Chapter 3 - Mathematical Description and Analysis of Systems

Selected Solutions

1. Show that a system with excitation, $\mathrm{x}(t)$, and response, $\mathrm{y}(t)$, described by

$$\mathrm{y}(t) = \mathrm{u}(\mathrm{x}(t))$$

is non-linear, time invariant, stable and non-invertible.

Homogeneity:
Let $\mathrm{x}_1(t) = \mathrm{g}(t)$. Then $\mathrm{y}_1(t) = \mathrm{u}(\mathrm{g}(t))$.
Let $\mathrm{x}_2(t) = K\,\mathrm{g}(t)$. Then $\mathrm{y}_2(t) = \mathrm{u}(K\,\mathrm{g}(t)) \neq K\,\mathrm{y}_1(t) = K\,\mathrm{u}(\mathrm{g}(t))$.
Not homogeneous

Additivity:
Let $\mathrm{x}_1(t) = \mathrm{g}(t)$. Then $\mathrm{y}_1(t) = \mathrm{u}(\mathrm{g}(t))$.
Let $\mathrm{x}_2(t) = \mathrm{h}(t)$. Then $\mathrm{y}_2(t) = \mathrm{u}(\mathrm{h}(t))$.
Let $\mathrm{x}_3(t) = \mathrm{g}(t) + \mathrm{h}(t)$.
Then $\mathrm{y}_3(t) = \mathrm{u}(\mathrm{g}(t) + \mathrm{h}(t)) \neq \mathrm{y}_1(t) + \mathrm{y}_2(t) = \mathrm{u}(\mathrm{g}(t)) + \mathrm{u}(\mathrm{h}(t))$
Not additive

Since it is not homogeneous and not additive, it is not linear.

It is also not incrementally linear because incremental changes in the excitation do not produce proportional incremental changes in the response.

It is statically non-linear because it is non-linear without memory (lack of memory proven below).

Time Invariance:
Let $\mathrm{x}_1(t) = \mathrm{g}(t)$. Then $\mathrm{y}_1(t) = \mathrm{u}(\mathrm{g}(t))$.
Let $\mathrm{x}_2(t) = \mathrm{g}(t - t_0)$.
Then $\mathrm{y}_2(t) = \mathrm{u}(\mathrm{g}(t - t_0)) = \mathrm{y}_1(t - t_0)$.
Time Invariant

Stability:
The unit step function can only have the values, zero or one, therefore any bounded (or unbounded) excitation produces a bounded response.
Stable

Causality:
The response at any time, $t = t_0$, depends only on the excitation at time, $t = t_0$ and not on any future values.

Causal

Memory:
The response at any time, $t = t_0$, depends only on the excitation at time, $t = t_0$ and not on any past values.
System has no memory.

Invertibility:
There are many value of the excitation that all cause a response of zero and there are many values of the excitation that all cause a response of one. Therefore the system is not invertible.

2. Show that a system with excitation, $x(t)$, and response, $y(t)$, described by

$$y(t) = x(t-5) - x(3-t)$$

is linear but not causal and not invertible.

Causality:
At time, $t = 0$, $y(0) = x(-5) - x(3)$. Therefore the response at time, $t = 0$, depends on the excitation at a later time, $t = 3$.
Not Causal

Memory:
At time, $t = 0$, $y(0) = x(-5) - x(3)$. Therefore the response at time, $t = 0$, depends on the excitation at a previous time, $t = -5$.
System has memory.

Invertibility:
A counterexample will demonstrate that the system is not invertible. Let the excitation be a constant, K. Then the response is $y(t) = K - K = 0$. This is the response, no matter what K is. Therefore when the response is a constant zero, the excitation cannot be determined.
Not Invertible.

3. Show that a system with excitation, $x(t)$, and response, $y(t)$, described by

$$y(t) = x\left(\frac{t}{2}\right)$$

is linear, time variant and non-causal.

Time Invariance:
Let $x_1(t) = g(t)$. Then $y_1(t) = g\left(\frac{t}{2}\right)$.
Let $x_2(t) = g(t - t_0)$. Then $y_2(t) = g\left(\frac{t}{2} - t_0\right) \neq y_1(t - t_0) = g\left(\frac{t - t_0}{2}\right)$.
Time Variant

Causality:

At time, $t=-2$, $y(-2)=x(-1)$. Therefore the response at time, $t=-2$, depends on the excitation at a later time, $t=-1$.
Not Causal

Memory:
At time, $t=2$, $y(2)=x(1)$. Therefore the response at time, $t=2$, depends on the excitation at a previous time, $t=1$.
System has memory.

Invertibility:
The system excitation at any arbitrary time, $t=t_0$, is uniquely determined by the system response at time, $t=2t_0$.
Invertible.

4. Show that a system with excitation, $x(t)$, and response, $y(t)$, described by

$$y(t)=\cos(2\pi t)\,x(t)$$

is time variant, BIBO stable, static and non-invertible.

Time Invariance:
Let $x_1(t)=g(t)$. Then $y_1(t)=\cos(2\pi t)\,g(t)$.
Let $x_2(t)=g(t-t_0)$. Then $y_2(t)=\cos(2\pi t)\,g(t-t_0)\neq y_1(t-t_0)=\cos(2\pi(t-t_0))\,g(t-t_0)$.
Time Variant

Invertibility:
This system is not invertible because when the cosine function is zero the unique relationship between x and y is lost; any x produces the same y, zero.
Not Invertible.

5. Show that a system whose response is the magnitude of its excitation is non-linear, BIBO stable, causal and non-invertible.

$$y(t)=|x(t)|$$

Invertibility:
Any response, y, can be caused by either x or –x.
Not Invertible.

6. Show that the system in Figure E6 is linear, time invariant, BIBO unstable and dynamic.

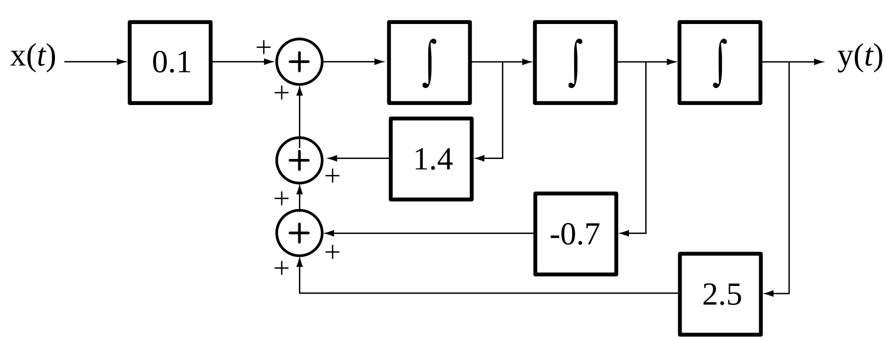


Figure E6 A CT system

The differential equation of the system is $10\,y'''(t) - 14\,y''(t) + 7\,y'(t) - 25\,y(t) = x(t)$.

Homogeneity:
Let $x_1(t) = g(t)$. Then $10\,y_1'''(t) - 14\,y_1''(t) + 7\,y_1'(t) - 25\,y_1(t) = g(t)$.
Let $x_2(t) = K\,g(t)$. Then $10\,y_2'''(t) - 14\,y_2''(t) + 7\,y_2'(t) - 25\,y_2(t) = K\,g(t)$.
If we multiply the first equation by *K*, we get

$$10K\,y_1'''(t) - 14K\,y_1''(t) + 7K\,y_1'(t) - 25K\,y_1(t) = K\,g(t)$$

Therefore

$$10K\,y_1'''(t) - 14K\,y_1''(t) + 7K\,y_1'(t) - 25K\,y_1(t) = 10\,y_2'''(t) - 14\,y_2''(t) + 7\,y_2'(t) - 25\,y_2(t)$$

This can only be true for all time for an arbitrary excitation if $y_2(t) = K\,y_1(t)$.
Homogeneous

Additivity:
Let $x_1(t) = g(t)$. Then $10\,y_1'''(t) - 14\,y_1''(t) + 7\,y_1'(t) - 25\,y_1(t) = g(t)$.
Let $x_2(t) = h(t)$. Then $10\,y_2'''(t) - 14\,y_2''(t) + 7\,y_2'(t) - 25\,y_2(t) = h(t)$.
Let $x_3(t) = g(t) + h(t)$. Then $10\,y_3'''(t) - 14\,y_3''(t) + 7\,y_3'(t) - 25\,y_3(t) = g(t) + h(t)$
Adding the first two equations,

$$10\left[y_1'''(t) + y_2'''(t)\right] - 14\left[y_1''(t) + y_2''(t)\right] + 7\left[y_1'(t) + y_2'(t)\right] - 25\left[y_1(t) | + y_2(t)\right] = g(t) + h(t)$$

Therefore

$$10\left[y_1'''(t) + y_2'''(t)\right] - 14\left[y_1''(t) + y_2''(t)\right] + 7\left[y_1'(t) + y_2'(t)\right] - 25\left[y_1(t) | + y_2(t)\right]$$
$$= 10\,y_3'''(t) - 14\,y_3''(t) + 7\,y_3'(t) - 25\,y_3(t)$$

$$10\left[y_1(t) | + y_2(t)\right]''' - 14\left[y_1(t) | + y_2(t)\right]'' + 7\left[y_1(t) | + y_2(t)\right]' - 25\left[y_1(t) | + y_2(t)\right]$$
$$= 10\,y_3''(t) - 14\,y_3''(t) + 7\,y_3'(t) - 25\,y_3(t)$$

This can only be true for all time for an arbitrary excitation if $y_3(t) = y_1(t) + y_2(t)$.
Additive

Since it is homogeneous and additive, it is also linear.

Time Invariance:
Let $x_1(t) = g(t)$. Then $10\,y_1'''(t) - 14\,y_1''(t) + 7\,y_1'(t) - 25\,y_1(t) = g(t)$.
Let $x_2(t) = g(t - t_0)$.
Then $10\,y_2''(t) - 14\,y_2''(t) + 7\,y_2'(t) - 25\,y_2(t) = g(t - t_0)$.

The first equation can be written as

$$10\,y_1'''(t - t_0) - 14\,y_1''(t - t_0) + 7\,y_1'(t - t_0) - 25\,y_1(t - t_0) = g(t - t_0)$$

Therefore

$$10\,y_1'''(t-t_0)-14\,y_1''(t-t_0)+7\,y_1'(t-t_0)-25\,y_1(t-t_0)$$
$$=10\,y_2'''(t)-14\,y_2''(t)+7\,y_2'(t)-25\,y_2(t)$$

This can only be true for all time for an arbitrary excitation if $y_2(t)=y_1(t-t_0)$.
Time Invariant

Stability:

The characteristic equation is $10\lambda^3-14\lambda^2+7\lambda-25=0$. The eigenvalues are

$$\lambda_1=1.7895$$
$$\lambda_2=-0.1948 + j1.1658$$
$$\lambda_3=-0.1948 - j1.1658$$

So the homogeneous solution is of the form,

$$y(t)=K_1e^{1.7895t}+K_2e^{(-0.1948 + j1.1658)t}+K_3e^{(-0.1948 - j1.1658)t}.$$

If there is no excitation, but the zero-excitation response is not zero, the response will grow without bound as time increases.
Unstable

Causality:
The system equation can be rewritten as

$$y(t)=\frac{1}{10}\left[\begin{array}{l}\int_{-\infty}^{t}\int_{-\infty}^{\lambda_3}\int_{-\infty}^{\lambda_2}x(\lambda_1)d\lambda_1d\lambda_2d\lambda_3+25\int_{-\infty}^{t}\int_{-\infty}^{\lambda_3}\int_{-\infty}^{\lambda_2}y(\lambda_1)d\lambda_1d\lambda_2d\lambda_3\\ \qquad -7\int_{-\infty}^{t}\int_{-\infty}^{\lambda_2}y(\lambda_1)d\lambda_1d\lambda_2+14\int_{-\infty}^{t}y(\lambda_1)d\lambda_1\end{array}\right]$$

So the response at any time, $t=t_0$, depends on the excitation at times, $t<t_0$ and not on any future values.
Causal

Memory:
The response at any time, $t=t_0$, depends on the excitation at times, $t<t_0$.
System has memory.

Invertibility:

The system equation,

$$10\,y'''(t)-14\,y''(t)+7\,y'(t)-25\,y(t)=x(t)$$

expresses the excitation in terms of the response and its derivatives. Therefore the excitation is uniquely determined by the response.
Invertible.

7. Show that the system of Figure E7 is non-linear, BIBO stable, static and non-invertible. (The response of an analog multiplier is the product of its two excitations.)

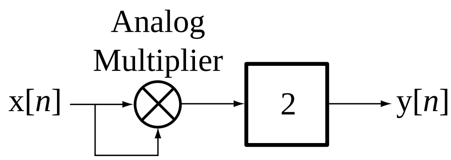


Figure E7 A DT system

8. Show that a system with excitation, $\mathrm{x}[n]$, and response, $\mathrm{y}[n]$, described by

$$\mathrm{y}[n] = n\,\mathrm{x}[n],$$

is linear, time variant and static.

9. Show that the system of Figure E9 is linear, time-invariant, BIBO unstable and dynamic.

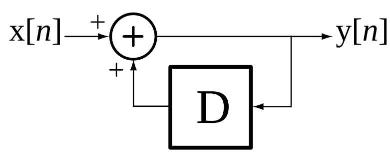


Figure E9 A DT system

$$\mathrm{y}[n] = \mathrm{x}[n] + \mathrm{y}[n-1]$$

$$\mathrm{y}[n-1] = \mathrm{x}[n-1] + \mathrm{y}[n-2]$$

$$\mathrm{y}[n] = \mathrm{x}[n] + \mathrm{x}[n-1] + \mathrm{y}[n-2]$$

Then, by induction,

$$\mathrm{y}[n] = \mathrm{x}[n] + \mathrm{x}[n-1] + \cdots + \mathrm{x}[n-k] + \cdots = \sum_{k=0}^{\infty} \mathrm{x}[n-k]$$

Let $m = n - k$. Then

$$\mathrm{y}[n] = \sum_{m=n}^{-\infty} \mathrm{x}[m] = \sum_{m=-\infty}^{n} \mathrm{x}[m]$$

Homogeneity:

Let $\mathrm{x}_1[n] = \mathrm{g}[n]$. Then $\mathrm{y}_1[n] = \sum_{m=-\infty}^{n} \mathrm{g}[m]$

Let $\mathrm{x}_2[n] = K\,\mathrm{g}[n]$. Then $\mathrm{y}_2[n] = \sum_{m=-\infty}^{n} K\,\mathrm{g}[m] = K \sum_{m=-\infty}^{n} \mathrm{g}[m] = K\,\mathrm{y}_1[n]$.

Homogeneous.

Let $\mathrm{x}_1[n] = \mathrm{g}[n]$. Then $\mathrm{y}_1[n] = \sum_{m=-\infty}^{n} \mathrm{g}[m]$

Let $x_2[n] = h[n]$. Then $y_2[n] = \sum_{m=-\infty}^{n} h[m]$

Let $x_3[n] = g[n] + h[n]$.

