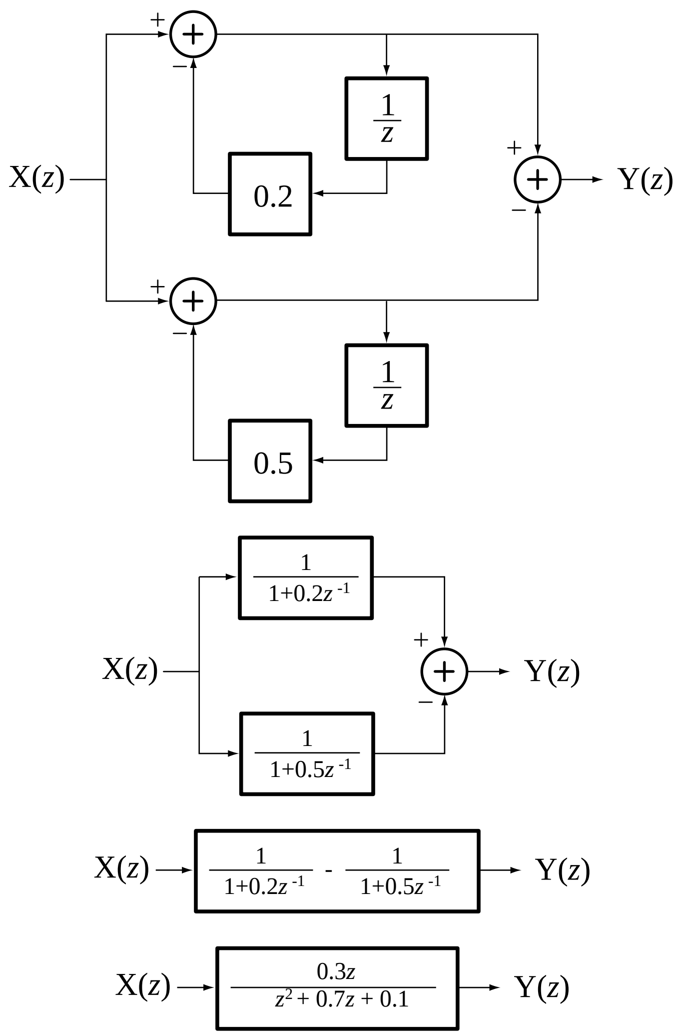
# Chapter 12 - *z* Transform Analysis of Signals and Systems

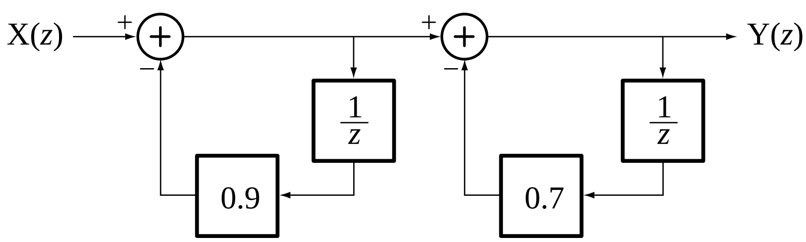
## Solutions

1. Find the transfer functions for these systems by block diagram reduction.

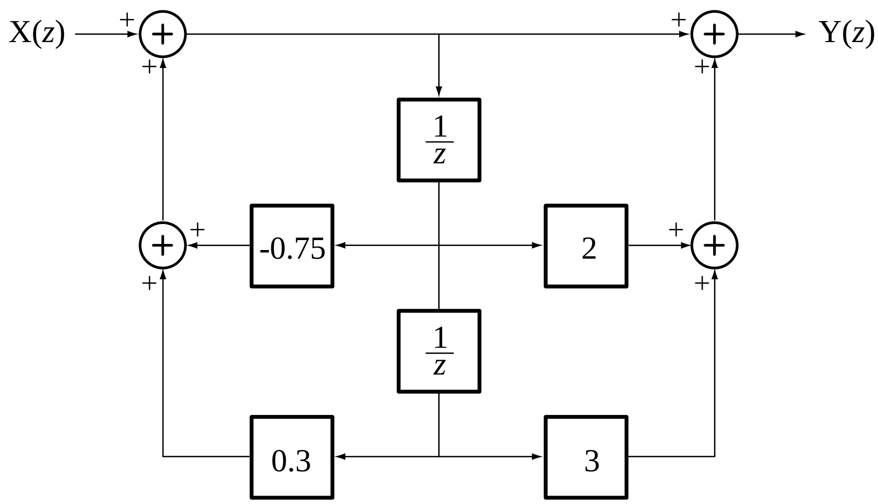
(a)




(b)



(c)




2. Evaluate the stability of the systems with each of these transfer functions.

(a) $\mathrm{H}(z) = \dfrac{z}{z-2}$

(b) $\mathrm{H}(z) = \dfrac{z}{z^2 - \frac{7}{8}}$

(c) $\mathrm{H}(z) = \dfrac{z}{z^2 - \frac{3}{2}z + \frac{9}{8}}$ Poles at $z = \frac{3}{4} \pm j\frac{3}{4}$. Both outside the unit circle. Unstable.

(d) $\mathrm{H}(z) = \dfrac{z^2 - 1}{z^3 - 2z^2 + 3.75z - 0.5625}$

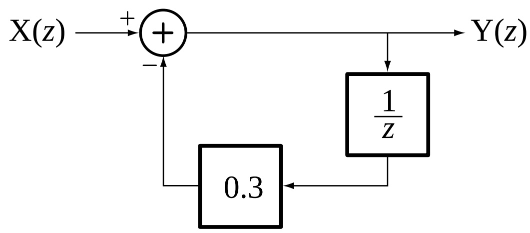
3. A feedback DT system has a transfer function,

$$\mathrm{H}(z) = \frac{K}{1 + K\dfrac{z}{z-0.9}} \quad .$$

For what range of $K$'s is this system stable?

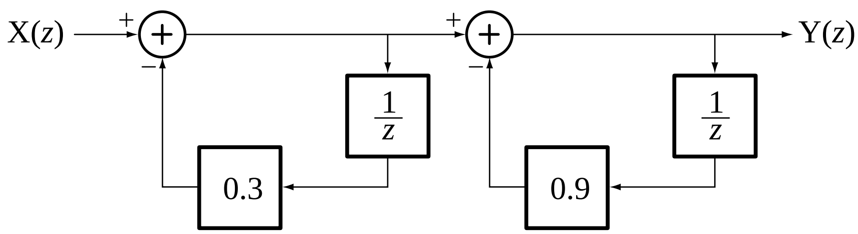
4. Find the overall transfer functions of these systems in the form of a single ratio of polynomials in $z$.

(a)




$$\mathrm{H}(z) = \frac{1}{1+0.3z^{-1}} = \frac{z}{z+0.3}$$

(b)




5. Find the DT-domain responses, $\mathrm{y}[n]$, of the systems with these transfer functions to the unit sequence excitation, $\mathrm{x}[n] = \mathrm{u}[n]$.

(a) $\mathrm{H}(z) = \frac{z}{z-1}$

$$\mathrm{Y}(z) = \frac{z}{z-1}\frac{z}{z-1} = z\frac{z}{(z-1)^2}$$

$$\mathrm{ramp}[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{(z-1)^2}$$

$$\mathrm{ramp}[n+1] \xleftrightarrow{\mathcal{Z}} z\left[\frac{z}{(z-1)^2} - \underbrace{\mathrm{ramp}[0]}_{=0}\right]$$

$$\mathrm{y}[n] = \mathrm{ramp}[n+1] \xleftrightarrow{\mathcal{Z}} z\frac{z}{(z-1)^2}$$

(b) $\mathrm{H}(z) = \dfrac{z-1}{z-\frac{1}{2}}$

6. Find the DT-domain responses, $\mathrm{y}[n]$, of the systems with these transfer functions to the excitation, $\mathrm{x}[n] = \cos\left(\frac{2\pi n}{8}\right)\mathrm{u}[n]$. Then show that the steady-state response is the same

as would have been obtained by using DTFT analysis with an excitation, $\mathrm{x}[n] = \cos\left(\frac{2\pi n}{8}\right)$.

