + f_c) - X_2(f - f_c)\right]\right\}$$

The frequency response is modeled by

$$H(f) \cong Ae^{-j2\pi f t_g} \begin{cases} e^{j\phi_0} & , \quad f_c - W < f < f_c + W \\ e^{-j\phi_0} & , \quad -f_c - W < f < -f_c + W \end{cases}$$

then the Fourier transform of the response $y(t)$ is

$$Y(f) \cong H(f)X(f) = (A/2)\begin{Bmatrix} X_1(f - f_c)e^{-j(2\pi f t_g - \phi_0)} + X_1(f + f_c)e^{-j(2\pi f t_g + \phi_0)} \\ + jX_2(f + f_c)e^{-j(2\pi f t_g + \phi_0)} - jX_2(f - f_c)e^{-j(2\pi f t_g - \phi_0)} \end{Bmatrix}$$

Signal Distortion in Transmission

$$\mathrm{Y}(f) \cong \mathrm{H}(f)\mathrm{X}(f) = (A/2)\left\{ \begin{array}{l} \mathrm{X}_1(f-f_c)e^{-j(2\pi f t_g - \phi_0)} + \mathrm{X}_1(f+f_c)e^{-j(2\pi f t_g + \phi_0)} \\ +j\mathrm{X}_2(f+f_c)e^{-j(2\pi f t_g + \phi_0)} - j\mathrm{X}_2(f-f_c)e^{-j(2\pi f t_g - \phi_0)} \end{array} \right\}$$

Inverse Fourier transforming, using the time and frequency shifting properties,

$$\mathrm{y}(t) \cong (A/2)\left\{ \begin{array}{l} e^{j\phi_0}\,\mathrm{x}_1(t-t_g)e^{j2\pi f_c t} + e^{-j\phi_0}\,\mathrm{x}_1(t-t_g)e^{-j2\pi f_c t} \\ +je^{-j\phi_0}\,\mathrm{x}_2(t-t_g)e^{-j2\pi f_c t} - je^{j\phi_0}\,\mathrm{x}_2(t-t_g)e^{j2\pi f_c t} \end{array} \right\}$$

$$\mathrm{y}(t) \cong (A/2)\left\{ \begin{array}{l} \mathrm{x}_1(t-t_g)\left[e^{j(2\pi f_c t + \phi_0)} + e^{-j(2\pi f_c t + \phi_0)}\right] \\ +\mathrm{x}_2(t-t_g)\,j\left[e^{-j(2\pi f_c t + \phi_0)} - e^{j(2\pi f_c t + \phi_0)}\right] \end{array} \right\}$$

$$\mathrm{y}(t) \cong A\left\{\mathrm{x}_1(t-t_g)\cos(2\pi f_c t + \phi_0) + \mathrm{x}_2(t-t_g)\sin(2\pi f_c t + \phi_0)\right\}$$

$$\mathrm{y}(t) \cong A\left\{\mathrm{x}_1(t-t_g)\cos(2\pi f_c(t-t_d)) + \mathrm{x}_2(t-t_g)\sin(2\pi f_c(t-t_d))\right\}$$

where $t_d = -\dfrac{\phi_0}{2\pi f_c} = -\dfrac{\measuredangle \mathrm{H}(f_c)}{2\pi f_c}$ is known as the **phase** or **carrier delay**.

Signal Distortion in Transmission

From the approximate form of the system frequency response

$$\mathrm{H}(f) \cong Ae^{-j2\pi f t_{\mathrm{g}}} \begin{cases} e^{j\phi_0} \quad , \quad f_c - W < f < f_c + W \\ e^{-j\phi_0} \quad , \quad -f_c - W < f < -f_c + W \end{cases}$$

we get

$$\measuredangle \mathrm{H}(f) \cong \begin{cases} -2\pi f t_{\mathrm{g}} + \phi_0 \quad , \quad f_c - W < f < f_c + W \\ -2\pi f t_{\mathrm{g}} - \phi_0 \quad , \quad -f_c - W < f < -f_c + W \end{cases}$$

If we differentiate both sides w.r.t. f we get

$$\frac{d}{df}\left(\measuredangle \mathrm{H}(f)\right) \cong -2\pi t_{\mathrm{g}} \quad , \quad f_c - W < |f| < f_c + W$$

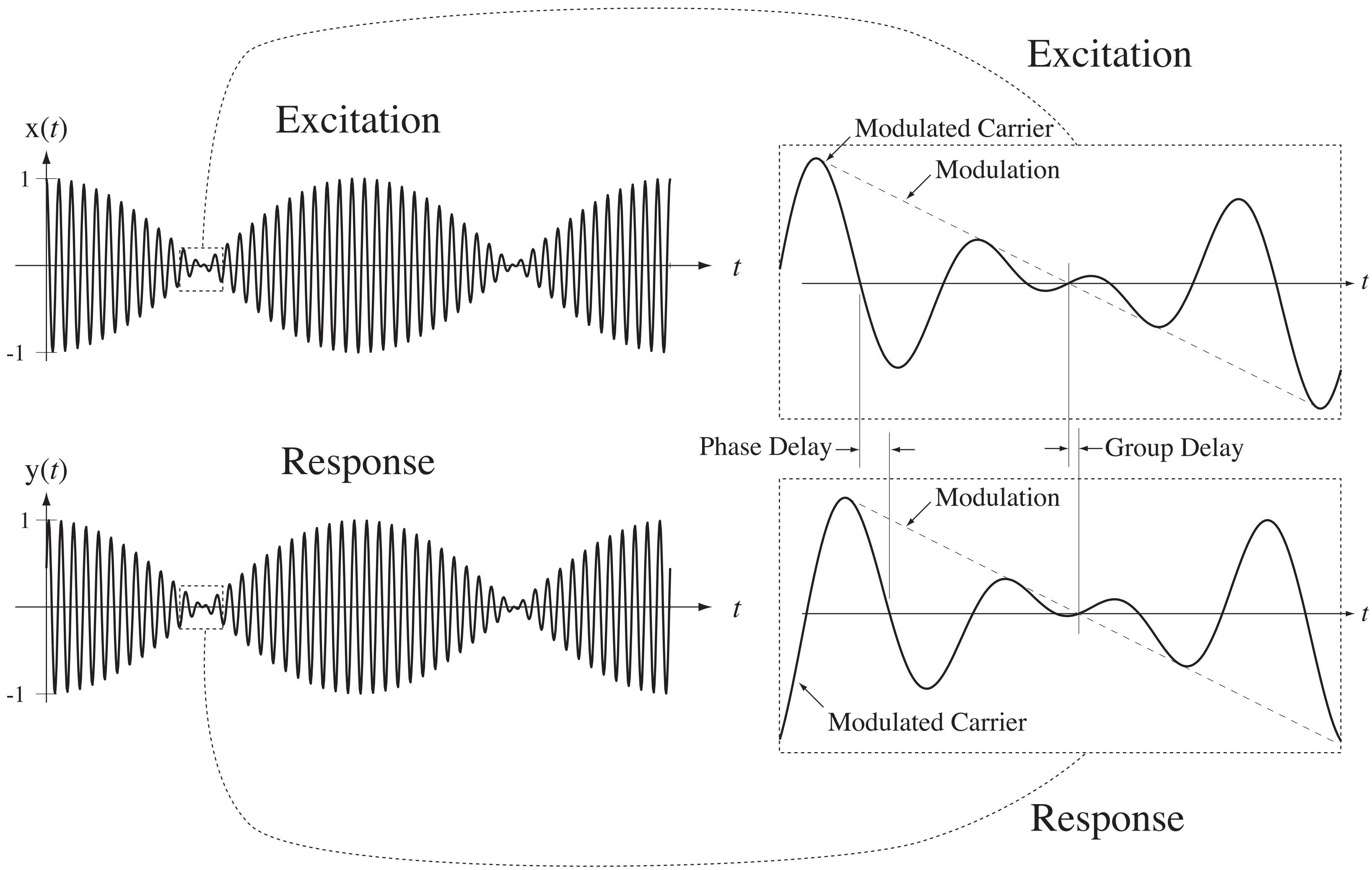
or

$$t_{\mathrm{g}} \cong -\frac{1}{2\pi}\frac{d}{df}\left(\measuredangle \mathrm{H}(f)\right) \; , \quad f_c - W < |f| < f_c + W$$

t_{g} is known as the **group delay**.

Signal Distortion in Transmission

Phase and Group Delay

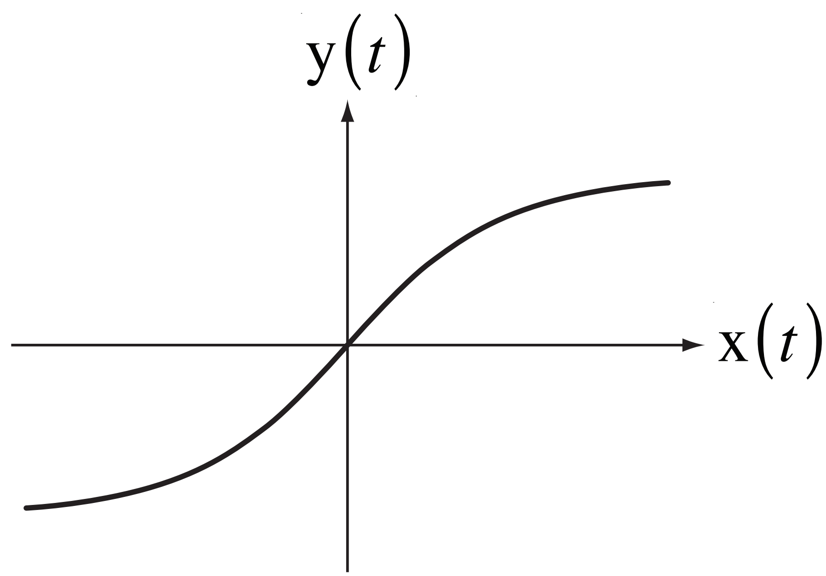


Signal Distortion in Transmission

Linear distortion can be corrected (theoretically) by an **equalization** network. If the communication channel's frequency response is $\text{H}_C(f)$ and it is followed by an equalization network with frequency response $\text{H}_{eq}(f)$ then the overall frequency response is $\text{H}(f) = \text{H}_C(f)\text{H}_{eq}(f)$ and the overall frequency response will be distortionless if $\text{H}(f) = \text{H}_C(f)\text{H}_{eq}(f) = Ke^{-j\omega t_d}$. Therefore, the frequency response of the equalization network should be $\text{H}_{eq}(f) = \dfrac{Ke^{-j\omega t_d}}{\text{H}_C(f)}$. It is very rare in practice that this can be done exactly but in many cases an excellent approximation can be made that greatly reduces linear distortion.

