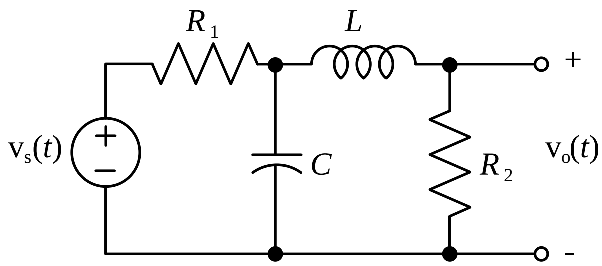
Chapter 10 - Laplace Transform Analysis of Signals and Systems

Selected Solutions

1. For each circuit write the transfer function between the indicated excitation and indicated response. Express each transfer function in the standard form,

$$\mathrm{H}(s) = A\frac{s^N + b_{N-1}s^{N-1} + \cdots + b_2 s^2 + b_1 s + b_0}{s^D + a_{D-1}s^{D-1} + \cdots + a_2 s^2 + a_1 s + a_0} .$$

(a) Excitation: $\mathrm{v}_s(t)$ Response: $\mathrm{v}_o(t)$



$$\mathrm{H}(s) = \frac{\mathrm{Z}_{RLC}(s)}{\mathrm{Z}_{RLC}(s) + R_1}\frac{R_2}{sL + R_2}$$

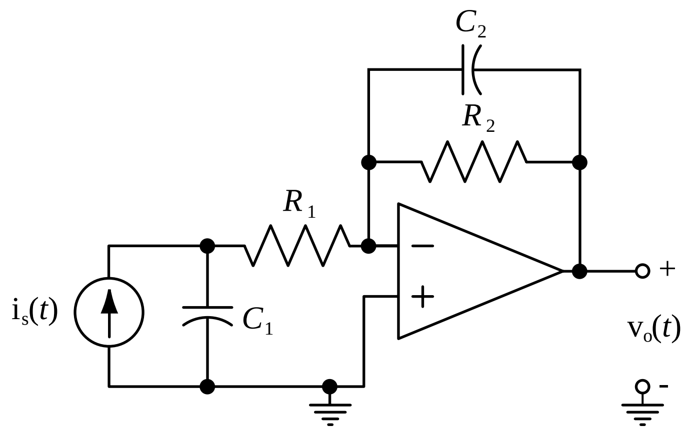
where

$$\mathrm{Z}_{RLC}(s) = \frac{(R_2 + sL)\frac{1}{sC}}{R_2 + sL + \frac{1}{sC}} = \frac{sL + R_2}{s^2LC + sR_2C + 1} = \frac{1}{C}\frac{s + \frac{R_2}{L}}{s^2 + s\frac{R_2}{L} + \frac{1}{LC}}$$

Combine and simplify, to get

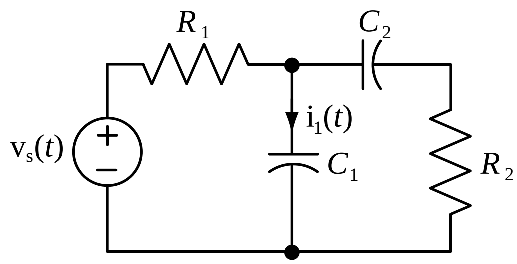
$$\mathrm{H}(s) = \frac{R_2}{R_1LC}\frac{1}{s^2 + s\left(\frac{1}{R_1C} + \frac{R_2}{L}\right) + \frac{R_2 + R_1}{R_1LC}}$$

(b) Excitation: $\mathrm{i}_s(t)$ Response: $\mathrm{v}_o(t)$



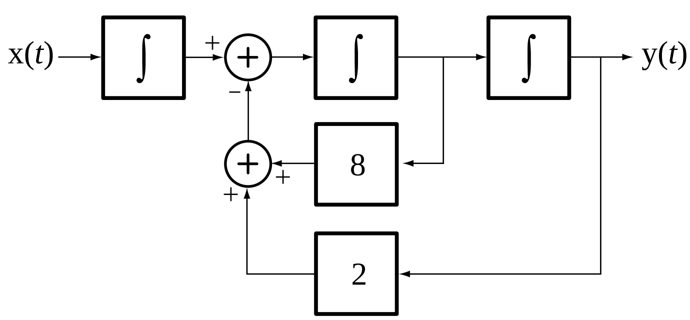
Use the fact that the voltage at both op-amp inputs must be zero and the fact that no current can flow into the op-amp input terminals. Find the current t flowing toward the op-amp through resistor, R_1, by current division. All that current must flow through the feedback network.

(c) Excitation: $v_s(t)$ Response: $i_1(t)$



2. For each block diagram write the transfer function between the excitation , $x(t)$, and the response, $y(t)$.

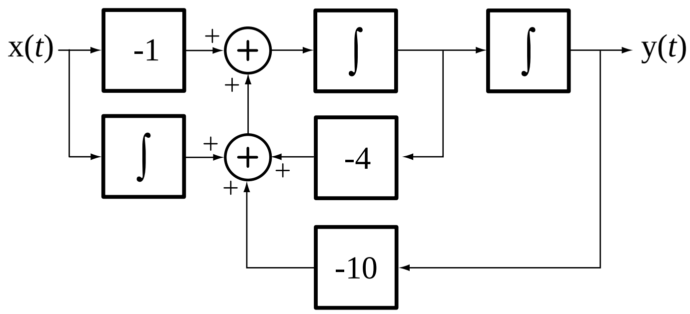
(a)



$$s^2 Y(s) = \frac{X(s)}{s} - [8s Y(s) + 2 Y(s)]$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s(s^2 + 8s + 2)} = \frac{1}{s^3 + 8s^2 + 2s}$$

(b)



3. Evaluate the stability of each of these system transfer functions.

(a) $\mathrm{H}(s) = -\frac{100}{s+200}$

(b) $\mathrm{H}(s) = \frac{80}{s-4}$ Pole at $s=4$. Unstable.

(c) $\mathrm{H}(s) = \frac{6}{s(s+1)}$ Poles at $s=0$ and $s=-1$. Marginally stable (therefore unstable)

(d) $\mathrm{H}(s) = -\frac{15s}{s^2+4s+4}$ Double pole at $s=-2$. Stable.

(e) $\mathrm{H}(s) = 3\frac{s-10}{s^2+4s+29}$

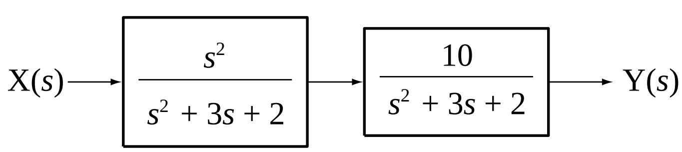
(f) $\mathrm{H}(s) = 3\frac{s^2+4}{s^2-4s+29}$

(g) $\mathrm{H}(s) = \frac{1}{s^2+64}$

(h) $\mathrm{H}(s) = \frac{10}{s^3+4s^2+29s}$

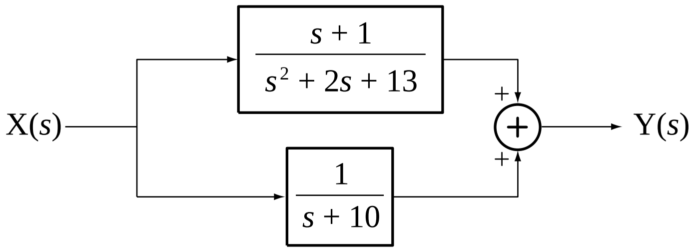
4. Find the overall transfer functions of these systems in the form of a single ratio of polynomials in *s*.

(a)



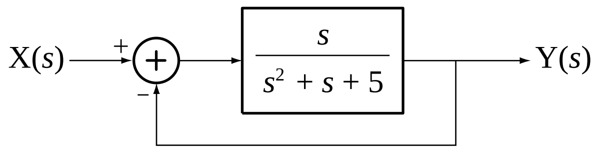
Multiply transfer functions.

(b)



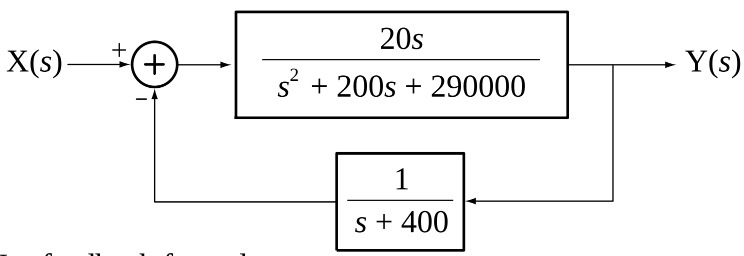
Add transfer functions.

(c)



Use feedback formula.

(d)



Use feedback formula.

5. In the feedback system below, find the overall system transfer function for these values of forward-path gain, K.

$$\mathrm{H}(s) = \frac{K}{1+0.1K}$$

(a) $K = 10^6$ $\quad \mathrm{H}(s) = \dfrac{10^6}{1+10^5} \cong 10$

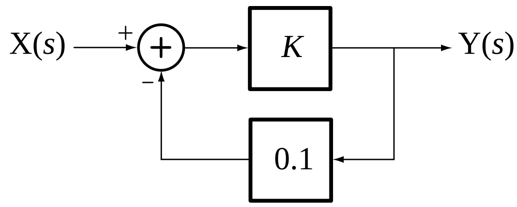
(b) $K = 10^5$

(c) $K = 10$

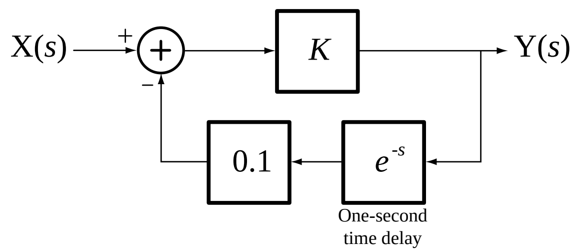
(d) $K = 1$

(e) $K = -1$

(f) $K = -10$



6. In the feedback system below, plot the response of the system to a unit step, for the time period, $0 < t < 10$, then write the expression for the overall system transfer function and draw a pole-zero diagram, for these values of *K*.



$$\mathrm{H}(s) = \frac{K}{1 + 0.1e^{-s}K}$$

(a) $K = 20$ $\qquad \mathrm{H}(s) = 20\frac{1}{1 + 2e^{-s}}$

This is not the usual ratio of polynomials in *s* so we cannot find the inverse transform in the usual way. Synthetically divide the denominator into the numerator to yield the infinite series,

$$\mathrm{H}(s) = 20\left(1 - 2e^{-s} + 4e^{-2s} - 8e^{-3s} + \cdots\right)$$

Then inverse transform term-by-term.

$$\mathrm{h}(t) = 20\left[\delta(t) - 2\delta(t-1) + 4\delta(t-2) - 8\delta(t-3) + \cdots\right]$$

The step response is the integral of the impulse response.

$$\mathrm{h}_{-1}(t) = 20\left[\mathrm{u}(t) - 2\,\mathrm{u}(t-1) + 4\,\mathrm{u}(t-2) - 8\,\mathrm{u}(t-3) + \cdots\right]$$

Poles at $1 + 2e^{-s} = 0$. Solving for *s*,

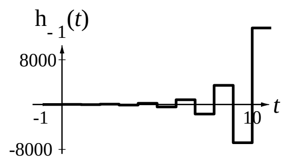
$$e^{-s} = -\frac{1}{2}$$

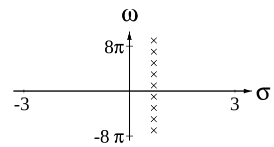
The term, e^{-s}, is, in general, complex because *s* is complex. There are multiple solutions for *s* in the complex plane. The logarithm of a complex number, $z = re^{j\theta}$, is

$$\log(z) = \ln(r) + j(\theta + 2n\pi).$$

where "log" means the generalized complex log (base e) and "ln" means the natural log (base e) of a real number, in this case, r.

$s = \log(-2) = \ln(2) + j\pi(2n+1)$ - Unstable

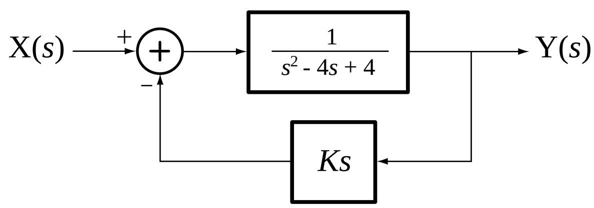




(b) $K = 10$ (c) $K = 1$ (d) $K = -1$

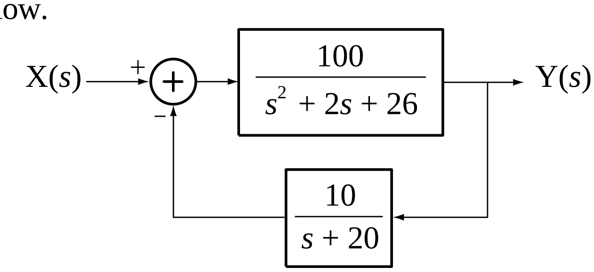
(e) $K = -10$ (f) $K = -20$

7. For what range of values of K is the system below stable? Plot the step responses for $K = 0$, $K = 4$ and $K = 8$.



Use the feedback formula to get the overall transfer function. Then solve for the pole locations as a function of K and then see what values of K will put the poles in the open left half-plane.