(a) $\mathrm{H}(z) = \frac{z}{z-0.9}$

$$\mathrm{Y}(z) = \frac{z}{z-0.9}\frac{z\left[z-\cos(\Omega_0)\right]}{z^2-2z\cos(\Omega_0)+1} = z\left[\frac{0.3232}{z-0.9}+\frac{0.6768z+0.3591}{z^2-2z\cos(\Omega_0)+1}\right]$$

Using the result derived in the text,

$$\mathrm{y}[n] = \mathcal{Z}^{-1}\left(z\frac{\mathrm{N}_1(z)}{\mathrm{D}(z)}\right)+\left|\mathrm{H}(p_1)\right|\cos\left(\Omega_0 n+\angle\mathrm{H}(p_1)\right)\mathrm{u}[n]$$

where $p_1 = e^{j\frac{\pi}{4}}$

$$\mathrm{y}[n] = \mathcal{Z}^{-1}\left(z\frac{0.3232}{z-0.9}\right)+\left|\frac{e^{j\frac{\pi}{4}}}{e^{j\frac{\pi}{4}}-0.9}\right|\cos\left(\Omega_0 n+\angle\frac{e^{j\frac{\pi}{4}}}{e^{j\frac{\pi}{4}}-0.9}\right)\mathrm{u}[n]$$

$$\mathrm{y}[n] = 0.3232(0.9)^n\,\mathrm{u}[n]+1.3644\cos\left(\frac{\pi}{4}n-1.0517\right)\mathrm{u}[n]$$

Using the DTFT,

$$\mathrm{Y}(j\Omega) = \frac{e^{j\Omega}}{e^{j\Omega}-0.9}\frac{1}{2}\left[\mathrm{comb}\left(\frac{\Omega-\frac{\pi}{4}}{2\pi}\right)+\mathrm{comb}\left(\frac{\Omega+\frac{\pi}{4}}{2\pi}\right)\right]$$

Although it is algebraically tedious, this expression can be simplified and inverse transformed into

$$\mathrm{y}[n] = 0.6768\cos\left(\frac{2\pi n}{8}\right)+1.1847\sin\left(\frac{2\pi n}{8}\right)$$

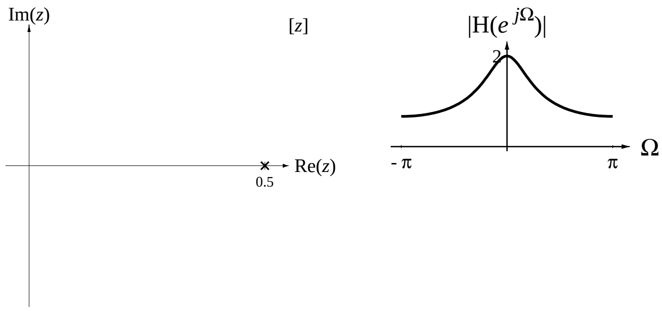
or

$$\mathrm{y}[n] = 1.3644\cos\left(\frac{2\pi n}{8}-1.0518\right)\text{. Check.}$$

(b) $\mathrm{H}(z) = \frac{z^2}{z^2-1.6z+0.63}$

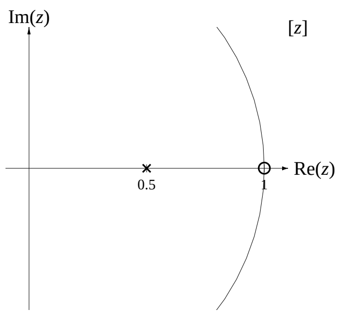
7. Sketch the magnitude frequency response of these systems from their pole-zero diagrams.

(a)

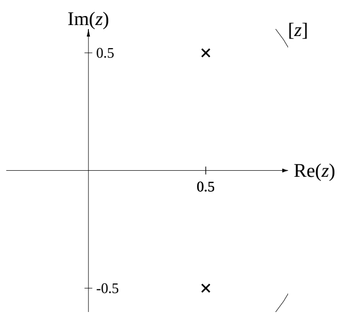



One pole and no zeros. Non-zero at $\Omega = 0$. Vector from pole to point on the unit circle gets longer as $\Omega$ moves from 0 to $\pi$ making the magnitude of the frequency response smaller.

(b)




(c)




8. Use the Jury stability test to determine which of these transfer functions are for unstable systems.

(a) $\mathrm{H}(z) = \dfrac{z^2 - z}{z^3 - 0.25z^2 - 0.6528z + 0.2083}$

The Jury array is

$$\begin{array}{ccccc} 1 & 0.2083 & -0.6528 & -0.25 & 1 \\ 2 & 1 & -0.25 & -0.6528 & 0.2083 \\ 3 & -0.9566 & 0.114 & 0.6007 & \end{array} .$$

$$\mathrm{D}(1) = 1 - 0.25 - 0.6528 + 0.2083 = 0.3055 > 0$$
$$(-1)^3 \mathrm{D}(-1) = (-1)^3(-1 - 0.25 + 0.6528 + 0.2083) = 0.3889 > 0$$
$$|-0.9566| > |0.6007|$$

Stable

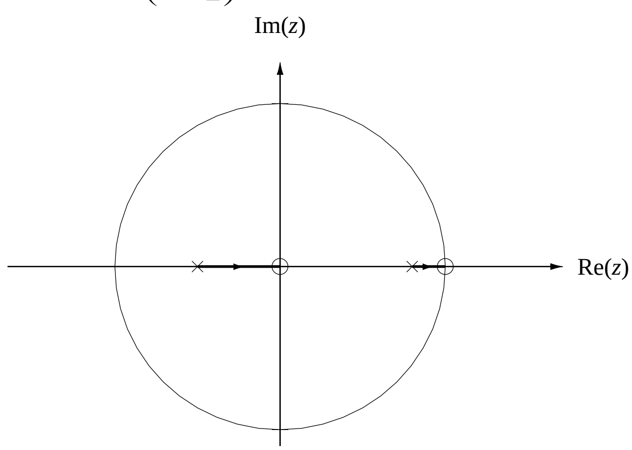
(b) $\mathrm{H}(z)=\dfrac{z-1}{z^4-0.9z^3-0.65z^2+0.873z}$

(c) $\mathrm{H}(z)=\dfrac{z}{z^4-1.5z^3+0.5z^2+0.25z-0.25}$

9. Draw a root locus for each system with the given forward and feedback path transfer functions.

(a) $\mathrm{H}_1(z)=K\dfrac{z-1}{z+\frac{1}{2}}$ , $\mathrm{H}_2(z)=\dfrac{4z}{z-0.8}$

$$\mathrm{T}(z)=4K\frac{z-1}{z+\frac{1}{2}}\frac{z}{z-0.8}=4K\frac{z(z-1)}{\left(z+\frac{1}{2}\right)(z-0.8)}$$




(b) $\mathrm{H}_1(z)=K\dfrac{z-1}{z+\frac{1}{2}}$ , $\mathrm{H}_2(z)=\dfrac{4}{z-0.8}$

(c) $\mathrm{H}_1(z)=K\dfrac{z}{z-\frac{1}{4}}$ , $\mathrm{H}_2(z)=\dfrac{z+\frac{1}{5}}{z-\frac{3}{4}}$

(d) $\mathrm{H}_1(z)=K\dfrac{z}{z-\frac{1}{4}}$ , $\mathrm{H}_2(z)=\dfrac{z+2}{z-\frac{3}{4}}$

(e) $\mathrm{H}_1(z)=K\dfrac{1}{z^2-\frac{1}{3}z-\frac{2}{9}}$ , $\mathrm{H}_2(z)=1$

10. Using the impulse-invariant design method, design a DT system to approximate the CT systems with these transfer functions at the sampling rates specified. Compare the impulse and unit step (or sequence) responses of the CT and DT systems.