Signal Distortion in Transmission

Communication systems can also have nonlinear distortion caused by elements in the system that are **statically nonlinear**. In that case the excitation and response are related through a **transfer characteristic** of the form $\mathrm{y}(t) = \mathrm{T}(\mathrm{x}(t))$. For example, some amplifiers experience a "soft" saturation in which the ratio of the response to the excitation decreases with an increase in the excitation level.



Signal Distortion in Transmission

The transfer characteristic is usually not a simple known function but can often be closely approximated by a polynomial curve fit of the form $y(t) = a_1 x(t) + a_2 x^2(t) + a_3 x^3(t) + \cdots$. The Fourier transform of $y(t)$ is

$$Y(f) = a_1 X(f) + a_2 X(f) * X(f) + a_3 X(f) * X(f) * X(f) + \cdots$$

In a linear system if the excitation is bandlimited, the response has the same band limits. The response cannot contain frequencies not present in the excitation. But in a nonlinear system of this type if $X(f)$ contains a range of frequencies, $X(f) * X(f)$ contains a greater range of frequencies and $X(f) * X(f) * X(f)$ contains a still greater range of frequencies.

Signal Distortion in Transmission

If $\mathrm{X}(f) * \mathrm{X}(f)$ contains frequencies that are all outside the range of $\mathrm{X}(f)$ then a filter can be used to eliminate them. But often $\mathrm{X}(f) * \mathrm{X}(f)$ contains frequencies both inside and outside that range, and those inside the range cannot be filtered out without affecting the spectrum of $\mathrm{X}(f)$. As a simple example of the kind of nonlinear distortion that can occur let $\mathrm{x}(t) = A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t)$ and let $\mathrm{y}(t) = \mathrm{x}^2(t)$. Then

$$\mathrm{y}(t) = \left[A_1 \cos(\omega_1 t) + A_2 \cos(\omega_2 t)\right]^2$$

$$= A_1^2 \cos^2(\omega_1 t) + A_2^2 \cos^2(\omega_2 t) + 2 A_1 A_2 \cos(\omega_1 t)\cos(\omega_2 t)$$

$$= \left(A_1^2 / 2\right)\left[1 + \cos(2\omega_1 t)\right] + \left(A_2^2 / 2\right)\left[1 + \cos(2\omega_2 t)\right]$$

$$+ A_1 A_2 \left[\cos\left((\omega_1 - \omega_2)t\right) + \cos\left((\omega_1 + \omega_2)t\right)\right]$$

Signal Distortion in Transmission

$$\mathrm{y}(t) = \left(A_1^2/2\right)\left[1+\cos\left(2\omega_1 t\right)\right] + \left(A_2^2/2\right)\left[1+\cos\left(2\omega_2 t\right)\right]$$
$$+ A_1 A_2\left[\cos\left(\left(\omega_1-\omega_2\right)t\right)+\cos\left(\left(\omega_1+\omega_2\right)t\right)\right]$$

$\mathrm{y}(t)$ contains frequencies $2\omega_1$, $2\omega_2$, $\omega_1 - \omega_2$ and $\omega_1 + \omega_2$. The frequencies $\omega_1 - \omega_2$ and $\omega_1 + \omega_2$ are called **intermodulation distortion products**. When the excitation contains more frequencies (which it usually does) and the nonlinearity is of higher order (which it often is), many more intermodulation distortion products occur. All systems have nonlinearities and intermodulation disortion will occur. But, by careful design, it can often be reduced to a negligible level.

Transmission Loss and Decibels

Communication systems affect the power of a signal. If the signal power at the input is P_{in} and the signal power at the output is P_{out}, the **power gain** g of the system is $g = P_{out} / P_{in}$. It is very common to express this gain in **decibels**. A decibel is one-tenth of a **bel**, a unit named in honor of Alexander Graham Bell. The system gain g expressed in decibels would be $g_{dB} = 10\log_{10}\left(P_{out} / P_{in}\right)$.

g	0.1	1	10	100	1000	10,000	100,000
g_{dB}	-10	0	10	20	30	40	50

Because gains expressed in dB are logarithmic, they compress the range of numbers. If two systems are cascaded, the overall power gain is the product of the two individual power gains $g = g_1 g_2$. The overall power gain expressed in dB is the sum of the two power gains expressed in dB , $g_{dB} = g_{1,dB} + g_{2,dB}$.

Transmission Loss and Decibels

The decibel was defined based on a power ratio, but it is often used to indicate the power of a single signal. Two common types of power indication of this type are **dBW** and **dBm**. dBW is the power of a signal with reference to one watt. That is, a one watt signal would have a power expressed in dBW of 0 dBW. dBm is the power of a signal with reference to one milliwatt. A 20 mW signal would have a power expressed in dBm of 13.0103 dBm. Signal power gain as a function of frequency is the square of the magnitude of frequency response $\left|\mathrm{H}(f)\right|^2$. Frequency response magnitude is often expressed in dB also. $\left|\mathrm{H}(f)\right|_{dB} = 10\log_{10}\left(\left|\mathrm{H}(f)\right|^2\right) = 20\log_{10}\left(\left|\mathrm{H}(f)\right|\right)$.

Transmission Loss and Decibels

A communication system generally consists of components that amplify a signal and components that attenuate a signal. Any cable, optical or copper, attenuates the signal as it propagates. Also there are noise processes in all cables and amplifiers that generate random noise. If the power level gets too low, the signal power becomes comparable to the noise power and the fidelity of analog signals is degraded too far or the detection probability for digital signals becomes too low. So, before that signal level is reached, we must boost the signal power back up to transmit it further. Amplifiers used for this purpose are called **repeaters**.

Transmission Loss and Decibels

On a signal cable of 100's or 1000's of kilometers many repeaters will be needed. How many are needed depends on the **attenuation** per kilometer of the cable and the power gains of the repeaters. Attenuation will be symbolized by $L = 1/g = P_{in} / P_{out}$ or $L_{dB} = -g_{dB} = 10\log 10\left(P_{in} / P_{out}\right)$, (L for "loss".) For optical and copper cables the attenuation is typically exponential and $P_{out} = 10^{-\alpha l/10} P_{in}$ where l is the length of the cable and α is the **attenuation coefficient** in dB/unit length. Then $L = 10^{\alpha l/10}$ and $L_{dB} = \alpha l$.

Filters and Filtering

An **ideal** bandpass filter has the frequency response

$$\mathrm{H}(f) = \begin{cases} Ke^{-j\omega t_d} & , f_l \le |f| \le f_h \\ 0 & , \text{otherwise} \end{cases}$$

where f_l is the lower cutoff frequency and f_h is the upper cutoff frequency and K and t_d are constants. The filter's bandwidth is $B = f_h - f_l$. An ideal lowpass filter has the same frequency response but with $f_l = 0$ and $B = f_h$. An ideal highpass filter has the same frequency response but with $f_h \to \infty$ and $B \to \infty$. These filters are called ideal because they cannot actually be built. They cannot be built because they are non-causal. But they are useful fictions for introducing in a simplified way some of the concepts of communication systems.

Filters and Filtering

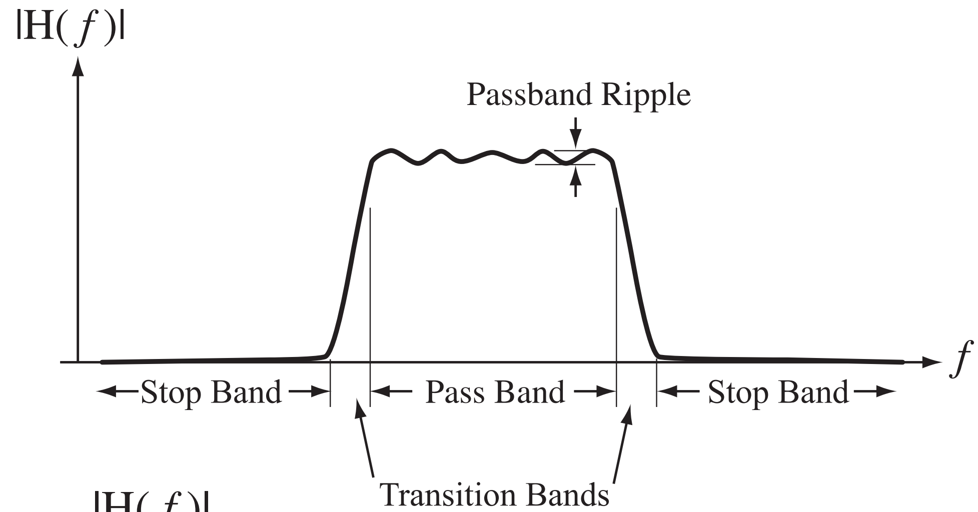
Strictly speaking a signal cannot be both bandlimited and timelimited. But many signals are almost bandlimited and timelimited. That is, many signals have very little signal energy outside a defined bandwidth and, at the same time, very little signal energy outside a defined time range. A good example of this is a Gaussian pulse

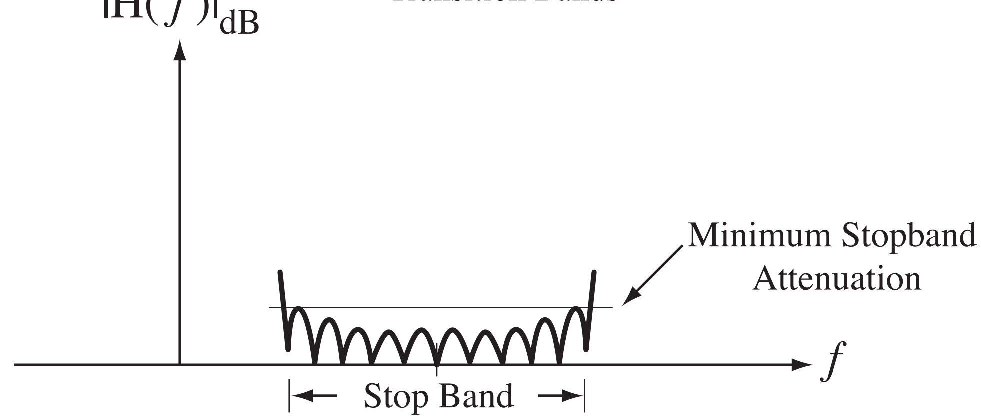
$$\mathrm{x}(t) = e^{-\pi t^2} \xleftrightarrow{\mathcal{F}} \mathrm{X}(f) = e^{-\pi f^2}$$

Strictly speaking, this signal is not bandlimited or timelimited. The total signal energy of this signal is $1/\sqrt{2}$. 99% of its energy lies in the time range $-0.74 < t < 0.74$ and in the frequency range $-0.74 < f < 0.74$. So in many practical calculations this signal could be considered both bandlimited and timelimited with very little error.