8. Plot the impulse response and the pole-zero diagram for the forward-path and the overall system below.



$$H_1(s) = 100\frac{1}{s^2+2s+26} = 100\frac{1}{(s+1)^2+25} \Rightarrow h_1(t) = 20e^{-t}\sin(5t)u(t)$$

Poles at $s = -1 \pm j5$.

$$H(s) = 100\frac{s+20}{s^3+22s^2+66s+1520}$$

Poles at $s = -22.12, 0.0612 \pm j8.29$.

When the denominator gets to third degree in s, finding the poles becomes tedious unless there is an obvious factorization. The MATLAB function, `residue`, can be used to find the poles and to help do the partial-fraction expansion.

$$\mathrm{H}(s) = -\frac{0.3785}{s+22.12} + 0.3785\left[\frac{s-0.0612}{(s-0.0612)^2+68.71} + \frac{242.02}{8.29}\frac{8.29}{(s-0.0612)^2+68.71}\right]$$

$$\mathrm{h}(t) = \left\{-0.3785e^{-22.12t} + 0.3785e^{0.0612t}\left[\cos(8.29t) + 29.19\sin(8.29t)\right]\right\}\mathrm{u}(t)$$

9. Using the Routh-Hurwitz method, evaluate the stability of the system whose transfer function is

$$\mathrm{H}(s) = \frac{s^3+3s+10}{s^5+2s^4+10s^3+4s^2+8s+20}$$

5	1	10	8
4	2	4	20
3	8	–2	0
2	$\frac{9}{2}$	20	0
1	$-\frac{338}{9}$	0	0
0	20	0	0

This result indicates that there are two poles in the RHP. That is confirmed by finding the poles which lie at

$$s = -0.9432 + j2.9107$$
$$s = -0.9432 - j2.9107$$
$$s = 0.5502 + j1.207$$
$$s = 0.5502 - j1.207$$
$$s = -1.214$$

10. Using the Routh-Hurwitz stability test, evaluate the stability of the system whose transfer function is of the general form,

$$\mathrm{H}(s) = \frac{\mathrm{N}(s)}{s^3+a_2s^2+a_1s+a_0}\ .$$

What are the relations among a_2, a_1 and a_0 that ensure stability?

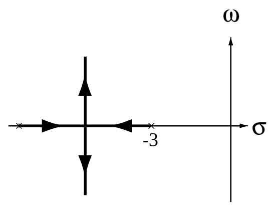
3	1	a_1
2	a_2	a_0
1	$-\frac{a_0-a_1a_2}{a_2}$	0
0	a_0	0

$$-\frac{a_0 - a_1 a_2}{a_2}$$

For stability, $a_2 > 0,\ -\dfrac{a_0 - a_1 a_2}{a_2} > 0 \Rightarrow a_1 a_2 - a_0 > 0 \Rightarrow a_1 a_2 > a_0,\ \ a_0 > 0.$

11. Plot the root locus for each of the systems which have these loop transfer functions and identify the transfer functions that are stable for all positive real values of *K*.

(a) $\mathrm{T}(s) = \dfrac{K}{(s+3)(s+8)}$ Stable for positive *K*

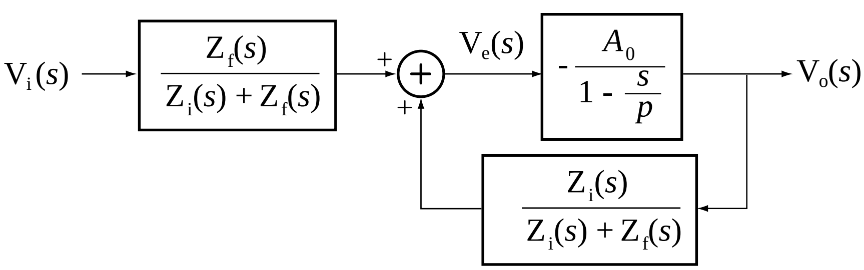


(b) $\mathrm{T}(s) = \dfrac{Ks}{(s+3)(s+8)}$

(c) $\mathrm{T}(s) = \dfrac{Ks^2}{(s+3)(s+8)}$

(d) $\mathrm{T}(s) = \dfrac{K}{(s+1)(s^2+4s+8)}$

12. Use the block diagram, of an inverting amplifier using an operational amplifier,



,

with $A_0 = 10^4$, $p = -2000\pi$, $Z_f = 10\,\mathrm{k}\Omega$ and $Z_i = 1\,\mathrm{k}\Omega$, to find the gain and phase margins of the amplifier.

$$\mathrm{T}(s) = \frac{0.5712 \times 10^7}{s + 6283}$$

Finding the zero-dB frequency,

$$\left|T\left(j\omega_{0dB}\right)\right| = \left|\frac{0.5712\times10^{7}}{j\omega_{0dB}+6283}\right| = 1$$

$$\frac{0.5712\times10^{7}}{\sqrt{\omega_{0dB}^{2}+(6283)^{2}}} = 1 \Rightarrow \omega_{0dB} = 5.712\times10^{6} \Rightarrow f_{0dB} = 909.1\,\text{kHz}$$

At that frequency the phase of the loop transfer function is

$$\angle T(s) = \angle\left(\frac{0.5712\times10^{7}}{j5.712\times10^{6}+6283}\right) = -\frac{\pi}{2}\text{ or }-90°$$

So the phase margin is 90°. The phase approaches 90° as the frequency approaches infinity. The loop transfer function magnitude approaches zero as the frequency approaches infinity. The loop transfer function magnitude in dB therefore approaches minus infinity. That means the gain margin is infinite.

13. Plot the unit step and ramp responses of unity-gain feedback systems with these forward-path transfer functions.

(a) $\mathrm{H}_1(s) = \dfrac{100}{s+10}$ $\mathrm{H}(s) = \dfrac{100}{s+110}$

Unit step response:

$$\mathrm{H}_{-1}(s) = \frac{100}{s(s+110)} = \frac{10}{11}\left(\frac{1}{s}-\frac{1}{s+110}\right) \Rightarrow \mathrm{h}_{-1}(t) = \frac{10}{11}\left(1-e^{-110t}\right)\mathrm{u}(t)$$

Unit ramp response:

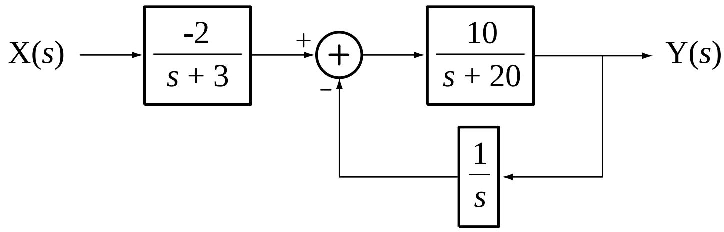
$$\mathrm{H}_{-2}(s) = \frac{100}{s^{2}(s+110)} = \frac{10}{11}\left(\frac{1}{s^{2}}-\frac{\frac{1}{110}}{s}+\frac{\frac{1}{110}}{s+110}\right) \Rightarrow \mathrm{h}_{-2}(t) = \frac{10}{11}\left(t-\frac{1-e^{-110t}}{110}\right)\mathrm{u}(t)$$

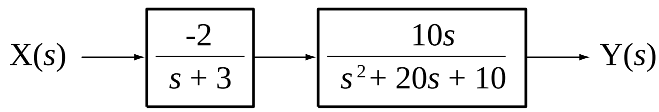
(b) $\mathrm{H}_1(s) = \dfrac{100}{s(s+10)}$ (c) $\mathrm{H}_1(s) = \dfrac{100}{s^{2}(s+10)}$

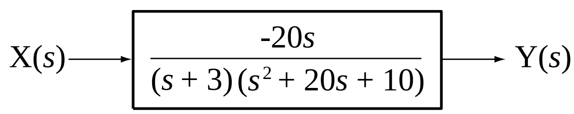
(d) $\mathrm{H}_1(s) = \dfrac{20}{(s+2)(s+6)}$

14. Reduce these block diagrams to a single block by block-diagram reduction. Check the answer using Mason's theorem.

(a)







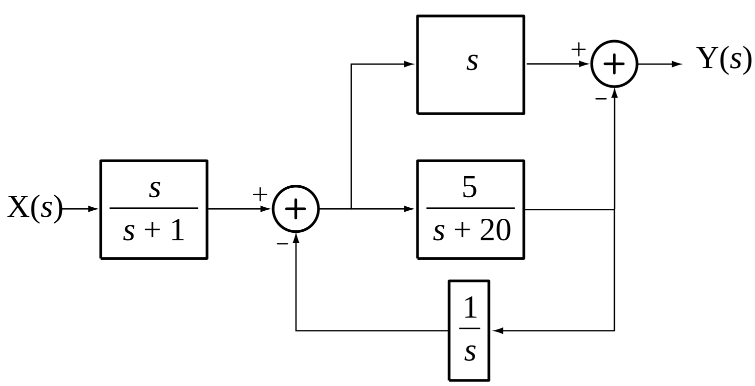
Mason's Theorem:

$$\mathrm{P}_1(s) = \frac{-20}{(s+3)(s+20)} \quad \text{and} \quad \mathrm{T}_1(s) = \frac{10}{s(s+20)}$$

$$\text{and} \quad \Delta(s) = 1 + \frac{10}{s(s+20)} \quad \text{and} \quad \Delta_1(s) = 1$$

$$\mathrm{H}(s) = \frac{\sum_{i=1}^{N_p} \mathrm{P}_i(s)\Delta_i(s)}{\Delta(s)} = \frac{\dfrac{-20}{(s+3)(s+20)}}{1 + \dfrac{10}{s(s+20)}} = \frac{-20s}{(s+3)\left(s^2+20s+10\right)}$$. Check.

(b)



15. Find the responses of the systems with these transfer functions to a unit-step and a suddenly-applied, unit-amplitude, 1 Hz cosine. Also find the responses to a true unit-amplitude, 1 Hz cosine (not suddenly applied) using the CTFT and compare to the steady-state part of the total solution found using the Laplace transform.

Use the results of the chapter in which if a cosine is suddenly applied to a system whose transfer function is $\mathrm{H}(s) = \frac{\mathrm{N}(s)}{\mathrm{D}(s)}$, the response is

$$y(t) = \mathcal{L}^{-1}\left(\frac{N_1(s)}{D(s)}\right) + |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0))u(t)$$

where $\frac{N_1(s)}{D(s)}$ is the partial fraction involving the transfer function poles.

(a) $H(s) = \frac{1}{s}$

Unit-step response:

$$H_{-1}(s) = \frac{1}{s^2} \Rightarrow h_{-1}(t) = t\,u(t) = \text{ramp}(t)$$

Unit-amplitude 1 Hz cosine response:

$$Y(s) = \frac{1}{s}\frac{s}{s^2 + (2\pi)^2} = \frac{1}{2\pi}\frac{2\pi}{s^2 + (2\pi)^2} \Rightarrow y(t) = \frac{\sin(2\pi t)u(t)}{2\pi}$$

Using the CTFT:

$$H(j\omega) = \frac{1}{j\omega}$$

$$Y(j\omega) = \frac{1}{j\omega}\pi[\delta(\omega - 2\pi) + \delta(\omega + 2\pi)] = \pi\left[\frac{\delta(\omega - 2\pi)}{j\omega} + \frac{\delta(\omega + 2\pi)}{j\omega}\right]$$

$$Y(j\omega) = \pi\left[\frac{\delta(\omega - 2\pi)}{j2\pi} + \frac{\delta(\omega + 2\pi)}{-j2\pi}\right]$$

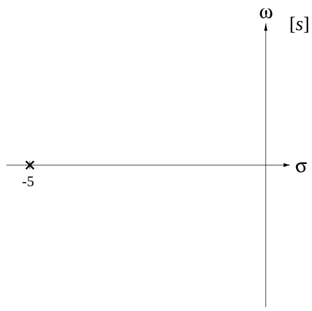
$$Y(j\omega) = \frac{j\pi}{2\pi}[\delta(\omega + 2\pi) - \delta(\omega - 2\pi)] \Rightarrow y(t) = \frac{\sin(2\pi t)}{2\pi}$$. Check.

(b) $H(s) = \frac{s}{s+1}$ (c) $H(s) = \frac{s}{s^2 + 2s + 40}$

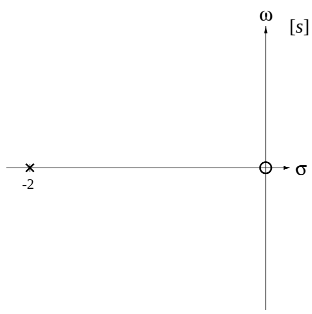
(d) $H(s) = \frac{s^2 + 2s + 40}{s^2}$

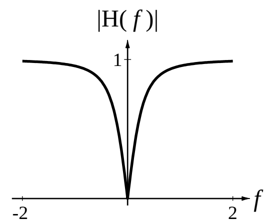
16. For each pole-zero diagram sketch the approximate frequency response magnitude.