Then $y_3[n] = \sum_{m=-\infty}^{n} (g[m] + h[m]) = \sum_{m=-\infty}^{n} g[m] + \sum_{m=-\infty}^{n} h[m] = y_1[n] + y_2[n]$.

Additive.

Since the system is homogeneous and additive it is also linear.

The system is also incrementally linear because it is linear.

The system is not statically non-linear because it is linear.

Time Invariance:

Let $x_1[n] = g[n]$. Then $y_1[n] = \sum_{m=-\infty}^{n} g[m]$.

Let $x_2[n] = g[n - n_0]$. Then $y_2[n] = \sum_{m=-\infty}^{n} g[m - n_0]$.

The first equation can be rewritten as

$$y_1[n - n_0] = \sum_{m=-\infty}^{n-n_0} g[m]$$

Let $m = q - n_0$. Then

$$y_1[n - n_0] = \sum_{q=-\infty}^{n} g[q - n_0] = y_2[n]$$

Time invariant

Stability:

If the excitation is a constant, the response increases without bound.

Also the solution of the homogeneous difference equation is $y_h[n] = K(1)^n = K$. Therefore the eigenvalue is 1 whose magnitude is not less than 1 and the system must be BIBO unstable.

Unstable

Causality:

At any discrete time, $n = n_0$, the response depends only on the excitation at that discrete time and previous discrete times.

Causal.

Memory:

At any discrete time, $n = n_0$, the response depends on the excitation at that discrete time and previous discrete times.

System has memory.

Invertibility:

Inverting the functional relationship,

$$y[n] = \sum_{m=-\infty}^{n} x[m] \ .$$

Invertible.
Taking the first backward difference of both sides of the original system equation,

$$y[n] - y[n-1] = \sum_{m=-\infty}^{n} x[m] - \sum_{m=-\infty}^{n-1} x[m]$$

$$x[n] = y[n] - y[n-1]$$

The excitation is uniquely determined by the response.
Invertible.

10. Show that a system with excitation, $x[n]$, and response, $y[n]$, described by

$$y[n] = \text{rect}(x[n]) \ ,$$

is non-linear, time invariant and non-invertible.

11. Show that the system of Figure E11 is non-linear, time-invariant, static and invertible.

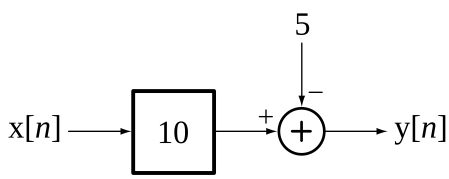


Figure E11 A DT system

$$y[n] = 10\,x[n] - 5 \ ,$$

The system is incrementally linear because the only deviation from linearity is caused by the presence of the non-zero, zero-excitation response.

Invertibility:
Solving the system equation for the excitation as a function of the response,

$$x[n] = \frac{y[n] + 5}{10}$$

Invertible.

12. Show that the system of Figure E12 is time-invariant, BIBO stable, and causal.

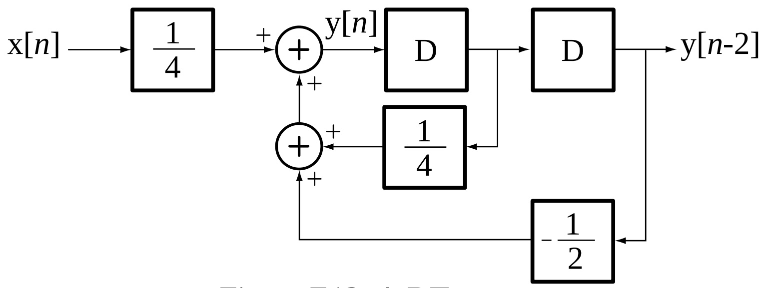


Figure E12 A DT system

Homogeneity:

Let $x_1[n] = g[n]$. Then $4\,y_1[n] - y_1[n-1] + 2\,y_1[n-2] = g[n]$
Let $x_2[n] = K\,g[n]$. Then $4\,y_2[n] - y_2[n-1] + 2\,y_2[n-2] = K\,g[n]$
Multiply the first equation by K. $4K\,y_1[n] - K\,y_1[n-1] + 2K\,y_1[n-2] = K\,g[n]$
Then, equating results,

$$4\,y_2[n] - y_2[n-1] + 2\,y_2[n-2] = 4K\,y_1[n] - K\,y_1[n-1] + 2K\,y_1[n-2]$$

If this equation is to be satisfied for all n,

$$y_2[n] = K\,y_1[n].$$

Homogeneous.

Additivity:

Let $x_1[n] = g[n]$. Then $4\,y_1[n] - y_1[n-1] + 2\,y_1[n-2] = g[n]$
Let $x_2[n] = h[n]$. Then $4\,y_2[n] - y_2[n-1] + 2\,y_2[n-2] = h[n]$
Let $x_3[n] = g[n] + h[n]$. Then $4\,y_3[n] - y_3[n-1] + 2\,y_3[n-2] = g[n] + h[n]$

Add the two first two equations.

$$4\left(y_1[n] + y_2[n]\right) - \left(y_1[n-1] + y_2[n-1]\right) + 2\left(y_1[n-2] + y_2[n-2]\right) = g[n] + h[n]$$

Then, equating results,

$$\begin{aligned} &4\,y_3[n] - y_3[n-1] + 2\,y_3[n-2] \\ &\quad = 4\left(y_1[n] + y_2[n]\right) - \left(y_1[n-1] + y_2[n-1]\right) + 2\left(y_1[n-2] + y_2[n-2]\right) \end{aligned}$$

If this equation is to be satisfied for any arbitrary excitation for all n,

$$y_3[n] = y_1[n] + y_2[n].$$

Additive.

Since the system is both homogeneous and additive, it is linear.

Since the system is linear it is also incrementally linear.

Since the system is linear, it is not statically non-linear.

Time Invariance:

Let $x_1[n] = g[n]$. Then $4\,y_1[n] - y_1[n-1] + 2\,y_1[n-2] = g[n]$
Let $x_2[n] = g[n-n_0]$. Then $4\,y_2[n] - y_2[n-1] + 2\,y_2[n-2] = g[n-n_0]$
We can re-write the first equation as

$$4\,y_1[n-n_0] - y_1[n-n_0-1] + 2\,y_1[n-n_0-2] = g[n-n_0]$$

Then, equating results,

$$4\,y_1[n-n_0] - y_1[n-n_0-1] + 2\,y_1[n-n_0-2] = 4\,y_2[n] - y_2[n-1] + 2\,y_2[n-2]$$

If this equation is to be satisfied for any arbitrary excitation for all n, then

$$y_2[n] = y_1[n-n_0].$$

Time Invariant.

Stability:
The eigenvalues of the system homogeneous solution are found from the characteristic equation,

$$4\alpha^2 - \alpha + 2 = 0 \;.$$

They are

$$\alpha_{1,2} = 0.125 \pm j0.696 \text{ or } \alpha_{1,2} = 0.7071 e^{\pm j1.3931}$$

Therefore the homogeneous solution is of the form,

$$y_h[n] = K_{h1}(0.7071)^n e^{+j1.3931n} + K_{h2}(0.7071)^n e^{-j1.3931n}$$

and, as n approaches infinity the homogeneous solution approaches zero and the total solution approaches the particular solution. The particular solution is bounded because it consists of functions of the same form as the excitation and all its unique differences and the excitation is bounded in the BIBO stability test. Therefore if x is bounded, so is y.
Stable.

Causality:
We can rearrange the system equation into

$$y[n] = \frac{1}{4}\left(x[n] + y[n-1] + 2\,y[n-2]\right)$$

showing that the response at time, n, depends on the excitation at time, n, and the response at previous times. It does not depend on any future values of the excitation.
Causal.

Memory:
The response depends on past values of the response.
The system has memory.

Invertibility:

The original system equation, $4\,\mathrm{y}[n]-\mathrm{y}[n-1]+2\,\mathrm{y}[n-2]=\mathrm{x}[n]$, expresses the excitation in terms of the response.
Invertible.

13. Find the impulse responses of these systems.

(a) $\mathrm{y}[n]=\mathrm{x}[n]-\mathrm{x}[n-1]$

The impulse response is very easily found by direct iteration to be $\mathrm{h}[n]=\delta[n]-\delta[n-1]$.

Also, using linearity and superposition, the impulse response of this system is the same as the impulse response of the system, $\mathrm{y}[n]=\mathrm{x}[n]$ minus the impulse response of the system, $\mathrm{y}[n]=\mathrm{x}[n-1]$. The impulse response of the first system is $\mathrm{h}_1[n]=\delta[n]$ and the impulse response of the second system is exactly the same except delayed by 1 in discrete time or $\mathrm{h}_1[n]=\delta[n-1]$. The overall impulse resopnse is therefore $\mathrm{h}[n]=\mathrm{h}_1[n]-\mathrm{h}_2[n]=\delta[n]-\delta[n-1]$, as before.

(b) $25\,\mathrm{y}[n]+6\,\mathrm{y}[n-1]+\mathrm{y}[n-2]=\mathrm{x}[n]$

The homogeneous solution is

$$\mathrm{y}_h[n]=K_1\left(\frac{-3+j4}{25}\right)^n+K_2\left(\frac{-3-j4}{25}\right)^n$$

and, after discrete-time, $n=0$, this is the total solution because the excitation is zero. The first two values of the impulse response are (by direct iteration),

$$\mathrm{y}[0]=\frac{1}{25}\quad\text{and}\quad\mathrm{y}[1]=-\frac{6}{625}\ .$$

Solving for the constants,

$$\begin{array}{c}\dfrac{1}{25}=K_1+K_2\\ -\dfrac{6}{625}=K_1\left(\dfrac{-3+j4}{25}\right)+K_2\left(\dfrac{-3-j4}{25}\right)\end{array}\Rightarrow\begin{array}{c}K_1=\dfrac{4+j3}{200}\\ K_2=\dfrac{4-j3}{200}\end{array}$$

Then the impulse response is

$$\mathrm{h}[n]=\frac{4+j3}{200}\left(\frac{-3+j4}{25}\right)^n+\frac{4-j3}{200}\left(\frac{-3-j4}{25}\right)^n$$

$$\mathrm{h}[n]=\frac{(4+j3)(-3+j4)^n+(4-j3)(-3-j4)^n}{200(25)^n}$$

$$\mathrm{h}[n] = \frac{(4+j3)5^n e^{j2.214n} + (4-j3)5^n e^{-j2.214n}}{200(25)^n} = \frac{4\left(e^{j2.214n} + e^{-j2.214n}\right) + j3\left(e^{j2.214n} - e^{-j2.214n}\right)}{200(5)^n}$$

$$\mathrm{h}[n] = \frac{4\cos(2.214n) - 3\sin(2.214n)}{100(5)^n}$$

Then, using

$$A\cos(x) + B\sin(x) = \sqrt{A^2 + B^2}\cos\left(x - \tan^{-1}\left(\frac{B}{A}\right)\right)$$

$$\mathrm{h}[n] = \frac{\cos(2.214n + 0.644)}{20(5)^n}$$

(c) $4\,\mathrm{y}[n] - 5\,\mathrm{y}[n-1] + \mathrm{y}[n-2] = \mathrm{x}[n]$

(d) $2\,\mathrm{y}[n] + 6\,\mathrm{y}[n-2] = \mathrm{x}[n] - \mathrm{x}[n-2]$

The impulse response is the difference of the response, $\mathrm{h}_1[n]$ to a unit impulse at time, $n = 0$, and the response, $\mathrm{h}_2[n]$, to a unit impulse at time, $n = 2$.

14. Sketch $\mathrm{g}[n]$. To the extent possible find analytical solutions. Where possible, compare analytical solutions with the results of using the MATLAB command, conv, to do the convolution.

(a) $\mathrm{g}[n] = \mathrm{u}[n] * \mathrm{u}[n] = \sum_{m=-\infty}^{\infty} \mathrm{u}[m]\mathrm{u}[n-m] = \sum_{m=0}^{\infty} \mathrm{u}[n-m] = \sum_{m=-\infty}^{0} \mathrm{u}[m-n] = \mathrm{ramp}[n+1]$

(b) $\mathrm{g}[n] = \mathrm{u}[n+2] * \mathrm{rect}_3[n] = \sum_{m=-\infty}^{\infty} \mathrm{u}[m+2]\mathrm{rect}_3[n-m] = \sum_{m=-2}^{\infty} \mathrm{rect}_3[n-m]$

(c) $\mathrm{g}[n] = \mathrm{rect}_2[n] * \mathrm{rect}_2[n] = \sum_{-\infty}^{\infty} \mathrm{rect}_2[m]\mathrm{rect}_2[n-m] = \sum_{-2}^{2} \mathrm{rect}_2[n-m]$

(d) $\mathrm{g}[n] = \mathrm{rect}_2[n] * \mathrm{rect}_4[n]$

(e) $\mathrm{g}[n] = 3\delta[n-4] * \left(\frac{3}{4}\right)^n \mathrm{u}[n]$

Using $A\delta[n-n_0] * \mathrm{g}[n] = A\,\mathrm{g}[n-n_0]$

$$g[n] = 3\left(\frac{3}{4}\right)^{n-4} u[n-4]$$

(f) $g[n] = 2\,\mathrm{rect}_4[n] * \left(\frac{7}{8}\right)^n u[n]$

(g) $g[n] = \mathrm{rect}_3[n] * \mathrm{comb}_{14}[n] = \mathrm{rect}_3[n] * \sum_{m=-\infty}^{\infty} \delta[n-14m] = \sum_{m=-\infty}^{\infty} \mathrm{rect}_3[n-14m]$

15. Given the excitations, $x[n]$, and the impulse responses, $h[n]$, find closed-form expressions for and plot the system responses, $y[n]$.

(a) $x[n] = e^{j\frac{2\pi n}{32}}$, $h[n] = (0.95)^n u(n)$

$$y[n] = h[n] * x[n] = \sum_{m=-\infty}^{\infty} e^{j\frac{2\pi m}{32}} (0.95)^{n-m} u[n-m]$$

$$y[n] = \sum_{m=-\infty}^{n} e^{j\frac{2\pi m}{32}} (0.95)^{n-m} = (0.95)^n \sum_{m=-\infty}^{n} \left(\frac{e^{j\frac{2\pi}{32}}}{0.95}\right)^m$$

Making the change of variable, $q = -m$

$$y[n] = (0.95)^n \sum_{-q=-\infty}^{-n} \left(\frac{e^{j\frac{2\pi}{32}}}{0.95}\right)^{-q} = (0.95)^n \sum_{q=-n}^{\infty} \left(0.95 e^{-j\frac{2\pi}{32}}\right)^q$$

Using $\sum_{n=k}^{\infty} r^n = \frac{r^k}{1-r}$, $|r| < 1$ from Appendix A,

$$y[n] = (0.95)^n \frac{\left(0.95 e^{-j\frac{2\pi}{32}}\right)^{-n}}{1 - 0.95 e^{-j\frac{2\pi}{32}}} = \frac{e^{j\frac{2\pi}{32}n}}{1 - 0.95 e^{-j\frac{2\pi}{32}}} = 5.0632 e^{j\left(\frac{2\pi}{32}n - 1.218\right)}$$

(b) $x[n] = \sin\left(\frac{2\pi n}{32}\right)$, $h[n] = (0.95)^n u[n]$

From part (a), the response to $x[n] = e^{j\frac{2\pi n}{32}}$ is $y[n] = 5.0632 e^{j\left(\frac{2\pi}{32}n - 1.218\right)}$. Since

$$\sin\left(\frac{2\pi n}{32}\right) = \frac{e^{j\frac{2\pi n}{32}} - e^{-j\frac{2\pi n}{32}}}{j2},$$

by applying linearity and superposition, the response to $\mathrm{x}[n]=\sin\left(\frac{2\pi n}{32}\right)$ is

$$\mathrm{y}[n]=\frac{5.0632e^{j\left(\frac{2\pi}{32}n-1.218\right)}-5.0632e^{-j\left(\frac{2\pi}{32}n-1.218\right)}}{j2}$$

16. Given the excitations, $\mathrm{x}[n]$, and the impulse responses, $\mathrm{h}[n]$, use MATLAB to plot the system responses, $\mathrm{y}[n]$.