Filters and Filtering

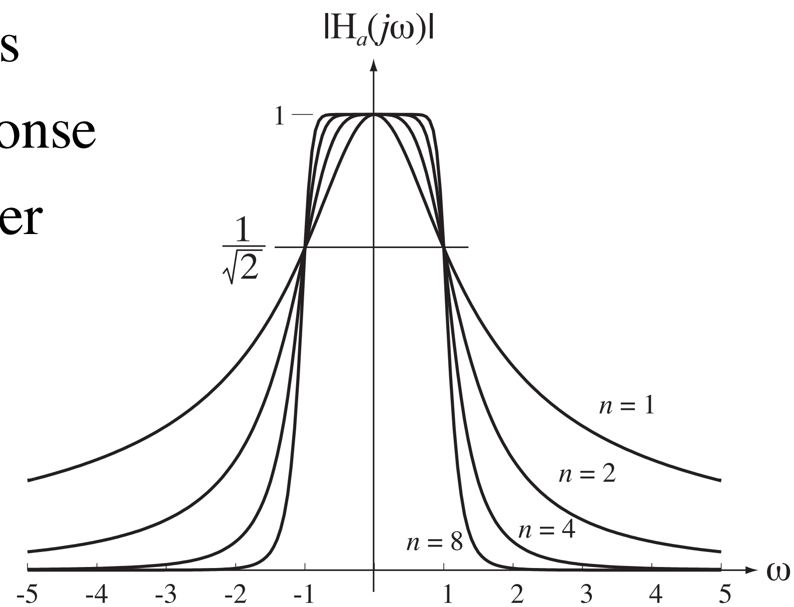
Real filters cannot have constant amplitude response and linear phase response in their passbands like ideal filters.





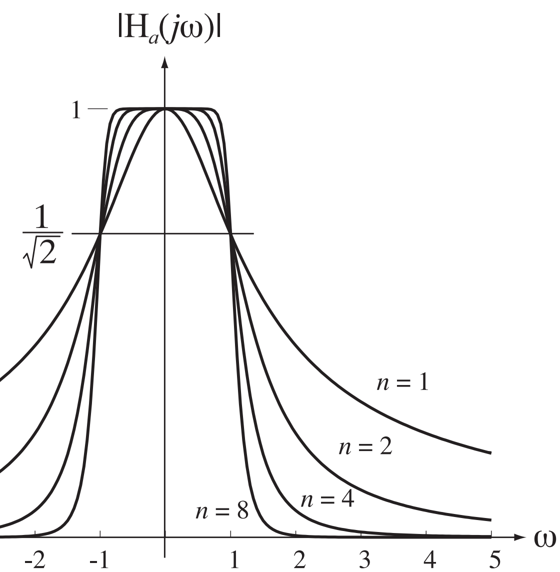
Filters and Filtering

There are many types of standardized filters. One very common and useful one is the **Butterworth** filter. The frequency response of a lowpass Butterworth filter is of the form $\left|\mathrm{H}(f)\right| = \dfrac{1}{\sqrt{1+(f/B)^{2n}}}$ where n is the **order** of the filter. As the order is increased, its magnitude response approaches that of an ideal filter, constant in the passband and zero outside the passband. (Below is illustrated the magnitude frequency response of a normalized lowpass Butterworth filter with a corner frequency of 1 radian/s.)



Filters and Filtering

The Butterworth filter is said to be **maximally flat** in its passband. It is given this description because the first n derivatives of its magnitude frequency response are all zero at $f = 0$ (for a lowpass filter). The passband of a lowpass Butterworth filter is defined as the frequency at which its magnitude frequency response is reduced from its maximum by a factor of $1/\sqrt{2}$. This is also known as its **half-power** bandwidth because, at this frequency the power gain of the filter is half its maximum value.



Filters and Filtering

The step response of a filter is

$$\mathrm{h}_{-1}(t) = \int_{-\infty}^{\infty} \mathrm{h}(\lambda)\mathrm{u}(t-\lambda)\,d\lambda = \int_{-\infty}^{t} \mathrm{h}(\lambda)\,d\lambda$$

($\mathrm{g}(t)$ in the book). That is, the step response is the integral of the impulse response. The impulse response of a unity-gain ideal lowpass filter with no delay is $\mathrm{h}(t) = 2B\,\mathrm{sinc}(2Bt)$ where B is its bandwidth. Its step response is therefore

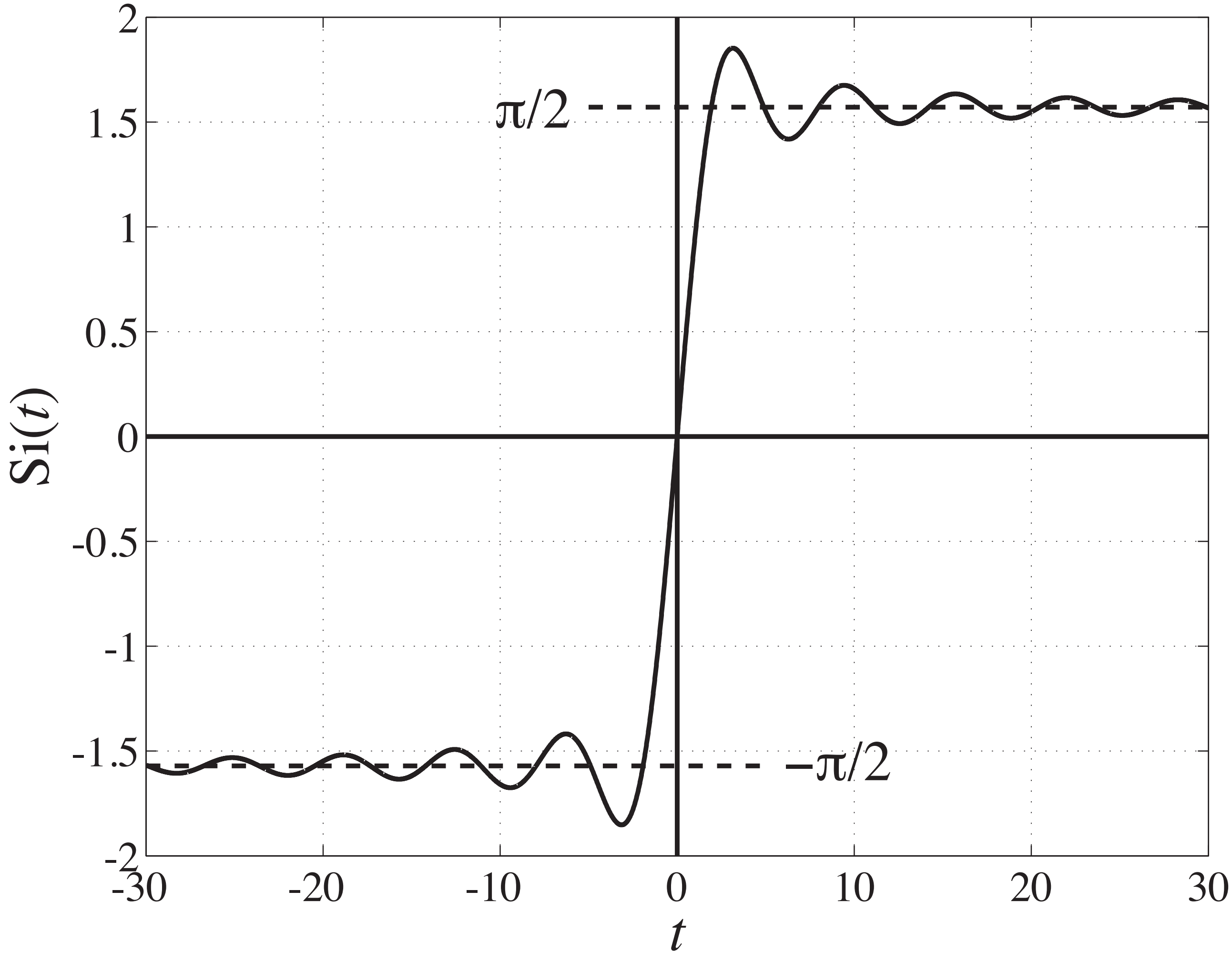
$$\mathrm{h}_{-1}(t) = \int_{-\infty}^{t} 2B\,\mathrm{sinc}(2B\lambda)\,d\lambda = 2B\left[\int_{-\infty}^{0} \mathrm{sinc}(2B\lambda)\,d\lambda + \int_{0}^{t} \mathrm{sinc}(2B\lambda)\,d\lambda\right]$$

This result can be further simplified by using the definition of the **sine integral function**

$$\mathrm{Si}(\theta) \triangleq \int_{0}^{\theta} \frac{\sin(\alpha)}{\alpha}\,d\alpha = \pi \int_{0}^{\theta/\pi} \mathrm{sinc}(\lambda)\,d\lambda$$

Filters and Filtering

The Sine Integral Function



Filters and Filtering

$$h_{-1}(t) = 2B\left[\int_{-\infty}^{0} \text{sinc}(2B\lambda)\,d\lambda + \int_{0}^{t} \text{sinc}(2B\lambda)\,d\lambda\right]$$

Let $2B\lambda = \alpha$. Then $h_{-1}(t) = \int_{-\infty}^{0} \text{sinc}(\alpha)\,d\alpha + \int_{0}^{2Bt} \text{sinc}(\alpha)\,d\alpha$.