(a)

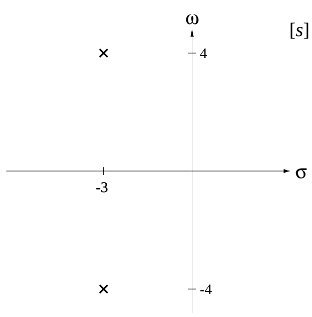


(b)

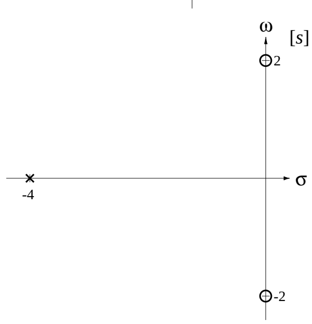




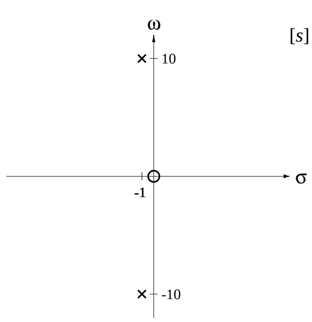
(c)



(d)



(e)



17. Using only a calculator, find the transfer function of a third-order ($n = 3$) lowpass Butterworth filter with cutoff frequency, $\omega_c = 1$, and unity gain at zero frequency.

The poles must be on a semicircle of radius, one, in the LHP. One pole is on the real axis and the spacing between poles is $\frac{\pi}{3}$ radians.

$$H(j\omega) = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

18. Using MATLAB, find the transfer function of an eighth-order lowpass Butterworth filter with cutoff frequency, $\omega_c = 1$, and unity gain at zero frequency.

```
»[z,p,k] = buttap(8) ;
»z

z =

     []

»p

p =

  -0.1951 + 0.9808i
  -0.5556 + 0.8315i
  -0.8315 + 0.5556i
  -0.9808 + 0.1951i
  -0.9808 - 0.1951i
  -0.8315 - 0.5556i
  -0.5556 - 0.8315i
  -0.1951 - 0.9808i

»k

k =

   1.00000000000000
```

$$H(s) = \frac{1}{\left\{\begin{bmatrix}(s+0.1951+j0.9808)(s+0.1951-j0.9808)\\(s+0.5556+j0.8315)(s+0.5556+j0.8315)\\(s+0.8315+j0.5556)(s+0.8315-j0.5556)\\(s+0.9808+j0.1951)(s+0.9808+j0.1951)\end{bmatrix}\right\}}$$

$$H(s) = \frac{1}{(s^2 + 0.3902s + 1)(s^2 + 1.1112s + 1)(s^2 + 1.663s + 1)(s^2 + 1.9616s + 1)}$$

Multiplying the first factor by the last and then the two middle factors,

$$H(s) = \frac{1}{(s^4 + 2.3518s^3 + 2.7654s^2 + 2.3518s + 1)(s^4 + 2.7741s^3 + 3.8478s^2 + 2.7741s + 1)}$$

$$H(s) = \frac{1}{s^8 + 5.126s^7 + 13.1371s^6 + 21.8462s^5 + 25.6884s^4 + 21.8462s^3 + 13.1371s^2 + 5.126s + 1}$$

19. Find the transfer functions of these Butterworth filters.

(a) Second-order highpass with a cutoff frequency of 20 kHz and a passband gain of 5.

The transfer function of the normalized filter is

$$H_{norm}(s) = \frac{1}{s^2 + 1.414s + 1}$$

$$f_c = 20\text{kHz} \Rightarrow \omega_c = 1.257 \times 10^5$$

Making the transformation, $s \to \frac{\omega_c}{s}$,

$$H(s) = \frac{s^2}{s^2 + 1.777 \times 10^5 s + 1.579 \times 10^{10}}$$

The passband is high frequencies (approaching infinity). Therefore $H(j\infty) = 1$ and we need to multiply the transfer function by five to get the required gain.

$$H(s) = \frac{5s^2}{s^2 + 1.777 \times 10^5 s + 1.579 \times 10^{10}}$$

(b) Third-order bandpass with a center frequency of 5 kHz, a -3 dB bandwidth of 500 Hz and a passband gain of 1.

This exercise is straightforward but the algebra is long and tedious.

$$H(s) = \frac{3.1 \times 10^{10} K s^3}{s^6 + 6283s^5 + 2.97 \times 10^9 s^4 + 1.24 \times 10^{13} s^3 + 2.93 \times 10^{18} s^2 + 6.09 \times 10^{21} s + 9.542 \times 10^{26}}$$

(c) Fourth-order bandstop with a center frequency of 10 MHz, a -3 dB bandwidth of 50 kHz and a passband gain of 1.

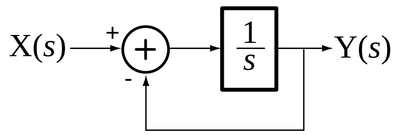
More tedious algebra.

$$H(s) = \frac{s^8 + 1.579 \times 10^{16} s^6 + 9.351 \times 10^{31} s^4 + 2.461 \times 10^{47} s^2 + 2.429 \times 10^{62}}{\left[\begin{array}{l} s^8 + 8.2093 \times 10^5 s^7 + 1.579 \times 10^{16} s^6 + 9.7223 \times 10^{21} s^5 + 9.35 \times 10^{31} s^4 \\ + 3.8377 \times 10^{37} s^3 + 2.4607 \times 10^{47} s^2 + 5.0499 \times 10^{52} s + 2.4285 \times 10^{62} \end{array}\right]}$$

20. Draw canonical system diagrams of the systems with these transfer functions.

(a) $\mathrm{H}(s) = \dfrac{1}{s+1}$

$$\frac{\mathrm{Y}(s)}{\mathrm{X}(s)} = \frac{1}{s+1} \Rightarrow \mathrm{X}(s) = s\,\mathrm{Y}(s) + \mathrm{Y}(s) \Rightarrow s\,\mathrm{Y}(s) = \mathrm{X}(s) - \mathrm{Y}(s)$$

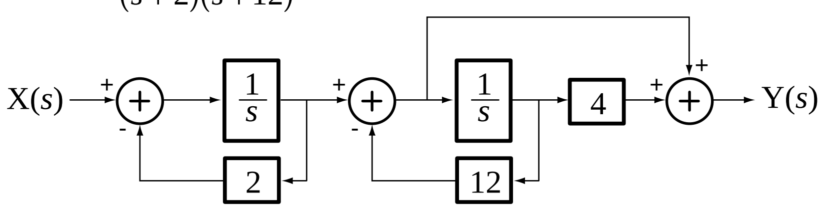


(b) $\mathrm{H}(s) = 4\dfrac{s+3}{s+10}$

21. Draw cascade system diagrams of the systems with these transfer functions.

(a) $\mathrm{H}(s) = \dfrac{s}{s+1}$

(b) $\mathrm{H}(s) = \dfrac{s+4}{(s+2)(s+12)}$



(c) $\mathrm{H}(s) = \dfrac{20}{s\left(s^2+5s+10\right)}$

22. Draw parallel system diagrams of the systems with these transfer functions.

(a) $\mathrm{H}(s) = \dfrac{-12}{s^2+3s+10}$

(b) $\mathrm{H}(s) = \dfrac{2s^2}{s^2+12s+32}$

23. Write state equations and output equations for the circuit of Figure E23 with the inductor current, $\mathrm{i}_L(t)$, and capacitor voltage, $\mathrm{v}_C(t)$, as the state variables and the voltage at the input, $\mathrm{v}_i(t)$, as the excitation and the voltage at the output, $\mathrm{v}_L(t)$, as the response.

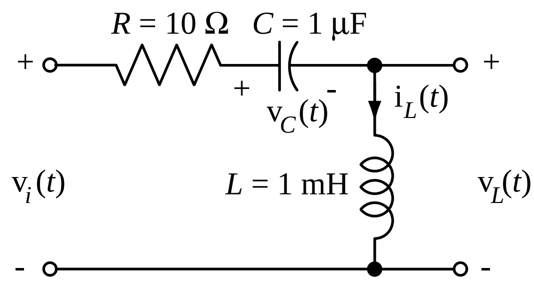


Figure E23 An RLC circuit

$$\mathrm{v}_i(t) = R\,\mathrm{i}_L(t) + \mathrm{v}_C(t) + L\,\mathrm{i}'_L(t)$$

$$\mathrm{v}'_C(t) = \frac{1}{C}\mathrm{i}_L(t)$$

$$\mathrm{v}_L(t) = \mathrm{v}_i(t) - R\,\mathrm{i}_L(t) - \mathrm{v}_C(t)$$

State equations,

$$\begin{bmatrix} \mathrm{v}'_C(t) \\ \mathrm{i}'_L(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} \mathrm{v}_C(t) \\ \mathrm{i}_L(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} \mathrm{v}_i(t)$$

$$\mathrm{v}_L(t) = \begin{bmatrix} -1 & -R \end{bmatrix} \begin{bmatrix} \mathrm{v}_C(t) \\ \mathrm{i}_L(t) \end{bmatrix} + \mathrm{v}_i(t)$$

24. Write state equations and output equations for the circuit of Figure E24 with the inductor current, $\mathrm{i}_L(t)$, and capacitor voltage, $\mathrm{v}_C(t)$, as the state variables and the current at the input, $\mathrm{i}_i(t)$, as the excitation and the voltage at the output, $\mathrm{v}_R(t)$, as the response.

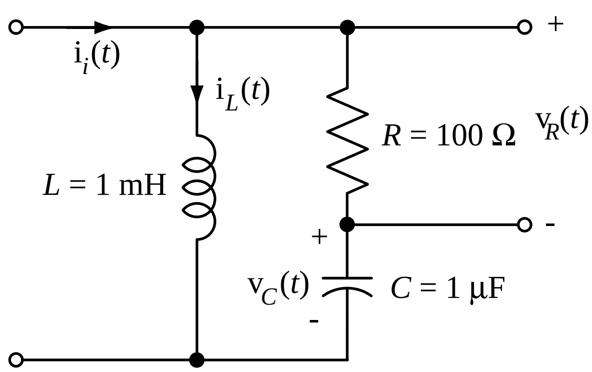


Figure E24 An RLC circuit

25. From the system transfer function,

$$\mathrm{H}(s) = \frac{s(s+3)}{s^2 + 2s + 9},$$

write a set of state equations and output equations using a minimum number of states.

Let $\mathrm{Y}(s)$ be a state, $\mathrm{Q}_1(s)$ and let $s\mathrm{Y}(s)$ be a state, $\mathrm{Q}_2(s)$. Then the state equations are

$$\begin{bmatrix} s\mathrm{Q}_1(s) \\ s\mathrm{Q}_2(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -9 & -2 \end{bmatrix} \begin{bmatrix} \mathrm{Q}_1(s) \\ \mathrm{Q}_2(s) \end{bmatrix} + \begin{bmatrix} 0 \\ s^2 + 3s \end{bmatrix} \mathrm{X}(s)$$

$$\mathrm{Y}(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{Q}_1(s) \\ \mathrm{Q}_2(s) \end{bmatrix}$$

26. Write state equations and output equations for the system whose block diagram is in Figure E26 using the responses of the integrators as the state variables.

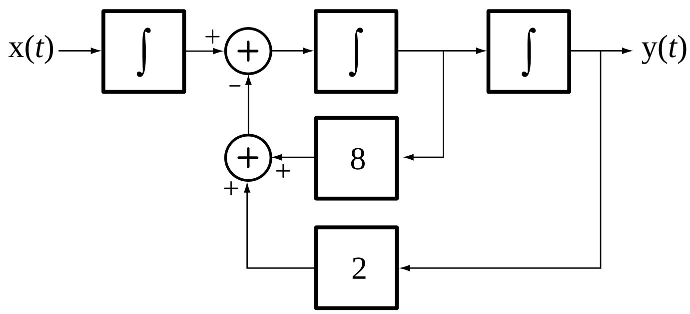


Figure E26 A system

27. A system is excited by the signal, $\mathrm{x}(t) = 3\mathrm{u}(t)$, and the response is $\mathrm{y}(t) = 0.961e^{-1.5t}\sin(3.122t)\mathrm{u}(t)$. Write a set of state equations and output equations using a minimum number of states.

$$\mathrm{X}(s) = \frac{3}{s}$$

$$\mathrm{Y}(s) = 0.961\frac{3.122}{(s+1.5)^2 + 3.122^2} = \frac{3}{s^2 + 3s + 12} \quad .$$

Then the transfer function is

$$\mathrm{H}(s) = \frac{\mathrm{Y}(s)}{\mathrm{X}(s)} = \frac{\frac{3}{s^2 + 3s + 12}}{\frac{3}{s}} = \frac{s}{s^2 + 3s + 12} \quad .$$

Let $\mathrm{Y}(s)$ be a state, $\mathrm{Q}_1(s)$ and let $s\mathrm{Y}(s)$ be a state, $\mathrm{Q}_2(s)$. Then the state equations are

$$\begin{bmatrix} s\mathrm{Q}_1(s) \\ s\mathrm{Q}_2(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -12 & -3 \end{bmatrix} \begin{bmatrix} \mathrm{Q}_1(s) \\ \mathrm{Q}_2(s) \end{bmatrix} + \begin{bmatrix} 0 \\ s \end{bmatrix} \mathrm{X}(s)$$

$$\mathrm{Y}(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{Q}_1(s) \\ \mathrm{Q}_2(s) \end{bmatrix}$$

28. A system is described by the differential equation,

$$y''(t) + 4y'(t) + 7y(t) = 10\cos(200\pi t)u(t) \ .$$

Write a set of state equations and output equations for this system with two states.