(a) $\mathrm{H}(s) = \dfrac{6}{s+6}$ , $f_s = 4$ Hz

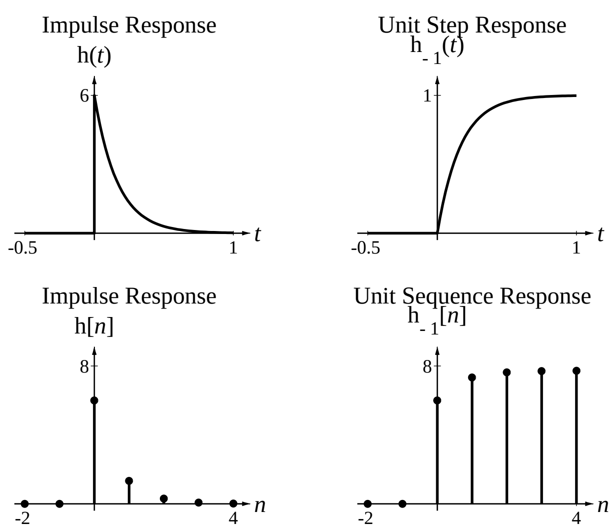
$$\mathrm{h}(t) = 6e^{-6t}\,\mathrm{u}(t) \Rightarrow \mathrm{h}[n] = 6e^{-\frac{3}{2}n}\,\mathrm{u}[n] \Rightarrow \mathrm{H}(z) = \frac{6z}{z - e^{-\frac{3}{2}}} = \frac{6z}{z - 0.2231}$$

Unit step response: $\mathrm{H}_{-1}(s) = \dfrac{1}{s}\dfrac{6}{s+6} = \dfrac{1}{s} - \dfrac{1}{s+6} \Rightarrow \mathrm{h}_{-1}(t) = \left(1 - e^{-6t}\right)\mathrm{u}(t)$

Unit sequence response:

$$\mathrm{H}_{-1}(z) = \frac{z}{z-1}\frac{6z}{z-0.2231} = z\left(\frac{7.723}{z-1} - \frac{1.723}{z-0.2231}\right)$$

$$\mathrm{h}_{-1}[n] = \left[7.723 - 1.723(0.2231)^n\right]\mathrm{u}[n]$$




(b) $\mathrm{H}(s) = \dfrac{6}{s+6}$ , $f_s = 20$ Hz

11. Using the impulse-invariant and step-invariant design methods, design digital filters to approximate analog filters with these transfer functions. In each case choose a sampling frequency which is 10 times the magnitude of the distance of the farthest pole or zero from the origin of the "$s$" plane. Graphically compare the step responses of the digital and analog filters.

(a) $\mathrm{H}(s) = \dfrac{2}{s^2 + 3s + 2}$

Sampling Rate: $f_s = \frac{10}{\pi} = 3.183 \Rightarrow T_s = 0.31416$

Impulse Invariant:

DT Impulse Response: $\mathrm{h}[n] = 2\left[(0.7304)^n - (0.5335)^n\right]\mathrm{u}[n]$

$z$-Domain Transfer Function:

$$\mathrm{H}(z) = 0.3938\frac{z}{z^2 - 1.264z + 0.3897}$$

$z$ Transform of Step Response:

$$\mathrm{H}_{-1}(z) = 0.3938\left(\frac{7.955}{z-1} - \frac{10.05}{z-0.7306} + \frac{3.092}{z-0.5334}\right)$$

DT Step Response:

$$\mathrm{h}_{-1}[n] = \left[3.1312 - 3.9575(0.7304)^{n-1} + 1.22(0.53348)^{n-1}\right]\mathrm{u}[n-1]$$

$$\mathrm{h}_{-1}[n] = \left[3.1312 - 5.4183(0.7304)^{n} + 2.287(0.53348)^{n}\right]\mathrm{u}[n]$$

(Last form is correct because $\mathrm{h}_{-1}[0] = 0$.)

Step Invariant:

Laplace Transform of Step Response:

$$\mathrm{H}_{-1}(s) = \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}$$

CT Step Response: $\mathrm{h}_{-1}(t) = \left(1 - 2e^{-t} + e^{-2t}\right)\mathrm{u}(t)$

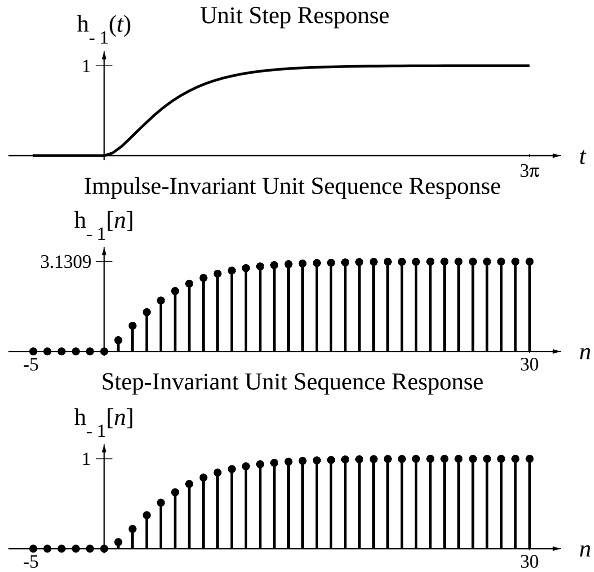
DT Step Response: $\mathrm{h}_{-1}[n] = \left(1 - 2(0.7304)^n + (0.5335)^n\right)\mathrm{u}[n]$

$z$ Transform of Step Response:

$$\mathrm{H}_{-1}(z) = \frac{z}{z-1} - 2\frac{z}{z-0.7304} + \frac{z}{z-0.5335}$$

Transfer Function:

$$\mathrm{H}(z) = 0.0726\frac{z+0.7314}{z^2 - 1.2639z + 0.3897}$$




(b) $$\mathrm{H}(s)=\frac{6s}{s^2+13s+40}=\frac{6s}{(s+5)(s+8)}=-\frac{10}{s+5}+\frac{16}{s+8}$$

(c) $$\mathrm{H}(s)=\frac{250}{s^2+10s+250}$$

DT Step Response:

$$\mathrm{h}_{-1}[n]=\begin{bmatrix}24.348\,\mathrm{u}[n-1]-36.245(0.8198)^n\sin(0.5959n)\\ +35.55(0.8198)^{n-1}\sin(0.5959(n-1))\,\mathrm{u}[n-1]\end{bmatrix}\mathrm{u}[n]$$

Alternate solution:

$$\mathrm{h}_{-1}[n]=\left\{24.348-24.348(0.81982)^n\left[\cos(0.59594n)+0.01413\sin(0.59594n)\right]\right\}\mathrm{u}[n]$$

12. Using the difference-equation method and all backward differences, design digital filters to approximate analog filters with these transfer functions. . In each case, if a sampling frequency is not specified choose a sampling frequency which is 10 times the magnitude of the distance of the farthest pole or zero from the origin of the "$s$" plane. Graphically compare the step responses of the digital and analog filters.

