Chapter 11 - The *z* Transform

Selected Solutions

1. Using the definition of the *z* transform and/or the transform pairs,

$$\alpha^n \mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \quad , \quad |z| > |\alpha|$$

and

$$\sin(\Omega_0 n)\mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z\sin(\Omega_0)}{z^2 - 2z\cos(\Omega_0)+1} = \frac{\sin(\Omega_0)z^{-1}}{1-2\cos(\Omega_0)z^{-1}+z^{-2}} \quad , \quad |z| > 1 \ ,$$

find the *z* transforms of these DT signals.

(a) $\mathrm{x}[n] = \mathrm{u}[n]$

Using $\alpha^n \mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}}$, $|z| > |\alpha|$, let $\alpha = 1$. Then

$$\mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-1} = \frac{1}{1-z^{-1}} \quad , \quad |z| > 1$$

(b) $\mathrm{x}[n] = e^{-10n}\mathrm{u}[n]$

(c) $\mathrm{x}[n] = e^n \sin(n)\mathrm{u}[n]$

$$\mathrm{x}[n] = e^n \frac{e^{jn} - e^{-jn}}{j2}\mathrm{u}[n] = \frac{e^{(1+j)n} - e^{(1-j)n}}{j2}\mathrm{u}[n]$$

(d) $\mathrm{x}[n] = \delta[n]$

2. Sketch the region of convergence (if it exists) in the *z* plane, of the bilateral *z* transform of these DT signals.

(a) $\mathrm{x}[n] = \mathrm{u}[n] + \mathrm{u}[-n]$

$$\mathrm{x}[n] \xleftrightarrow{\mathcal{Z}} \sum_{n=-\infty}^{\infty} \left(\mathrm{u}[n] + \mathrm{u}[-n]\right) z^{-n} = \sum_{n=-\infty}^{\infty} \mathrm{u}[n] z^{-n} + \sum_{n=-\infty}^{\infty} \mathrm{u}[-n] z^{-n}$$

$$\mathrm{x}[n] \xleftrightarrow{\mathcal{Z}} \sum_{n=0}^{\infty} z^{-n} + \sum_{n=-\infty}^{0} z^{-n} = \sum_{n=0}^{\infty} z^{-n} + \sum_{n=0}^{\infty} z^{n}$$

$$\sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}} \;,\; |z|>1 \qquad \text{and} \qquad \sum_{n=0}^{\infty} z^{n} = \frac{1}{1-z} \;,\; |z|<1$$

The two regions of convergence do not overlap. Therefore the bilateral transform does not exist.

(b) $\mathrm{x}[n] = \mathrm{u}[n] - \mathrm{u}[n-10]$

$$\mathrm{u}[n] - \mathrm{u}[n-10] \xleftrightarrow{\mathcal{Z}} \frac{z^9 + z^8 + z^7 + z^6 + z^5 + z^4 + z^3 + z^2 + z + 1}{z^9}$$

3. Using the time-shifting property, find the z transforms of these signals.

(a) $\mathrm{x}[n] = \mathrm{u}[n-5]$

(b) $\mathrm{x}[n] = \mathrm{u}[n+2]$

Using $\mathrm{g}[n+n_0] \xleftrightarrow{\mathcal{Z}} z^{n_0}\left(\mathrm{G}(z) - \sum_{m=0}^{n_0-1} \mathrm{g}[m] z^{-m}\right) \;,\; n_0 > 0$

$$\mathrm{u}[n+2] \xleftrightarrow{\mathcal{Z}} z^2\left(\frac{z}{z-1} - \sum_{m=0}^{1} \mathrm{u}[m] z^{-m}\right) = z^2\left[\frac{z}{z-1} - \left(1+z^{-1}\right)\right] = z\left(\frac{z^2}{z-1} - z - 1\right)$$

$$\mathrm{u}[n+2] \xleftrightarrow{\mathcal{Z}} z\frac{z^2 - z(z-1) - (z-1)}{z-1} = \frac{z}{z-1} \;,\; |z|>1$$

Which is the same as the z transform of the unit step, as it should be.