29. A system is described by the state equations and output equations,

$$\begin{bmatrix} q_1'(t) \\ q_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_1(t) \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

and

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix}$$

With excitation, $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\delta(t) \\ u(t) \end{bmatrix}$ and initial conditions, $\begin{bmatrix} q_1(0^+) \\ q_2(0^+) \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. Find the system response vector, $\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$.

$$\Phi(s) = \begin{bmatrix} \frac{s}{s^2+2s-3} & \frac{1}{s^2+2s-3} \\ \frac{3}{s^2+2s-3} & \frac{s+2}{s^2+2s-3} \end{bmatrix}$$

$$\mathbf{Q}(s) = \begin{bmatrix} \frac{-\frac{5}{2}}{s+3} + \frac{\frac{3}{2}}{s-1} \\ \frac{\frac{5}{2}}{s+3} + \frac{\frac{9}{2}}{s-1} - \frac{2}{s} \end{bmatrix}$$

$$\frac{s+2}{s^2+2s-3}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -\frac{5}{2}e^{-3t} + \frac{3}{2}e^{t} \\ \frac{5}{2}e^{-3t} + \frac{9}{2}e^{t} - 2 \end{bmatrix} u(t) = \begin{bmatrix} 5e^{-3t} + 27e^{t} - 10 \\ 15e^{-3t} + 15e^{t} - 8 \end{bmatrix} u(t)$$

30. A system is described by the vector state equation and output equation,

$$\mathbf{q}'(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}\mathbf{x}(t)$$

and

$$\mathbf{y}(t) = \mathbf{C}\mathbf{q}(t) + \mathbf{D}\mathbf{x}(t),$$

where $\mathbf{A}=\begin{bmatrix}-1 & -3\\ 2 & -7\end{bmatrix}$, $\mathbf{B}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $\mathbf{C}=\begin{bmatrix}2 & -3\\ 0 & 4\end{bmatrix}$ and $\mathbf{D}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$. Define two new states, in terms of the old states, for which the **A** matrix is diagonal and re-write the state equations.

The eigenvalues can be found from the **A** matrix. They are the roots of $s^2+8s+13$ which are $s=-2.2679$ and $s=-5.7321$. So the matrix of eigenvalues is

$$\Lambda=\begin{bmatrix}-2.2679 & 0\\ 0 & -5.7321\end{bmatrix}.$$

The equation to solve for the transformation matrix that diagonalizes the system is $\Lambda\mathbf{T}=\mathbf{TA}$. Therefore **T** is the matrix of eigenvectors for the matrix, **A**. A normalized **T** is

$$\mathbf{T}=\begin{bmatrix}0.8446 & -0.5354\\ -0.3893 & 0.9211\end{bmatrix}.$$

To verify that this solution is correct,

$$\begin{bmatrix}-2.2679 & 0\\ 0 & -5.7321\end{bmatrix}\begin{bmatrix}0.8446 & -0.5354\\ -0.3893 & 0.9211\end{bmatrix}=\begin{bmatrix}0.8446 & -0.5354\\ -0.3893 & 0.9211\end{bmatrix}\begin{bmatrix}-1 & -3\\ 2 & -7\end{bmatrix}$$

$$\begin{bmatrix}-1.9155 & 1.2143\\ 2.2315 & -5.2798\end{bmatrix}=\begin{bmatrix}-1.9155 & 1.2143\\ 2.2315 & -5.2798\end{bmatrix}\ \text{Check.}$$

Using

$$\mathbf{q}_2'(t)=\mathbf{T}\mathbf{q}_1'(t)$$

we get

$$\mathbf{q}_2'(t)=\begin{bmatrix}0.8446 & -0.5354\\ -0.3893 & 0.9211\end{bmatrix}\mathbf{q}_1'(t).$$

Using

$$\mathbf{q}_2'(t)=\mathbf{TA}_1\mathbf{T}^{-1}\mathbf{q}_2(t)+\mathbf{TB}_1\mathbf{x}(t)=\mathbf{A}_2\mathbf{q}_2(t)+\mathbf{B}_2\mathbf{x}(t)$$

we get

$$\mathbf{q}_2'(t)=\begin{bmatrix}0.8446 & -0.5354\\ -0.3893 & 0.9211\end{bmatrix}\begin{bmatrix}-1 & -3\\ 2 & -7\end{bmatrix}\begin{bmatrix}0.8446 & -0.5354\\ -0.3893 & 0.9211\end{bmatrix}^{-1}\mathbf{q}_2(t)+\begin{bmatrix}0.8446 & -0.5354\\ -0.3893 & 0.9211\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\mathbf{x}(t)$$

or

$$\mathbf{q}_2'(t)=\begin{bmatrix}-2.2679 & 0\\ 0 & -5.7321\end{bmatrix}\mathbf{q}_2(t)+\begin{bmatrix}0.8446 & -0.5354\\ -0.3893 & 0.9211\end{bmatrix}\mathbf{x}(t)$$

31. For the original state equations and output equations of Exercise 30 write a differential-equation description of the system.

    The original state equations are

$$\begin{bmatrix} q_1'(t) \\ q_2'(t) \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 2 & -7 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

and

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} .$$

From the output equation,

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} q_1'(t) \\ q_2'(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$$

or

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix} \left\{ \begin{bmatrix} -1 & -3 \\ 2 & -7 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \right\} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -8 & 15 \\ 8 & -28 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} .$$

Solving the output equations for the states,

$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \right\}$$

or

$$\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{8} \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} .$$

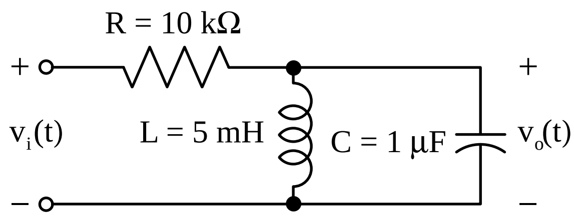
Then

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -8 & 15 \\ 8 & -28 \end{bmatrix} \left\{ \begin{bmatrix} \frac{1}{2} & \frac{3}{8} \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \right\} + \begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -4 & \frac{3}{4} \\ 4 & -4 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}$$

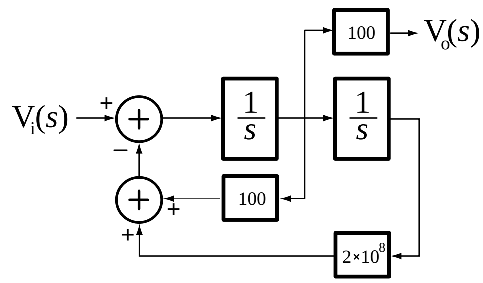
$$y_1'(t) = -4\,y_1(t) + \frac{3}{4}y_2(t) + 6\,x_1(t) - 3\,x_2(t) + x_1'(t)$$
$$y_2'(t) = 4\,y_1(t) - 4\,y_2(t) - 4\,x_1(t) + 4\,x_2(t)$$

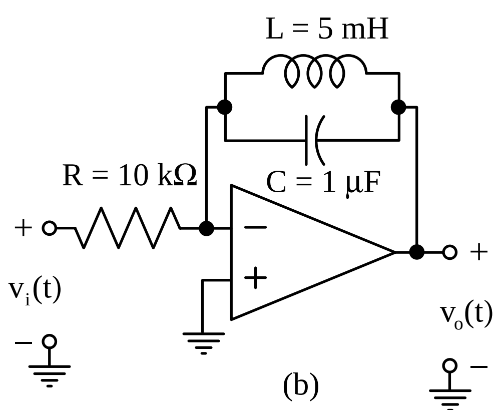
32. Find the *s*-domain transfer functions for the circuits below and then draw block diagrams for them as systems with $V_i(s)$ as the excitation and $V_o(s)$ as the response.



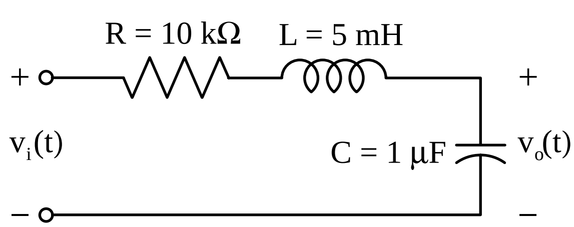
(a)

$$H(s) = 100\frac{s}{s^2 + 100s + 2\times10^8}$$

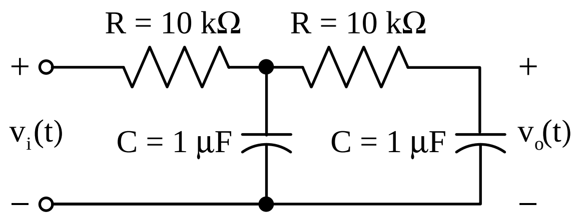




(b)



(c)



(d)

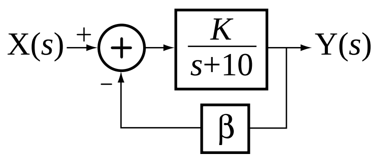
33. Determine whether the systems with these transfer functions are stable, marginally stable or unstable.

(a) $\mathrm{H}(s) = \dfrac{s(s+2)}{s^2+8}$ (b) $\mathrm{H}(s) = \dfrac{s(s-2)}{s^2+8}$

(c) $\mathrm{H}(s) = \dfrac{s^2}{s^2+4s+8}$ (d) $\mathrm{H}(s) = \dfrac{s^2}{s^2-4s+8}$

(e) $\mathrm{H}(s) = \dfrac{s}{s^3+4s^2+8s}$

34. Find the expression for the overall system transfer function of the system below.



$$\mathrm{H}(s) = \frac{K}{s+10+\beta K}$$

Pole at $s = -10 - \beta K$

(a) Let $\beta = 1$. For what values of K is the system stable?

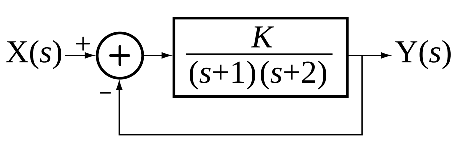
Pole at $s = -10 - K$

System is stable for $K > -10$

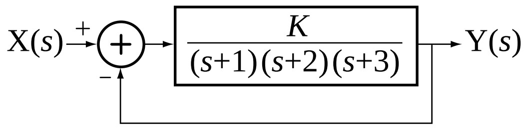
(b) Let $\beta = -1$. For what values of K is the system stable?

(c) Let $\beta = 10$. For what values of K is the system stable?

35. Find the expression for the overall system transfer function of the system below. For what positive values of K is the system stable?



36. Find the expression for the overall system transfer function of the system below. Using MATLAB plot the paths of the poles of the overall system transfer function as a function of *K*. For what positive values of *K* is the system stable?



$$H(s) = \frac{Y(s)}{X(s)} = \frac{K}{(s+1)(s+2)(s+3)+K} = \frac{K}{(s^2+3s+2)(s+3)+K} = \frac{K}{s^3+6s^2+11s+6+K}$$

Although this denominator can be factored it is probably easier just to numerically explore the pole locations versus the value of *K*. A MATLAB program was written to graph the pole locations for a range of *K*'s and it was found that when $K = 60$ the poles just touch the ω axis, indicating marginal stability. When $K = 60$, the poles are at

$$s = -6\ ,\ \pm j3.3166\ .$$

Therefore for $0 < K < 60$ the system is stable.

37. Thermocouples are used to measure temperature in many industrial processes. A thermocouple is usually mechanically mounted inside a "thermowell", a metal sheath which protects it from damage by vibration, bending stress, or other forces. One effect of the thermowell is that its thermal mass slows the effective time response of the thermocouple/thermowell combination compared witht the inherent time response of the thermocouple alone. Let the actual temperature on the outer surface of the thermowell in Kelvins be $T_s(t)$ and let the voltage developed by the thermocouple in response to temperature be $v_t(t)$. The response of the thermocouple to a one-Kelvin step change in the thermowell outer-surface temperature from T_1 to T_1+1 is

$$v_t(t) = K\left[T_1 + \left(1 - e^{-\frac{t}{0.2}}\right)u(t)\right]$$

where *K* is the thermocouple temperature-to-voltage conversion constant.

(a) Let the conversion constant be $K = 40\frac{\mu V}{K}$. Design an active filter which processes the thermocouple voltage and compensates for its time lag making the overall system have a response to a one-Kelvin step thermowell-surface temperature change that is itself a step of voltage of 1mV.