(a) $\mathrm{x}[n]=\mathrm{u}[n]-\mathrm{u}[n-8]$, $\mathrm{h}[n]=\sin\left(\frac{2\pi n}{8}\right)(\mathrm{u}[n]-\mathrm{u}[n-8])$

$$\mathrm{y}[n]=\mathrm{h}[n]*\mathrm{x}[n]=\sum_{m=-\infty}^{\infty}\sin\left(\frac{2\pi m}{8}\right)(\mathrm{u}[m]-\mathrm{u}[m-8])(\mathrm{u}[n-m]-\mathrm{u}[n-m-8])$$

$$\mathrm{y}[n]=\sum_{m=-\infty}^{\infty}\sin\left(\frac{2\pi m}{8}\right)\begin{pmatrix}\mathrm{u}[m]\mathrm{u}[n-m]-\mathrm{u}[m]\mathrm{u}[n-m-8]\\-\mathrm{u}[m-8]\mathrm{u}[n-m]+\mathrm{u}[m-8]\mathrm{u}[n-m-8]\end{pmatrix}$$

$$\mathrm{y}[n]=\sum_{m=0}^{n}\sin\left(\frac{2\pi m}{8}\right)-\sum_{m=0}^{n-8}\sin\left(\frac{2\pi m}{8}\right)-\sum_{m=8}^{n}\sin\left(\frac{2\pi m}{8}\right)+\sum_{m=8}^{n-8}\sin\left(\frac{2\pi m}{8}\right)$$

For $n<0$, all the summations are zero because the factors, $\mathrm{u}[n-m]$ and $\mathrm{u}[n-m-8]$ are zero in the summation range, $0<m<n$, and $\mathrm{y}[n]=0$.

For $n>15$,

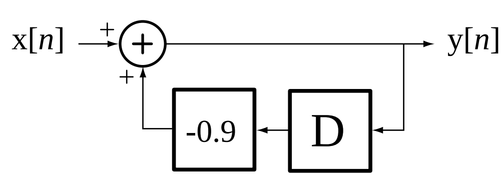
$$\mathrm{y}[n]=\sum_{m=n-7}^{n}\sin\left(\frac{2\pi m}{8}\right)-\sum_{m=n-7}^{n}\sin\left(\frac{2\pi m}{8}\right)=0\ .$$

So the response is only non-zero for $0\le m<16$ (and can be zero at some points within that range).

(b) $\mathrm{x}[n]=\sin\left(\frac{2\pi n}{8}\right)(\mathrm{u}[n]-\mathrm{u}[n-8])$, $\mathrm{h}[n]=-\sin\left(\frac{2\pi n}{8}\right)(\mathrm{u}[n]-\mathrm{u}[n-8])$

17. Which of these systems are BIBO stable?

(a)

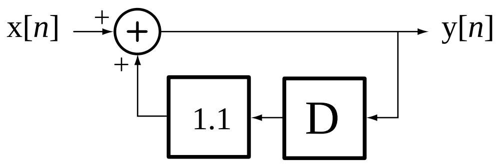


The system equation is

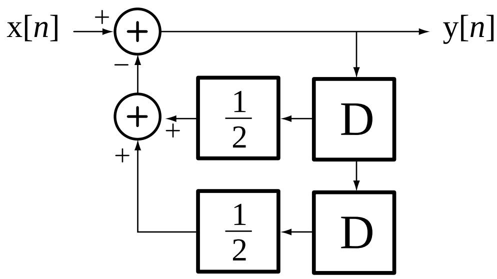
$$\mathrm{y}[n]=\mathrm{x}[n]-0.9\,\mathrm{y}[n-1]$$

The eigenvalue is $\alpha = -0.9$. Its magnitude is less than one, therefore the system is stable.

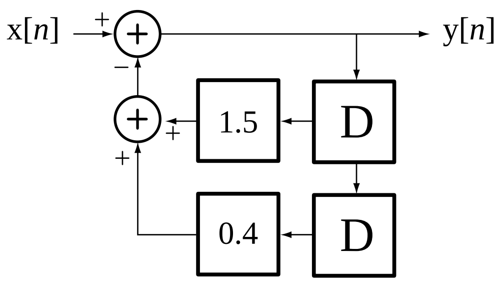
(b)



(c)

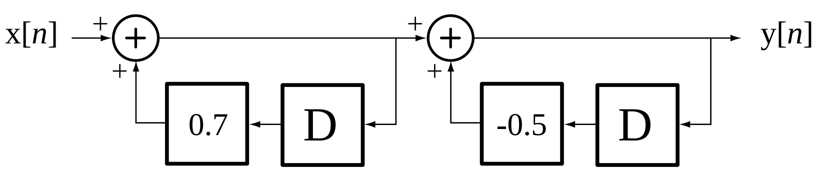


(d)



18. Find and plot the unit-sequence responses of these systems.

(a)



$$\mathrm{h}[n] = \mathrm{h}_1[n] * \mathrm{h}_2[n]$$

$$\mathrm{h}_1[n] = (0.7)^n \mathrm{u}[n] \quad \text{and} \quad \mathrm{h}_2[n] = (-0.5)^n \mathrm{u}[n]$$

$$\mathrm{h}[n] = \sum_{m=-\infty}^{\infty} (0.7)^m \mathrm{u}[m](-0.5)^{n-m} \mathrm{u}[n-m]$$

Simplify this expression as much as possible by letting the unit sequency functions modify the summation limits and then apply the formular for the summation of a geometric series,

$$\sum_{n=0}^{N-1} r^n = \begin{cases} 1 & , \quad |r| = 1 \\ \dfrac{1-r^N}{1-r} & , \quad \text{otherwise} \end{cases}$$

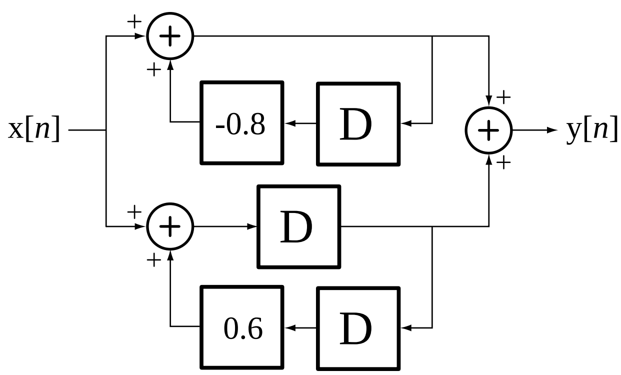
to get

$$\mathrm{h}[n] = (-0.5)^n \frac{1-(-1.4)^{n+1}}{2.4} \mathrm{u}[n]$$

Then convolve the impulse response with the unit sequence to get the overall response and use some of the same techniques to find a simple closed-form expression for the response.

$$\mathrm{y}[n] = 0.4167\left\{0.6667\left(1-(-0.5)^{n+1}\right) + 4.6667\left(1-(0.7)^{n+1}\right)\right\}\mathrm{u}[n]$$

(b)



$$\mathrm{h}[n] = \mathrm{h}_1[n] + \mathrm{h}_2[n]$$

$$\mathrm{h}[n] = \left[(-0.8)^n + 0.6455\left(\sqrt{0.6}\right)^n - 0.6455\left(-\sqrt{0.6}\right)^n\right]\mathrm{u}[n]$$

Then convolve the unit sequence with the impulse response to get the overall system response,

$$\mathrm{y}[n] = \left[\frac{1-(-0.8)^{n+1}}{1.8} + 0.6455\frac{1-\left(\sqrt{0.6}\right)^{n+1}}{0.2254} - 0.6455\frac{1-\left(-\sqrt{0.6}\right)^{n+1}}{1.7746}\right]\mathrm{u}[n]$$

19. Find the impulse responses of these systems:

(a) $\mathrm{y}'(t) + 5\,\mathrm{y}(t) = \mathrm{x}(t)$

Follow the example in the text.

(b) $\mathrm{y}''(t) + 6\,\mathrm{y}'(t) + 4\,\mathrm{y}(t) = \mathrm{x}(t)$

$\mathrm{h}''(t) + 6\,\mathrm{h}'(t) + 4\,\mathrm{h}(t) = \delta(t)$

For $t < 0$, $\mathrm{h}(t) = 0$.

For $t > 0$, $\mathrm{h}_h(t) = K_1 e^{-5.23t} + K_2 e^{-0.76t}$

Since the highest derivative of "x" is two less than the highest derivative of "y", the general solution is of the form,

$$\mathrm{h}(t) = \left(K_1 e^{-5.23t} + K_2 e^{-0.76t}\right)\mathrm{u}(t)$$

(See the discussion in the text of what the solution form must be for different derivatives of x and y.) Integrating the differential equation once from $t = 0^-$ to $t = 0^+$,

$$\mathrm{h}'(0^+) - \mathrm{h}'(0^-) + 6\left[\mathrm{h}(0^+) - \mathrm{h}(0^-)\right] + 4\int_{0^-}^{0^+} \mathrm{h}(t)\,dt = \int_{0^-}^{0^+} \delta(t)\,dt = 1$$

We know that the impulse response cannot contain an impulse because its second derivative would be a triplet and there is no triplet excitation. We also know that the impulse response cannot be discontinuous at time, $t = 0$, because if it were the second derivative would be a doublet and there is no doublet excitation. Therefore,

$$\mathrm{h}'(0^+) - \mathrm{h}'(0^-) = 1 \Rightarrow \mathrm{h}'(0^+) = 1$$

This requirement, along with the requirement that the solution be continuous at time, $t = 0$, leads to the two equations,

$$\mathrm{h}'(0^+) = 1 = \left[-5.23K_1 e^{-5.23t} - 0.76K_2 e^{-0.76t}\right]_{t=0^+} = -5.23K_1 - 0.76K_2$$

and

$$\mathrm{h}(0^+) = 0 = K_1 + K_2 \ .$$

(This second equation can also be found by integrating the differential equation twice from from $t = 0^-$ to $t = 0^+$.)

Solving,

$$K_1 = -0.2237 \text{ and } K_2 = 0.2237$$

Then the total impulse response is

$$\mathrm{h}(t) = 0.2237\left(e^{-0.76t} - e^{-5.23t}\right)\mathrm{u}(t) \ .$$

(c) $\quad 2\,\mathrm{y}'(t) + 3\,\mathrm{y}(t) = \mathrm{x}'(t)$

(d) $\quad 4\,\mathrm{y}'(t) + 9\,\mathrm{y}(t) = 2\,\mathrm{x}(t) + \mathrm{x}'(t)$

The homogeneous solution is $\mathrm{y}_h(t) = K_h e^{-\frac{9}{4}t}$. The impulse response is of the form,

$$\mathrm{h}(t) = K_h e^{-\frac{3}{2}t}\mathrm{u}(t) + K_i\delta(t) \ .$$

The solution is $\mathrm{h}(t) = -\frac{1}{16} e^{-\frac{9}{4}t} \mathrm{u}(t) + \frac{1}{4}\delta(t)$

20. Sketch $\mathrm{g}(t)$.

(a) $\mathrm{g}(t) = \mathrm{rect}(t) * \mathrm{rect}(t) = \int_{-\infty}^{\infty} \mathrm{rect}(\tau)\mathrm{rect}(t-\tau)\,d\tau = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{rect}(t-\tau)\,d\tau$

Probably the easiest way to find this solution is graphically through the "flipping and shifting" process. When the second rectangle is flipped, it looks exactly the same because it is an even function. This is the "zero shift" position, the $t = 0$ position. At this position the two rectangles coincide and the area under the product is one. If t is increased from this position the two rectangles no longer coincide and the area under the product is reduced linearly until at $t = 1$ the area goes to zero. Exactly the same thing happens for decreases in t until it gets to -1. The convolution is obviously a unit triangle function. This fact is the reason the unit triangle function was defined as it was, so it could simply be the convolution of a unit rectangle with itself.

This convolution can also be done analytically.

For $t < -1$, in the range of integration, $-\frac{1}{2} < \tau < \frac{1}{2}$, the rect function is zero and the convolution integral is zero.

For $t > 1$, in the range of integration, $-\frac{1}{2} < \tau < \frac{1}{2}$, the rect function is zero and the convolution integral is zero.

For $-1 < t < 0$. Since the rect function is even we can say that $\mathrm{rect}(t-\tau) = \mathrm{rect}(\tau - t)$. This is a rectangle extending in τ from $t - \frac{1}{2}$ to $t + \frac{1}{2}$. For t's in the range, $-1 < t < 0$, $t - \frac{1}{2}$ is always less than or equal to the lower limit, $\tau = -\frac{1}{2}$, so the integral is from $-\frac{1}{2}$ to $t + \frac{1}{2}$.

$$\mathrm{g}(t) = \int_{-\frac{1}{2}}^{t+\frac{1}{2}} \mathrm{rect}(\tau - t)\,d\tau$$

This is simply the accumulation of the area under a rectangle and therefore increases linearly from a minimum of zero for $t = -1$ to a maximum of one for $t = 0$.

For $0 < t < 1$. This is also rectangle extending in τ from $t - \frac{1}{2}$ to $t + \frac{1}{2}$. For t's in the range, $0 < t < 1$, $t + \frac{1}{2}$ is always greater than or equal to the upper limit, $\tau = \frac{1}{2}$, so the integral is from $t - \frac{1}{2}$ to $\frac{1}{2}$.

$$\mathrm{g}(t) = \int_{t-\frac{1}{2}}^{\frac{1}{2}} \mathrm{rect}(\tau - t)\, d\tau$$

This is also the accumulation of the area under a rectangle and decreases linearly from a maximum of one for $t = 0$ to a minimum of zero for $t = 1$.

(b) $\mathrm{g}(t) = \mathrm{rect}(t) * \mathrm{rect}\left(\frac{t}{2}\right)$

This convolution is easily done graphically.

(c) $\mathrm{g}(t) = \mathrm{rect}(t-1) * \mathrm{rect}\left(\frac{t}{2}\right)$

(d) $\mathrm{g}(t) = [\mathrm{rect}(t-5) + \mathrm{rect}(t+5)] * [\mathrm{rect}(t-4) + \mathrm{rect}(t+4)]$

Break this convolution down into the sum of four simpler convolutions.