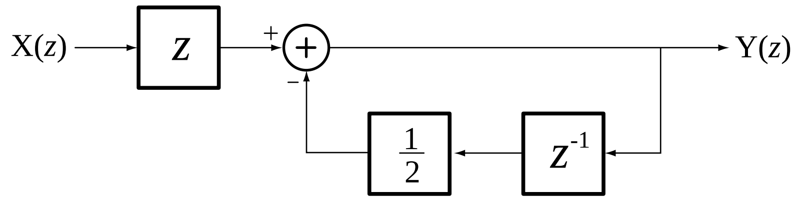
(c) $\mathrm{x}[n] = \left(\frac{2}{3}\right)^n \mathrm{u}[n+2]$

4. Draw system diagrams for these transfer functions.

(a) $\mathrm{H}(z) = \dfrac{z^2}{z + \frac{1}{2}}$

$$\mathrm{Y}(z) = z\,\mathrm{X}(z) - \frac{z^{-1}}{2}\mathrm{Y}(z)$$

This system is non-causal and cannot be built as a real-time system. But it can be diagrammed for use as an "off-line" system in which all the past and future excitation values are available.



This is not a unique solution. Many other correct diagrams could be drawn.

(b) $\mathrm{H}(z) = \dfrac{z}{z^2 + z + 1}$

5. Using the change-of-scale property, find the z transform of

$$\mathrm{x}[n] = \sin\left(\frac{2\pi n}{32}\right)\cos\left(\frac{2\pi n}{8}\right)\mathrm{u}[n] \ .$$

$$\mathrm{x}[n] = \frac{e^{j\frac{2\pi n}{32}} - e^{-j\frac{2\pi n}{32}}}{j2}\cos\left(\frac{2\pi n}{8}\right)\mathrm{u}[n]$$

The algebra is in this exercise is straightforward , but long and tedious.

$$\sin\left(\frac{2\pi n}{32}\right)\cos\left(\frac{2\pi n}{8}\right)\mathrm{u}[n] \stackrel{z}{\longleftrightarrow} z\frac{0.1379z^2 - 0.3827z + 0.1379}{z^4 - 2.7741z^3 + 3.8478z^2 - 2.7741z + 1}$$

6. Using the z-domain-differentiation property find the z transform of

$$\mathrm{x}[n] = n\left(\frac{5}{8}\right)^n \mathrm{u}[n] \ .$$

7. Using the convolution property, find the z transforms of these signals.

(a) $\mathrm{x}[n] = (0.9)^n \mathrm{u}[n] * \mathrm{u}[n]$

(b) $\mathrm{x}[n] = (0.9)^n \mathrm{u}[n] * (0.6)^n \mathrm{u}[n]$

8. Using the differencing property and the z transform of the unit sequence, find the z transform of the DT unit impulse and verify your result by checking the z-transform table.

9. Find the z transform of

$$\mathrm{x}[n] = \mathrm{u}[n] - \mathrm{u}[n-10]$$

and, using that result and the differencing property, find the z transform of

$$x[n] = \delta[n] - \delta[n-10] \ .$$

Compare this result with the z transform found directly by applying the time-shifting property to a DT impulse.

$$u[n] - u[n-10] \overset{z}{\longleftrightarrow} \frac{z}{z-1} - z^{-10}\frac{z}{z-1}$$

$$\delta[n] - \delta[n-10] \overset{z}{\longleftrightarrow} \frac{z}{z-1} - z^{-1}\frac{z}{z-1} - \left[z^{-10}\frac{z}{z-1} - z^{-11}\frac{z}{z-1}\right]$$

Simplify.

10. Using the accumulation property, find the z transforms of these signals.

(a) $x[n] = \text{ramp}[n]$

Use $x[n] = \sum_{m=0}^{n} u[m-1]$ and $u[n-1] \overset{z}{\longleftrightarrow} \frac{1}{z-1}$

(b) $x[n] = \sum_{m=0}^{n} \left(u[m] - u[m-5]\right)$

11. Using the final-value theorem, find the final value of functions that are the inverse z transforms of these functions (if the theorem applies).

(a) $X(z) = \frac{z}{z-1}$

The inverse transform is the unit sequence and its limit exists as t approaches infinity. Applying the final value theorem,

$$\lim_{n\to\infty} g[n] = \lim_{z\to 1}(z-1)G(z) \ ,$$

we get

$$\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)\frac{z}{z-1} = 1$$

and this checks with our understanding of the unit sequence function.