The unit step response of the thermocouple-thermowell combination is

$$v_t(t) = K\left(1 - e^{-5t}\right)u(t)\ .$$

The impulse response is the derivative of the step response,

$$h_t(t) = 5Ke^{-5t}\,u(t)\ .$$

The transfer function is the transform of the impulse response,

$$\mathrm{H}_t(s) = \frac{5K}{s+5} \ .$$

The desired overall frequency response is

$$\mathrm{H}(s) = \frac{1\text{mV}}{\text{K}} \ .$$

(Here K is the kelvin not the thermocouple gain, K.) Therefore the transfer function of the compensating active filter is

$$\mathrm{H}_f(s) = \frac{\mathrm{H}(s)}{\mathrm{H}_t(s)} = \frac{10^{-3}}{\frac{5K}{s+5}} = 2\times10^{-4}\frac{s+5}{K} \ .$$

This is a system with a real zero creating a corner frequency, $\omega_c = 5$. This can be synthesized by the circuit of Figure S37 .

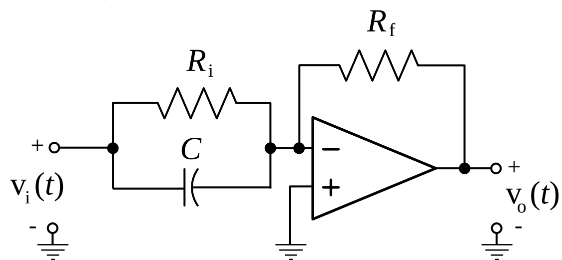


Figure S37 Thermocouple-thermowell compensator

Choose resistor and capacitor values to locate the poles and zeros in the proper places.

(b) Suppose that the thermocouple also is subject to electromagnetic interference (EMI) from nearby high-power electrical equipment. Let the EMI be modeled as a sinusoid with an amplitude of $20\,\mu\text{V}$ at the thermocouple terminals. Calculate the response of the thermocouple-filter combination to EMI frequencies of 1 Hz, 10 Hz and 60 Hz. How big is the apparent temperature fluctuation caused by the EMI in each case?

At 1 Hz: $\mathrm{H}_f(j2\pi) = 5(j2\pi + 5) = 40.14\angle 51°$

So the response to the $20\,\mu\text{V}$ excitation is about $800\,\mu\text{V}$ which is equivalent to about 0.8 K.

At 10 Hz:

At 60 Hz:

38. A laser operates on the fundamental principle that a pumped medium amplifies a travelling light beam as it propagates through the medium. Without mirrors a laser becomes a single-pass travelling wave amplifier (Figure E38-1).

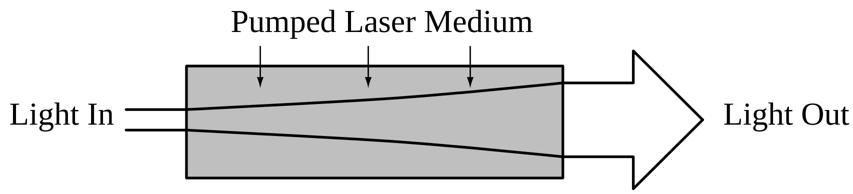


Figure E38 -1 A one-pass travelling-wave light amplifier

This is a system without feedback. If we now place mirrors at each end of the pumped medium, we introduce feedback into the system.

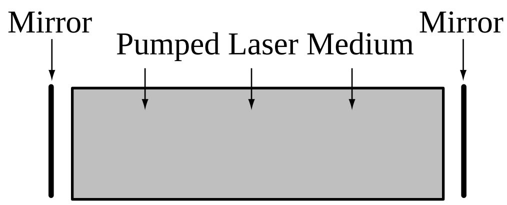


Figure E38-2 A regenerative travelling-wave amplifier

When the gain of the medium becomes large enough the system oscillates creating a coherent output light beam. That is laser operation. If the gain of the medium is less that that required to sustain oscillation, the system is known as a regenerative travelling-wave amplifier (RTWA).

Let the electric field of a light beam incident on the RTWA from the left be the excitation of the system, $E_{inc}(s)$, and let the electric fields of the reflected light, $E_{refl}(s)$, and the transmitted light, $E_{trans}(s)$, be the responses of the system (Figure E38-3) .

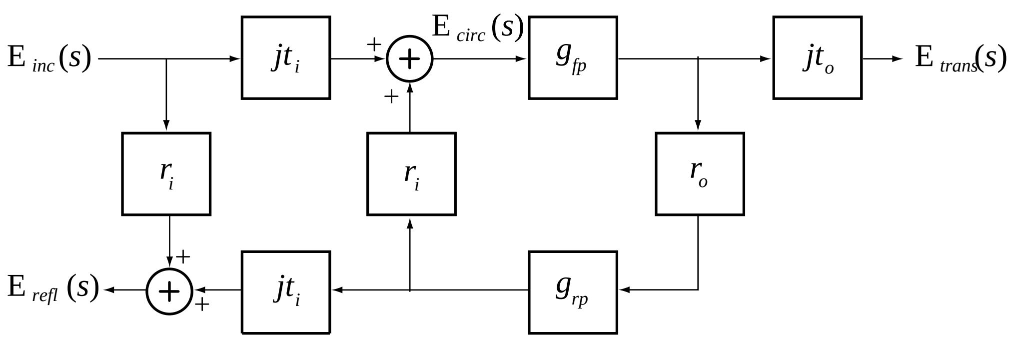


Figure E38-3 Block diagram of an RTWA

Let the system parameters be as follows:

Electric field reflectivity of the input mirror, $r_i = 0.99$

Electric field transmissivity of the input mirror, $t_i = \sqrt{1 - r_i^2}$

Electric field reflectivity of the output mirror, $r_o = 0.98$

Electric field transmissivity of the output mirror, $t_o = \sqrt{1 - r_o^2}$

Forward and reverse path electric field gains, $g_{fp}(s) = g_{rp}(s) = 1.01e^{-10^{-9}s}$

Find an expression for the frequency response, $\frac{E_{trans}(f)}{E_{inc}(f)}$, and plot its magnitude over the frequency range, $3\times10^{14} \pm 5\times10^{8}$ Hz.

$$E_{circ}(s) = jt_i E_{inc}(s) + g_{fp}(s) r_o g_{rp}(s) r_i E_{circ}(s)$$

$$E_{circ}(s) = \frac{jt_i}{1 - g_{fp}(s) r_o g_{rp}(s) r_i} E_{inc}(s)$$

$$E_{trans}(s) = jt_o g_{fp}(s) E_{circ}(s)$$

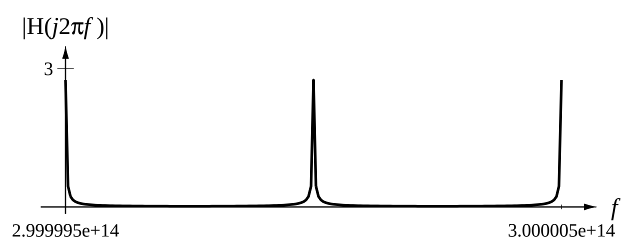
$$E_{trans}(s) = -\frac{t_i t_o g_{fp}(s)}{1 - r_o r_i g_{fp}^2(s)} E_{inc}(s)$$

The transfer function is

$$H(s) = \frac{E_{trans}(s)}{E_{inc}(s)} = -\frac{t_i t_o g_{fp}(s)}{1 - r_o r_i g_{fp}^2(s)}$$

and the frequency response is

$$H(j\omega) = -\frac{t_i t_o g_{fp}(j\omega)}{1 - r_o r_i g_{fp}^2(j\omega)} .$$



39. A classical example of the use of feedback is the phase-locked loop used to demodulate frequency-modulated signals (Figure E39) .

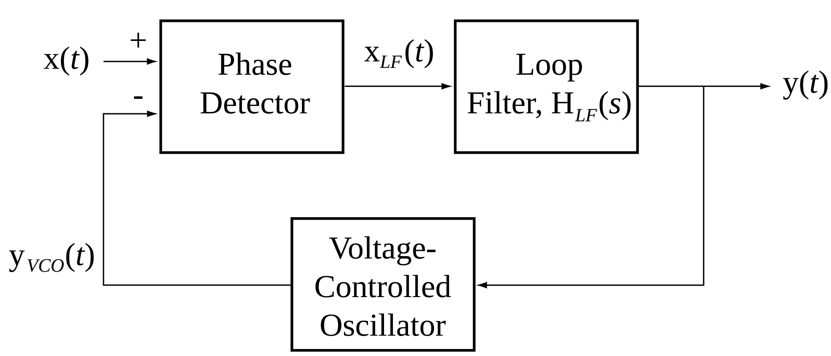


Figure E39 A phase-locked loop

The excitation, $\mathrm{x}(t)$, is a frequency-modulated sinusoid. The phase detector detects the phase difference between the excitation and the signal produced by the voltage-controlled oscillator. The response of the phase detector is a voltage proportional to phase difference. The loop filter filters that voltage. Then the loop filter response controls the frequency of the voltage-controlled oscillator. When the excitation is at a constant frequency and the loop is "locked" the phase difference between the two phase-detector excitation signals is zero. (In an actual phase detector the phase difference is 90° at lock. But that is not significant in this analysis since that only causes is a 90° phase shift and has no impact on system performance or stability.) As the frequency of the excitation, $\mathrm{x}(t)$, varies, the loop detects the accompanying phase variation and tracks it. The overall response signal, $\mathrm{y}(t)$, is a signal proportional to the frequency of the excitation.

The actual excitation, in a system sense, of this system is not $\mathrm{x}(t)$, but rather *the phase of* $\mathrm{x}(t)$, $\phi_x(t)$, because the phase detector detects differences in phase, not voltage. Let the frequency of $\mathrm{x}(t)$ be $\mathrm{f}_x(t)$. The relation between phase and frequency can be seen by examining a sinusoid. Let $\mathrm{x}(t) = A\cos(2\pi f_0 t)$. The phase of this cosine is $2\pi f_0 t$ and, for a simple sinusoid (f_0 constant), it increases linearly with time. The frequency is f_0, the derivative of the phase. Therefore the relation between phase and frequency for a frequency-modulated signal is

$$\mathrm{f}_x(t) = \frac{1}{2\pi}\frac{d}{dt}\left(\phi_x(t)\right).$$

Let the frequency of the excitation be 100 MHz. Let the transfer function of the voltage-controlled oscillator be $10^8\,\frac{\mathrm{Hz}}{\mathrm{V}}$. Let the transfer function of the loop filter be

$$\mathrm{H}_{LF}(s) = \frac{1}{s + 1.2\times 10^5} \quad .$$

Let the transfer function of the phase detector be $1\frac{\mathrm{V}}{\mathrm{radian}}$. If the frequency of the excitation signal suddenly changes to 100.001MHz, plot the change in the output signal, $\Delta\,\mathrm{y}(t)$.

Let $\phi_{diff}(t)$ be phase difference between $\mathrm{x}(t)$ and $\mathrm{y}_{VCO}(t)$. Then the following are the relations among the signals,

$$\mathrm{X}_{LF}(s) = \Phi_{diff}(s)$$

$$\mathrm{Y}(s) = \frac{\mathrm{X}_{LF}(s)}{s + 1.2\times 10^5}$$

$$\mathrm{F}_{VCO}(s) = 10^4\ \mathrm{Y}(s)$$

$$\Phi_{VCO}(s) = 2\pi\frac{\mathrm{F}_{VCO}(s)}{s}$$

$$\Phi_{diff}(s) = \Phi_x(s) - \Phi_{VCO}(s)$$

Combining equations,

$$X_{VCO}(s)=\frac{\frac{X_{LF}(s)}{s+1.2\times10^5}}{s}=\frac{\Phi_{diff}(s)}{s\left(s+1.2\times10^5\right)}$$

$$F_{VCO}(s)=10^4\,\frac{\Phi_{diff}(s)}{s+1.2\times10^5}$$

$$\Phi_{VCO}(s)=2\pi\frac{10^8\,\frac{\Phi_{diff}(s)}{s+1.2\times10^5}}{s}=2\pi\times10^4\,\frac{\Phi_{diff}(s)}{s\left(s+1.2\times10^5\right)}$$

$$\Phi_{diff}(s)=\Phi_x(s)-2\pi\times10^8\,\frac{\Phi_{diff}(s)}{s\left(s+1.2\times10^5\right)}$$

$$\Phi_{diff}(s)\left[1+\frac{2\pi\times10^8}{s\left(s+1.2\times10^5\right)}\right]=\Phi_x(s)$$

$$\frac{\Phi_{diff}(s)}{\Phi_x(s)}=\frac{1}{1+\frac{2\pi\times10^8}{s\left(s+1.2\times10^5\right)}}=\frac{s\left(s+1.2\times10^5\right)}{s\left(s+1.2\times10^5\right)+2\pi\times10^8}$$

$$\frac{Y(s)}{\Phi_x(s)}=\frac{s}{s\left(s+1.2\times10^5\right)+2\pi\times10^8}$$

$$F_x(s)=\frac{s\Phi_x(s)}{2\pi}$$

$$\frac{Y(s)}{F_x(s)}=\frac{2\pi}{s\left(s+1.2\times10^5\right)+2\pi\times10^8}\ .$$

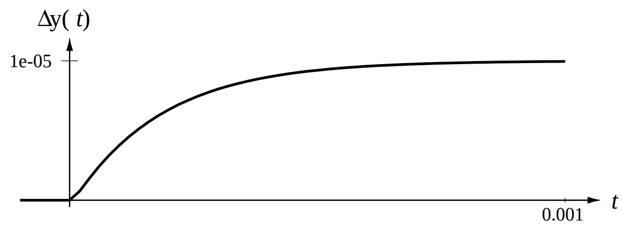
In steady state with no frequency modulation and a frequency of 100 MHz, $y(t)=1$.