Using the fact that sinc is an even function, $\int_{-\infty}^{0} \text{sinc}(\alpha)\,d\alpha = \int_{0}^{\infty} \text{sinc}(\alpha)\,d\alpha$.

Then, using $\text{Si}(\theta) = \pi \int_{0}^{\theta/\pi} \text{sinc}(\alpha)\,d\alpha$ and $\text{Si}(\infty) = \pi/2$, we get

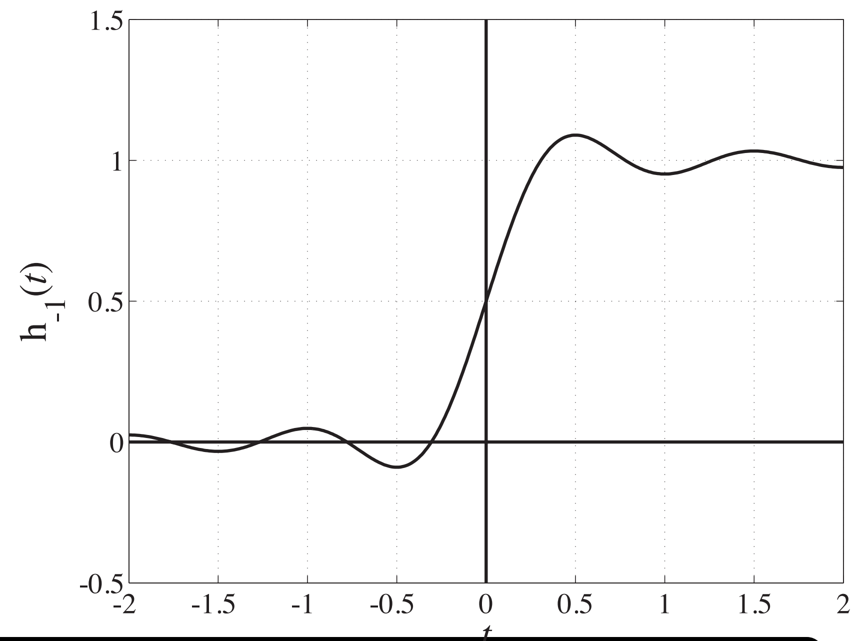
$$h_{-1}(t) = \frac{\text{Si}(\infty)}{\pi} + \frac{1}{\pi}\text{Si}(2\pi Bt) = \frac{1}{2} + \frac{1}{\pi}\text{Si}(2\pi Bt)$$

Filters and Filtering

$$h_{-1}(t) = \frac{1}{2} + \frac{1}{\pi}\text{Si}(2\pi B t)$$

This step response has **precursors**, **overshoot**, and **oscillations** (**ringing**). **Risetime** is defined as the time required to move from 10% of the final value to 90% of the final value. For this ideal lowpass filter the rise time is $0.44/B$. The rise time for a single-pole, lowpass filter is $0.35/B$.

Step response of an Ideal Lowpass Filter with $B = 1$ →

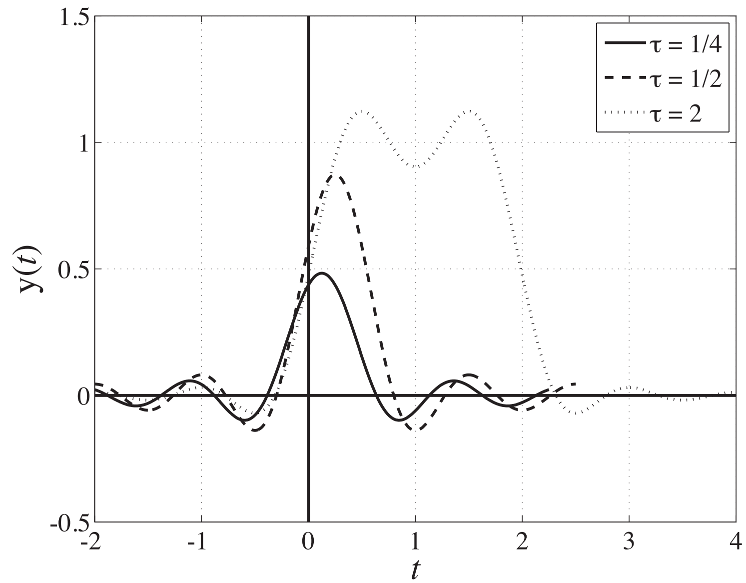


Filters and Filtering

The response of an ideal lowpass filter to a rectangular pulse of width τ is

$$y(t) = h_{-1}(t) - h_{-1}(t-\tau) = \frac{1}{\pi}\left[\mathrm{Si}(2\pi Bt) - \mathrm{Si}(2\pi B(t-\tau))\right].$$

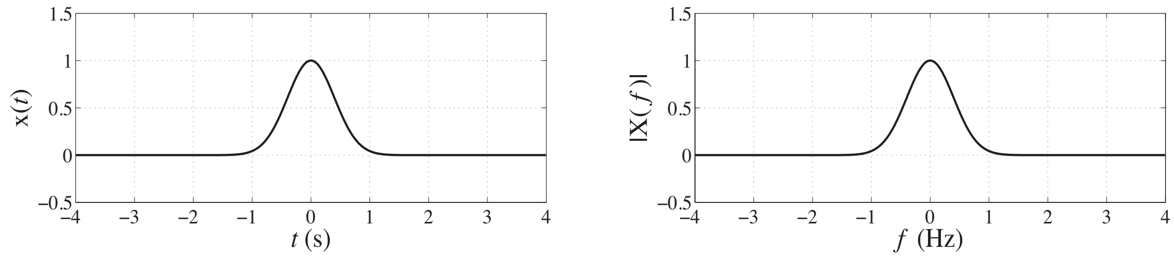
From the graph (in which $B = 1$) we see that, to reproduce the rectangular pulse shape, even very crudely, requires a bandwidth much greater than $1/\tau$. If we have a pulse train with pulse widths τ and spaces between pulses also τ and we want to simply detect whether or not a pulse is present at some time, we will need at least $B \geq 1/2\tau$. If the bandwidth is any lower the overlap between pulses makes them very hard to resolve.



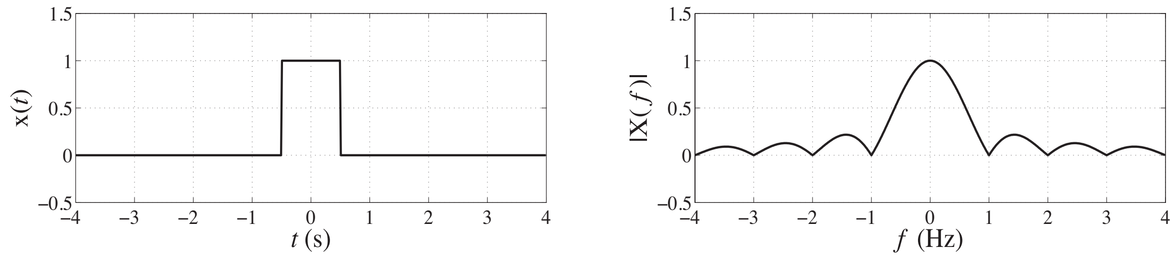
Pulse Width and Bandwidth

Pulses (and their Fourier transforms) can have many shapes

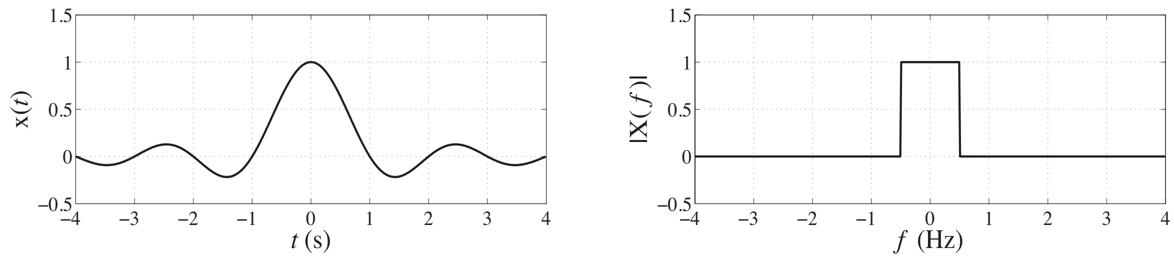
Infinite pulse width
and bandwidth



Finite pulse width and
infinite bandwidth

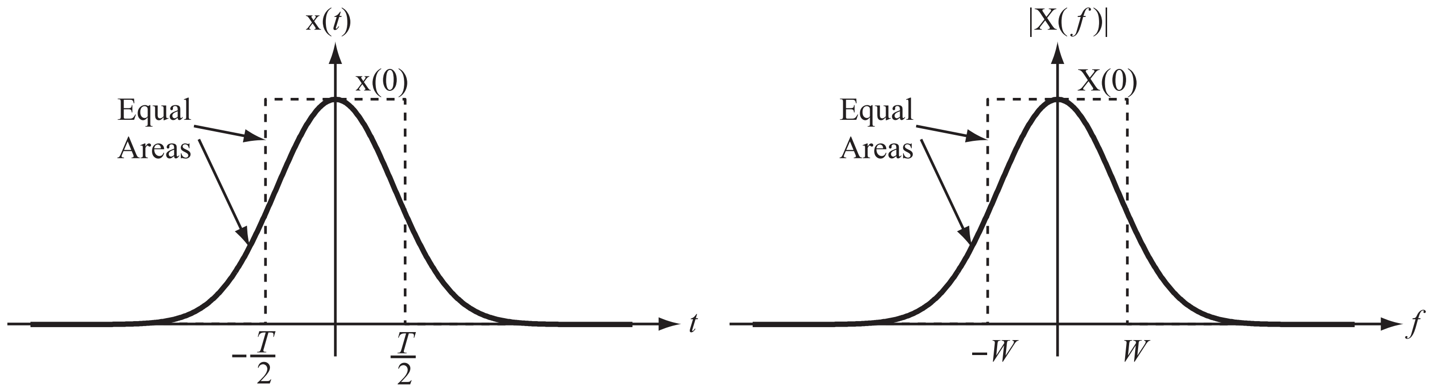


Infinite pulse width and
finite bandwidth



Pulse Width and Bandwidth

We need a practical general relationship between pulse width and bandwidth.