(b) $X(z) = z\dfrac{2z - \frac{7}{4}}{z^2 - \frac{7}{4}z + \frac{3}{4}}$

12. Find the inverse z transforms of these functions in series form by synthetic division.

(a) $\mathrm{X}(z) = \dfrac{z}{z - \frac{1}{2}}$

$$\begin{array}{r} 1 + \frac{1}{2z} + \frac{1}{4z^2} + \cdots + \frac{1}{(2z)^k} + \cdots \\ z - \frac{1}{2} \overline{\big) z \qquad\qquad\qquad\qquad\qquad} \\ \underline{z - \frac{1}{2}} \\ \frac{1}{2} \\ \underline{\frac{1}{2} - \frac{1}{4z}} \\ \frac{1}{4z} \cdots \end{array}$$

(b) $\mathrm{X}(z) = \dfrac{z-1}{z^2 - 2z + 1}$

Notice that $\mathrm{X}(z) = \dfrac{z-1}{(z-1)^2} = \dfrac{1}{z-1} = z^{-1}\dfrac{z}{z-1} \Rightarrow \mathrm{x}[n] = \mathrm{u}[n-1]$

13. Find the inverse z transforms of these functions in closed form using partial fraction expansions, a z transform table and the properties of the z transform.

(a) $\mathrm{X}(z) = \dfrac{1}{z\left(z - \frac{1}{2}\right)} = -\dfrac{2}{z} + \dfrac{2}{z - \frac{1}{2}} = 2\left(\dfrac{1}{z - \frac{1}{2}} - \dfrac{1}{z}\right)$

$$\mathrm{x}[n] = 2\left[\left(\frac{1}{2}\right)^{n-1} - \delta[n-1]\right]\mathrm{u}[n-1]$$

Alternate Solution:

$$\mathrm{X}(z) = z^{-2}\frac{z}{z - \frac{1}{2}} \Rightarrow \mathrm{x}[n] = \left(\frac{1}{2}\right)^{n-2}\mathrm{u}[n-2]$$

These solutions look different but are they? Check a few values to see.

(b) $\mathrm{X}(z) = \dfrac{z^2}{\left(z - \frac{1}{2}\right)\left(z - \frac{3}{4}\right)}$

(c) $\mathrm{X}(z) = \dfrac{z^2}{z^2 + 1.8z + 0.82}$

Recognize this as being similar to the forms,

$$\alpha^n \sin(\Omega_0 n)\mathrm{u}[n] \xleftrightarrow{z} \frac{z\alpha \sin(\Omega_0)}{z^2 - 2\alpha z \cos(\Omega_0) + \alpha^2} \quad , \quad |z| > |\alpha|$$

and

$$\alpha^n \cos(\Omega_0 n)\mathrm{u}[n] \xleftrightarrow{z} \frac{z[z - \alpha \cos(\Omega_0)]}{z^2 - 2\alpha z \cos(\Omega_0) + \alpha^2} \quad , \quad |z| > |\alpha|$$

where $\alpha = 0.9055$ and $\Omega_0 = 3.031$.

$$\mathrm{x}[n] = (0.9055)^n [\cos(3.031n) - 9.03\sin(3.031n)]\mathrm{u}[n]$$

14. Using the z transform, find the total solutions to these difference equations with initial conditions, for discrete time, $n \geq 0$.

(a) $2\,\mathrm{y}[n+1] - \mathrm{y}[n] = \sin\left(\dfrac{2\pi n}{16}\right)\mathrm{u}[n] \quad , \quad \mathrm{y}[0] = 1$

Z-transform the equation.

$$2z(\mathrm{Y}(z) - \mathrm{y}[0]) - \mathrm{Y}(z) = \frac{z\sin\left(\frac{\pi}{8}\right)}{z^2 - 2z\cos\left(\frac{\pi}{8}\right) + 1}$$

Then solve for Y and find the inverse transform of Y.

$$\mathrm{Y}(z) = \frac{0.2934}{z - 0.5} - \frac{0.2934z - 0.5868}{z^2 - 1.8478z + 1} + \frac{z}{z - 0.5}$$