21. Sketch these functions.

(a) $\mathrm{g}(t) = \mathrm{rect}(4t)$

(b) $\mathrm{g}(t) = \mathrm{rect}(4t) * 4\delta(t)$

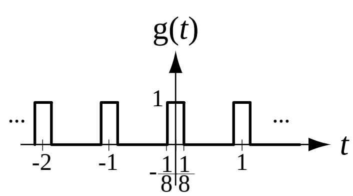
(c) $\mathrm{g}(t) = \mathrm{rect}(4t) * 4\delta(t-2)$

(d) $\mathrm{g}(t) = \mathrm{rect}(4t) * 4\delta(2t)$

Don't forget the scaling property of the CT impulse.

(e) $\mathrm{g}(t) = \mathrm{rect}(4t) * \mathrm{comb}(t)$

Convolution with a comb is relatively easy because it is simply convolution with a periodic sequence of impulses.



(f) $\mathrm{g}(t) = \mathrm{rect}(4t) * \mathrm{comb}(t-1)$

This result is identical to the result of part (e).

(g) $\mathrm{g}(t) = \mathrm{rect}(4t) * \mathrm{comb}(2t)$

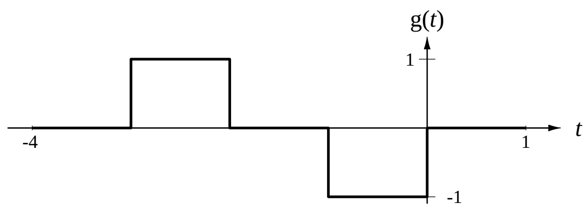
Don't forget the scaling property of the CT impulses in the comb function. The average value of $\mathrm{g}(t)$ is 1/4.

(h) $\mathrm{g}(t) = \mathrm{rect}(t) * \mathrm{comb}(2t)$

This is the sum of multiple rectangle functions periodically repeated.

22. Plot these convolutions.

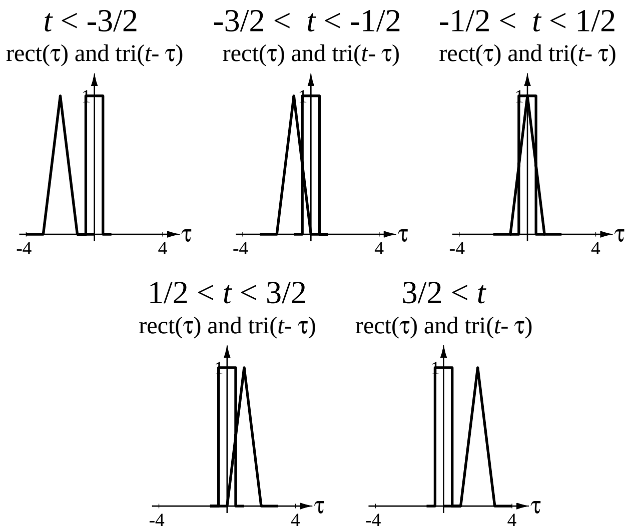
(a) $\mathrm{g}(t) = \mathrm{rect}\left(\frac{t}{2}\right) * [\delta(t+2) - \delta(t+1)] = \mathrm{rect}\left(\frac{t+2}{2}\right) - \mathrm{rect}\left(\frac{t+1}{2}\right)$



(b) $\mathrm{g}(t) = \mathrm{rect}(t) * \mathrm{tri}(t)$

This is a challenging convolution because it is not so simple to do graphically (although you can get a rough idea of what it looks like that way) and it is tedious analytically.

$$\mathrm{g}(t) = \int_{-\infty}^{\infty} \mathrm{rect}(\tau)\,\mathrm{tri}(t-\tau)\,d\tau = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{tri}(t-\tau)\,d\tau$$



If $t < -\frac{3}{2}$, $\mathrm{g}(t) = 0$.

If $-\frac{3}{2} < t < -\frac{1}{2}$, $\mathrm{g}(t) = \int_{-\frac{1}{2}}^{t+1} \left(1 - \left|\underbrace{\tau - t}_{>0}\right|\right) d\tau = \int_{-\frac{1}{2}}^{t+1} \left(1 - (\tau - t)\right) d\tau = \left[\tau - \frac{\tau^2}{2} + \tau t\right]_{-\frac{1}{2}}^{t+1}$

$$\mathrm{g}(t) = \left[t + 1 - \frac{(t+1)^2}{2} + (t+1)t - \left(-\frac{1}{2}\right) + \frac{\left(-\frac{1}{2}\right)^2}{2} - \left(-\frac{1}{2}\right)t\right]$$

$$\mathrm{g}(t) = \frac{t^2}{2} + \frac{3t}{2} + \frac{9}{8}$$

If $-\frac{1}{2} < t < \frac{1}{2}$, $\mathrm{g}(t) = \int_{-\frac{1}{2}}^{t} \left(1 - \left|\underbrace{\tau - t}_{<0}\right|\right) d\tau + \int_{t}^{\frac{1}{2}} \left(1 - \left|\underbrace{\tau - t}_{>0}\right|\right) d\tau$

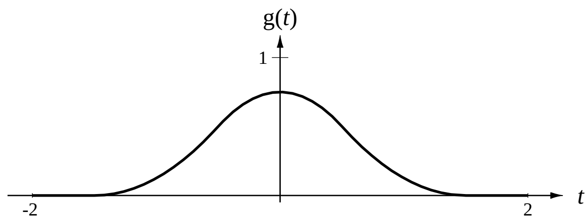
$$\mathrm{g}(t) = \int_{-\frac{1}{2}}^{t} \left(1 - (t - \tau)\right) d\tau + \int_{t}^{\frac{1}{2}} \left(1 - (\tau - t)\right) d\tau = \left[\tau - \tau t + \frac{\tau^2}{2}\right]_{-\frac{1}{2}}^{t} + \left[\tau - \frac{\tau^2}{2} + \tau t\right]_{t}^{\frac{1}{2}}$$

$$\mathrm{g}(t) = \left[t - t^2 + \frac{t^2}{2} + \frac{1}{2} - \frac{t}{2} - \frac{1}{8}\right] + \left[\frac{1}{2} - \frac{1}{8} + \frac{t}{2} - t + \frac{t^2}{2} - t^2\right]$$

$$\mathrm{g}(t) = \frac{3}{4} - t^2$$

By symmetry, $\mathrm{g}(t) = \mathrm{g}(-t)$ and

$$g(t) = \begin{cases} 0 \ , & |t| > \frac{3}{2} \\ \frac{t^2}{2} - \frac{3|t|}{2} + \frac{9}{8} \ , & \frac{1}{2} < |t| < \frac{3}{2} \\ \frac{3}{4} - t^2 \ , & |t| < \frac{1}{2} \end{cases}$$



(c) $g(t) = e^{-t} u(t) * e^{-t} u(t)$

(d) $g(t) = \left[\text{tri}\left(2\left(t+\frac{1}{2}\right)\right) - \text{tri}\left(2\left(t-\frac{1}{2}\right)\right)\right] * \frac{1}{2}\text{comb}\left(\frac{t}{2}\right)$

(e) $g(t) = \left[\text{tri}\left(2\left(t+\frac{1}{2}\right)\right) - \text{tri}\left(2\left(t-\frac{1}{2}\right)\right)\right] * \text{comb}(t)$

This is a very complicated way of saying $g(t) = 0$. Can you determine this without going throught the whole process of convolving them?

23. A system has an impulse response, $h(t) = 4e^{-4t} u(t)$. Find and plot the response of the system to the excitation, $x(t) = \text{rect}\left(2\left(t - \frac{1}{4}\right)\right)$.

    Express the rectangle as a difference between two unit steps to simplify the problem.

24. Change the system impulse response in Exercise 23 to $h(t) = \delta(t) - 4e^{-4t} u(t)$ and find and plot the response to the same excitation, $x(t) = \text{rect}\left(2\left(t - \frac{1}{4}\right)\right)$.

25. Find the impulse responses of the two systems in Figure E25. Are these systems BIBO stable?

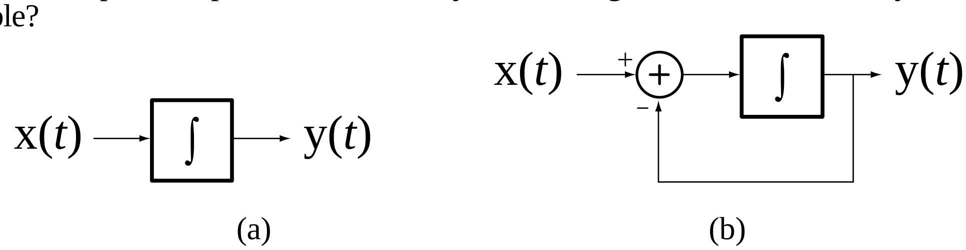


Figure E25 Two single-integrator systems

(a) $y'(t) = x(t) \Rightarrow h(t) = u(t)$

A CT system is BIBO stable if its impulse response is absolutely integrable.

Impulse response is not absolutely integrable. BIBO unstable.

26. Find the impulse response of the system in Figure E26. Is this system BIBO stable?

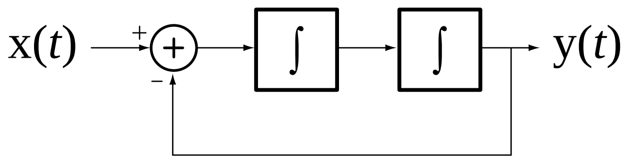


Figure E26 A double-integrator system

27. In the circuit of Figure E27 the excitation is $v_i(t)$ and the response is $v_o(t)$.

(a) Find the impulse response in terms of R and L.

(b) If $R = 10\ \text{k}\Omega$ and $L = 100\ \mu\text{H}$ graph the unit step response.

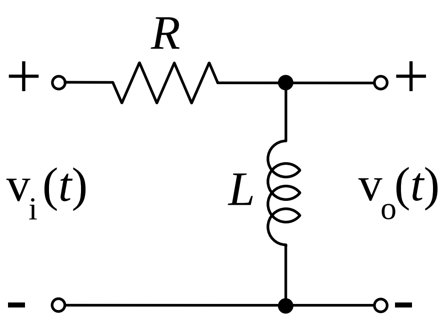


Figure E27 An RL circuit

$$v_i(t) = R\,i(t) + v_o(t)$$

$$i(t) = \frac{v_i(t) - v_o(t)}{R}$$

$$v_o(t) = L\frac{d}{dt}(i(t)) = \frac{L}{R}\left[v_i'(t) - v_o'(t)\right]$$

$$v_i(t) = \left[v_i(t) - v_o(t)\right] + \frac{L}{R}\left[v_i'(t) - v_o'(t)\right]$$

$$\frac{L}{R}v_o'(t) + v_o(t) = \frac{L}{R}v_i'(t)$$

$$\frac{L}{R}\mathrm{h}'(t)+\mathrm{h}(t)=\frac{L}{R}\delta'(t)$$

$$\mathrm{h}(t)=0 \quad , \quad t<0$$

For times, $t>0$, the solution is the homogeneous solution,

$$\mathrm{h}(t)=K_h e^{-\frac{R}{L}t} \quad , \quad t>0$$

Since the highest derivatives on both sides of this differential equation are the same the impulse response contains an impulse and is of the form,

$$\mathrm{h}(t)=K_\delta \delta(t)+K_h e^{-\frac{R}{L}t}\mathrm{u}(t)$$

Integrating both sides of the differential equation from 0^- to 0^+,

$$\frac{L}{R}\left[\underbrace{\mathrm{h}(0^+)}_{=K_h}-\underbrace{\mathrm{h}(0^-)}_{=0}\right]+\underbrace{\int_{0^-}^{0^+}\mathrm{h}(t)\,dt}_{=K_\delta}=\frac{L}{R}\left[\underbrace{\delta(0^+)}_{=0}-\underbrace{\delta(0^-)}_{=0}\right]\Rightarrow\frac{L}{R}K_h+K_\delta=0$$

Integrating both sides of the differential equation a second time from 0^- to 0^+,

$$\frac{L}{R}\underbrace{\int_{0^-}^{0^+}\mathrm{h}(t)\,dt}_{=K_\delta}+K_\delta\underbrace{\int_{0^-}^{0^+}\mathrm{u}(t)\,dt}_{=0}=\frac{L}{R}\left[\underbrace{\mathrm{u}(0^+)}_{=1}-\underbrace{\mathrm{u}(0^-)}_{=0}\right]\Rightarrow K_\delta=1$$

Then, from the first integration, $K_h=-\frac{R}{L}$ and $\mathrm{h}(t)=\delta(t)-\frac{R}{L}e^{-\frac{R}{L}t}\mathrm{u}(t)$

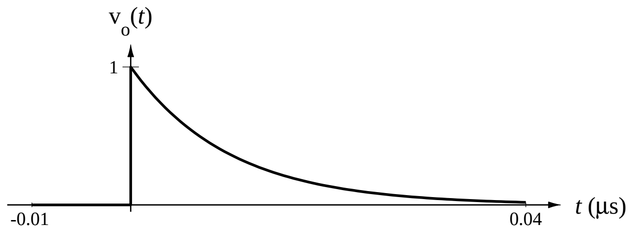
The unit-step response, $\mathrm{h}_{-1}(t)$ is the integral of the impulse response,

$$\mathrm{h}_{-1}(t)=\int_{-\infty}^{t}\left[\delta(\lambda)-\frac{R}{L}e^{-\frac{R}{L}\lambda}\mathrm{u}(\lambda)\right]d\lambda=\int_{0^-}^{t}\left[\delta(\lambda)-\frac{R}{L}e^{-\frac{R}{L}\lambda}\right]d\lambda$$

For $t<0$ the integral is obviously zero. Therefore $\mathrm{h}_{-1}(t)=0 \quad , \quad t<0$

$$\mathrm{h}_{-1}(t)=1-\frac{R}{L}\int_{0^-}^{t}e^{-\frac{R}{L}\lambda}d\lambda=1-\frac{R}{L}\left(-\frac{L}{R}\right)\left[e^{-\frac{R}{L}\lambda}\right]_{0^-}^{t}=1+e^{-\frac{R}{L}t}-1=e^{-\frac{R}{L}t} \quad , \quad t>0$$

$$\mathrm{h}_{-1}(t)=e^{-\frac{R}{L}t}\mathrm{u}(t)=e^{-10^8 t}\mathrm{u}(t) \quad \text{, Step response}$$



28. Find the impulse response of the system in Figure E28 and evaluate its BIBO stability.

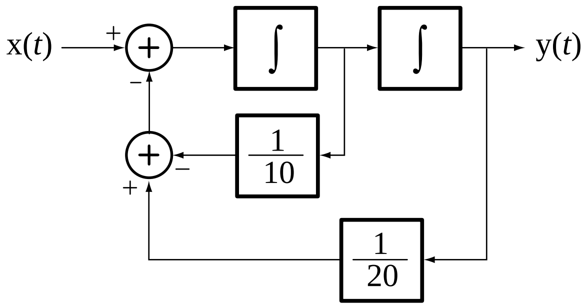


Figure E28 A two-integrator system

$$\mathrm{y}''(t) = \mathrm{x}(t) + \frac{1}{10}\mathrm{y}'(t) - \frac{1}{20}\mathrm{y}(t)$$

29. Find the impulse response of the system in Figure E**Error! Reference source not found.** and evaluate its BIBO stability.