The response to a step frequency change of 1 kHz, $\Delta f_x(t)=1000\,u(t)$, is

$$\Delta Y(s)=\frac{2\pi}{s\left(s+1.2\times10^5\right)+2\pi\times10^8}\frac{1000}{s}=\frac{2000\pi}{s^2\left(s+1.2\times10^5\right)+2\pi\times10^8 s}$$

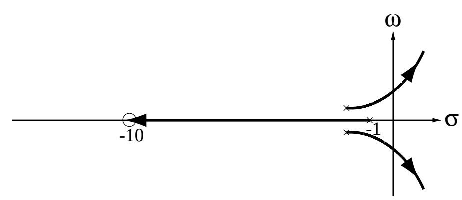
$$\Delta Y(s)=\frac{10^{-5}}{s}+\frac{5.033\times10^{-7}}{s+1.145\times10^5}-\frac{1.05\times10^{-5}}{s+5487}$$

$$\Delta y(t)=\left(10^{-5}+5.033\times10^{-7}e^{-1.145\times10^5 t}-1.05\times10^{-5}e^{-5487t}\right)u(t)$$



40. Plot the root locus for each of the systems which have these loop transfer functions and identify the transfer functions that are stable for all positive real values of *K*.

(a) $$\mathrm{T}(s)=\frac{K(s+10)}{(s+1)(s^2+4s+8)}$$ Unstable for some positive real *K*.



(b) $$\mathrm{T}(s)=\frac{K(s^2+10)}{(s+1)(s^2+4s+8)}$$

(c) $$\mathrm{T}(s)=\frac{K}{s^3+37s^2+332s+800}$$

(d) $$\mathrm{T}(s)=\frac{K(s-4)}{s+4}$$

(e) $$\mathrm{T}(s)=\frac{K(s-4)}{(s+4)^2}$$

(f) $$\mathrm{T}(s)=\frac{K(s+6)}{(s+5)(s+9)(s^2+4s+12)}$$

41. The circuit below is a simple approximate model of an operational amplifier with the inverting input grounded.

(a) Define the excitation of the circuit as the current of a current source applied to the non-inverting input and define the response as the voltage developed between the non-inverting input and ground. Find the transfer function and graph its frequency response. This transfer function is the input impedance.

$$\mathrm{Z}_i(s)=R_i=1\mathrm{M}\Omega$$

(b) Define the excitation of the circuit as the current of a current source applied to the *output* and define the response as the voltage developed between the *output* and ground with the non-inverting input grounded. Find the transfer function and graph its frequency response. This transfer function is the output impedance.

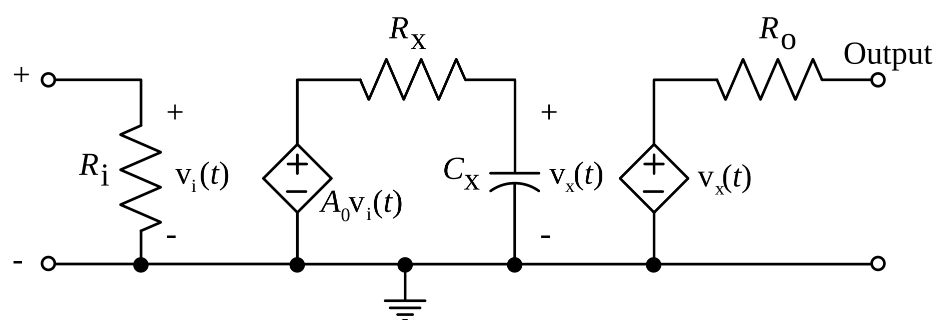
$$Z_i(s) = R_o = 10\Omega$$

(c) Define the excitation of the circuit as the voltage of a voltage source applied to the non-inverting input and define the response as the voltage developed between the output and ground. Find the transfer function and graph its frequency response. This transfer function is the voltage gain.

$$V_o(s) = V_X(s) = A_0 V_i(s) \frac{\frac{1}{sC_x}}{\frac{1}{sC_x} + R_x} = A_0 V_i(s) \frac{1}{sR_xC_x + 1}$$

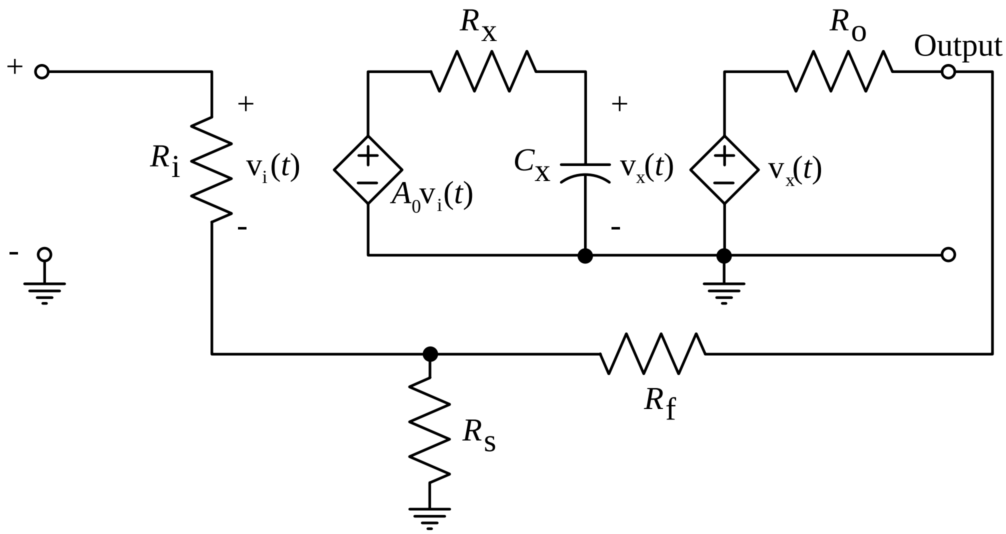
$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{A_0}{R_xC_x} \frac{1}{s + \frac{1}{R_xC_x}} = 1.25 \times 10^7 \frac{1}{s + 125}$$

The corner frequency is approximately 20 Hz.



$$R_i = 1\text{M}\Omega\ ,\ R_x = 1\text{k}\Omega\ ,\ C_x = 8\mu\text{F}\ ,\ R_o = 10\Omega\ ,\ A_0 = 10^6$$

42. Change the circuit of Exercise 41 to the circuit below. This is a feedback circuit which establishes a positive closed-loop voltage gain of the overall amplifier. Repeat steps (a), (b) and (c) of Problem #6 for the feedback circuit and compare the results. What are the important effects of feedback for this circuit?



$R_i = 1\text{M}\Omega$, $R_x = 1\text{k}\Omega$, $C_x = 8\mu\text{F}$, $R_o = 10\Omega$, $A_0 = 10^6$, $R_f = 10\text{k}\Omega$, $R_s = 5\text{k}\Omega$

(a) $$Z_i(s) = \frac{V_i(s) + V_s(s)}{I_i(s)} = \frac{R_i I_i(s) + V_s(s)}{I_i(s)} = R_i + \frac{V_s(s)}{I_i(s)}$$

$$V_s(s)G_s + [V_s(s) - V_o(s)]G_f = I_i(s)$$

$$V_x(s) = A_0 \frac{1}{sR_xC_x + 1} V_i(s) = A_0 \frac{R_i}{sR_xC_x + 1} I_i(s)$$

$$V_o(s)\left(G_o + G_f\right) - V_s(s)G_f - V_x(s)G_o = 0$$

Combining the last two equations,

$$V_o(s)\left(G_o + G_f\right) - V_s(s)G_f - A_0 \frac{R_i}{sR_xC_x + 1} I_i(s)G_o = 0$$

Then

$$\begin{bmatrix} G_s + G_f & -G_f \\ -G_f & G_o + G_f \end{bmatrix} \begin{bmatrix} V_s(s) \\ V_o(s) \end{bmatrix} = \begin{bmatrix} 1 \\ A_0 \frac{R_i}{sR_xC_x + 1} G_o \end{bmatrix} I_i(s)$$

$$\Delta = \left(G_s + G_f\right)\left(G_o + G_f\right) - G_f^2 = G_sG_o + G_sG_f + G_fG_o$$

$$\frac{V_s(s)}{I_i(s)} = \frac{1}{\Delta} \begin{vmatrix} 1 & -G_f \\ A_0 \frac{R_i}{sR_xC_x + 1} G_o & G_o + G_f \end{vmatrix} = \frac{G_o + G_f + A_0 \frac{R_iG_fG_o}{sR_xC_x + 1}}{G_sG_o + G_sG_f + G_fG_o}$$

$$\frac{\mathrm{V}_s(s)}{\mathrm{I}_i(s)} = \frac{\frac{1}{R_o} + \frac{1}{R_f} + A_0 \frac{R_i \frac{1}{R_f}\frac{1}{R_o}}{sR_xC_x + 1}}{\frac{1}{R_s}\frac{1}{R_o} + \frac{1}{R_s}\frac{1}{R_f} + \frac{1}{R_f}\frac{1}{R_o}} = \frac{R_sR_f + R_sR_0 + A_0 \frac{R_iR_s}{sR_xC_x + 1}}{R_f + R_o + R_s}$$

$$Z_i(s) = R_i + \frac{R_sR_f + R_sR_0 + A_0 \frac{R_iR_s}{sR_xC_x + 1}}{R_f + R_o + R_s}$$

Substituting in numbers,

$$Z_i(s) = 10^6 + \frac{50\times 10^6 + 50{,}000 + 10^6 \frac{5\times 10^9}{8\times 10^{-3}s + 1}}{15{,}010}$$

At low frequencies,

$$\mathrm{Z}_i(s) \cong \frac{10^{12}}{3}$$

This input impedance is much higher than in the open-loop case.

(b) Ground the input terminal for this calculation.

$$\mathrm{Z}_o(s) = \frac{\mathrm{V}_o(s)}{\mathrm{I}_o(s)} = \frac{V_s(s) + V_f(s)}{\mathrm{I}_o(s)}$$

$$\left[\mathrm{V}_o(s) - \mathrm{V}_s(s)\right]G_f + \left[\mathrm{V}_o(s) - \mathrm{V}_x(s)\right]G_o = \mathrm{I}_o(s)$$

$$\left[\mathrm{V}_s(s) - \mathrm{V}_o(s)\right]G_f + \mathrm{V}_s(s)\left(G_s + G_i\right) = 0$$

Since the non-inverting input is grounded we can write

$$\mathrm{V}_x(s) = -\mathrm{V}_s(s)A_0 \frac{1}{sR_xC_x + 1}$$

Combining equations and solving

$$\mathrm{V}_o(s)\left(G_f + G_o\right) + \left(\frac{A_0G_o}{sR_xC_x + 1} - G_f\right)\mathrm{V}_s(s) = \mathrm{I}_o(s)$$

$$-\mathrm{V}_o(s)G_f + \left(G_f + G_s + G_i\right)\mathrm{V}_s(s) = 0$$

$$\begin{bmatrix} G_f + G_o & \dfrac{A_0 G_o}{sR_x C_x + 1} - G_f \\ -G_f & G_f + G_s + G_i \end{bmatrix} \begin{bmatrix} \mathrm{V}_o(s) \\ \mathrm{V}_s(s) \end{bmatrix} = \begin{bmatrix} \mathrm{I}_o(s) \\ 0 \end{bmatrix}$$

$$\Delta = \left(G_f + G_o\right)\left(G_f + G_s + G_i\right) + G_f\left(\frac{A_0 G_o}{sR_x C_x + 1} - G_f\right)$$

$$\Delta = G_f\left(G_s + G_i\right) + G_o\left(G_f + G_s + G_i\right) + \frac{A_0 G_o G_f}{sR_x C_x + 1}$$

$$\mathrm{V}_o(s) = \frac{1}{\Delta}\begin{vmatrix} I_o(s) & \dfrac{A_0 G_o}{sR_x C_x + 1} - G_f \\ 0 & G_f + G_s + G_i \end{vmatrix} = \frac{G_f + G_s + G_i}{\Delta} I_o(s)$$

$$\mathrm{Z}_o(s) = \frac{G_f + G_s + G_i}{G_f\left(G_s + G_i\right) + G_o\left(G_f + G_s + G_i\right) + \dfrac{A_0 G_o G_f}{sR_x C_x + 1}}$$

$$\mathrm{Z}_o(s) = R_o \frac{R_s R_i + R_f R_i + R_f R_s}{R_o\left(R_i + R_s\right) + R_s R_i + R_f R_i + R_f R_s + R_s R_i \dfrac{A_0}{sR_x C_x + 1}}$$

$$\mathrm{Z}_o(s) = 10 \frac{10^6\left(R_s + R_f\right) + R_f R_s}{10\left(10^6 + R_s\right) + 10^6\left(R_s + R_f\right) + R_f R_s + 10^6 R_s \dfrac{10^6}{8\times10^{-3}s + 1}}$$

Substituting in numbers,

$$\mathrm{Z}_o(s) \cong 10 \frac{15.05\times10^9}{15.06\times10^9 + \dfrac{5\times10^{15}}{8\times10^{-3}s + 1}} \cong \frac{1}{0.1 + \dfrac{5\times10^6}{8\times10^{-3}s + 1}}$$

At low frequencies

$$\mathrm{Z}_o(s) \cong 0.2\times10^{-6}$$

This output impedance is much lower than in the open-loop case.