Use $\sin(\Omega_0 n)\mathrm{u}[n] \xleftrightarrow{z} \dfrac{z\sin(\Omega_0)}{z^2 - 2z\cos(\Omega_0) + 1} \quad , \quad |z| > 1$

and

$$\cos(\Omega_0 n)\mathrm{u}[n] \xleftrightarrow{z} \frac{z[z - \cos(\Omega_0)]}{z^2 - 2z\cos(\Omega_0) + 1} \quad , \quad |z| > 1$$

and identify $\Omega_0 = 0.3926$. Y can be manipulated into the form,

$$Y(z) = \frac{0.2934}{z-0.5} + \frac{z}{z-0.5} - 0.2934z^{-1}\left(\frac{z^2-0.9239z}{z^2-1.8478z+1} - 2.812\frac{0.3827z}{z^2-1.8478z+1}\right)$$

Then

$$y[n] = 0.2934\left(\frac{1}{2}\right)^{n-1} u[n-1] + \left(\frac{1}{2}\right)^n u[n] - 0.2934\left[\cos\left(\frac{\pi}{8}(n-1)\right) - 2.812\sin\left(\frac{\pi}{8}(n-1)\right)\right]u[n-1]$$

(b) $5y[n+2] - 3y[n+1] + y[n] = (0.8)^n u[n]$, $y[0] = -1$, $y[1] = 10$

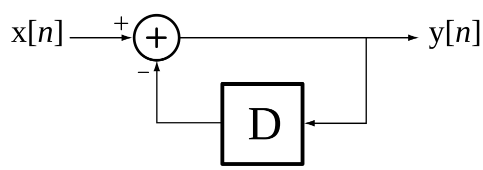
$$Y(z) = \frac{0.4444}{z-0.8} - \left(1 - 9.5556z^{-1}\left[\frac{z^2-0.3z}{z^2-0.6z+0.2} + 0.9325\frac{0.3317z}{z^2-0.6z+0.2}\right]\right)$$

$$y[n] = 0.4444(0.8)^n u[n] - \left\{\delta[n] - 9.5556(0.4472)^{n-1}\begin{bmatrix}\cos(0.8355(n-1)) \\ +0.9325\sin(0.8355(n-1))\end{bmatrix}u[n-1]\right\}$$

This is not the only way to find the inverse transform and other solution forms can be correct, but they should be equivalent to this one.

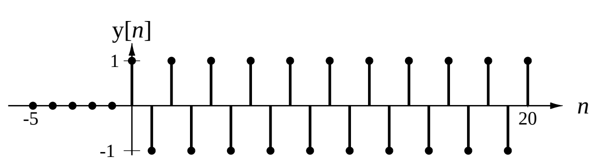
15. From each block diagram, write the difference equation and find and sketch the response, $y[n]$, of the system for discrete time, $n \geq 0$, assuming no initial energy storage in the system and impulse excitation, $x[n] = \delta[n]$.

(a)

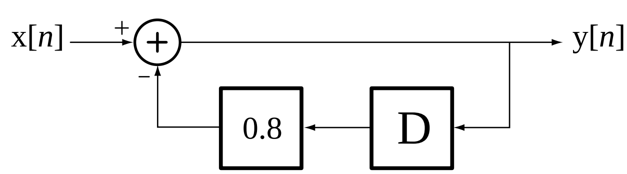


$$y[n] + y[n-1] = \delta[n]$$

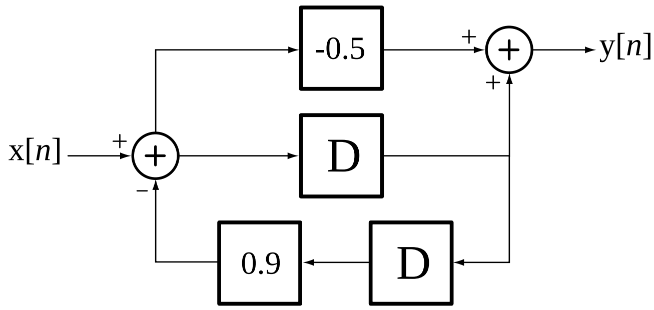
$$Y(z) = \frac{1}{1+z^{-1}} = \frac{z}{z+1} \Rightarrow y[n] = (-1)^n u[n]$$



(b)



(c)