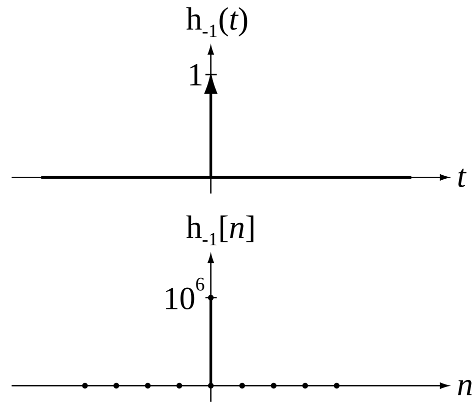
(a) $\mathrm{H}(s)=s,\quad f_s=1\mathrm{MHz}$

$$\frac{\mathrm{Y}(s)}{\mathrm{X}(s)}=s$$

$$\mathrm{H}(z)=\left.\frac{\mathrm{Y}(s)}{\mathrm{X}(s)}\right|_{s\to\frac{1-z^{-1}}{T_s}}=\left.s\right|_{s\to\frac{1-z^{-1}}{T_s}}=\frac{1-z^{-1}}{T_s}=10^6\,\frac{z-1}{z}$$

$$\mathrm{H}_{-1}(s) = 1 \Rightarrow \mathrm{h}_{-1}(t) = \delta(t)$$

$$\mathrm{H}_{-1}(z) = 10^6 \Rightarrow \mathrm{h}_{-1}[n] = 10^6 \delta[n]$$




(b) $\mathrm{H}(s) = \frac{1}{s}$ , $f_s = 1\text{kHz}$

(c) $\mathrm{H}(s) = \frac{2}{s^2 + 3s + 2}$

13. Using the matched-z-transform method, design digital filters to approximate analog filters with these transfer functions. In each case, if a sampling frequency is not specified choose a sampling frequency which is 10 times the magnitude of the distance of the farthest pole or zero from the origin of the "$s$" plane (unless all poles or zeros are at the origin, in which case the sampling rate will not matter, in this method). Graphically compare the step responses of the digital and analog filters.

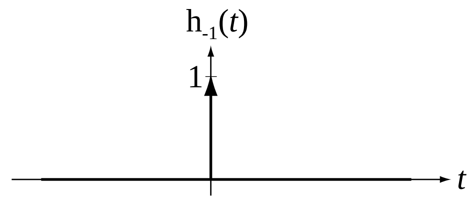
(a) $\mathrm{H}(s) = s$

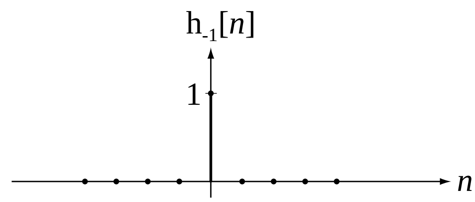
Zero at $s = 0$. Transformation is $s - a \to 1 - e^{aT} z^{-1}$. Therefore

$$\mathrm{H}(z) = 1 - z^{-1} = \frac{z-1}{z}$$

$$\mathrm{H}_{-1}(s) = \frac{1}{s} s = 1 \Rightarrow \mathrm{h}_{-1}(t) = \delta(t)$$

$$\mathrm{H}_{-1}(z) = \frac{z}{z-1} \frac{z-1}{z} = 1 \Rightarrow \mathrm{h}_{-1}[n] = \delta[n]$$




(b) $\mathrm{H}(s)=\frac{1}{s}$

(c) $\mathrm{H}(s)=\frac{2s}{s^2+10s+25}=\frac{2s}{(s+5)^2}$ , Double pole at $s=-5$.

14. Using the bilinear-z-transform method, design digital filters to approximate analog filters with these transfer functions In each case choose a sampling frequency which is 10 times the magnitude of the distance of the farthest pole or zero from the origin of the "*s*" plane. Graphically compare the step responses of the digital and analog filters.

(a) $\mathrm{H}(s)=\frac{s-10}{s+10}$

$$\mathrm{h}(t)=\left[\delta(t)-20e^{-10t}\right]\mathrm{u}(t)$$

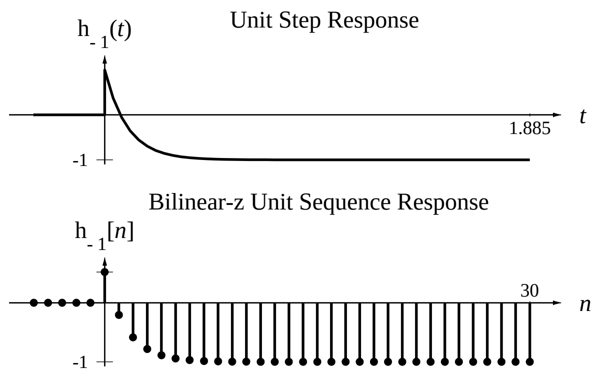
$$\mathrm{H}_{-1}(s)=\frac{1}{s}\frac{s-10}{s+10}=-\frac{1}{s}+\frac{2}{s+10}\Rightarrow \mathrm{h}_{-1}(t)=\left(2e^{-10t}-1\right)\mathrm{u}(t)$$

$$\mathrm{H}(z)=\left.\frac{\mathrm{Y}(s)}{\mathrm{X}(s)}\right|_{s\to\frac{2}{T_s}\frac{z-1}{z+1}}=0.5219\frac{z-1.9161}{z-0.5219}$$

$$\mathrm{H}_{-1}(z)=0.5219\frac{z}{z-1}\frac{z-1.9161}{z-0.5219}=0.5219z\frac{z-1.9161}{z^2-1.522z+0.5219}$$

$$\mathrm{H}_{-1}(z)=0.5219\left(\frac{2.914z}{z-0.5218}-\frac{1.914z}{z-1}\right)$$

$$\mathrm{h}_{-1}[n]=\left(1.521(0.5218)^n-1\right)\mathrm{u}[n]$$




(b) $$\mathrm{H}(s) = \frac{10}{s^2 + 11s + 10}$$

(c) $$\mathrm{H}(s) = \frac{3s}{s^2 + 11s + 10}$$

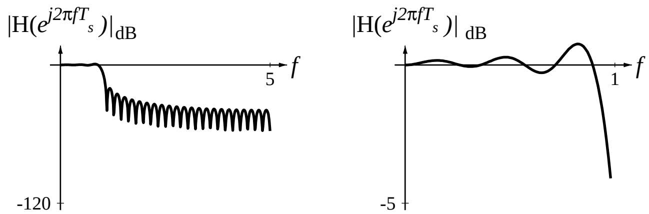
15. Design a digital-filter approximation to each of these ideal analog filters by sampling a truncated version of the impulse response and using the specified window. In each case choose a sampling frequency which is 10 times the highest frequency passed by the analog filter. Choose the delays and truncation times such that no more than 1% of the signal energy of the impulse response is truncated. Graphically compare the magnitude frequency responses of the digital and ideal analog filters using a dB magnitude scale versus linear frequency.

```
fc = 1 ; type = 'LP' ; fs = 10*fc ;
%
%     Lowpass, Rectangular Window
%
h = FIRDF(type,fc,fs,'RE',0.01) ;
N = length(h) ; Ts = 1/fs ; n = [0:N-1]' ;
F = [0:0.001:1/2]' ; [H,F] = DTFT(n,h,F) ;
subplot(2,2,1) ;
p = xyplot(F*fs,20*log10(abs(H)),[0,fs/2,-120,0],'\itf ',...
                  '|H(\ite^{{\itj}2{\pi}{\itf}{\itT}_s} )|',...
                  'Times',18,'Times',14,...
                  'Lowpass - Rectangular Window','Times',24,'n','c') ;
subplot(2,2,2) ;
p = xyplot(F*fs,20*log10(abs(H)),[0,fs/10,-5,0],'\itf ',...
                  '|H(\ite^{{\itj}2{\pi}{\itf}{\itT}_s} )|',...
                  'Times',18,'Times',14,...
                  'Lowpass - Rectangular Window','Times',24,'n','c') ;
%
%     Lowpass, von Hann Window
%
h = FIRDF(type,fc,fs,'VH',0.01) ;
N = length(h) ; Ts = 1/fs ; n = [0:N-1]' ;
F = [0:0.001:1/2]' ; [H,F] = DTFT(n,h,F) ;
subplot(2,2,3) ;
p = xyplot(F*fs,20*log10(abs(H)),[0,fs/2,-120,20],'\itf ',...
                  '|H(\ite^{{\itj}2{\pi}{\itf}{\itT}_s} )|',...
                  'Times',18,'Times',14,...
```