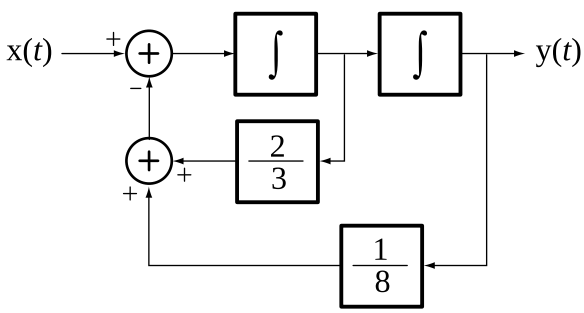


Figure E**Error! Reference source not found.** A two-integrator system

30. Plot the amplitudes of the responses of the systems of Exercise 19 to the excitation, $e^{j\omega t}$, as a function of radian frequency, ω.

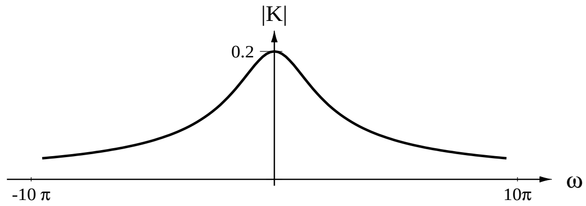
(a) $\mathrm{y}'(t) + 5\,\mathrm{y}(t) = \mathrm{x}(t)$

First realize that the excitation, $e^{j\omega t}$, is periodic, that is, it has always existed and will always exist repeating periodically. Therefore there is no homogeneous solution to worry about. If the system is stable it died out a long time ago and if the system is not stable, this exercise has no useful physical interpretation. So the solution is simply the particular solution of the differential equation of the form,

$$\mathrm{y}_p(t) = Ke^{j\omega t} \quad .$$

Putting that into the differential equation and solving,

$$K\omega e^{j\omega t} + 5Ke^{j\omega t} = e^{k\omega t} \Rightarrow K = \frac{1}{j\omega + 5}$$



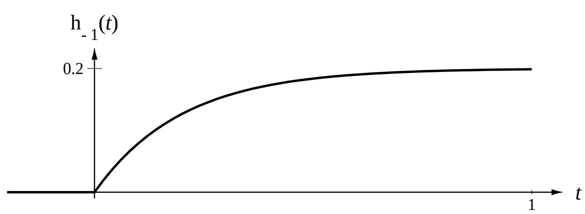
31. Plot the responses of the systems of Exercise 19 to a unit-step excitation.

(a) $\mathrm{h}(t) = e^{-5t}\,\mathrm{u}(t)$

$$\mathrm{h}_{-1}(t) = \int_{-\infty}^{t} \mathrm{h}(\lambda)d\lambda = \int_{-\infty}^{t} e^{-5\lambda}\,\mathrm{u}(\lambda)d\lambda = \int_{0}^{t} e^{-5\lambda}d\lambda = -\frac{1}{5}\left[e^{-5\lambda}\right]_0^t = \frac{1}{5}\left(1 - e^{-5t}\right) \;\;,\;\; t > 0$$

$$\mathrm{h}_{-1}(t)0 \;\;,\;\; t < 0$$

$$\mathrm{h}_{-1}(t) = \frac{1}{5}\left(1 - e^{-5t}\right)\mathrm{u}(t)$$



32. A CT system is described by the block diagram in Figure E32.

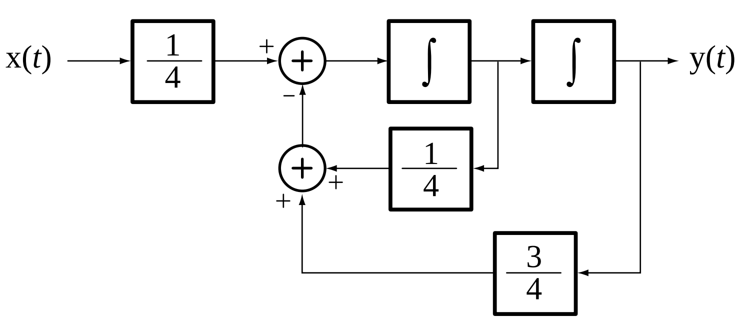


Figure E32 A CT system

Classify the system as to homogeneity, additivity, linearity, time-invariance, stability, causality, memory, and invertibility.

33. A system has a response that is the cube of its excitation. Classify the system as to homogeneity, additivity, linearity, time-invariance, stability, causality, memory, and invertibility.

$$\mathrm{y}(t) = \mathrm{x}^3(t)$$

Invertibility:

Solve $\mathrm{y}(t) = \mathrm{x}^3(t)$ for $\mathrm{x}(t)$. $\mathrm{x}(t) = \mathrm{y}^{\frac{1}{3}}(t)$. The cube root operation is multiple valued. Therefore the system is not invertible, unless we assume that the excitation must be real-valued. In that case, the response does determine the excitation because for any real y there is only one real cube root.

34. A CT system is described by the differential equation,

$$t\,\mathrm{y}'(t) - 8\,\mathrm{y}(t) = \mathrm{x}(t) \ .$$

Classify the system as to linearity, time-invariance and stability.

Stability:

The homogeneous solution to the differential equation is of the form,

$$t\,\mathrm{y}'(t) = 8\,\mathrm{y}(t)$$

To satisfy this equation the derivative of "y" times "t" must be of the same functional form as "y" itself. This is satisfied by a homogeneous solution of the form,

$$\mathrm{y}(t) = Kt^8$$

If there is no excitation, but the zero-excitation response is not zero, the response will increase without bound as time increases.
Unstable

35. A CT system is described by the equation,

$$\mathrm{y}(t) = \int_{-\infty}^{\frac{t}{3}} \mathrm{x}(\lambda)\,d\lambda \ .$$

Classify the system as to time-invariance, stability and invertibility.

Time Invariance:

Let $\mathrm{x}_1(t) = \mathrm{g}(t)$. Then $\mathrm{y}_1(t) = \int_{-\infty}^{\frac{t}{3}} \mathrm{g}(\lambda)\,d\lambda$.

Let $\mathrm{x}_2(t) = \mathrm{g}(t - t_0)$.

Then $\mathrm{y}_2(t) = \int_{-\infty}^{\frac{t}{3}} \mathrm{g}(\lambda - t_0)d\lambda = \int_{-\infty}^{\frac{t}{3}-t_0} \mathrm{g}(u)du \neq \mathrm{y}_1(t-t_0) = \int_{-\infty}^{\frac{t-t_0}{3}} \mathrm{g}(\lambda)d\lambda$.
Time Variant

Stability:

If $\mathrm{x}(t)$ is a constant, *K*, then $\mathrm{y}(t) = \int_{-\infty}^{\frac{t}{3}} K d\lambda = K\int_{-\infty}^{\frac{t}{3}} d\lambda$ and, as $t \to \infty$, $\mathrm{y}(t)$ increases without bound.
Unstable

Invertibility:

Differentiate both sides of $\mathrm{y}(t) = \int_{-\infty}^{\frac{t}{3}} \mathrm{x}(\lambda)d\lambda$ w.r.t. t yielding $\mathrm{y}'(t) = \mathrm{x}\left(\frac{t}{3}\right)$. Then it follows that $\mathrm{x}(t) = \mathrm{y}'(3t)$.
Invertible.

36. A CT system is described by the equation,

$$\mathrm{y}(t) = \int_{-\infty}^{t+3} \mathrm{x}(\lambda)d\lambda \ .$$

Classify the system as to linearity, causality and invertibility.

37. Show that the system described by $\mathrm{y}(t) = \mathrm{Re}(\mathrm{x}(t))$ is additive but not homogeneous. (Remember, if the excitation is multiplied by any *complex* constant and the system is homogeneous, the response must be multiplied by that same complex constant.)

38. Graph the magnitude and phase of the complex-sinusoidal response of the system described by

$$\mathrm{y}'(t) + 2\,\mathrm{y}(t) = e^{-j2\pi ft}$$

as a function of cyclic frequency, *f*.

Similar to Exercise 30.

39. A DT system is described by

$$\mathrm{y}[n] = \sum_{m=-\infty}^{n+1} \mathrm{x}[m] \ .$$

Classify this system as to time invariance, BIBO stability and invertibility.

Time Invariance:

Let $\mathrm{x}_1[n] = \mathrm{g}[n]$. Then $\mathrm{y}_1[n] = \sum_{m=-\infty}^{n+1} \mathrm{g}[m]$.

Let $\mathrm{x}_2[n] = \mathrm{g}[n - n_0]$. Then $\mathrm{y}_2[n] = \sum_{m=-\infty}^{n+1} \mathrm{g}[m - n_0]$.

The first equation can be rewritten as

$$\mathrm{y}_1[n-n_0] = \sum_{m=-\infty}^{n-n_0+1} \mathrm{g}[m] = \sum_{q=-\infty}^{n+1} \mathrm{g}[q-n_0] = \mathrm{y}_2[n]$$

Time invariant

Invertibility:
Inverting the functional relationship,

$$\mathrm{y}[n] = \sum_{m=-\infty}^{n+1} \mathrm{x}[m] \ .$$

Taking the first backward difference of both sides of the original system equation,

$$\mathrm{y}[n] - \mathrm{y}[n-1] = \sum_{m=-\infty}^{n+1} \mathrm{x}[m] - \sum_{m=-\infty}^{n+1-1} \mathrm{x}[m]$$

$$\mathrm{x}[n+1] = \mathrm{y}[n] - \mathrm{y}[n-1]$$

The excitation is uniquely determined by the response.
Invertible.

40. A DT system is described by

$$n\,\mathrm{y}[n] - 8\,\mathrm{y}[n-1] = \mathrm{x}[n] \ .$$

Classify this system as to time invariance, BIBO stability and invertibility.

Stability:

The homogeneous equation is

$$n\,\mathrm{y}[n] = 8\,\mathrm{y}[n-1]$$

or

$$\mathrm{y}[n] = \frac{8}{n}\mathrm{y}[n-1] \ .$$

Thus, as n increases without bound, $\mathrm{y}[n]$ must be decreasing because it is $\frac{8}{n}$ times its previous value and $\frac{8}{n}$ approaches zero. Rearranging the original equation,

$$\mathrm{y}[n] = \frac{\mathrm{x}[n]}{n} + \frac{8}{n}\mathrm{y}[n-1] \ .$$

For any bounded excitation, $\mathrm{x}[n]$, as n gets larger, $\frac{\mathrm{x}[n]}{n}$ must be bounded and $\frac{8}{n}\mathrm{y}[n-1]$ must be getting smaller because it is a decreasing fraction of its previous value. Therefore for a bounded excitation, the response is bounded.
Stable.

41. A DT system is described by

$$y[n] = \sqrt{x[n]}\ .$$

Classify this system as to linearity, BIBO stability, memory and invertibility.

Invertibility:
Inverting the functional relationship,

$$x[n] = y^2[n]\ .$$

Invertible.

42. Graph the magnitude and phase of the complex-sinusoidal response of the system described by

$$y[n] + \frac{1}{2}y[n-1] = e^{-j\Omega n}$$

as a function of Ω.

This is the steady-state solution so all we need is the particular solution of the difference equation. The equation can be written as

$$y[n] + \frac{1}{2}y[n-1] = \left(e^{-j\Omega}\right)^n = \alpha^n$$

where

$$\alpha = e^{-j\Omega}$$

The particular solution has the form,

$$y[n] = K\alpha^n\ .$$

43. Find the impulse response, $h[n]$, of the system in Figure E43.

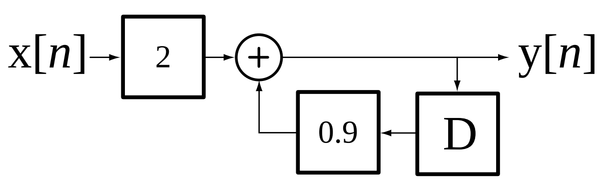


Figure E43 DT system block diagram

$$y[n] = 2\,x[n] + 0.9\,y[n-1]$$

or

$$y[n] - 0.9\,y[n-1] = 2\,x[n]$$

The homogeneous solution (for $n \geq 0$) is of the form,

$$y[n] = K_h\alpha^n$$

therefore the characteristic equation is

$$K_h\alpha^n - 0.9K_h\alpha^{n-1} = 0\ .$$

and the eigenvalue is $\alpha = 0.9$ and, therefore, $y[n] = K_h(0.9)^n$

We can find an initial condition to evaluate the constant, K_h, by directly solving the difference equation for $n = 0$.

$$y[0] = 2\,x[0] + 0.9\,y[-1] = 2 \ .$$

Therefore

$$2 = K_h(0.9)^0 \Rightarrow K_h = 2 \ .$$

Therefore the total solution is

$$y[n] = 2(0.9)^n$$

which is the impulse response.

44. Find the impulse responses of these systems.

(a) $3\,y[n] + 4\,y[n-1] + y[n-2] = x[n] + x[n-1]$

(b) $\frac{5}{2}y[n] + 6\,y[n-1] + 10\,y[n-2] = x[n]$

45. Plot $g[n]$. Use the MATLAB `conv` function if needed.

(a) $g[n] = \text{rect}_1[n] * \sin\left(\frac{2\pi n}{9}\right)$

Write out the convolution sum. Then use

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

write out the entire summation and simplify what you get. You should ultimately get

$$g[n] = 2.5321\sin\left(\frac{2\pi n}{9}\right)$$

(b) $g[n] = \text{rect}_2[n] * \sin\left(\frac{2\pi n}{9}\right)$

(c) $g[n] = 0$

(d) $g[n] = \text{rect}_3[n] * \text{rect}_3[n] * \text{comb}_{14}[n]$

First convolve the two rectangles. Then convolve the result with the comb, thereby periodically repeating it.

(e) Similar to (d) but with a different result.

(f) $\quad \mathrm{g}[n] = 2\cos\left(\frac{2\pi n}{7}\right) * \left(\frac{7}{8}\right)^n \mathrm{u}[n]$

Write the convolution sum. Express the cos in exponential form and combine with other terms. Then use

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad , \quad |\mathrm{r}| < 1$$

to put the result in closed form, and simplify using

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

Finally use

$$A\cos(x) + B\sin(x) = \sqrt{A^2+B^2}\cos\left(x - \tan^{-1}\left(\frac{B}{A}\right)\right)$$

and you should get

$$\mathrm{g}[n] = 2.434\cos\left(\frac{2\pi n}{7} - 0.9845\right)$$

(g) $\quad \mathrm{g}[n] = \frac{\mathrm{sinc}\left(\frac{n}{4}\right)}{2\sqrt{2}} * \frac{\mathrm{sinc}\left(\frac{n}{4}\right)}{2\sqrt{2}}$

In the absence of the transform methods which have not been covered yet, this convolution must be done numerically. This will be relatively simple to do analytically using transform methods.