Let the rectangular pulse approximate the general pulse with the same height and area. Then $T\,\mathrm{x}(0) = \int_{-\infty}^{\infty} |\mathrm{x}(t)|\,dt \geq \int_{-\infty}^{\infty} \mathrm{x}(t)\,dt = \mathrm{X}(0)$.

Let the rectangular bandwidth approximate the general pulse bandwidth with the same height and area. Then $2W\,\mathrm{X}(0) = \int_{-\infty}^{\infty} |\mathrm{X}(f)|\,df \geq \int_{-\infty}^{\infty} \mathrm{X}(f)\,df = \mathrm{x}(0)$.

Pulse Width and Bandwidth

Now we have the relationships

$$\frac{\mathrm{x}(0)}{\mathrm{X}(0)} \geq \frac{1}{T} \text{ and } 2W \geq \frac{\mathrm{x}(0)}{\mathrm{X}(0)}$$

which combine to $2W \geq \dfrac{1}{T}$ or $W \geq \dfrac{1}{2T}$. This is a handy, practical rule of thumb for the approximate bandwidth of a pulse.

Quadrature Filters and Hilbert Transforms

A **quadrature filter** is an allpass network that shifts the phase of positive frequency components by $-90°$ and negative frequency components by $+90°$. Its frequency response is therefore

$$\mathrm{H}_Q(f) = \begin{cases} -j & , f > 0 \\ j & , f < 0 \end{cases} = -j\,\mathrm{sgn}(f).$$

Its magnitude is one at all frequencies, therefore an even function of f and its phase is an odd function of f. The inverse Fourier transform of $\mathrm{H}_Q(f)$ is the impulse response $\mathrm{h}_Q(t) = 1/\pi t$. The **Hilbert transform** $\hat{\mathrm{x}}(t)$ of a signal $\mathrm{x}(t)$ is defined as the response of a quadrature filter to $\mathrm{x}(t)$. That is $\hat{\mathrm{x}}(t) = \mathrm{x}(t) * \mathrm{h}_Q(t) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} \dfrac{\mathrm{x}(\lambda)}{t-\lambda}\,d\lambda.$

$$\mathscr{F}\left(\hat{\mathrm{x}}(t)\right) = -j\,\mathrm{sgn}(f)\,\mathrm{X}(f)$$

Quadrature Filters and Hilbert Transforms

The impulse response of a quadrature filter $\mathrm{h}_Q(t) = 1/\pi t$ is non-causal. That means it is physically unrealizable. Some important properties of the Hilbert transform are

1. The Fourier transforms of a signal and its Hilbert transform have the same magnitude. Therefore the signal and its Hilbert transform have the same signal energy.
2. If $\hat{\mathrm{x}}(t)$ is the Hilbert transform of $\mathrm{x}(t)$ then $-\mathrm{x}(t)$ is the Hilbert transform of $\hat{\mathrm{x}}(t)$.
3. A signal $\mathrm{x}(t)$ and its Hilbert transform are orthogonal on the entire real line. That means for energy signals $\int_{-\infty}^{\infty} \mathrm{x}(t)\hat{\mathrm{x}}(t)\,dt = 0$ and for power signals $\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathrm{x}(t)\hat{\mathrm{x}}(t)\,dt = 0$.

Quadrature Filters and Hilbert Transforms

$\mathrm{g}(t)$	$\hat{\mathrm{g}}(t)$
$a_1\,\mathrm{g}_1(t)+a_2\,\mathrm{g}_2(t);\ a_1,a_2\in\mathbb{C}$	$a_1\,\hat{\mathrm{g}}_1(t)+a_2\,\hat{\mathrm{g}}_2(t)$
$\mathrm{h}(t-t_0)$	$\hat{\mathrm{h}}(t-t_0)$
$\mathrm{h}(at);a\neq 0$	$\mathrm{sgn}(a)\hat{\mathrm{h}}(at)$
$\frac{d}{dt}(\mathrm{h}(t))$	$\frac{d}{dt}(\hat{\mathrm{h}}(t))$
$\delta(t)$	$\frac{1}{\pi t}$
e^{jt}	$-je^{jt}$
e^{-jt}	je^{-jt}
$\cos(t)$	$\sin(t)$
$\mathrm{rect}(t)$	$\frac{1}{\pi}\ln\left\|\frac{2t+1}{2t-1}\right\|$
$\mathrm{sinc}(t)$	$(\pi t/2)\mathrm{sinc}^2(t/2)=\sin(\pi t/2)\mathrm{sinc}(t/2)$
$\frac{1}{1+t^2}$	$\frac{t}{1+t^2}$

Analytic Signals and Complex Envelopes

An analytic signal $x_p(t)$ corresponding to a real signal $x(t)$ is defined by $x_p(t) = x(t) + j\hat{x}(t)$. The envelope of a signal $x(t)$ is defined as the magnitude of the analytic signal $x_p(t)$. It follows that

$$X_p(f) = X(f) + j \times (-j)\operatorname{sgn}(f)X(f) = X(f)\left[1 + \operatorname{sgn}(f)\right] = 2X(f)u(f)$$

Therefore $X_p(f) = \begin{cases} 2X(f) & , f > 0 \\ 0 & , f < 0 \end{cases}$. Similarly,

$x_n(t) = x(t) - j\hat{x}(t)$ and $X_n(f) = 2X(f)u(-f) = \begin{cases} 0 & , f > 0 \\ 2X(f) & , f < 0 \end{cases}$.

Analytic Signals and Complex Envelopes

The complex envelope of a real signal $\mathrm{x}(t)$ is defined as

$\tilde{\mathrm{x}}(t) = \mathrm{x}_p(t)e^{-j2\pi f_0 t}$ where f_0 is a reference frequency chosen for convenience. Therefore $\mathrm{x}_p(t) = \tilde{\mathrm{x}}(t)e^{j2\pi f_0 t} = \mathrm{x}(t) + j\hat{\mathrm{x}}(t)$

and $\mathrm{x}(t) = \mathrm{Re}\left(\tilde{\mathrm{x}}(t)e^{j2\pi f_0 t}\right)$ and $\hat{\mathrm{x}}(t) = \mathrm{Im}\left(\tilde{\mathrm{x}}(t)e^{j2\pi f_0 t}\right)$.

$$\mathrm{x}(t) = \mathrm{Re}\left(\tilde{\mathrm{x}}(t)\left(\cos(2\pi f_0 t) + j\sin(2\pi f_0 t)\right)\right)$$

$$\mathrm{x}(t) = \mathrm{Re}\begin{pmatrix} \mathrm{Re}(\tilde{\mathrm{x}}(t))\cos(2\pi f_0 t) + j\,\mathrm{Im}(\tilde{\mathrm{x}}(t))\cos(2\pi f_0 t) \\ + j\mathrm{Re}(\tilde{\mathrm{x}}(t))\sin(2\pi f_0 t) + \underbrace{j \times j}_{=-1}\mathrm{Im}(\tilde{\mathrm{x}}(t))\sin(2\pi f_0 t) \end{pmatrix}$$

$$\mathrm{x}(t) = \mathrm{x}_R(t)\cos(2\pi f_0 t) - \mathrm{x}_I(t)\sin(2\pi f_0 t)$$

where $\mathrm{x}_R(t) = \mathrm{Re}(\tilde{\mathrm{x}}(t))$ and $\mathrm{x}_I(t) = \mathrm{Im}(\tilde{\mathrm{x}}(t))$,

$\tilde{\mathrm{x}}(t) = \mathrm{x}_R(t) + j\mathrm{x}_I(t)$, $\mathrm{x}_R(t)$ is the "in-phase" component of $\mathrm{x}(t)$ and $\mathrm{x}_I(t)$ is the "quadrature" component of $\mathrm{x}(t)$.

Analytic Signals and Complex Envelopes

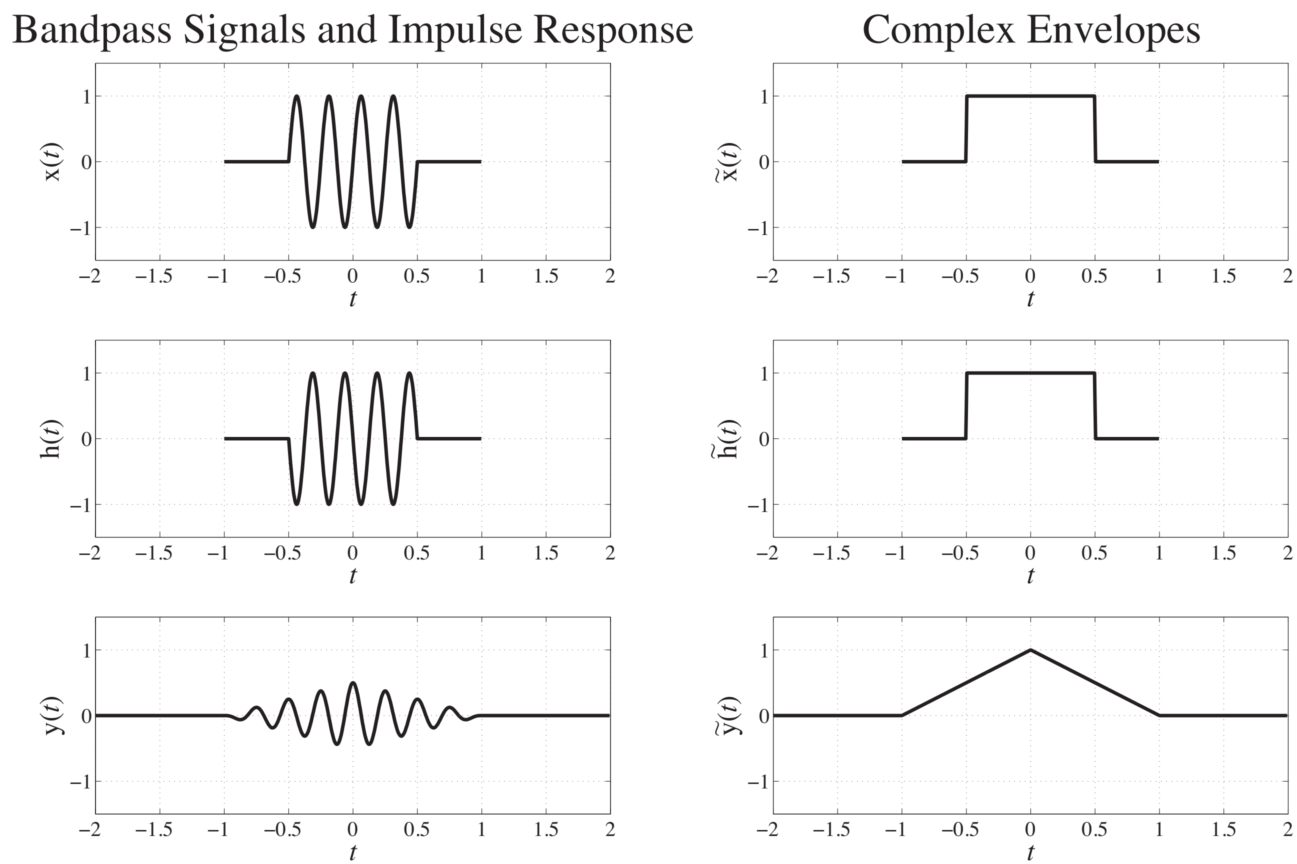
It can be shown (page 89 in the text) that if a system has a bandpass response with impulse response $\mathrm{h}(t)$ and it is excited by a bandpass signal $\mathrm{x}(t)$, that the complex envelope of the system response is $\tilde{\mathrm{y}}(t) = \tilde{\mathrm{x}}(t) * \tilde{\mathrm{h}}(t) = \mathcal{F}^{-1}\left(\tilde{\mathrm{X}}(f)\tilde{\mathrm{H}}(f)\right)$ and the system response is $\mathrm{y}(t) = \frac{1}{2}\mathrm{Re}\left(\tilde{\mathrm{y}}(t)e^{j2\pi f_0 t}\right)$. (The term "bandpass" means that there is a finite-width band of frequencies, including $f = 0$, in which the Fourier magnitude spectrum is zero or, as a practical matter, small enough to be considered negligible.)