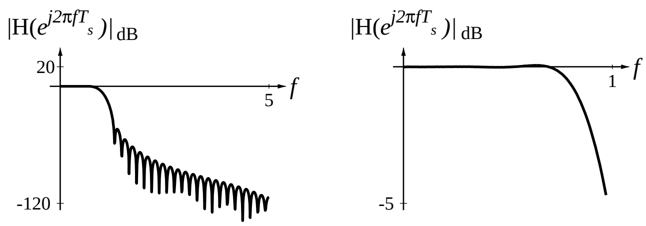
```
                  'Lowpass - Von Hann Window','Times',24,'n','c') ;
subplot(2,2,4) ;
p = xyplot(F*fs,20*log10(abs(H)),[0,fs/10,-5,0],'\itf ',...
                  '|H(\ite^{{\itj}2{\pi}{\itf}{\itT}_s} )|',...
                  'Times',18,'Times',14,...
                  'Lowpass - Von Hann Window','Times',24,'n','c') ;


%     Function to design a digital filter using truncation of the
%     impulse response to approximate the ideal impulse response.  The
%     user specifies the type of ideal filter,
%
%     LP    -     lowpass
%     BP    -     bandpass
%
%     the cutoff frequency(s) fcs (a scalar for LP and HP a 2-vector
%     for BP and BS), the sampling rate, fs, the type of window,
%
%     RE    -     rectangular
%     VH    -     von Hann
%     BA    -     Bartlett
%     HA    -     Hamming
%     BL    -     Blackman
%
%     and the allowable truncation error,
%     err, as a fraction of the total impulse response signal energy.
%
%     The function returns the filter coefficients as a vector.
%
%     function h = FIRDF(type,fcs,fs,window,err)
%
function h = FIRDF(type,fcs,fs,window,err)

      if fs == 0 | fs == inf | fs == -inf,
            disp('Sampling rate is unusable') ;
      else
            fs
            Ts = 1/fs ;
            type = upper(type) ; window = upper(window) ;
            switch type
                  case 'LP',
                        fc = fcs(1) ; zc1 = 1/(2*fc) ;
                        N = 200*zc1/Ts ; n = [0:N]' ;
                        Etotal = sum(sinc(2*fc*n*Ts).^2)
                        E = 0 ; N = 1 ;
                        while abs(E-Etotal) > Etotal*err,
                              N = 2*N ;
                              n = [0:N]' ;
                              E = sum(sinc(2*fc*n*Ts).^2) ;
                        end
                        delN = floor(N/10) ;
                        while abs(E-Etotal)<Etotal*err & abs(delN)>1,
                              N = N-delN ;
                              n = [0:N]' ;
                              E = sum(sinc(2*fc*n*Ts).^2) ;
                              delN = floor(delN/2) ;
                        end
                        N = ceil(N/2)*2 ; %     Make number of pts even
```

```
                        nmid = (N/2-1) ;
                        w = makeWindow(window,n,N) ;
                        h = w.*sinc(2*fc*(n-nmid)*Ts) ;
                        h = h/sum(h) ;
                case 'BP'
                        fl = fcs(1) ; fh = fcs(2) ; fmid = (fh + fl)/2 ;
                        df = abs(fh-fl) ; zc1 = 1/df ;
                        N = 200*zc1/Ts ; n = [0:N]' ;
                        Etotal = sum((2*df*sinc(df*n*Ts).*...
                                        cos(2*pi*fmid*n*Ts)).^2) ;
                        E = 0 ; N = 1 ;
                        while abs(E-Etotal) > Etotal*err,
                                N = 2*N ;
                                n = [0:N]' ;
                                E = sum((2*df*sinc(df*n*Ts).*...
                                        cos(2*pi*fmid*n*Ts)).^2) ;
                        end
                        delN = floor(N/10) ;
                        while abs(E-Etotal)<Etotal*err & abs(delN)>1,
                                N = N-delN ;
                                n = [0:N]' ;
                                E = sum((2*df*sinc(df*n*Ts).*...
                                        cos(2*pi*fmid*n*Ts)).^2) ;
                                delN = floor(delN/2) ;
                        end
                        N = ceil(N/2)*2 ; %     Make number of pts even
                        nmid = (N/2-1) ; n = [0:N-1]' ;
                        w = makeWindow(window,n,N) ;
                        h = w.*2.*df.*sinc(df*(n-nmid)*Ts).*...
                                        cos(2*pi*fmid*(n-nmid)*Ts) ;
                        h = h/sum(h.*cos(2*pi*fmid*(n-nmid)*Ts)) ;
        end
end

function w = makeWindow(window,n,N)
        switch window
                case 'RE'
                        w = ones(N,1) ;
                case 'VH'
                        w = (1-cos(2*pi*n/(N-1)))/2 ;
                case 'BA'
                        w = 2*n/(N-1).*(0<=n & n<=(N-1)/2) + ...
                                (2-2*n/(N-1)).*((N-1)/2<=n & n<N) ;
                case 'HA'
                        w = 0.54 - 0.46*cos(2*pi*n/(N-1)) ;
                case 'BL'
                        w = 0.42 - 0.5*cos(2*pi*n/(N-1)) + ...
                                        0.08*cos(4*pi*n/(N-1)) ;
                otherwise
                        w = ones(N,1) ;
        end
```

(a) Lowpass, $f_c = 1\text{ Hz}$, Rectangular window

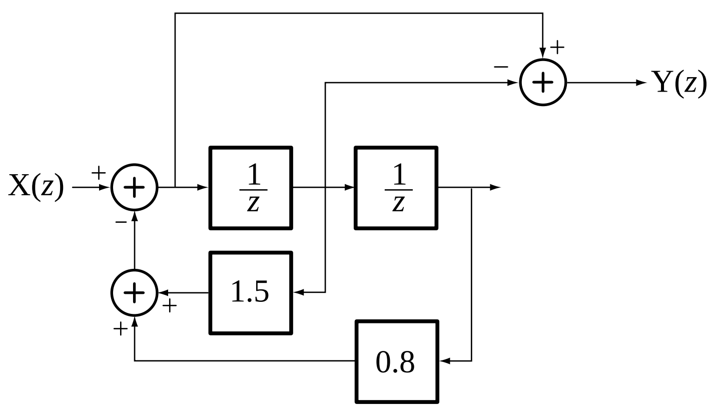



(b) Lowpass, $f_c = 1\text{ Hz}$, von Hann window




16. Draw a canonical-form block diagram for each of these system transfer functions.

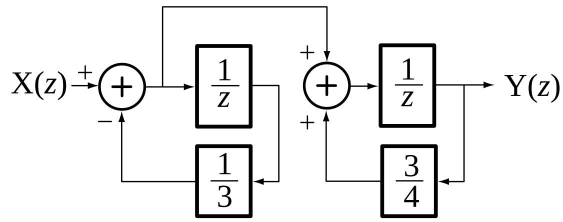
(a) $$\mathrm{H}(z) = \frac{z(z-1)}{z^2 + 1.5z + 0.8}$$




(b) $$\mathrm{H}(z) = \frac{z^2 - 2z + 4}{\left(z - \frac{1}{2}\right)\left(2z^2 + z + 1\right)}$$

17. Draw a cascade-form block diagram for each of these system transfer functions.

(a) $$\mathrm{H}(z) = \frac{z}{\left(z + \frac{1}{3}\right)\left(z - \frac{3}{4}\right)}$$

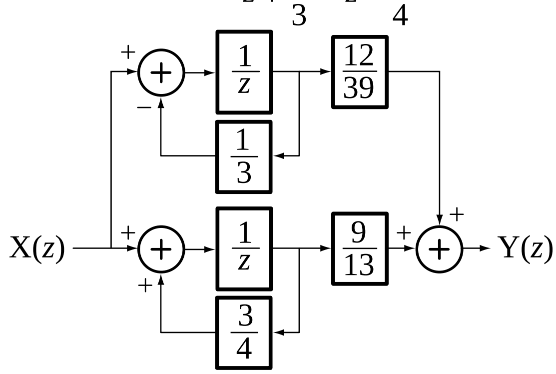



(b) $\mathrm{H}(z)=\dfrac{z-1}{4z^3+2z^2+2z+3}$

18. Draw a parallel-form block diagram for each of these system transfer functions.

(a) $$\mathrm{H}(z)=\frac{z}{\left(z+\frac{1}{3}\right)\left(z-\frac{3}{4}\right)}$$

$$\mathrm{H}(z)=\frac{\frac{12}{39}}{z+\frac{1}{3}}+\frac{\frac{36}{52}}{z-\frac{3}{4}}$$