46. Find the impulse responses of the subsystems in Figure E 46 and then convolve them to find the impulse response of the cascade connection of the two subsystems. You may find this formula for the summation of a finite series useful,

$$\sum_{n=0}^{N-1} \alpha^n = \begin{cases} N & , \ \alpha = 1 \\ \dfrac{1-\alpha^N}{1-\alpha} & , \ \alpha \neq 1 \end{cases} .$$

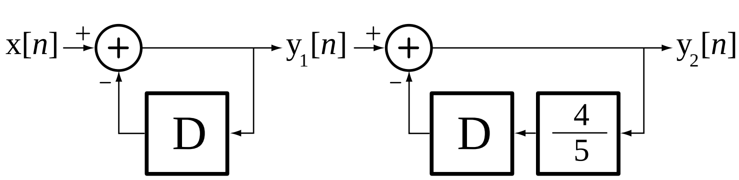
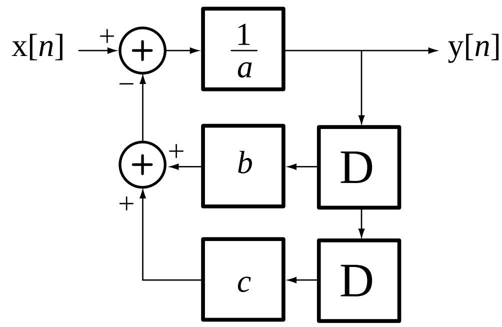


Figure E 46 Two cascaded subsystems

47. For the system of Exercise 43, let the excitation, $\mathrm{x}[n]$, be a unit-amplitude complex sinusoid of DT cyclic frequency, F. Plot the amplitude of the response complex sinusoid versus F over the range, $-1 < F < 1$.

48. In the second-order DT system below what is the relationship between a, b and c that ensures that the system is stable?



$$\mathrm{y}[n] = \frac{\mathrm{x}[n] - \left(b\,\mathrm{y}[n-1] + c\,\mathrm{y}[n-2]\right)}{a}$$

Stability is determined by the eigenvalues of the homogeneous solution.

$$a\,\mathrm{y}[n] + b\,\mathrm{y}[n-1] + c\,\mathrm{y}[n-2] = 0$$

The eigenvalues are

$$\alpha_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For stability the magnitudes of all the eigenvalues must be less than one. Therefore

$$\left| -\frac{b}{2a} + \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} \right| < 1 \quad \text{and} \quad \left| -\frac{b}{2a} - \sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} \right| < 1$$

$$\left| -\frac{b}{2a} + \frac{1}{2a}\sqrt{b^2 - 4ac} \right| < 1 \quad \text{and} \quad \left| -\frac{b}{2a} - \frac{1}{2a}\sqrt{b^2 - 4ac} \right| < 1$$

$$\left| -b + \sqrt{b^2 - 4ac} \right| < |2a| \quad \text{and} \quad \left| -b - \sqrt{b^2 - 4ac} \right| < |2a|$$

If $b^2 - 4ac < 0$,

$$\left| -b + j\sqrt{4ac - b^2} \right| < |2a| \quad \text{and} \quad \left| -b - j\sqrt{4ac - b^2} \right| < |2a|$$

In either case

$$b^2 + 4ac - b^2 < 4a^2$$

or

$$ac < a^2$$

From the requirement, $b^2 - 4ac < 0$ we know that ac must be positive. Then we can divide both sides by the positive number, ac, yielding

$$\frac{a}{c} > 1 \quad .$$

If $b^2 - 4ac \geq 0$,

$$\left(-b + \sqrt{b^2 - 4ac}\right)^2 < 4a^2 \quad \text{and} \quad \left(-b - \sqrt{b^2 - 4ac}\right)^2 < 4a^2$$

$$b^2 - 2b\sqrt{b^2 - 4ac} + b^2 - 4ac < 4a^2 \quad \text{and} \quad b^2 + 2b\sqrt{b^2 - 4ac} + b^2 - 4ac < 4a^2$$

$$-2b\sqrt{b^2 - 4ac} < 4a^2 - 2b^2 + 4ac \quad \text{and} \quad 2b\sqrt{b^2 - 4ac} < 4a^2 - 2b^2 + 4ac$$

$$-b\sqrt{b^2 - 4ac} < 2a^2 - b^2 + 2ac \quad \text{and} \quad b\sqrt{b^2 - 4ac} < 2a^2 - b^2 + 2ac$$

Taken together, these two requirements lead to

$$2a^2 - b^2 + 2ac > \left|b\sqrt{b^2 - 4ac}\right| \geq 0$$

$$2a(a + c) \geq b^2$$

and

$$\left(2a^2 - b^2 + 2ac\right)^2 > b^2\left(b^2 - 4ac\right)$$

$$4a^4 + b^4 + 4a^2c^2 - 4a^2b^2 - 4ab^2c + 8a^3c > b^4 - 4ab^2c$$

$$4a^2\left(a^2 + c^2 - b^2 + 2ac\right) > 0$$

$$a^2 + c^2 - b^2 + 2ac > 0$$

$$a^2 - 2ac + c^2 > b^2 - 4ac$$

$$(a - c)^2 > b^2 - 4ac$$

49. Given the excitations, $\mathrm{x}[n]$, and the impulse responses, $\mathrm{h}[n]$, find closed-form expressions for and plot the system responses, $\mathrm{y}[n]$.

(a) $\mathrm{x}[n] = \mathrm{u}[n] \quad , \quad \mathrm{h}[n] = n\left(\frac{7}{8}\right)^n \mathrm{u}[n]$

(Hint: Differentiate $\sum_{n=0}^{N-1} r^n = \begin{cases} \frac{1 - r^N}{1 - r} & , \ r \neq 1 \\ N & , \ r = 1 \end{cases}$ with respect to r.)

$$y[n] = h[n] * x[n] = \sum_{m=-\infty}^{\infty} m\left(\frac{7}{8}\right)^m u[m]u[n-m] = \sum_{m=0}^{n} m\left(\frac{7}{8}\right)^m$$

Differentiating $\sum_{n=0}^{N-1} r^n = \begin{cases} \dfrac{1-r^N}{1-r} & , \ r \neq 1 \\ N & , \ r = 1 \end{cases}$ with respect to r,

$$\sum_{n=0}^{N-1} n r^{n-1} = \frac{(1-r)(-Nr^{N-1}) - (1-r^N)(-1)}{(1-r)^2} \quad , \ r \neq 1$$

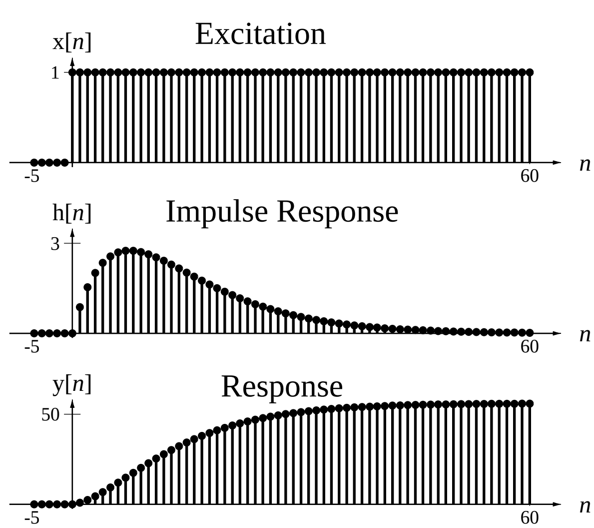
$$r\sum_{n=0}^{N-1} n r^{n-1} = r\frac{-Nr^{N-1} + Nr^N + 1 - r^N}{(1-r)^2} \quad , \ r \neq 1$$

$$\sum_{n=0}^{N-1} n r^{n} = r\frac{Nr^{N-1}(r-1) + 1 - r^N}{(1-r)^2} \quad , \ r \neq 1$$

$$y[n] = \frac{7}{8}\frac{(n+1)\left(\frac{7}{8}\right)^n\left(\frac{7}{8}-1\right) + 1 - \left(\frac{7}{8}\right)^{n+1}}{\left(1-\frac{7}{8}\right)^2} u[n]$$

$$y[n] = 56\left[(n+1)\left(\frac{7}{8}\right)^n\left(-\frac{1}{8}\right) + 1 - \left(\frac{7}{8}\right)^{n+1}\right] u[n]$$

$$y[n] = 56\left[1 - \left(\frac{7}{8}\right)^n\left(\frac{n}{8}+1\right)\right] u[n]$$



(b) $\mathrm{x}[n]=\mathrm{u}[n]$, $\mathrm{h}[n]=\frac{4}{7}\delta[n]-\left(-\frac{3}{4}\right)^n \mathrm{u}[n]$

50. A CT function is non-zero over a range of its argument from 0 to 4. It is convolved with a function which is non-zero over a range of its argument from -3 to -1. What is the non-zero range of the convolution of the two?

Imagine any two functions with finite non-zero width and convolve.

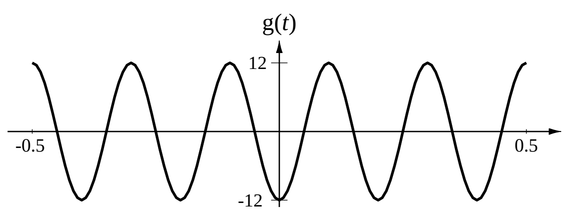
51. What function convolved with $-2\cos(t)$ would produce $6\sin(t)$?

Think of a sine as a shifted cosine. There are multiple correct answers to this exercise.

52. Sketch these functions.

(a)

$$\mathrm{g}(t)=3\cos(10\pi t)*4\delta\left(t+\frac{1}{10}\right)=12\cos\left(10\pi\left(t+\frac{1}{10}\right)\right)=12\cos(10\pi t+\pi)=-12\cos(10\pi t)$$



(b) $\mathrm{g}(t)=\mathrm{tri}(2t)*\mathrm{comb}(t)$

(c) $\mathrm{g}(t)=[\mathrm{tri}(2t)-\mathrm{rect}(t-1)]*\mathrm{comb}\left(\frac{t}{2}\right)$

(d) $\mathrm{g}(t)=\left[\mathrm{tri}\left(\frac{t}{4}\right)\mathrm{comb}(t)\right]*\mathrm{comb}\left(\frac{t}{8}\right)$

(e) $\mathrm{g}(t)=\mathrm{sinc}(4t)*\frac{1}{2}\mathrm{comb}\left(\frac{t}{2}\right)$

The result should look like a Dirichlet function. It is a Dirichlet function written in a different form.

(f) $\mathrm{g}(t)=e^{-2t}\,\mathrm{u}(t)*\frac{1}{4}\left[\mathrm{comb}\left(\frac{t}{4}\right)-\mathrm{comb}\left(\frac{t-2}{4}\right)\right]$

(g) This result looks like a full-wave rectified sinusoid.

(h) $\mathrm{g}(t)=\left[\mathrm{sinc}(2t)*\frac{1}{2}\mathrm{comb}\left(\frac{t}{2}\right)\right]\mathrm{rect}\left(\frac{t}{4}\right)$

53. Find the signal power of these signals.

(a) $\mathrm{x}(t) = \mathrm{rect}(t) * \mathrm{comb}\left(\frac{t}{4}\right)$

$$\mathrm{rect}(t) * \mathrm{comb}\left(\frac{t}{4}\right) = 4\sum_{n=-\infty}^{\infty} \mathrm{rect}(t-4n)$$

This is a periodic signal whose period, T, is 4. Between -T/2 and +T/2, there is one rectangle whose height is 4 and whose width is 1. Therefore, between -T/2 and +T/2, the square of the signal is

$$[4\,\mathrm{rect}(t)]^2 = 16\,\mathrm{rect}^2(t) \text{ and } P = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} 16\,\mathrm{rect}^2(t)\,dt = \frac{16}{4}\int_{-2}^{2} \mathrm{rect}^2(t)\,dt = 4$$

(b) $\mathrm{x}(t) = \mathrm{tri}(t) * \mathrm{comb}\left(\frac{t}{4}\right)$

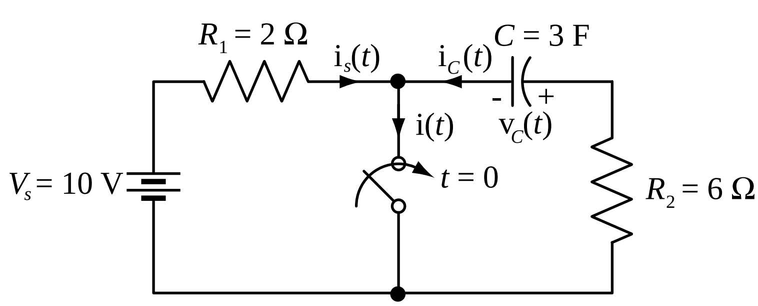
Remember, the square of a triangle function is not triangular.

54. A rectangular voltage pulse which begins at $t = 0$, is 2 seconds wide and has a height of 0.5 V drives an RC lowpass filter in which $R = 10\ \mathrm{k}\Omega$ and $C = 100\ \mu\mathrm{F}$.

(a) Sketch the voltage across the capacitor versus time.
(b) Change the pulse duration to 0.2 s and the pulse height to 5 V and repeat.
(c) Change the pulse duration to 2 ms and the pulse height to 500 V and repeat.
(d) Change the pulse duration to 2 µs and the pulse height to 500 kV and repeat.

The solutions in this problem approach the impulse response of the system.

55. Write the differential equation for the voltage, $\mathrm{v}_C(t)$, in the circuit below for time, $t > 0$, then find an expression for the current, $\mathrm{i}(t)$, for time, $t > 0$.



$$\mathrm{i}(t) = \mathrm{i}_s(t) + \mathrm{i}_C(t) \quad , \quad \mathrm{i}_s(t) = \frac{V_s}{R_1} \quad , \quad \mathrm{i}_C(t) = C\frac{d}{dt}(\mathrm{v}_C(t))$$

$$\mathrm{v}_C(t) + \mathrm{i}_C(t)R_2 = 0 \quad , \quad \mathrm{v}_C(t) + R_2C\frac{d}{dt}(\mathrm{v}_C(t)) = 0$$

56. The water tank in Figure E56 is filled by an inflow, $\mathrm{x}(t)$, and is emptied by an outflow, $\mathrm{y}(t)$. The outflow is controlled by a valve which offers resistance, *R*, to the flow of water out of the tank. The water depth in the tank is $\mathrm{d}(t)$ and the surface area of the water is *A*, independent of depth (cylindrical tank). The outflow is related to the water depth (head) by

$$\mathrm{y}(t) = \frac{\mathrm{d}(t)}{R} \ .$$

The tank is 1.5 m high with a 1m diameter and the valve resistance is $10\frac{\mathrm{s}}{\mathrm{m}^2}$.

(a) Write the differential equation for the water depth in terms of the tank dimensions and valve resistance.

(b) If the inflow is $0.05\frac{\mathrm{m}^3}{s}$, at what water depth will the inflow and outflow rates be equal, making the water depth constant?

(c) Find an expression for the depth of water versus time after $1\,\mathrm{m}^3$ of water is dumped into an empty tank.