Analytic Signals and Complex Envelopes

In the previous two slides a real signal $\mathrm{x}(t)$ was related to its complex envelope $\tilde{\mathrm{x}}(t)$ by $\mathrm{x}(t) = \mathrm{Re}\left(\tilde{\mathrm{x}}(t) e^{j2\pi f_0 t}\right)$ and a real system impulse response $\mathrm{h}(t)$ was related to its complex envelope $\tilde{\mathrm{h}}(t)$ by $\mathrm{h}(t) = \mathrm{Re}\left(\tilde{\mathrm{h}}(t) e^{j2\pi f_0 t}\right)$. But then when $\mathrm{x}(t)$ is applied to the system and the response is $\mathrm{y}(t)$, we found $\tilde{\mathrm{y}}(t) = \tilde{\mathrm{x}}(t) * \tilde{\mathrm{h}}(t)$ and related it to $\mathrm{y}(t)$ by $\mathrm{y}(t) = \frac{1}{2}\mathrm{Re}\left(\tilde{\mathrm{y}}(t) e^{j2\pi f_0 t}\right)$. Where did the factor of $\frac{1}{2}$ come from? It can be seen in the derivation on page 89. But it can also be seen in concept by looking at what happens when we convolve a bandpass signal and a bandpass impulse response and compare that to convolving the corresponding complex envelopes.

Analytic Signals and Complex Envelopes

Let $\mathrm{x}(t)=\Pi(t)\sin(8\pi t)$ and $\mathrm{h}(t)=-\Pi(t)\sin(8\pi t)$. The complex envelope of $\mathrm{x}(t)$ has twice the signal energy of $\mathrm{x}(t)$. The same is true for $\mathrm{h}(t)$. As a result, the complex envelope of $\mathrm{y}(t)$ has four times the signal energy of $\mathrm{y}(t)$.



Analytic Signals and Complex Envelopes

Example

Let $\mathrm{x}(t) = \mathrm{sinc}(t)\cos(4\pi t)$.

Then $\mathrm{X}(f) = \mathrm{rect}(f) * \frac{1}{2}\left[\delta(f-2) + \delta(f+2)\right]$

$\mathrm{X}(f) = \frac{1}{2}\left[\mathrm{rect}(f-2) + \mathrm{rect}(f+2)\right]$ and $\hat{\mathrm{X}}(f) = -j\,\mathrm{sgn}(f)\mathrm{X}(f)$.

$$\hat{\mathrm{X}}(f) = -\frac{j}{2}\mathrm{rect}(f-2) + \frac{j}{2}\mathrm{rect}(f+2)$$

$$= \mathrm{rect}(f) * \frac{j}{2}\left[\delta(f+2) - \delta(f-2)\right]$$

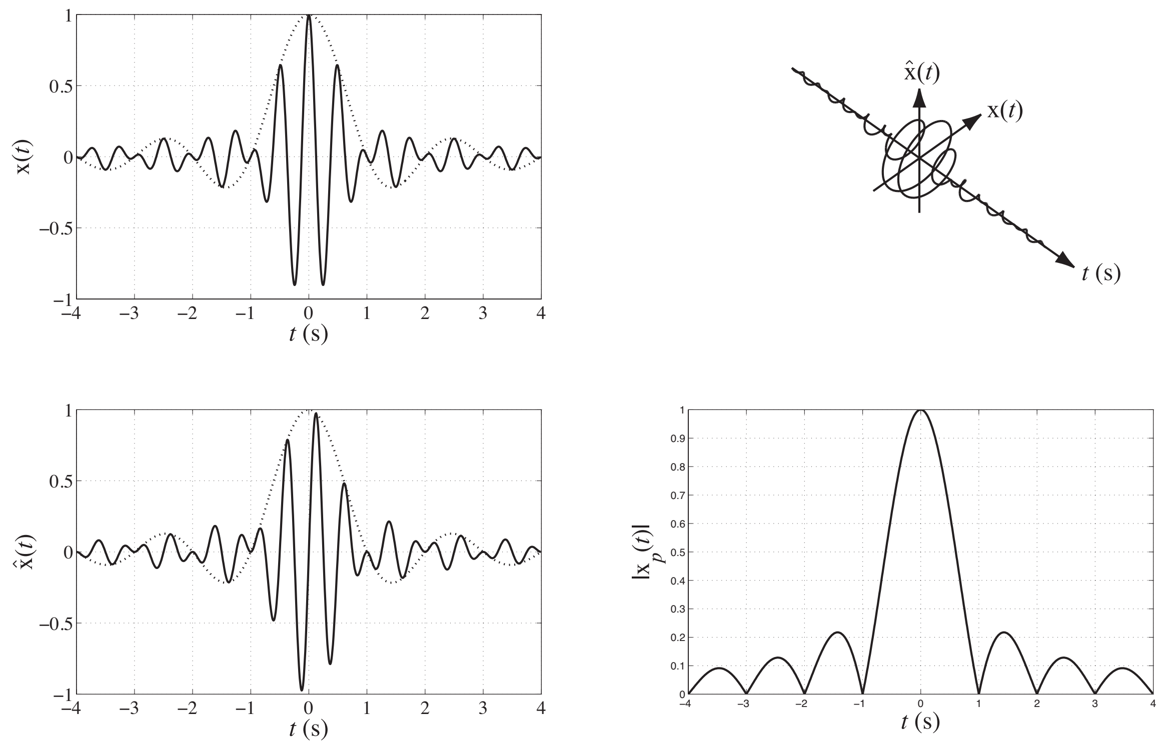
$\hat{\mathrm{x}}(t) = \mathrm{sinc}(t)\sin(4\pi t)$.

Analytic Signals and Complex Envelopes

$$x(t) = \text{sinc}(t)\cos(4\pi t) \quad \hat{x}(t) = \text{sinc}(t)\sin(4\pi t)$$

$$x_p(t) = \text{sinc}(t)\left[\cos(4\pi t) + j\sin(4\pi t)\right]$$

$|x_p(t)|$ is the envelope of $x(t)$. The concept of an envelope will be very useful later in the exploration of modulation techniques.



Analytic Signals and Complex Envelopes

Example 2.32 in the text :

Let $\mathrm{x}(t) = \Pi\left(\frac{t}{\tau}\right)\cos(2\pi f_0 t)$ and let $\mathrm{h}(t) = \alpha e^{-\alpha t}\,\mathrm{u}(t)\cos(2\pi f_0 t)$.

Find the system output signal $\mathrm{y}(t)$ using complex envelope techniques.

$$\mathrm{x}_p(t) = \mathrm{x}(t) + j\,\hat{\mathrm{x}}(t) = \Pi\left(\frac{t}{\tau}\right)\cos(2\pi f_0 t) + j\Pi\left(\frac{t}{\tau}\right)\sin(2\pi f_0 t)$$

$\mathrm{x}_p(t) = \Pi\left(\frac{t}{\tau}\right)e^{j2\pi f_0 t} \Rightarrow \tilde{\mathrm{x}}(t) = \mathrm{Re}\left(\mathrm{x}_p(t)e^{-j2\pi f_0 t}\right) = \Pi\left(\frac{t}{\tau}\right)$. Similarly,

$\tilde{\mathrm{h}}(t) = \alpha e^{-\alpha t}\,\mathrm{u}(t)$. Therefore $\tilde{\mathrm{y}}(t) = \tilde{\mathrm{x}}(t) * \tilde{\mathrm{h}}(t) = \Pi\left(\frac{t}{\tau}\right) * \alpha e^{-\alpha t}\,\mathrm{u}(t)$

$$\tilde{\mathrm{y}}(t) = \int_{-\infty}^{\infty} \Pi\left(\frac{\lambda}{\tau}\right)\alpha e^{-\alpha(t-\lambda)}\,\mathrm{u}(t-\lambda)\,d\lambda$$