(b) $\mathrm{H}(z)=\dfrac{8z^3-4z^2+5z+9}{7z^3+4z^2+z+2}$

19. For the system in Figure E19 write state equations and output equations.

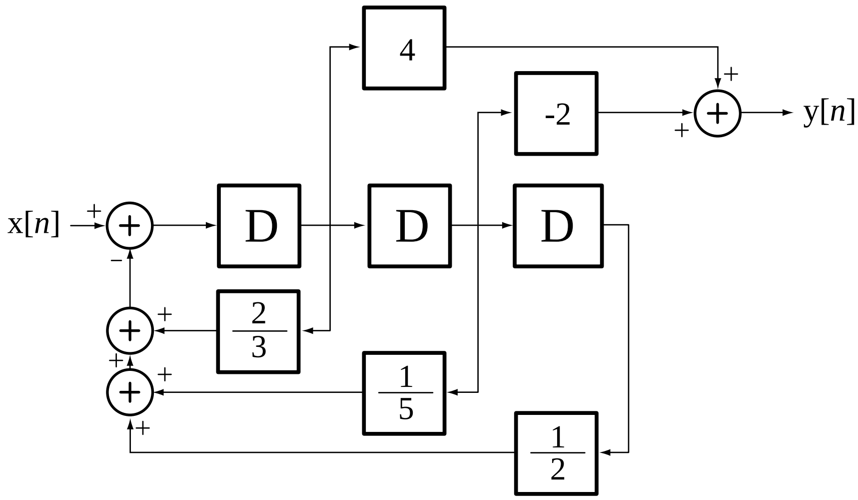



Figure E19 A 3-state DT system

Assigning states to the responses of delay elements in the order 1,2,3 right-to-left,

$$\begin{aligned}
q_1[n+1] &= q_2[n] \\
q_2[n+1] &= q_3[n] \\
q_3[n+1] &= x[n] - \frac{2}{3}q_3[n] - \frac{1}{5}q_2[n] - \frac{1}{2}q_1[n] \\
y[n] &= -2\,q_2[n] + 4\,q_3[n]
\end{aligned}$$

Writing these equations in standard state-space form,

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \\ q_3[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & -\frac{1}{5} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \\ q_3[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x[n] \quad ,$$

$$y[n] = \begin{bmatrix} 0 & -2 & 4 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \\ q_3[n] \end{bmatrix}$$

20. Write a set of state equations and output equations corresponding to these transfer functions.

(a) $H(z) = \dfrac{0.9z}{z^2 - 1.65z + 0.9}$

We can write a resursion relation directly from the transfer function.

$$y[n+2] = 0.9\,x[n+1] + 1.65\,y[n+1] - 0.9\,y[n]$$

or

$$y[n+1] = 0.9\,x[n] + 1.65\,y[n] - 0.9\,y[n-1]$$

Let $\mathrm{q}_1[n] = \mathrm{y}[n-1]$ and let $\mathrm{q}_2[n] = \mathrm{y}[n]$. Then

$$\begin{aligned} &\mathrm{q}_1[n+1] = \mathrm{q}_2[n] \\ &\mathrm{q}_2[n+1] = 0.9\,\mathrm{x}[n] + 1.65\,\mathrm{q}_2[n] - 0.9\,\mathrm{q}_1[n] \quad . \\ &\mathrm{y}[n] = \mathrm{q}_2[n] \end{aligned}$$

Writing the state equations in standard form,

$$\begin{bmatrix} \mathrm{q}_1[n+1] \\ \mathrm{q}_2[n+1] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0.9 & 1.65 \end{bmatrix} \begin{bmatrix} \mathrm{q}_1[n] \\ \mathrm{q}_2[n] \end{bmatrix} + \begin{bmatrix} 0 \\ 0.9 \end{bmatrix} \mathrm{x}[n]$$

$$\mathrm{y}[n] = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{q}_1[n] \\ \mathrm{q}_2[n] \end{bmatrix}$$

(b) $\mathrm{H}(z) = \dfrac{4(z-1)}{(z-0.9)(z-0.7)}$

21. Convert the difference equation,

$$10\,\mathrm{y}[n] + 4\,\mathrm{y}[n-1] + \mathrm{y}[n-2] + 2\,\mathrm{y}[n-3] = \cos\left(\frac{2\pi n}{16}\right)\mathrm{u}[n]$$

into a set of state equations and output equations.

22. Convert the state equations and output equation,

$$\begin{bmatrix} \mathrm{q}_1[n+1] \\ \mathrm{q}_2[n+1] \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{q}_1[n] \\ \mathrm{q}_2[n] \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \left(\frac{1}{3}\right)^n \mathrm{u}[n] \\ 0 \end{bmatrix}$$

$$\mathrm{y}[n] = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{q}_1[n] \\ \mathrm{q}_2[n] \end{bmatrix}$$

into a single difference equation. Using the relationships between the q's and y, we can write

$$\begin{bmatrix} \mathrm{y}[n+1] \\ \mathrm{y}[n] \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{y}[n] \\ \mathrm{y}[n-1] \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \left(\frac{1}{3}\right)^n \mathrm{u}[n] \\ 0 \end{bmatrix}$$

Mulitplying matrices and using only the top equation that results,

$$\mathrm{y}[n] + 2\,\mathrm{y}[n-1] + 5\,\mathrm{y}[n-2] = \left(\frac{1}{3}\right)^{n-1} \mathrm{u}[n-1] \ .$$

23. Find the responses of the system described by this set of state equations and output equations. (Assume the system is initially at rest.)

$$\begin{bmatrix} q_1[n+1] \\ q_1[n+1] \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_1[n] \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix} \mathrm{u}[n]$$

$$\begin{bmatrix} y_1[n] \\ y_1[n] \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_1[n] \end{bmatrix}$$

The transfer function is

$$\mathbf{H}(z) = \mathbf{C}[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D} = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} z-3 & -1 \\ 0 & z+2 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{H}(z) = \begin{bmatrix} \dfrac{z+20}{z^2 - z - 6} \\ \dfrac{8z+22}{z^2 - z - 6} \end{bmatrix}$$

The $z$ transform of the excitation vector (in this case, a scalar) is

$$\mathbf{X}(z) = \frac{z}{z-1} \quad .$$

Therefore the $z$-domain response vector is

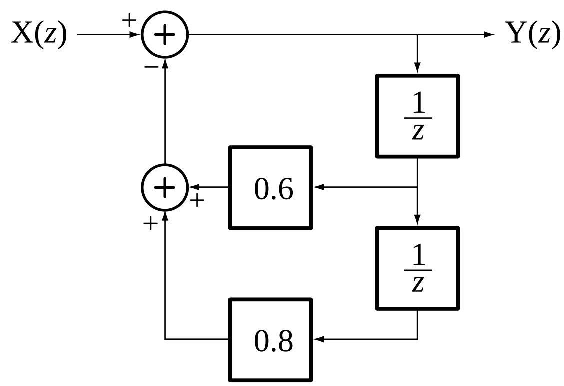
$$\mathbf{Y}(z) = \mathbf{H}(z)\mathbf{X}(z) = \begin{bmatrix} \dfrac{z+20}{z^2 - z - 6} \\ \dfrac{8z+22}{z^2 - z - 6} \end{bmatrix} \frac{z}{z-1}$$

$$\mathbf{Y}(z) = z \begin{bmatrix} \dfrac{2.3}{z-3} + \dfrac{1.2}{z+2} - \dfrac{3.5}{z-1} \\ \dfrac{4.6}{z-3} + \dfrac{0.4}{z+2} - \dfrac{5}{z-1} \end{bmatrix}$$