Designate the lower input to the last summer as $\mathrm{w}[n]$. Then the excitation of the delay feeding that point is $\mathrm{w}[n+1]$ and

$$\mathrm{x}[n] - 0.9\,\mathrm{w}[n-1] = \mathrm{w}[n+1]$$

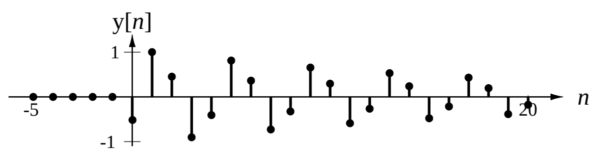
or

$$\mathrm{w}[n] + 0.9\,\mathrm{w}[n-2] = \mathrm{x}[n-1]$$

$$\mathrm{y}[n] = -0.5(\mathrm{x}[n] - 0.9\,\mathrm{w}[n-1]) + \mathrm{w}[n]$$

Then z-transform, eliminate W, solve for Y and inverse transform to find y.

$$\mathrm{y}[n] = -0.5\delta[n] + (0.9486)^n \left\{1.0541 \sin\left(\frac{n\pi}{2}\right)\mathrm{u}[n] + 0.5 \sin\left(\frac{(n-1)\pi}{2}\right)\mathrm{u}[n-1]\right\}$$

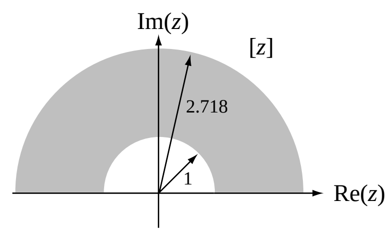


16. Sketch regions in the z plane corresponding to these regions in the s plane.

(a) $0 < \sigma < \dfrac{1}{T_s}$, $0 < \omega < \dfrac{\pi}{T_s}$, $z = e^{sT_s} = e^{(\sigma + j\omega)T_s}$

First find some corresponding points. Then the region should be obvious.

s	σ	$j\omega$	e^{sT_s}	z
0	0	0	e^0	1
$\frac{1}{T_s}$	$\frac{1}{T_s}$	0	e^1	2.718
$j\frac{\pi}{T_s}$	0	$j\frac{\pi}{T_s}$	$e^{j\pi}$	-1
$\frac{1}{T_s}+j\frac{\pi}{T_s}$	$\frac{1}{T_s}$	$j\frac{\pi}{T_s}$	$e^1 e^{j\pi}$	-2.718
$j\frac{\pi}{2T_s}$	0	$j\frac{\pi}{2T_s}$	$e^{j\frac{\pi}{2}}$	j
$\frac{1}{T_s}+j\frac{\pi}{2T_s}$	$\frac{1}{T_s}$	$j\frac{\pi}{2T_s}$	$e^1 e^{j\frac{\pi}{2}}$	$j2.718$



(b) $-\frac{1}{T_s} < \sigma < 0 \;,\; -\frac{\pi}{T_s} < \omega < 0$

(c) $-\infty < \sigma < \infty \;,\; 0 < \omega < \frac{2\pi}{T_s}$

17. Find the bilateral z transforms and ROC's of these signals.

(a) $x[n] = u[-n]$

$$u[n]u[-n] \xleftrightarrow{\mathcal{Z}} 1 = X_c(z) \;,\; \text{any } z$$

$$u[n]u[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-1} = X_{ac}\left(\frac{1}{z}\right) \;,\; |z| > 1$$

$$u[-n] \xleftrightarrow{\mathcal{Z}} \frac{\frac{1}{z}}{\frac{1}{z}-1} = \frac{1}{1-z} = X_{ac}(z) \;,\; |z| < 1$$

$$X(z) = X_c(z) - x[0] + X_{ac}(z) = 1 - 1 + \frac{1}{1-z} = \frac{1}{1-z} \;,\; |z| < 1$$

Alternate Solution:

$$\mathrm{X}(z) = \sum_{n=-\infty}^{\infty} \mathrm{x}[n] z^{-n} = \sum_{n=-\infty}^{\infty} \mathrm{u}[-n] z^{-n} = \sum_{n=-\infty}^{0} z^{-n} = \sum_{n=0}^{\infty} z^{n} = \frac{1}{1-z} \quad , \ |z| < 1$$