(d) If the tank is initially empty at time, $t = 0$, and the inflow is a constant $0.2\frac{\mathrm{m}^3}{s}$ after time, $t = 0$, at what time will the tank start to overflow?

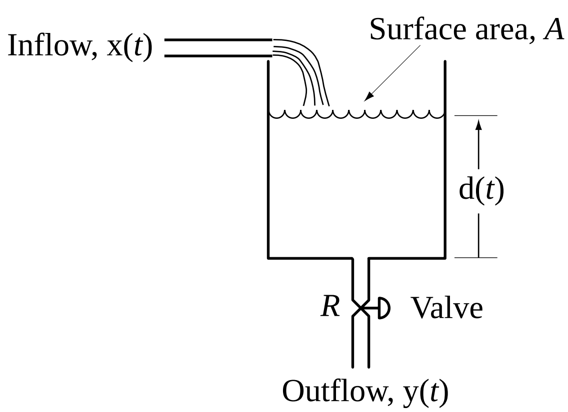


Figure E56 Water tank with inflow and outflow

(a)

$$\mathrm{y}(t) = \frac{\mathrm{d}(t)}{R}$$

The rate of change of water volume is the difference between the inflow rate and the outflow rate. (Be sure not to confuse d and *d* in this equation.)

$$\frac{d}{dt}\left(\underbrace{A\,\mathrm{d}(t)}_{\text{volume}}\right) = \mathrm{x}(t) - \mathrm{y}(t)$$

$$A\mathrm{d}'(t) = \mathrm{x}(t) - \frac{\mathrm{d}(t)}{R}$$

$$A\mathrm{d}'(t) + \frac{\mathrm{d}(t)}{R} = \mathrm{x}(t)$$

(b) For the water height to be constant, $\mathrm{d}'(t) = 0$.

(c) Dumping $1\ \mathrm{m}^3$ of water into an empty tank is exciting this system with a unit impulse of water inflow. Find the impulse response. It should come out to be

$$\mathrm{h}(t) = 1.273 e^{-\frac{t}{AR}} \mathrm{u}(t) \quad .$$

(d) The response to a step of flow is the convolution of the impulse response with the step excitation.

57. The suspension of a car can be modeled by the mass-spring-dashpot system of Figure E57 Let the mass, *m*, of the car be 1500 kg, let the spring constant, K_s, be $75000\ \frac{\mathrm{N}}{\mathrm{m}}$ and let the shock absorber (dashpot) viscosity coefficient, K_d, be $20000\ \frac{\mathrm{N}\cdot\mathrm{s}}{\mathrm{m}}$.

At a certain length, d_0, of the spring, it is unstretched and uncompressed and exerts no force. Let that length be 0.6 m.

(a) What is the distance, $\mathrm{y}(t) - \mathrm{x}(t)$, when the car is at rest?

(b) Define a new variable $\mathrm{z}(t) = \mathrm{y}(t) - \mathrm{x}(t) - \text{constant}$ such that, when the system is at rest, $\mathrm{z}(t) = 0$ and write a describing equation in z and x which describes an LTI system. Then find the impulse response.

(c) The effect of the car striking a curb can be modeled by letting the road surface height change discontinuously by the height of the curb, h_c. Let $h_c = 0.15$ m. Graph $\mathrm{z}(t)$ versus time after the car strikes a curb.

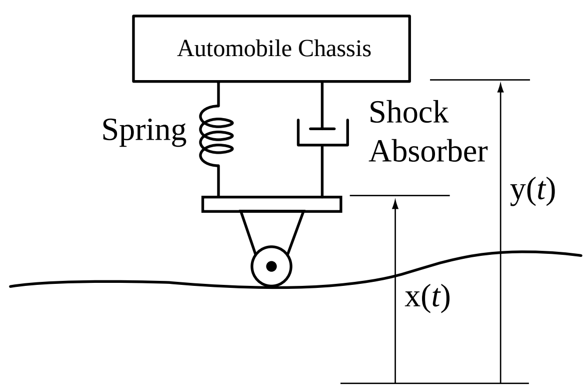


Figure E57 Car suspension model

Using the basic principle, $F = ma$, we can write

$$K_s\left[\mathrm{y}(t)-\mathrm{x}(t)-d_0\right]+K_d\frac{d}{dt}\left[\mathrm{y}(t)-\mathrm{x}(t)\right]+mg=-m\,\mathrm{y}''(t)$$

or

$$m\,\mathrm{y}''(t)+K_d\,\mathrm{y}'(t)+K_s\,\mathrm{y}(t)=K_d\,\mathrm{x}'(t)+K_s\,\mathrm{x}(t)+K_sd_0-mg\ .$$

(a) At rest all the derivatives are zero and

$$K_s\left(\mathrm{y}(t)-\mathrm{x}(t)-d_0\right)+mg=0\ .$$

Solving,

$$\mathrm{y}(t)-\mathrm{x}(t)=\frac{K_sd_0-mg}{K_s}=\frac{75000\times 0.6-1500\times 9.8}{75000}=0.404\ \mathrm{m}$$

(b) The describing equation is

$$m\,\mathrm{y}''(t)+K_d\,\mathrm{y}'(t)+K_s\,\mathrm{y}(t)=K_d\,\mathrm{x}'(t)+K_s\,\mathrm{x}(t)+K_sd_0-mg\ .$$

which can be rewritten as

$$m\,\mathrm{y}''(t)+K_d\left[\mathrm{y}'(t)-\mathrm{x}'(t)\right]+K_s\left[\mathrm{y}(t)-\mathrm{x}(t)\right]-K_sd_0+mg=0$$

or

$$m\,\mathrm{y}''(t)+K_d\left[\mathrm{y}'(t)-\mathrm{x}'(t)\right]+K_s\left[\mathrm{y}(t)-\mathrm{x}(t)-d_0+\frac{mg}{K_s}\right]=0$$

Let $\mathrm{z}(t)=\mathrm{y}(t)-\mathrm{x}(t)-d_0+\dfrac{mg}{K_s}$. Then $\mathrm{y}''(t)=\mathrm{z}''(t)+\mathrm{x}''(t)$ and

$$m\left[\mathrm{z}''(t)+\mathrm{x}''(t)\right]+K_d\,\mathrm{z}'(t)+K_s\,\mathrm{z}(t)=0$$

or

$$m\,\mathrm{z}''(t)+K_d\,\mathrm{z}'(t)+K_s\,\mathrm{z}(t)=-m\,\mathrm{x}''(t)$$

This equation is in a form which describes an LTI system. We can find its impulse response. After time, $t=0$, the impulse response is the homogenous solution. The eigenvalues are

$$\lambda_{1,2}=\frac{-K_d\pm\sqrt{K_d^2-4mK_s}}{2m}=-\frac{K_d}{2m}\pm\sqrt{\frac{K_d^2}{4m^2}-\frac{K_s}{m}}=-6.667\pm j2.357\ .$$

The homogeneous solution is

$$\mathrm{h}(t)=K_{h1}e^{\lambda_1 t}+K_{h2}e^{\lambda_2 t}=K_{h1}e^{(-6.667+j2.357)t}+K_{h2}e^{(-6.667-j2.357)t}\ .$$

Since the system is underdamped another (equivalent) form of homogeneous solution will be more convenient,

$$\mathrm{h}(t)=e^{-6.667t}\left[K_{h1}\cos(2.357t)+K_{h2}\sin(2.357t)\right]\ .$$

The impulse response can have a discontinuity at $t = 0$ and an impulse but no higher-order singularity there. Therefore the general form of the impulse response is

$$\mathrm{h}(t) = K\delta(t) + e^{-6.667t}\left[K_{h1}\cos(2.357t) + K_{h2}\sin(2.357t)\right]\mathrm{u}(t)$$

Integrating both sides of the describing equation between 0^- and 0^+,

$$m\left(\mathrm{h}'(0^+) - \mathrm{h}'(0^-)\right) + K_d\left(\mathrm{h}(0^+) - \mathrm{h}(0^-)\right) + K_s\int_{0^-}^{0^+}\mathrm{h}(t)dt = 0 \ .$$

(The integral of the doublet, which is the derivative of the impulse excitation, is zero.) Since the impulse response and all its derivatives are zero before time, $t = 0$, it follows then that

$$m\,\mathrm{h}'(0^+) + K_d\,\mathrm{h}(0^+) + K_s\int_{0^-}^{0^+}\mathrm{h}(t)dt = 0$$

and

$$m(-6.667K_{h1} + 2.357K_{h2}) + K_d K_{h1} + K_s K = 0 \ .$$

Integrating the describing equation a second time between 0^- and 0^+,

$$m\,\mathrm{h}(0^+) + K_d\int_{0^-}^{0^+}\mathrm{h}(t)dt = 0$$

or

$$mK_{h1} + K_d K = 0 \ .$$

Integrating the describing equation a third time,

$$m\int_{0^-}^{0^+}\mathrm{h}(t)dt = -m$$

or

$$mK = -m \Rightarrow K = -1 \ .$$

Solving for the other two constants, $K_{h1} = \frac{K_d}{m}$ and

$$m\left(-6.667\frac{K_d}{m} + 2.357K_{h2}\right) + K_d\frac{K_d}{m} - K_s = 0$$

or

$$K_{h2} = \frac{\frac{K_s}{m} - \frac{K_d^2}{m^2} + 6.667\frac{K_d}{m}}{2.357}$$

$\frac{K_d}{m}$

Therefore

$$\mathrm{h}(t) = -\delta(t) + e^{-6.667t}\left[13.333\cos(2.357t) - 16.497\sin(2.357t)\right]\mathrm{u}(t)$$

(c) The response to a step of size 0.15 is then the convolution,

$$\mathrm{z}(t) = 0.15\,\mathrm{u}(t) * \mathrm{h}(t)$$

or

$$\mathrm{z}(t) = 0.15\int_{-\infty}^{\infty}\left\{-\delta(\tau) + e^{-6.667\tau}\left[13.333\cos(2.357\tau) - 16.497\sin(2.357\tau)\right]\mathrm{u}(\tau)\right\}\mathrm{u}(t-\tau)\,d\tau$$

$$\mathrm{z}(t) = 0.15\int_{0^-}^{\infty}\left\{-\delta(\tau) + e^{-6.667\tau}\left[13.333\cos(2.357\tau) - 16.497\sin(2.357\tau)\right]\right\}\mathrm{u}(t-\tau)\,d\tau$$

For $t < 0$, $\mathrm{z}(t) = 0$.

For $t > 0$,

using

$$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx) - b\cos(bx)\right]$$

$$\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos(bx) + b\sin(bx)\right]$$

we get

$$\mathrm{z}(t) = -0.15\,\mathrm{u}(t) + 0.15\left[\begin{matrix}13.333\dfrac{e^{-6.667\tau}}{50}\left[-6.667\cos(2.357\tau) + 2.357\sin(2.357\tau)\right] \\ -16.497\dfrac{e^{-6.667\tau}}{50}\left[-6.667\sin(2.357\tau) - 2.357\cos(2.357\tau)\right]\end{matrix}\right]_{0^-}^{t}$$

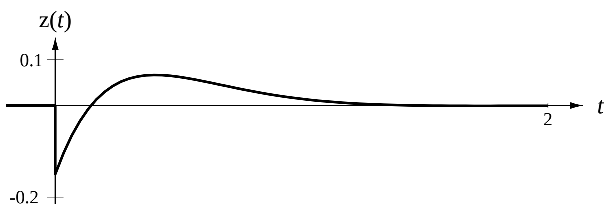
or

$$\mathrm{z}(t) = -0.15\,\mathrm{u}(t) + 0.15\left[\begin{matrix}13.333\dfrac{e^{-6.667t}}{50}\left[-6.667\cos(2.357t) + 2.357\sin(2.357t)\right] \\ -16.497\dfrac{e^{-6.667t}}{50}\left[-6.667\sin(2.357t) - 2.357\cos(2.357t)\right] \\ -13.333\dfrac{-6.667}{50} + 16.497\dfrac{-2.357}{50}\end{matrix}\right]$$

$$\mathrm{z}(t) = -0.15\,\mathrm{u}(t) + 0.15\left\{e^{-3.333t}\left[2.812\sin(2.357t) - \cos(2.357t)\right] + 1\right\}\mathrm{u}(t)$$

or

$$\mathrm{z}(t) = 0.15e^{-3.333t}\left[2.812\sin(2.357t) - \cos(2.357t)\right]\mathrm{u}(t)$$



58. As derived in the text, a simple pendulum is approximately described for small angles, θ, by the differential equation,

$$mL\theta''(t) + mg\theta(t) \cong \mathrm{x}(t)$$

where m is the mass of the pendulum, L is the length of the massless rigid rod supporting the mass and θ is the angular deviation of the pendulum from vertical.

(a) Find the general form of the impulse response of this system.

After time, $t = 0$ he impulse is an undamped sine function whose (radian) frequency is $\sqrt{\frac{g}{L}}$.

59. Pharmacokinetics is the study of how drugs are absorbed into, distributed through, metabolized by and excreted from the human body. Some drug processes can be approximately modeled by a "one compartment" model of the body in which V is the volume of the compartment, $\mathrm{C}(t)$ is the drug concentration in that compartment, k_e is a rate constant for excretion of the drug from the compartment and k_0 is the infusion rate at which the drug enters the compartment.

(a) Write a differential equation in which the infusion rate is the excitation and the drug concentration is the response.

(b) Let the parameter values be $k_e = 0.4\,\mathrm{hr}^{-1}$, $V = 20\ \mathrm{l}$ and $k_0 = 200\frac{\mathrm{mg}}{\mathrm{hr}}$ (where "l" is the symbol for "liter"). If the initial drug concentration is $\mathrm{C}(0) = 10\frac{\mathrm{mg}}{\mathrm{l}}$, plot the drug concentration as a function of time (in hours) for the first 10 hours of infusion. Find the solution as the sum of the zero-excitation response and the zero-state response.

(a) The differential equation equates the rate of increase of drug in the compartment to the difference between the rate of infusion and the rate of excretion.

$$V\frac{d}{dt}(\mathrm{C}(t)) = k_0 - Vk_e\,\mathrm{C}(t)$$

60. At the beginning of the year 2000, the country, Freedonia, had a population, p, of 100 million people. The birth rate is 4% per annum and the death rate is 2% per annum, compounded daily. That is, the births and deaths occur every day at a uniform fraction of the current population and the next day the number of births and deaths changes because the population changed the previous day. For example, every day the number of people who die is the fraction, $\frac{0.02}{365}$, of the total population at the end of the previous day (neglect leap-year effects). Every day 275 immigrants enter Freedonia.

(a) Write a difference equation for the population at the beginning of the nth day after January 1, 2000 with the immigration rate as the excitation of the system.

(b) By finding the zero-exctiation and zero-state responses of the system determine the population of Freedonia be at the beginning of the year 2050.

(b) The beginning of the year 2050 is the 18250th day.

61. A car rolling on a hill can be modeled as shown in Figure E61. The excitation is the force, $f(t)$, for which a positive value represents accelerating the car forward with the motor and a negative value represents slowing the car by braking action. As it rolls, the car experiences drag due to various frictional phenomena which can be approximately modeled by a coefficient, k_f, which multiplies the car's velocity to produce a force which tends to slow the car when it moves in either direction. The mass of the car is *m* and gravity acts on it at all times tending to make it roll down the hill in the absence of other forces. Let the mass, *m*, of the car be 1000 kg, let the friction coefficient, k_f, be $5\frac{\text{N}\cdot\text{s}}{\text{m}}$ and let the angle, θ, be $\frac{\pi}{12}$.