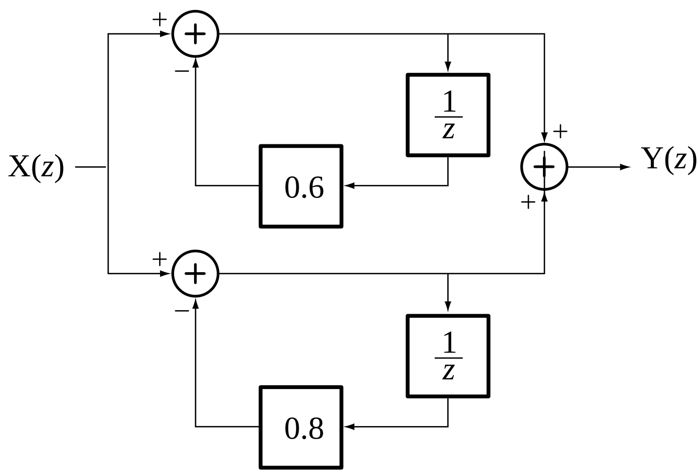
$$\mathbf{y}[n] = \begin{bmatrix} 2.3(3)^n + 1.2(-2)^n - 3.5 \\ 4.6(3)^n + 0.4(-2)^n - 5 \end{bmatrix} \mathrm{u}[n]$$

24. Find the overall transfer functions of these systems in the form of a single ratio of polynomials in $z$.

(a)




(b)

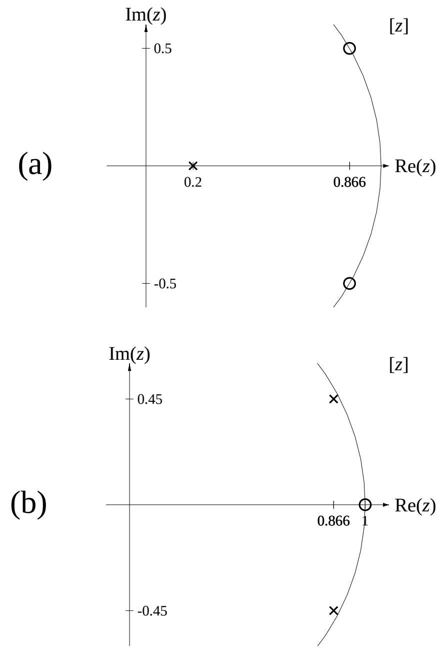



25. Find the DT-domain responses, $y[n]$, of the systems with these transfer functions to the unit sequence excitation, $x[n] = u[n]$.

(a) $$H(z) = \frac{z}{z^2 - 1.8z + 0.82}$$

(b) $$H(z) = \frac{z^2 - 1.932z + 1}{z(z - 0.95)}$$

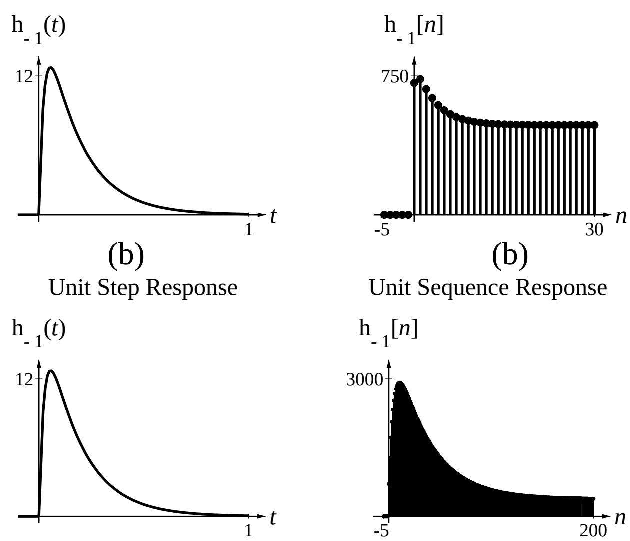
26. Sketch the magnitude frequency response of these systems from their pole-zero diagrams.




27. Using the impulse-invariant design method, design a DT system to approximate the CT systems with these transfer functions at the sampling rates specified. Compare the impulse and unit step (or sequence) responses of the CT and DT systems.

(a) $$\mathrm{H}(s) = \frac{712s}{s^2 + 46s + 240} \quad , \quad f_s = 20 \text{ Hz}$$

(b) $$\mathrm{H}(s) = \frac{712s}{s^2 + 46s + 240} \quad , \quad f_s = 200 \text{ Hz}$$



28. Using the impulse-invariant and step-invariant design methods, design digital filters to approximate analog filters with these transfer functions. In each case choose a sampling frequency which is 10 times the magnitude of the distance of the farthest pole or zero from the origin of the "*s*" plane. Graphically compare the step responses of the digital and analog filters.

(a) $\mathrm{H}(s) = \dfrac{16s}{s^2 + 10s + 250}$

(b) $\mathrm{H}(s) = \dfrac{s+4}{s^2 + 12s + 32}$

(c) $\mathrm{H}(s) = \dfrac{s^2+4}{s(s^2 + 12s + 32)} = \dfrac{0.125}{s} + \dfrac{2.125}{s+8} - \dfrac{1.25}{s+4}$

29. Using the difference-equation method and all backward differences, design digital filters to approximate analog filters with these transfer functions. In each case choose a sampling frequency which is 10 times the magnitude of the distance of the farthest pole or zero from the origin of the "*s*" plane. Graphically compare the step responses of the digital and analog filters.

(a) $\mathrm{H}(s) = \dfrac{s^2}{s^2 + 3s + 2}$

(b) $\mathrm{H}(s) = \dfrac{s+60}{s^2 + 120s + 2000}$

(c) $\mathrm{H}(s) = \dfrac{16s}{s^2 + 10s + 250}$

30. Using the direct substitution method, design digital filters to approximate analog filters with these transfer functions. In each case choose a sampling frequency which is 10 times the magnitude of the distance of the farthest pole or zero from the origin of the "*s*" plane (unless all poles or zeros are at the origin, in which case the sampling rate will not matter, in this method). Graphically compare the step responses of the digital and analog filters. (First printing of the text had "matched *z*-transform" instead of "direct substitution". These solutions are for direct substitution.)

(a) $\mathrm{H}(s) = \dfrac{s^2}{s^2 + 1100s + 10^5}$

$$\mathrm{h}(t) = \left[\delta(t) - 1111e^{-1000t} + 11.111e^{-100t}\right]\mathrm{u}(t)$$

$$\mathrm{H}_{-1}(s) = \frac{1}{9}\left(\frac{10}{s+1000} - \frac{1}{s+100}\right)$$

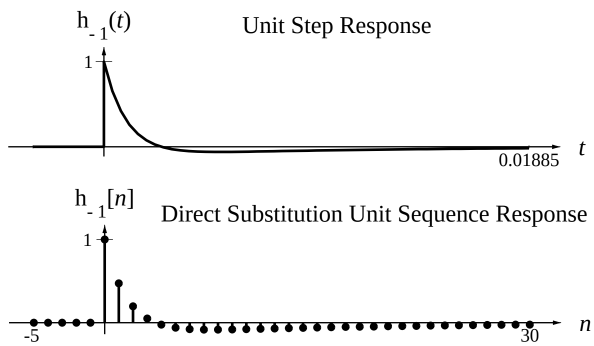
$$h_{-1}(t) = \frac{10e^{-1000t} - e^{-100t}}{9} u(t)$$

$$f_s = 1591.55 \text{ Hz} \quad \text{and} \quad T_s = 628.32\ \mu\text{s}$$

$$H(z) = \frac{(z-1)^2}{(z-0.9391)(z-0.5335)}$$

$$H_{-1}(z) = \frac{1.1474z}{z-0.5335} - \frac{0.1474z}{z-0.9391}$$

$$h_{-1}[n] = \left[1.1474(0.5335)^n - 0.1474(0.9391)^n\right]u[n]$$




(b) $$H(s) = \frac{s^2 + 100s + 5000}{s^2 + 120s + 2000}$$

(c) $$H(s) = \frac{s^2 + 4}{s(s^2 + 12s + 32)}$$

31. Using the bilinear-$z$-transform method, design digital filters to approximate analog filters with these transfer functions In each case choose a sampling frequency which is 10 times the magnitude of the distance of the farthest pole or zero from the origin of the "$s$" plane. Graphically compare the step responses of the digital and analog filters.