(b) $\mathrm{x}[n] = \alpha^n \mathrm{u}[-n]$

(c) $\mathrm{x}[n] = (0.5)^n \mathrm{u}[-n] + (0.3)^n \mathrm{u}[n]$

(d) $\mathrm{x}[n] = (-1.5)^n \cos\left(\frac{2\pi n}{8}\right) \mathrm{u}[-n]$

18. Using the definition of the z transform verify the z transforms of the following functions:

(a)

z-Transform Table Entry: $\mathrm{u}[n] \xleftrightarrow{z} \frac{z}{z-1} = \frac{1}{1-z^{-1}}$

$$\mathrm{X}(z) = \sum_{n=0}^{\infty} \mathrm{u}[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n} = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots$$

$$\begin{array}{r l} & 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots \\ z-1 \,\big) & z \\ & \underline{z-1} \\ & \quad 1 \\ & \underline{1 - \frac{1}{z}} \\ & \quad \frac{1}{z} \end{array}$$

Check.

(b) $\mathrm{x}[n] = \frac{n^2}{2!} \mathrm{u}[n]$

z-Transform Table Entry: $\frac{n^2}{2!} \mathrm{u}[n] \xleftrightarrow{z} \frac{z(z+1)}{2(z-1)^3} = \frac{1+z^{-1}}{2z(1-z^{-1})}$

(c) $\mathrm{x}[n] = n\alpha^n \mathrm{u}[n]$

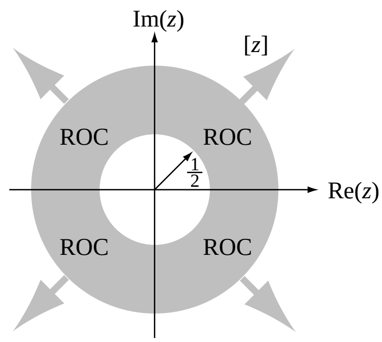
(d) $\mathrm{x}[n] = \alpha^n \sin(2\pi F_0 n) \mathrm{u}[n]$

19. Sketch the region of convergence (if it exists) in the z plane, of the bilateral z transform of these DT signals.

(a) $\mathrm{x}[n] = \left(\frac{1}{2}\right)^n \mathrm{u}[n]$

Using $\alpha^n \mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}}$, $|z| > |\alpha|$

$$\left(\frac{1}{2}\right)^n \mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\frac{1}{2}} = \frac{1}{1-\frac{1}{2}z^{-1}} \ , \ |z| > \frac{1}{2}$$



(b) $\mathrm{x}[n] = \left(\frac{5}{4}\right)^n \mathrm{u}[n] + \left(\frac{10}{7}\right)^n \mathrm{u}[-n]$

20. Using the time-shifting property, find the z transforms of these signals.

(a) $\mathrm{x}[n] = \left(\frac{2}{3}\right)^{n-1} \mathrm{u}[n-1]$

$$\left(\frac{2}{3}\right)^n \mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\frac{2}{3}}$$

$$\left(\frac{2}{3}\right)^{n-1} \mathrm{u}[n-1] \xleftrightarrow{\mathcal{Z}} z^{-1}\frac{z}{z-\frac{2}{3}} = \frac{1}{z-\frac{2}{3}}$$

(b) $\mathrm{x}[n] = \left(\frac{2}{3}\right)^n \mathrm{u}[n-1]$

$$\mathrm{x}[n] = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)^{n-1} \mathrm{u}[n-1]$$

(c) $$\mathrm{x}[n] = \sin\left(\frac{2\pi(n-1)}{4}\right)\mathrm{u}[n-1]$$

21. Draw system diagrams for these transfer functions.

(a) $$\mathrm{H}(z) = \frac{z\left(z+\frac{2}{3}\right)}{z^2 + \frac{2}{3}z + \frac{3}{4}}$$

(b) $$\mathrm{H}(z) = \frac{z^2}{(z-0.75)(z+0.1)(z-0.3)}$$

22. If the *z* transform of $\mathrm{x}[n]$ is $\mathrm{X}(z) = \frac{1}{z-\frac{3}{4}}$, and

$$\mathrm{Y}(z) = j\left[\mathrm{X}\left(e^{j\frac{\pi}{6}}z\right) - \mathrm{X}\left(e^{-j\frac{\pi}{6}}z\right)\right]$$

what is $\mathrm{y}[n]$?