(c) $\mathrm{V}_o(s)\left(G_o + G_f\right) - \mathrm{V}_x(s)G_o - \mathrm{V}_s(s)G_f = 0$

$$\mathrm{V}_s(s)\left(G_s + G_f + G_i\right) - \mathrm{V}_o(s)G_f - \mathrm{V}_{ex}(s)G_i = 0$$

where $\mathrm{V}_{ex}(s)$ is the overall excitation voltage with respect to ground

$$\mathrm{V}_x(s) = \left(\mathrm{V}_{ex}(s) - \mathrm{V}_s(s)\right) A_0 \frac{1}{sR_xC_x + 1}$$

$$\begin{bmatrix} G_o + G_f & -G_o & -G_f \\ -G_f & 0 & G_s + G_f + G_i \\ 0 & 1 & A_0 \dfrac{1}{sR_xC_x + 1} \end{bmatrix} \begin{bmatrix} \mathrm{V}_o(s) \\ \mathrm{V}_x(s) \\ \mathrm{V}_s(s) \end{bmatrix} = \begin{bmatrix} 0 \\ \mathrm{V}_{ex}(s) G_i \\ \mathrm{V}_{ex}(s) A_0 \dfrac{1}{sR_xC_x + 1} \end{bmatrix}$$

$$\Delta = -A_0 \frac{G_f G_o}{sR_xC_x + 1} - \left(G_o + G_f\right)\left(G_s + G_f + G_i\right) + G_f^2$$

$$\mathrm{V}_o(s) = \frac{1}{\Delta} \begin{vmatrix} 0 & -G_o & -G_f \\ \mathrm{V}_{ex}(s) G_i & 0 & G_s + G_f + G_i \\ \mathrm{V}_{ex}(s) A_0 \dfrac{1}{sR_xC_x + 1} & 1 & A_0 \dfrac{1}{sR_xC_x + 1} \end{vmatrix}$$

$$\mathrm{V}_o(s) = \frac{-\mathrm{V}_{ex}(s) G_i \left(A_0 \dfrac{-G_o}{sR_xC_x + 1} + G_f \right) - \mathrm{V}_{ex}(s) A_0 \dfrac{1}{sR_xC_x + 1} \left(G_s + G_f + G_i\right) G_o}{\Delta}$$

$$\mathrm{H}(s) = \frac{\mathrm{V}_o(s)}{\mathrm{V}_{ex}(s)} = \frac{G_i \left(A_0 \dfrac{-G_o}{sR_xC_x + 1} + G_f \right) + A_0 \dfrac{1}{sR_xC_x + 1} \left(G_s + G_f + G_i\right) G_o}{A_0 \dfrac{G_f G_o}{sR_xC_x + 1} + \left(G_o + G_f\right)\left(G_s + G_f + G_i\right) - G_f^2}$$

$$\mathrm{H}(s) = \frac{\dfrac{A_0}{sR_xC_x + 1} \left(G_s + G_f\right) G_o + G_i G_f}{A_0 \dfrac{G_f G_o}{sR_xC_x + 1} + G_o\left(G_s + G_f + G_i\right) + G_f\left(G_s + G_i\right)}$$

$$\mathrm{H}(s) = \frac{\dfrac{A_0}{sR_xC_x + 1} R_i\left(R_f + R_s\right) + R_o R_s}{A_0 \dfrac{R_s R_i}{sR_xC_x + 1} + \left(R_f R_i + R_s R_i + R_s R_f\right) + R_o\left(R_s + R_i\right)}$$

Substituting in numbers,

$$\mathrm{H}(s) = \frac{\dfrac{15 \times 10^{13}}{8 \times 10^{-3} s + 1} + 5 \times 10^4}{\dfrac{5 \times 10^{13}}{8 \times 10^{-3} s + 1} + 20 \times 10^7 + 15 \times 10^4} \cong \frac{\dfrac{15 \times 10^{13}}{8 \times 10^{-3} s + 1} + 5 \times 10^4}{\dfrac{5 \times 10^{13}}{8 \times 10^{-3} s + 1} + 20 \times 10^7}$$

$$\mathrm{H}(s) \cong \frac{400s + 15\times10^{13}}{1.6\times10^{6}s + 5\times10^{13}} \cong 2.5\times10^{-4}\,\frac{s + 3.75\times10^{11}}{s + 3.125\times10^{7}}$$

The low-frequency gain is

$$\mathrm{H}(0) \cong 2.5\times10^{-4}\,\frac{3.75\times10^{11}}{3.125\times10^{7}} = 3 = \frac{R_f + R_s}{R_s}$$

as it should be. The closed-loop gain has a pole at $s = -3.125\times10^{7}$ which sets a corner frequency of approximately 5 MHz. The open-loop corner frequency was approximately 20 Hz. So the bandwidth has been increased by a factor of approximately 250,000.

43. Plot the unit step and ramp responses of unity-gain feedback systems with these forward-path transfer functions.

(a) $\mathrm{H}_1(s) = \dfrac{20}{s(s+2)(s+6)}$

(b) $\mathrm{H}_1(s) = \dfrac{20}{s^2(s+2)(s+6)}$

(c) $\mathrm{H}_1(s) = \dfrac{100}{s^2 + 10s + 34}$

(d) $\mathrm{H}_1(s) = \dfrac{100}{s(s^2 + 10s + 34)}$

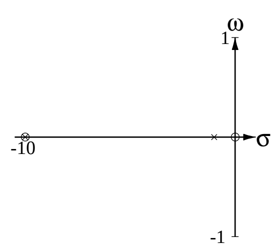
(e) $\mathrm{H}_1(s) = \dfrac{100}{s^2(s^2 + 10s + 34)}$

44. Draw pole-zero diagrams of these transfer functions.

(a) $\mathrm{H}(s) = \dfrac{(s+3)(s-1)}{s(s+2)(s+6)}$

(b) $\mathrm{H}(s) = \dfrac{s}{s^2 + s + 1}$

(c) $\mathrm{H}(s) = \dfrac{s(s+10)}{s^2 + 11s + 10}$



(Pole-zero cancellation.)

(d) $$\mathrm{H}(s) = \frac{1}{(s+1)(s^2+1.618s+1)(s^2+0.618s+1)}$$

45. A second-order system is excited by a unit step and the response is as illustrated in Figure E45 . Write an expression for the transfer function of the system.

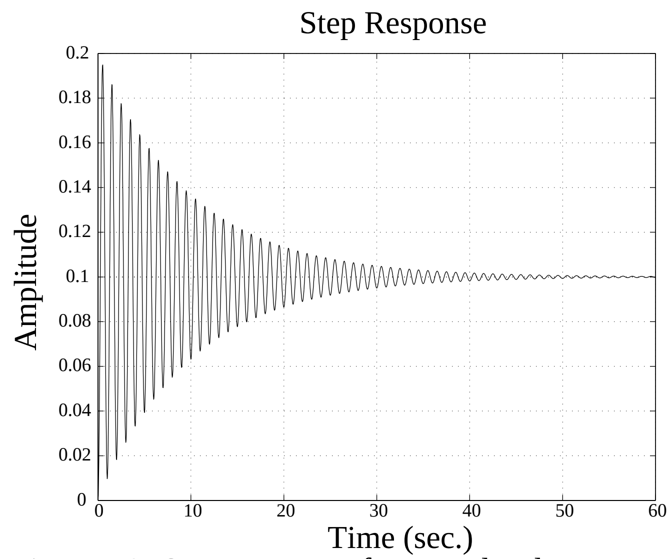


Figure E45 Step response of a second-order system

From the graph it is apparent that system is highly underdamped and that the final value of the step response is 0.1. So this second-order system has no zeros at zero. Therefore the general form of the transfer function of this second-order system is

$$\mathrm{H}(s) = \frac{A\omega_0^2}{s^2 + 2\zeta\omega_o s + \omega_0^2}$$

and $A = 0.1$. From the graph there are 10 ringing cycles of response between 0 and 10 seconds. Therefore the resonant frequency is approximately 1Hz or 2π radians per second. The impulse response is of the form,

$$\mathrm{h}(t) = Ke^{-\zeta\omega_0 t}\cos\left(\omega_0\sqrt{1-\zeta^2}\,t + \theta\right) .$$

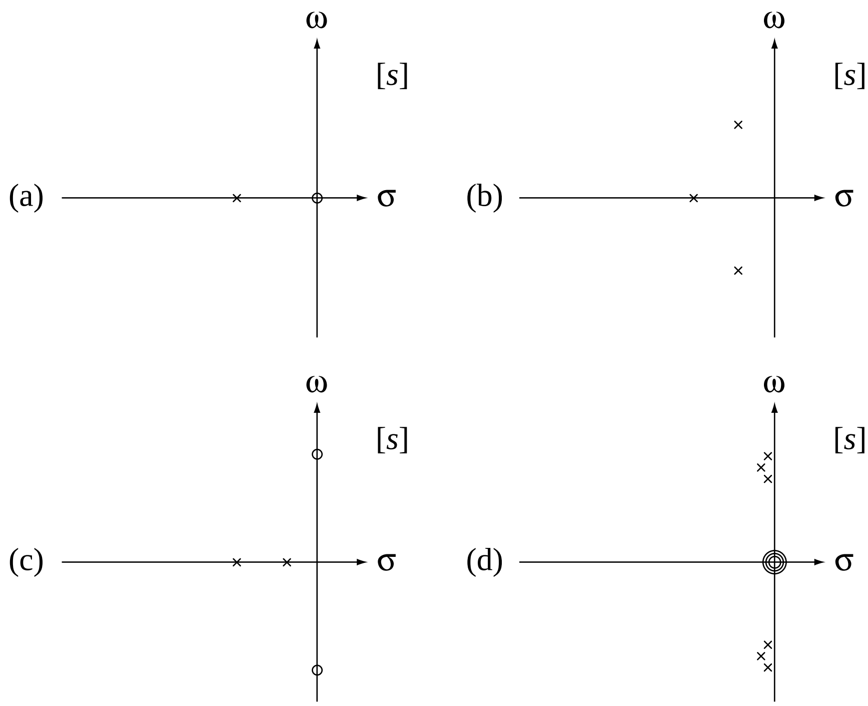
So the characteristic exponential decay has a time constant of $\tau = \frac{1}{\zeta\omega_0}$. From the graph, the time constant is approximately the time at which the ringing is at 36.8% of its maximum value. That is at about 10 seconds. Therefore

$$\zeta = \frac{1}{\omega_0\tau} = \frac{1}{20\pi} = 0.0159 .$$

So the transfer function is

$$\mathrm{H}(s) = \frac{3.948}{s^2 + 0.2s + 39.48} .$$

46. For each of the pole-zero plots below determine whether the frequency response is that of a practical lowpass, bandpass, highpass or bandstop filter.

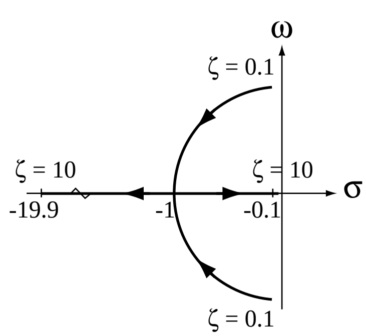


(a) Highpass

47. A system has a transfer function,

$$\mathrm{H}(s) = \frac{A}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \quad .$$

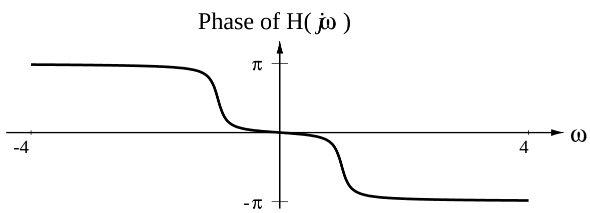
(a) Let $\omega_0 = 1$. Then let ζ vary continuously from 0.1 to 10 and plot in the s-plane the paths that the two poles take while ζ is varying between those limits.



(b) Find the real-valued functional form of the impulse response for the case, $\omega_0 = 1$ and $\zeta = 0.5$.

$$h(t) = \sqrt{\frac{4}{3}} A e^{-0.5t} \sin\left(\sqrt{\frac{3}{4}} t\right)$$

(c) Sketch the phase frequency response for the case, $\omega_0 = 1$ and $\zeta = 0.1$.