Analytic Signals and Complex Envelopes

$$\tilde{\mathrm{y}}(t)=\alpha\left[\begin{array}{l}\displaystyle\int_{-\infty}^{\infty}\mathrm{u}(\lambda+\tau/2)e^{-\alpha(t-\lambda)}\mathrm{u}(t-\lambda)\,d\lambda\\ \displaystyle-\int_{-\infty}^{\infty}\mathrm{u}(\lambda-\tau/2)e^{-\alpha(t-\lambda)}\mathrm{u}(t-\lambda)\,d\lambda\end{array}\right]$$

$$\tilde{\mathrm{y}}(t)=\left[\left(1-e^{-\alpha(t+\tau/2)}\right)\mathrm{u}(t+\tau/2)-\left(1-e^{-\alpha(t-\tau/2)}\right)\mathrm{u}(t-\tau/2)\right]$$

$$\mathrm{y}(t)=\frac{1}{2}\mathrm{Re}\left(\tilde{\mathrm{y}}(t)e^{j2\pi f_0 t}\right)$$

$$\mathrm{y}(t)=\frac{1}{2}\mathrm{Re}\left(\left(\left(1-e^{-\alpha(t+\tau/2)}\right)\mathrm{u}(t+\tau/2)-\left(1-e^{-\alpha(t-\tau/2)}\right)\mathrm{u}(t-\tau/2)\right)e^{j2\pi f_0 t}\right)$$

$$\mathrm{y}(t)=\frac{1}{2}\left[\left(1-e^{-\alpha(t+\tau/2)}\right)\mathrm{u}(t+\tau/2)-\left(1-e^{-\alpha(t-\tau/2)}\right)\mathrm{u}(t-\tau/2)\right]\cos(2\pi f_0 t)$$

Energy Spectral Density

According to Parseval's Theorem, the signal energy of a signal can be found directly from its Fourier transform, $E_x = \int_{-\infty}^{\infty} |\mathrm{X}(f)|^2 \, df$. $\mathrm{X}(f)$ indicates the variation of the amplitudes of the complex sinusoidal components of $\mathrm{x}(t)$ as a function of their frequencies, f. So the units are $\frac{\mathrm{V}}{\mathrm{Hz}}$ (if $\mathrm{x}(t)$ is a voltage signal). Therefore the units of $|\mathrm{X}(f)|^2$ must be $\left(\frac{\mathrm{V}}{\mathrm{Hz}}\right)^2$. When we integrate $|\mathrm{X}(f)|^2$ over all frequencies we get signal energy whose units are $\left(\frac{\mathrm{V}}{\mathrm{Hz}}\right)^2 \mathrm{Hz}$ or $\mathrm{V}^2 \cdot \mathrm{s}$, the units of signal energy.

Energy Spectral Density

Therefore $\left|X(f)\right|^2$ is the density of signal energy as a function of frequency. It is known as **Energy Spectral Density** (**ESD**). The term "spectral density" means "variation with respect to frequency". A common symbol for ESD is $G(f) = \left|X(f)\right|^2$.

Power Spectral Density

Energy Spectral Density is the variation of signal energy with frequency. It applies to energy signals. The corresponding quantity that applies to power signals is **Power Spectral Density** (**PSD**).

Power spectral density $\mathrm{S}(f)$ is defined by $P = \int_{-\infty}^{\infty} \mathrm{S}(f)\,df$. That is, its integral over all frequency yields total average signal power. Therefore $\mathrm{S}(f)$ indicates the variation of average signal power as a function of frequency.

Autocorrelation

Suppose we take the inverse Fourier transform of energy spectral density

$$\mathscr{F}^{-1}\left(G(f)\right) = \mathscr{F}^{-1}\left(\left|X(f)\right|^2\right) = \mathscr{F}^{-1}\left(X(f)X^*(f)\right)$$

$$= \underbrace{\mathscr{F}^{-1}\left(X(f)\right)}_{=x(t)} * \underbrace{\mathscr{F}^{-1}\left(X^*(f)\right)}_{=x(-t)}$$

So $\mathscr{F}^{-1}\left(G(f)\right) = x(t) * x(-t) = \int_{-\infty}^{\infty} x(\tau)x(t+\tau)\,d\tau.$

We can exchange the meanings of t and τ to form

$$\phi(\tau) = x(\tau) * x(-\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)\,dt$$

This integral is the area under the product of a function x and a version of x that has been shifted to the left by τ, as a function of the shift amount. This function is called **autocorrelation**.

Autocorrelation

The autocorrelation function indicates how similar a signal is to itself when shifted. When the shift is zero (τ=0), $\phi(0) = \int_{-\infty}^{\infty} \mathrm{x}^2(t)\,dt$ which is the signal energy. If the shift τ is small and the value of $\phi(\tau)$ does not change much, we say there is a strong correlation between x and the shifted version of x for small shifts. So a slowly changing $\phi(\tau)$ indicates that the signal still looks like itself even when shifted a significant amount. A quickly changing $\phi(\tau)$ indicates that even a small shift makes the signal look very different.

Autocorrelation

The definition $\phi(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)\,dt$ applies to energy signals.

For power signals, the definition of autocorrelation is $R(\tau) = \langle x(t)x(t+\tau) \rangle$ and $R(\tau) \xleftrightarrow{\mathcal{F}} S(f)$. Some properties of autocorrelation are

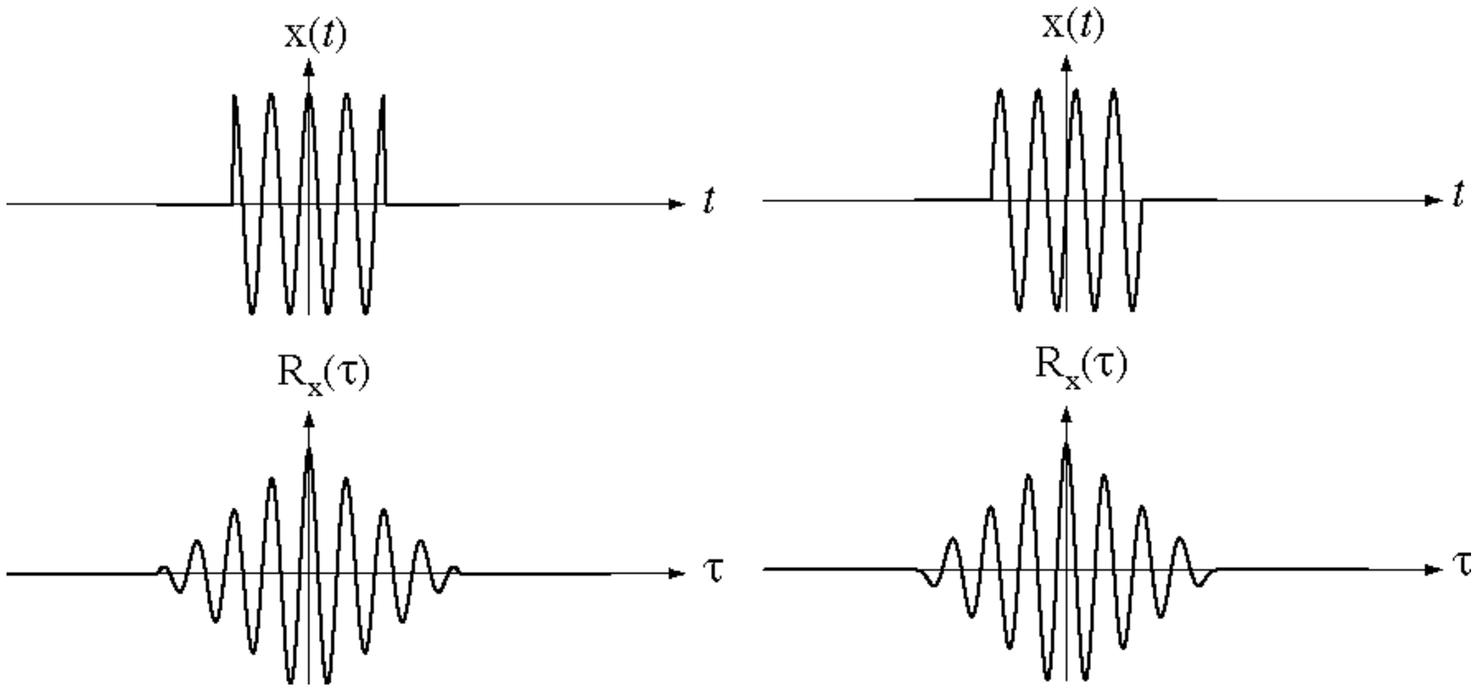
1. $R(0) = \int_{-\infty}^{\infty} S(f)\,df$ = total average signal power
2. $R(0) \geq R(\tau)$, autocorrelation can never exceed the signal power
3. $R(\tau)$ is always an even function, that is $R(\tau) = R(-\tau)$
4. $\mathcal{F}(R(\tau))$ is everywhere non-negative
5. If $x(t)$ is periodic then $R_x(\tau)$ is also, with the same period
6. If $x(t)$ contains no periodic components $\lim_{|\tau|\to\infty} R_x(\tau) = \langle x(t) \rangle^2$

Autocorrelation Examples

Autocorrelations of a cosine and sine "burst".

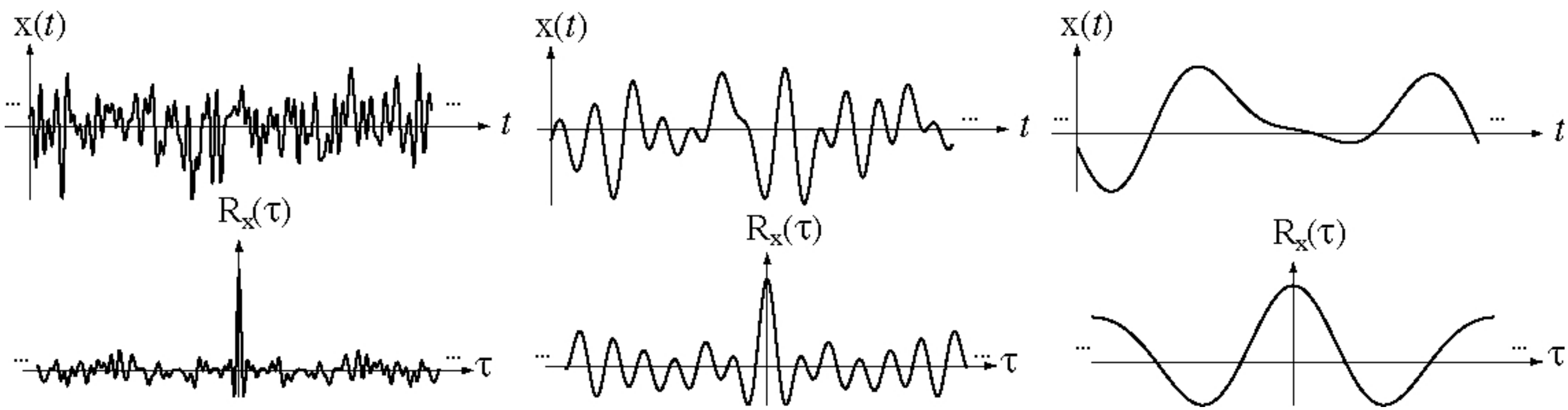
They are very similar but not exactly the same.