(a) Write a differential equation for this system with the force, $f(t)$, as the excitation and the position of the car, $y(t)$, as the response.

(b) If the nose of the car is initially at position, $y(0)=0$, with an initial velocity, $[y'(t)]_{t=0}=10\,\frac{\text{m}}{\text{s}}$, and no applied acceleration or braking force, graph the velocity of the car, $y'(t)$, for positive time.

(c) If a constant force, $f(t)$, of 200 N is applied to the car what is its terminal velocity ?

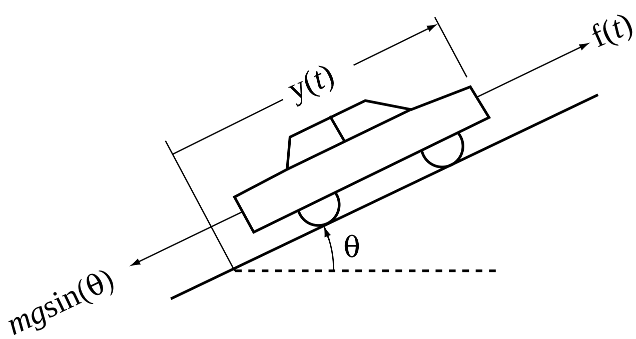


Figure E61 Car on an inclined plane

(a) Summing forces,

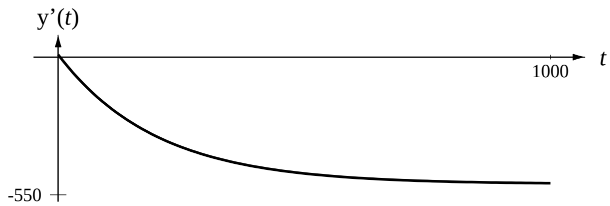
$$f(t) - mg\sin(\theta) - k_f\, y'(t) = m\, y''(t)$$

(b) The zero-excitation response can be found by setting the force, $f(t)$, to zero.

The homogeneous solution is $y_h(t) = K_{h1} + K_{h2}e^{-\frac{k_f}{m}t}$. The particular solution must be in the form of a linear function of *t*, to satisfy the differential equation.

$$\mathrm{y}(t) = 1.0346 \times 10^5 \left(1 - e^{-\frac{t}{200}}\right) - 507.28t$$

$$\mathrm{y}'(t) = \frac{1.0346 \times 10^5}{200}\left(e^{-\frac{t}{200}}\right) - 507.28 = 517.28 e^{-\frac{t}{200}} - 507.28 = 517.28\left(e^{-\frac{t}{200}} - 1\right) + 10$$



(c) The differential equation is

$$m\,\mathrm{y}''(t) + k_f\,\mathrm{y}'(t) + mg\sin(\theta) = \mathrm{f}(t)$$

We can re-write the equation as

$$m\,\mathrm{y}''(t) + k_f\,\mathrm{y}'(t) = \mathrm{f}(t) - mg\sin(\theta)$$

treating the force due to gravity as part of the excitation. Then the impulse response is the solution of

$$m\,\mathrm{h}''(t) + k_f\,\mathrm{h}'(t) = \delta(t)$$

which is of the form,

$$\mathrm{h}(t) = \left(K_{h1} + K_{h2} e^{-\frac{k_f}{m}t}\right)\mathrm{u}(t) \quad .$$

The impulse response is

$$\mathrm{h}(t) = \frac{1 - e^{-\frac{k_f}{m}t}}{k_f}\mathrm{u}(t) \quad .$$

Now, if we say that the force, $\mathrm{f}(t)$, is a step of size, 200 N, the excitation of the system is

$$\mathrm{x}(t) = 200\,\mathrm{u}(t) - mg\sin(\theta) \quad .$$

But this is going to cause a problem. The problem is that the term, $-mg\sin(\theta)$, is a constant, therefore presumed to have acted on the system for all time *before* time, $t = 0$. The implication from that is that the position at time, $t = 0$, is at infinity. Since we are only interested in the final velocity, not position, we can assume that the car was held in place at $\mathrm{y}(t) = 0$ until the force was applied and gravity was allowed to act on the car. That makes the excitation,

$$\mathrm{x}(t) = [200 - mg\sin(\theta)]\mathrm{u}(t)$$

and the response is

$$\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t) = [200 - mg\sin(\theta)]\mathrm{u}(t) * \frac{1 - e^{-\frac{k_f}{m}t}}{k_f}\mathrm{u}(t)$$

or

$$\mathrm{y}(t) = \frac{200 - mg\sin(\theta)}{k_f}\int_0^t \left(1 - e^{-\frac{k_f}{m}\tau}\right)d\tau$$

or

$$\mathrm{y}(t) = \frac{200 - mg\sin(\theta)}{k_f}\left[\tau + \frac{m}{k_f}e^{-\frac{k_f}{m}\tau}\right]_0^t = \frac{200 - mg\sin(\theta)}{k_f}\left(t + \frac{m}{k_f}e^{-\frac{k_f}{m}t} - \frac{m}{k_f}\right)$$

The terminal velocity is the derivative of position as time approaches infinity which, in this case is

$$\mathrm{y}'(+\infty) = \frac{200 - mg\sin(\theta)}{k_f} = \frac{200 - 2536.43}{5} = -467.3\ \frac{\mathrm{m}}{\mathrm{s}} \quad .$$

Obviously a force of 200 N is insufficient to move the car forward and its terminal velocity is negative indicating it is rolling backward down the hill.

62. A block of aluminum is heated to a temperature of 100 °C. It is then dropped into a flowing stream of water which is held at a constant temperature of 10°C. After 10 seconds the temperature of the ball is 60°C. (Aluminum is such a good heat conductor that its temperature is essentially uniform throughout its volume during the cooling process.) The rate of cooling is proportional to the temperature difference between the ball and the water.

(a) Write a differential equation for this system with the temperature of the water as the excitation and the temperature of the block as the response.

(b) Compute the time constant of the system.

(c) Find the impulse response of the system and, from it, the step response.

(d) If the same block is cooled to 0 °C and dropped into a flowing stream of water at 80 °C, at time, $t = 0$, at what time will the temperature of the block reach 75°C?

(a) The controlling differential equation is

$$\frac{d}{dt}\mathrm{T}_a(t) = K(\mathrm{T}_w - \mathrm{T}_a(t))$$

or

$$\frac{1}{K}\frac{d}{dt}\mathrm{T}_a(t) + \mathrm{T}_a(t) = \mathrm{T}_w$$

where T_a is the temperature of the aluminum ball and T_w is the temperature of the water.

(b) We can find the constant, K, by using the temperature after 10 seconds,

$$\mathrm{h}(t) = Ke^{-Kt}\mathrm{u}(t) = 0.0588e^{-0.0588t}\mathrm{u}(t).$$

(c) The unit step response is the integral of the impulse response,

$$h_{-1}(t) = \left(1 - e^{-0.0588t}\right)u(t) \ .$$

63. A well-stirred vat has been fed for a long time by two streams of liquid, fresh water at 0.2 cubic meters per second and concentrated blue dye at 0.1 cubic meters per second. The vat contains 10 cubic meters of this mixture and the mixture is being drawn from the vat at a rate of 0.3 cubic meters per second to maintain a constant volume. The blue dye is suddenly changed to red dye at the same flow rate. At what time after the switch does the mixture drawn from the vat contain a ratio of red to blue dye of 99:1?

Let the concentration of red dye be denoted by $C_r(t)$ and the concentration of blue dye be denoted by $C_b(t)$. The concentration of water is constant throughout at $\frac{2}{3}$. The rates of change of the dye concentrations are governed by

$$\frac{d}{dt}\left(VC_b(t)\right) = -C_b(t)\, f_{draw}$$

$$\frac{d}{dt}\left(VC_r(t)\right) = f_r - C_r(t)\, f_{draw}$$

where V is the constant volume, 10 cubic meters, f_{draw} is the flow rate of the draw from the vat and f_r is the flow rate of red dye into the tank. Solving the two differential equations,

$$C_b(t) = \frac{1}{3} e^{-\frac{f_{draw}}{V}t}$$

and

$$C_r(t) = \frac{1}{3}\left(1 - e^{-\frac{f_{draw}}{V}t}\right) .$$

Then the ratio of red to blue dye concentration is

$$\frac{C_r(t)}{C_b(t)} = \frac{\frac{1}{3}\left(1 - e^{-\frac{f_{draw}}{V}t}\right)}{\frac{1}{3} e^{-\frac{f_{draw}}{V}t}} = \frac{1 - e^{-\frac{f_{draw}}{V}t}}{e^{-\frac{f_{draw}}{V}t}} = e^{\frac{f_{draw}}{V}t} - 1 \ .$$

Setting that ratio to 99 and solving for t_{99},

$$99 = e^{\frac{0.3}{10}t_{99}} - 1 \Rightarrow t_{99} = 153.5 \text{ seconds}$$

64. Some large auditoriums have a noticeable echo or reverberation. While a little reverberation is desirable, too much is undesirable. Let the response of an auditorium to an acoustic impulse of sound be

$$\mathrm{h}(t)=\sum_{n=0}^{\infty}e^{-n}\delta\left(t-\frac{n}{5}\right) .$$

We would like to design a signal processing system that will remove the effects of reverberation. In later chapters on transform theory we will be able to show that the compensating system that can remove the reverberations has an impulse response of the form,

$$\mathrm{h}_c(t)=\sum_{n=0}^{\infty}\mathrm{g}[n]\delta\left(t-\frac{n}{5}\right) .$$

Find the function, $\mathrm{g}[n]$.

Removal of the reverberation is equivalent to making the overall impulse response, $\mathrm{h}_0(t)$, an impulse. That means that

$$\mathrm{h}_o(t)=\mathrm{h}(t)*\mathrm{h}_c(t)=\left[\sum_{n=0}^{\infty}e^{-n}\delta\left(t-\frac{n}{5}\right)\right]*\left[\sum_{m=0}^{\infty}\mathrm{g}[m]\delta\left(t-\frac{m}{5}\right)\right]=K\delta(t)$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}e^{-n}\delta\left(t-\frac{n}{5}\right)*\mathrm{g}[m]\delta\left(t-\frac{m}{5}\right)=K\delta(t)$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}e^{-n}\,\mathrm{g}[m]\delta\left(t-\frac{n+m}{5}\right)=K\delta(t)$$

$$\sum_{m=0}^{\infty}\mathrm{g}[m]\sum_{n=0}^{\infty}e^{-n}\delta\left(t-\frac{n+m}{5}\right)=K\delta(t)$$

$$\left\{\begin{array}{l}\mathrm{g}[0]\left[\delta(t)+e^{-1}\delta\left(t-\frac{1}{5}\right)+e^{-2}\delta\left(t-\frac{2}{5}\right)+\cdots\right]\\+\mathrm{g}[1]\left[\delta\left(t-\frac{1}{5}\right)+e^{-1}\delta\left(t-\frac{2}{5}\right)+e^{-2}\delta\left(t-\frac{3}{5}\right)+\cdots\right]\\+\mathrm{g}[2]\left[\delta\left(t-\frac{2}{5}\right)+e^{-1}\delta\left(t-\frac{3}{5}\right)+e^{-2}\delta\left(t-\frac{4}{5}\right)+\cdots\right]\\\vdots\end{array}\right\}=K\delta(t)$$

$$\begin{aligned}&\mathrm{g}[0]=K\\&\mathrm{g}[1]+e^{-1}\mathrm{g}[0]=0\Rightarrow\mathrm{g}[1]=-Ke^{-1}\\&\mathrm{g}[2]+e^{-1}\mathrm{g}[1]+e^{-2}\mathrm{g}[0]=0\Rightarrow\mathrm{g}[2]=Ke^{-2}-Ke^{-2}=0\\&\mathrm{g}[3]+e^{-1}\mathrm{g}[2]+e^{-2}\mathrm{g}[1]+e^{-3}\mathrm{g}[0]=0\Rightarrow\mathrm{g}[3]=Ke^{-3}-Ke^{-3}=0\\&\vdots\end{aligned}$$

So the compensating impulse response is

$$\mathrm{h}_c(t) = K\delta(t) - Ke^{-1}\delta\left(t - \frac{1}{5}\right)$$

and the function, g, is

$$\mathrm{g}[n] = K\delta[n] - Ke^{-1}\delta[n-1] \ .$$

65. Show that the area property and the scaling property of the convolution integral are in agreement by finding the area of $\mathrm{x}(at) * \mathrm{h}(at)$ and comparing it with the area of $\mathrm{x}(t) * \mathrm{h}(t)$.

66. The convolution of a function, $\mathrm{g}(t)$, with a doublet can be written as

$$\mathrm{g}(t) * \mathrm{u}_1(t) = \int_{-\infty}^{\infty} \mathrm{g}(\tau)\mathrm{u}_1(t-\tau)d\tau \ .$$

Integrate by parts to show that $\mathrm{g}(t) * \mathrm{u}_1(t) = \mathrm{g}'(t)$.

67. Derive the "sampling" property for a unit triplet. That is, find an expression for the integral,

$$\int_{-\infty}^{\infty} \mathrm{g}(t)\mathrm{u}_2(t)dt$$

which is analogous to the sampling property of the unit doublet, $-\mathrm{g}'(t) = \int_{-\infty}^{\infty} \mathrm{g}(t)\mathrm{u}_1(t)dt$.

In $-\mathrm{g}'(t) = \int_{-\infty}^{\infty} \mathrm{g}(t)\mathrm{u}_1(t)dt$, let $u = \mathrm{g}(t)$ and let $dv = \mathrm{u}_2(t)dt$. Then $du = \mathrm{g}'(t)dt$ and $v = \mathrm{u}_1(t)$ and

$$\int_{-\infty}^{\infty} \mathrm{g}(t)\mathrm{u}_2(t)dt = \underbrace{\mathrm{g}(t)\mathrm{u}_1(t)\Big|_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} \mathrm{u}_1(t)\mathrm{g}'(t)dt = -\int_{-\infty}^{\infty} \mathrm{u}_1(t)\mathrm{g}'(t)dt$$

Then, applying $-\mathrm{g}'(t) = \int_{-\infty}^{\infty} \mathrm{g}(t)\mathrm{u}_1(t)dt$, we get

$$\int_{-\infty}^{\infty} \mathrm{g}(t)\mathrm{u}_2(t)dt = \mathrm{g}''(t) \ .$$

68. Sketch block diagrams of the systems described by these equations. For the differential equation use only integrators in the block diagrams.

(a) $\mathrm{y}''(t) + 3\,\mathrm{y}'(t) + 2\,\mathrm{y}(t) = \mathrm{x}(t)$

(b) $6\,\mathrm{y}[n] + 4\,\mathrm{y}[n-1] - 2\,\mathrm{y}[n-2] + \mathrm{y}[n-3] = \mathrm{x}[n]$