(d) Find the -3 dB bandwidth for for the case, $\omega_0 = 1$ and $\zeta = 0.1$.

$$H(j\omega) = \frac{A}{(j\omega)^2 + 0.2 j\omega + 1} = \frac{A}{1 - \omega^2 + 0.2 j\omega}$$

The maximum value of the transfer function occurs at resonance, $\omega = \omega_0 = 1$. There

$$\left|H(j\omega_0)\right| = \left|\frac{A}{0.2 j\omega_0^2}\right| = 5A$$

At the -3 dB points the magnitude of the square of the transfer function is one-half of the square of this value.

$$\omega_{-3dB} = \pm 1.086, \pm 0.883$$

So the bandwidth is 0.0323 Hz.

(e) The Q of a system is a measure of how "sharp" its frequency response is near a resonance. It is defined as

$$Q = \frac{1}{2\zeta} \quad .$$

For very high-Q systems what is the relationship between Q, ω_0 and -3 dB bandwidth?

If the Q is very high that means that the damping factor is very low. The peak of the frequency response occurs at resonance and at that frequency,

$$\left|H(j\omega_0)\right| = \frac{A}{2\zeta\omega_0^2} = Q\frac{A}{\omega_0^2}$$

The -3 dB bandwidth is found by solving

$$\left|H(j\omega_{-3dB})\right|^2 = \left|\frac{A}{\omega_0^2 - \omega_{-3dB}^2 + j\frac{\omega_0}{Q}\omega_{-3dB}}\right|^2 = \frac{\left(Q\frac{A}{\omega_0^2}\right)^2}{2}$$

Solving,

$$\omega_{-3dB}^2 = \omega_0^2 \frac{\left(2-\frac{1}{Q^2}\right) \pm \sqrt{\frac{1}{Q^4}+\frac{4}{Q^2}}}{2} = \omega_0^2 \frac{\left(2-\frac{1}{Q^2}\right) \pm \frac{1}{Q}\sqrt{\frac{1}{Q^2}+4}}{2}$$

For very large Q,

$$\omega_{-3dB} \cong \omega_0\sqrt{1 \pm \frac{1}{Q}} \cong \omega_0\left(1 \pm \frac{1}{2Q}\right)$$

which means that the bandwidth is

$$\Delta\omega_{-3dB} \cong \frac{\omega_0}{Q} \ .$$

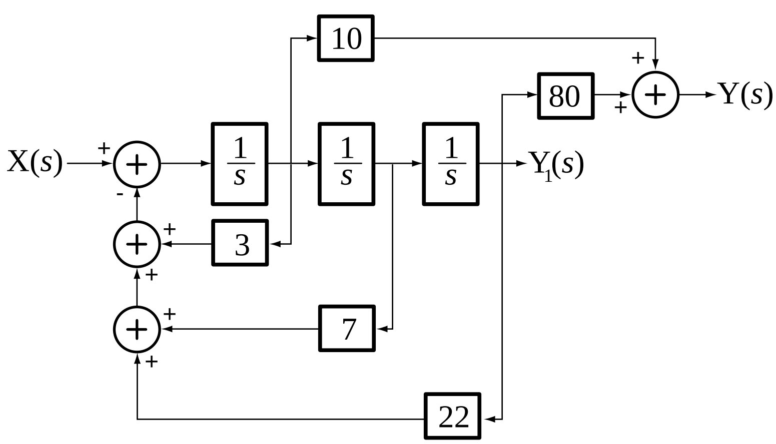
That is, for very-high-Q systems the -3 dB bandwidth is approximately the center frequency divided by the Q.

48. Draw canonical system diagrams of the systems with these transfer functions.

(a) $\mathrm{H}(s) = 10\frac{s^2+8}{s^3+3s^2+7s+22}$

$$s^3\,\mathrm{Y}_1(s) = \mathrm{X}(s) - 3s^2\,\mathrm{Y}_1(s) - 7s\,\mathrm{Y}_1(s) - 22\,\mathrm{Y}_1(s)$$

$$\mathrm{Y}(s) = 10s^2\,\mathrm{Y}_1(s) + 80\,\mathrm{Y}_1(s)$$



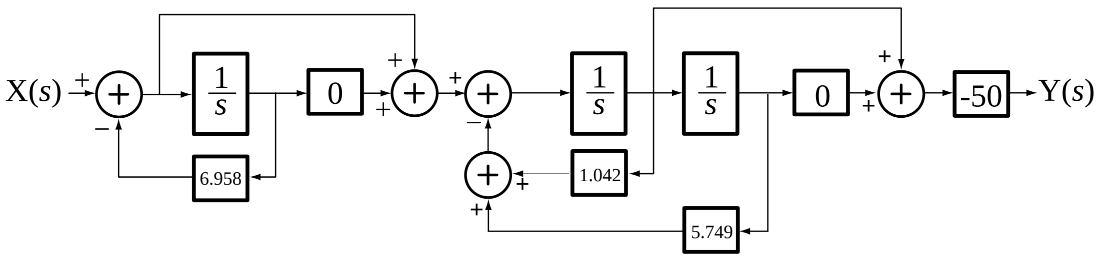
(b) $\mathrm{H}(s) = 10\frac{s+20}{(s+4)(s+8)(s+14)}$

49. Draw cascade system diagrams of the systems with these transfer functions.

(a) $$\mathrm{H}(s) = -50\frac{s^2}{s^3 + 8s^2 + 13s + 40}$$

Factoring the numerator and denominator,

$$\mathrm{H}(s) = -50\frac{s}{s + 6.958}\frac{s}{s^2 + 1.042s + 5.749}$$



(b) $$\mathrm{H}(s) = \frac{s^3}{s^3 + 18s^2 + 92s + 120}$$

50. Draw parallel system diagrams of the systems with these transfer functions.

(a) $$\mathrm{H}(s) = 10\frac{s^3}{s^3 + 4s^2 + 9s + 3} = 10 - \frac{39.9s + 73.84}{s^2 + 3.604s + 7.572} - \frac{0.09869}{s + 0.3962}$$

(b) $$\mathrm{H}(s) = \frac{5}{6s^3 + 77s^2 + 228s + 189} = \frac{0.01667}{s + 9} - \frac{0.15}{s + 2.333} + \frac{0.1333}{s + 1.5}$$

51. Write state equations and output equations for the circuit of Figure E51 with the two capacitor voltages, $\mathrm{v}_{C1}(t)$ and $\mathrm{v}_{C2}(t)$, as the state variables and the voltage at the input, $\mathrm{v}_i(t)$, as the excitation and the voltage, $\mathrm{v}_{R1}(t)$, as the response. Then, assuming the capacitors are initially uncharged, find the unit step response of the circuit.

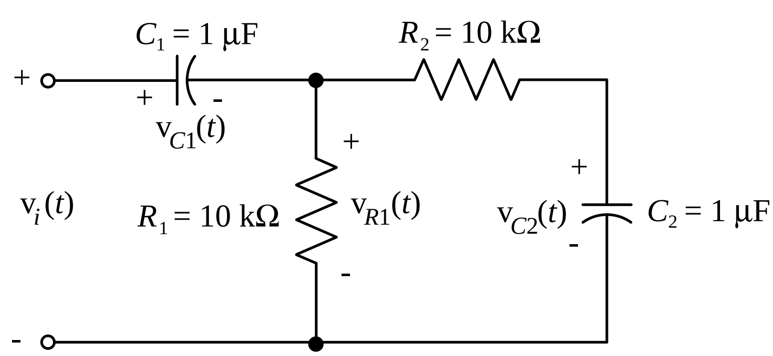


Figure E51 A second-order RC circuit

$$v'_{C1}(t) = \left(-\frac{1}{R_1C_1} - \frac{1}{R_2C_1}\right)v_{C1}(t) + \left(-\frac{1}{R_2C_1}\right)v_{C2}(t) + \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right)v_i(t)$$

$$v'_{C2}(t) = -\frac{v_{C1}(t)}{R_2C_2} - \frac{v_{C2}(t)}{R_2C_2} + \frac{v_i(t)}{R_2C_2}$$

and an output equation,

$$v_{R1}(t) = -v_{C1}(t) + v_i(t) \; .$$

Writing them in standard matrix form, (with numbers substituted)

$$\begin{bmatrix} v'_{C1}(t) \\ v'_{C2}(t) \end{bmatrix} = \begin{bmatrix} -200 & -100 \\ -100 & -100 \end{bmatrix} \begin{bmatrix} v_{C1}(t) \\ v_{C2}(t) \end{bmatrix} + \begin{bmatrix} 200 \\ 100 \end{bmatrix}$$

and

$$v_{R1}(t) = \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} v_{C1}(t) \\ v_{C2}(t) \end{bmatrix} + v_i(t) \; .$$

The transfer function is

$$\mathbf{H}(s) = \mathbf{C}[s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$$

which, in this case, is

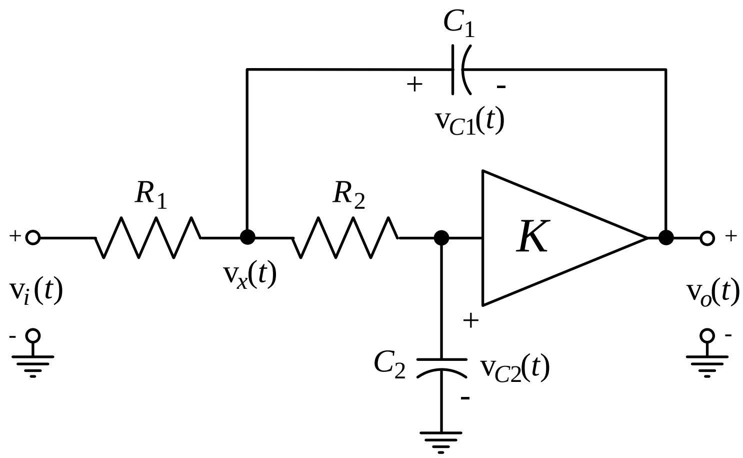
$$H(s) = \frac{s(s+100)}{s^2 + 300s + 10{,}000}$$

The step response is

$$h_{-1}(t) = \left(0.7326e^{-261.8t} + 0.2764e^{-38.2t}\right)u(t)$$

52. Write state equations and output equations for the circuit of Figure E52 with the two capacitor voltages, $v_{C1}(t)$ and $v_{C2}(t)$, as the state variables and the voltage at the input, $v_i(t)$, as the excitation and the voltage at the output, $v_o(t)$, as the response. Then, find and plot the response voltage for a unit step excitation assuming that the initial conditions are

$$\begin{bmatrix} v_{C1}(0) \\ v_{C2}(0) \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \; .$$



$$R_1 = 6.8\text{ k}\Omega \ , \ R_2 = 12\text{ k}\Omega$$
$$C_1 = 6.8\text{ nF} \ , \ C_2 = 6.8\text{ nF}$$
$$K = 3$$

Figure E52 A constant-*K* lowpass filter

$$\begin{bmatrix} R_1C_1 & R_1C_2 + R_2C_2 \\ R_1C_1 & R_1C_2 \end{bmatrix} \begin{bmatrix} v'_{C1}(t) \\ v'_{C2}(t) \end{bmatrix} = \begin{bmatrix} v_i(t) - v_{C2}(t) \\ v_i(t) - K\,v_{C2}(t) - v_{C1}(t) \end{bmatrix}$$

$$\begin{bmatrix} v'_{C1}(t) \\ v'_{C2}(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{R_1C_2 + R_2C_2}{R_1R_2C_1C_2} & \dfrac{R_1C_2 - K(R_1C_2 + R_2C_2)}{R_1R_2C_1C_2} \\ \dfrac{1}{R_2C_2} & \dfrac{K-1}{R_2C_2} \end{bmatrix} \begin{bmatrix} v_{C1}(t) \\ v_{C2}(t) \end{bmatrix} + \begin{bmatrix} \dfrac{1}{R_1C_1} \\ 0 \end{bmatrix} v_i(t)$$

$$v_o(t) = \begin{bmatrix} 0 & K \end{bmatrix} \begin{bmatrix} v_{C1}(t) \\ v_{C2}(t) \end{bmatrix}$$

Substituting numbers,

$$\Phi(s) = \frac{\begin{bmatrix} s - 24510 & -89389 \\ 12254 & s + 33881 \end{bmatrix}}{s^2 + 9371s + 2.65 \times 10^8}$$

$$\mathbf{Q}(s) = \begin{bmatrix} \dfrac{2s^2 + 61995s - 5.3 \times 10^8}{s(s^2 + 9371s + 2.65 \times 10^8)} \\ -\dfrac{s^2 + 9373s - 2.65 \times 10^8}{s(s^2 + 9371s + 2.65 \times 10^8)} \end{bmatrix}$$

$$Y(s) = \frac{3}{s} - 6\left(\frac{s+4686}{(s+4686)^2 + 2.43\times10^8} + \frac{4686}{1.559\times10^4}\frac{1.559\times10^4}{(s+4686)^2 + 2.43\times10^8}\right)$$

$$y(t) = \left[3 - 6e^{-4686t}\left(\cos\left(1.559\times10^4 t\right) + 0.3\sin\left(1.559\times10^4 t\right)\right)\right]u(t)$$