Notice that both are even functions, even though cosine is even and sine is odd.



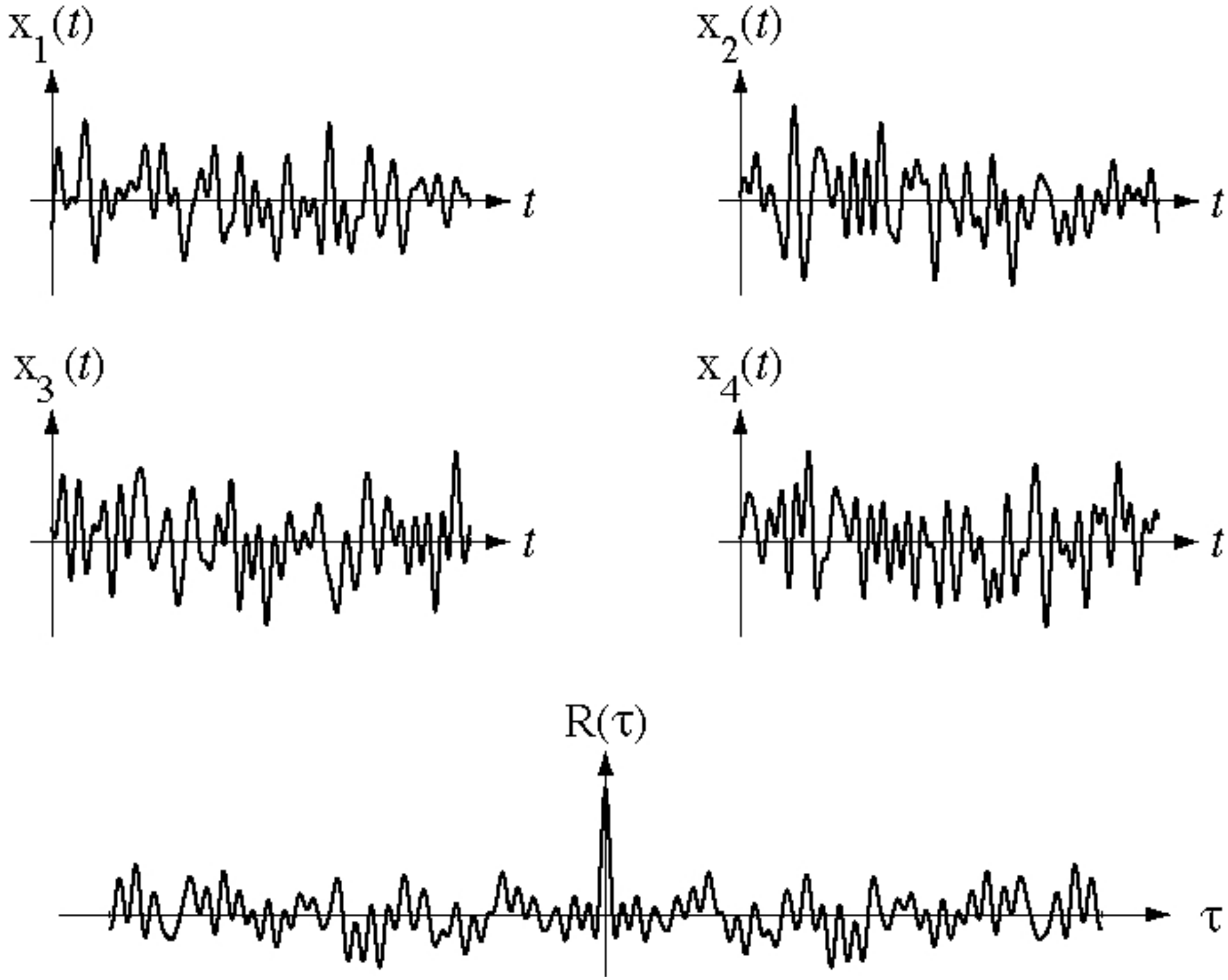
Autocorrelation Examples

Three random power signals with different frequency content and their autocorrelations.



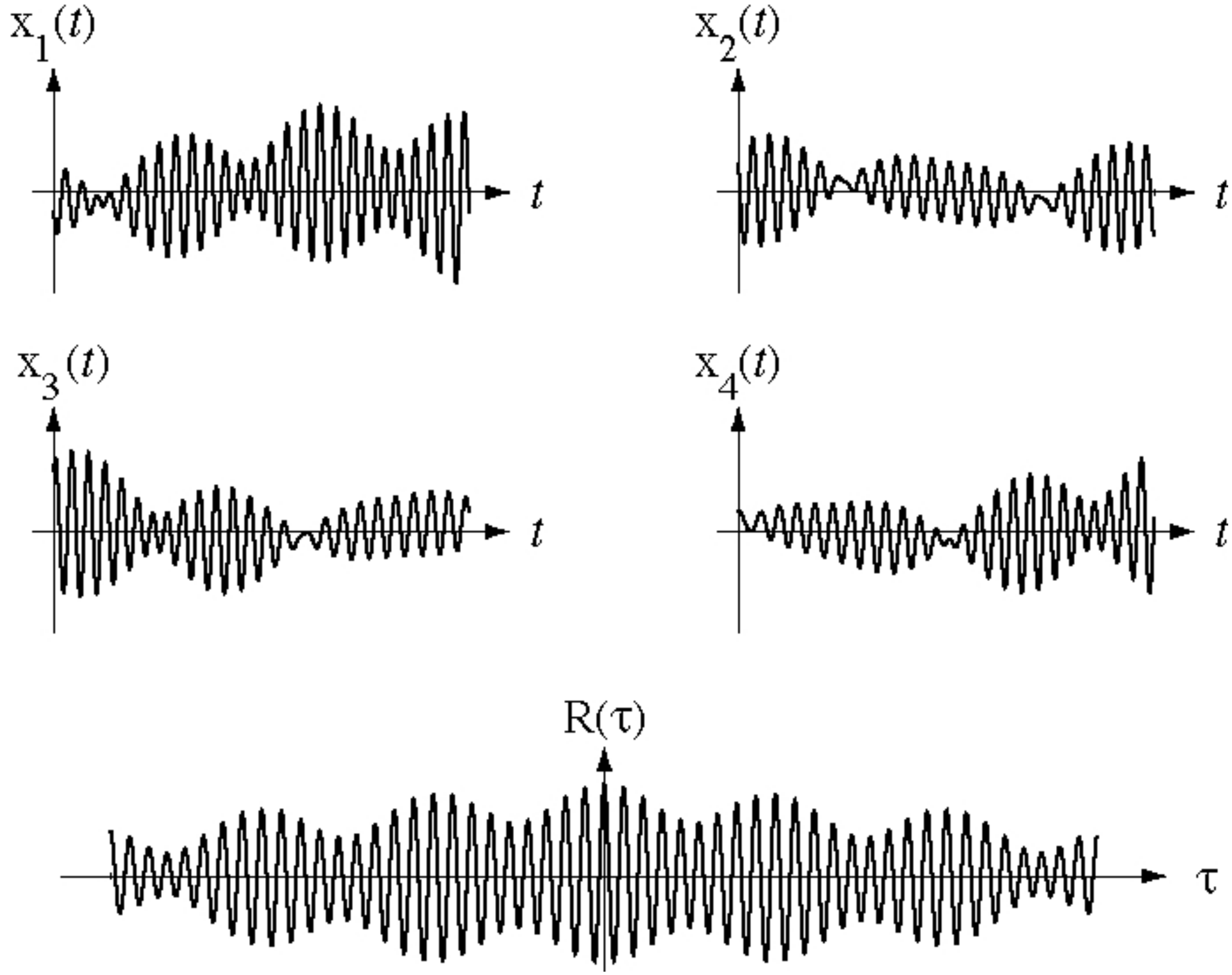
Autocorrelation Examples

Four Different Random Signals with Identical Autocorrelations



Autocorrelation Examples

Four Different Random Signals with Identical Autocorrelations



Sampling

Uniform sampling of a continuous-time signal can be represented by multiplying the signal by a periodic impulse, forming a signal consisting only of impulses.

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

where T_s is the time between samples. When sampling a signal, the salient question is always whether the original continuous-time signal can be recovered from the samples. The sampling theorem says that if the signal is sampled for all time at a rate greater than twice the highest frequency in the signal, the original signal can be recovered exactly from the samples.

Sampling

Sampling Theorem: If the signal is sampled for all time at a rate greater than twice the highest frequency in the signal, the original signal can be recovered exactly from the samples.

Practically speaking, a signal can never be sampled for all time. Also if a signal is not bandlimited its highest frequency is infinite, requiring an infinite sampling rate, and no real signal can be bandlimited because all real signals are time limited. Therefore the sampling theorem can never quite be satisfied. The sampling theorem really just serves as a limiting requirement to be approached but never reached in practice. Practically, sampling always yields an approximation, but one which can often be very good.

Sampling

The sampling rate that is twice the highest rate in a signal is called the **Nyquist rate** in honor of Harry Nyquist, one of the earliest contributors to sampling theory.

There is a variation on the sampling theorem for signals that are narrowband. That is, signals whose center frequency is much greater than the bandwidth. If the bandwidth is W and the highest frequency is f_u and the signal is sampled at a rate $f_s = 2f_u / m$ where m is the greatest integer in f_u / W, then the signal can be recovered from the samples by passing it through a bandpass filter.