23. Using the convolution property, find the *z* transforms of these signals.

(a) $$\mathrm{x}[n] = \sin\left(\frac{2\pi n}{8}\right)\mathrm{u}[n] * \mathrm{u}[n]$$

(b) $$\mathrm{x}[n] = \sin\left(\frac{2\pi n}{8}\right)\mathrm{u}[n] * (\mathrm{u}[n] - \mathrm{u}[n-8])$$

From part (a),

$$\sin\left(\frac{2\pi n}{8}\right)\mathrm{u}[n] * \mathrm{u}[n] \xleftrightarrow{z} \frac{z^2 \sin\left(\frac{\pi}{4}\right)}{z^3 - z^2\left[2\cos\left(\frac{\pi}{4}\right) - 1\right] + z\left[1 + 2\cos\left(\frac{\pi}{4}\right)\right] - 1}$$

24. Find the inverse *z* transforms of these functions in closed form using partial fraction expansions, a *z* transform table and the properties of the *z* transform.

(a) $$\mathrm{X}(z) = \frac{z-1}{z^2 + 1.8z + 0.82}$$

$$\mathrm{X}(z)=\frac{z}{z^2+1.8z+0.82}-\frac{1}{z^2+1.8z+0.82}$$

The denominators are of the form, $z^2-2\alpha\cos(\Omega_0)z+\alpha^2$ where $\alpha=0.9055$ and $\Omega_0=3.031$. Therefore

$$\mathrm{X}(z)=\frac{1}{\alpha\sin(\Omega_0)}\frac{z\alpha\sin(\Omega_0)}{z^2+1.8z+0.82}-\frac{z^{-1}}{\alpha\sin(\Omega_0)}\frac{z\alpha\sin(\Omega_0)}{z^2+1.8z+0.82}$$

or

$$\mathrm{X}(z)=10\frac{0.1z}{z^2+1.8z+0.82}-10z^{-1}\frac{0.1z}{z^2+1.8z+0.82}$$

Then, using

$$\alpha^n\sin(\Omega_0 n)\mathrm{u}[n]\xleftrightarrow{\mathcal{Z}}\frac{z\alpha\sin(\Omega_0)}{z^2-2\alpha z\cos(\Omega_0)+\alpha^2}\quad,\quad |z|>|\alpha|$$

$$\mathrm{x}[n]=10\left\{(0.9055)^n\sin(3.031n)\mathrm{u}[n]-(0.9055)^{n-1}\sin(3.031(n-1))\mathrm{u}[n-1]\right\}$$

Alternate Solution:

$$\mathrm{X}(z)=\frac{z-1}{z^2+1.8z+0.82}=\frac{0.5+j9.5}{z+0.9-j0.1}+\frac{0.5-j9.5}{z+0.9+j0.1}$$

$$\mathrm{x}[n]=\left[(0.5+j9.5)(-0.9+j0.1)^{n-1}+(0.5-j9.5)(-0.9-j0.1)^{n-1}\right]\mathrm{u}[n-1]$$

$$\mathrm{x}[n]=\begin{bmatrix}(0.5+j9.5)(0.9055)^{n-1}e^{j3.031(n-1)}\\+(0.5-j9.5)(0.9055)^{n-1}e^{-j3.031(n-1)}\end{bmatrix}\mathrm{u}[n-1]$$

$$\mathrm{x}[n]=\begin{bmatrix}0.5(0.9055)^{n-1}\left(e^{j3.031(n-1)}+e^{-j3.031(n-1)}\right)\\+j9.5(0.9055)^{n-1}\left(e^{j3.031(n-1)}-e^{-j3.031(n-1)}\right)\end{bmatrix}\mathrm{u}[n-1]$$

$$\mathrm{x}[n]=\left[(0.9055)^{n-1}\cos(3.031(n-1))-19(0.9055)^{n-1}\sin(3.031(n-1))\right]\mathrm{u}[n-1]$$

Although the two solutions above look different in analytical form, they are equivalent. That is, the graphs of these functions are identical.

(b) $\mathrm{X}(z)=\dfrac{z-1}{z(z^2+1.8z+0.82)}$

The answer to this part is the same as the answer to the previous part except delayed by 1 in discrete time.

(c) $$\mathrm{X}(z) = \frac{z^2}{z^2 - z + \frac{1}{4}}$$