(a) $$H(s) = \frac{s^2}{s^2 + 100s + 250000}$$

(b) $$H(s) = \frac{s^2 + 100s + 5000}{s^2 + 120s + 2000}$$

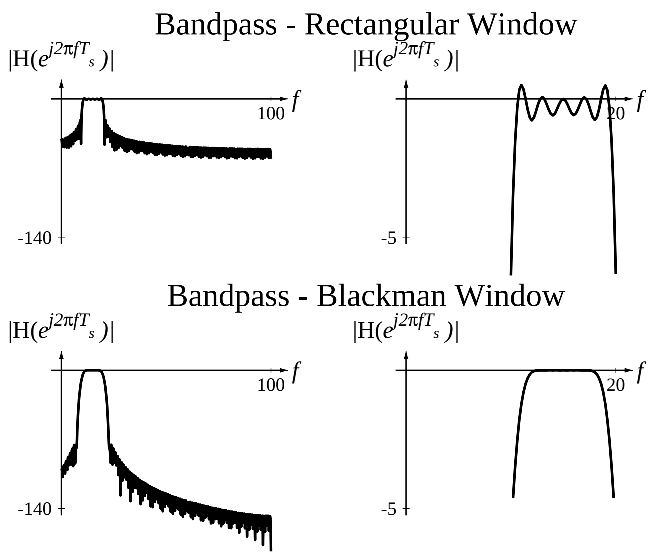
(c) $$H(s) = \frac{s^2 + 4}{s^2 + 12s + 32}$$

32. Design a digital-filter approximation to each of these ideal analog filters by sampling a truncated version of the impulse response and using the specified window. In each case choose a sampling frequency which is 10 times the highest frequency passed by the analog filter. Choose the delays and truncation times such that no more than 1% of the signal energy of the impulse response is truncated. Graphically compare the magnitude frequency responses of the digital and ideal analog filters using a dB magnitude scale versus linear frequency.

Refer to MATLAB code in Exercise 15.

(a) Bandpass, $f_{low} = 10$ Hz , $f_{high} = 20$ Hz, Rectangular window

(b) Bandpass, $f_{low} = 10$ Hz , $f_{high} = 20$ Hz, Blackman window




33. Draw a canonical-form block diagram for each of these system transfer functions.

(a) $$\mathrm{H}(z) = \frac{z^2}{2z^4 + 1.2z^3 - 1.06z^2 + 0.08z - 0.02}$$

(b) $$\mathrm{H}(z) = \frac{z^2\left(z^2 + 0.8z + 0.2\right)}{\left(2z^2 + 2z + 1\right)\left(z^2 + 1.2z + 0.5\right)}$$

34. Draw a cascade-form block diagram for each of these system transfer functions.

(a) $$\mathrm{H}(z) = \frac{z^2}{z^2 - 0.1z - 0.12} + \frac{z}{z - 1}$$

(b) $$\mathrm{H}(z)=\frac{\frac{z}{z-1}}{1+\frac{z}{z-1}\frac{z^2}{z^2-\frac{1}{2}}}$$

35. Draw a parallel-form block diagram for each of these system transfer functions.

(a) $$\mathrm{H}(z)=\left(1+z^{-1}\right)\frac{18}{(z-0.1)(z+0.7)}$$

(b) $$\mathrm{H}(z)=\frac{\frac{z}{z-1}}{1+\frac{z}{z-1}\frac{z^2}{z^2-\frac{1}{2}}}$$

36. Write a set of state equations and output equations corresponding to these transfer functions (which are for DT Butterworth filters).

(a) $$\mathrm{H}(z)=\frac{0.06746z^2+0.1349z+0.06746}{z^2-1.143z+0.4128}$$

(b) $$\mathrm{H}(z)=\frac{0.0201z^4-0.0402z^2+0.0201}{z^4-2.5494z^3+3.2024z^2-2.0359z+0.6414}$$

37. For the system in Figure E37 write state equations and response equations.

38. Find the response of the system in Exercise E37 to the excitation, $\mathrm{x}[n]=\mathrm{u}[n]$. (Assume that the system is initially at rest.)

39. A DT system is excited by a unit sequence and the response is

$$\mathrm{y}[n]=\left(8+2\left(\frac{1}{2}\right)^{n-1}-9\left(\frac{3}{4}\right)^{n-1}\right)\mathrm{u}[n-1]\ .$$

Write state equations and output equations for this system.

40. Define new states which transform this set of state equations and output equations into a set of diagonalized state equations and output equations and write the new state equations and output equations.

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \\ q_3[n+1] \end{bmatrix} = \begin{bmatrix} -0.4 & -0.1 & -0.2 \\ 0.3 & 0 & -0.2 \\ 1 & 0 & -1.3 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \\ q_3[n] \end{bmatrix} + \begin{bmatrix} 2 & -0.5 \\ 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 0.1\cos\left(\frac{2\pi n}{16}\right)u[n] \\ \left(\frac{3}{4}\right)^n u[n] \end{bmatrix}$$

$$\begin{bmatrix} y_1[n+1] \\ y_2[n+1] \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \\ q_3[n] \end{bmatrix}$$

The eigenvalue matrix is

$$\Lambda = \begin{bmatrix} -0.8365 + j0.0717 & 0 & 0 \\ 0 & -0.8365 - j0.0717 & 0 \\ 0 & 0 & -0.027 \end{bmatrix}.$$

The solution of the equation, $\Lambda\mathbf{T} = \mathbf{TA}$, for the transformation matrix, $\mathbf{T}$, is

$$\mathbf{T} = \begin{bmatrix} 0.8908 + j0.1086 & 0.1046 + j0.0219 & -0.4280 + j0.0098 \\ 0.8908 - j0.1086 & 0.1046 - j0.0219 & -0.4280 - j0.0098 \\ -0.2556 & -0.9481 & 0.1891 \end{bmatrix}.$$

The new state-variable vector is

$$\mathbf{q}_2[n] = \mathbf{T}\mathbf{q}_1[n]$$

and the new diagonalized state equations are

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \\ q_3[n+1] \end{bmatrix} = \begin{bmatrix} -0.8365 + j0.0717 & 0 & 0 \\ 0 & -0.8365 - j0.0717 & 0 \\ 0 & 0 & -0.027 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \\ q_3[n] \end{bmatrix} + \begin{bmatrix} 1.8862 + j0.2391 & -1.7294 - j0.0248 \\ 1.8862 - j0.2391 & -1.7294 + j0.0248 \\ -1.4592 & 0.6951 \end{bmatrix} \begin{bmatrix} 0.1\cos\left(\frac{2\pi n}{16}\right)u[n] \\ \left(\frac{3}{4}\right)^n u[n] \end{bmatrix}$$

$$\begin{bmatrix} y_1[n+1] \\ y_2[n+1] \end{bmatrix} = \begin{bmatrix} 1.3188 + j0.0988 & 1.3188 - j0.0988 & -0.4447 \\ -0.2682 + j0.0135 & -0.2682 - j0.0135 & -0.1521 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \\ q_3[n] \end{bmatrix}$$

41. Find the response of the system described by this set of state equations and output equations. (Assume the system is initially at rest.)

$$\begin{bmatrix} q_1[n+1] \\ q_2[n+1] \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{5} \\ 0 & \frac{7}{10} \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u[n] \\ \left(\frac{3}{4}\right)^n u[n] \end{bmatrix}$$

$$y[n] = \begin{bmatrix} 4 & -1 \end{bmatrix} \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} u[n] \\ \left(\frac{3}{4}\right)^n u[n] \end{bmatrix}$$