Chapter 4 - The Fourier Series

Selected Solutions

(In this solution manual, the symbol, ⊗, is used for periodic convolution because the preferred symbol which appears in the text is not in the font selection of the word processor used to create this manual.)

1. Using MATLAB plot each sum of complex sinusoids over the time period indicated.

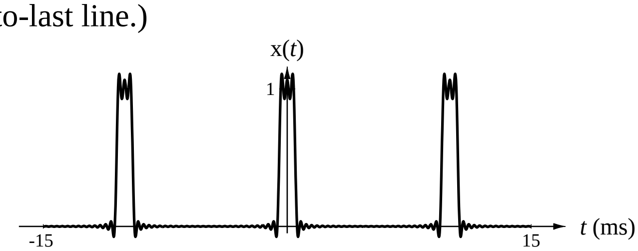
(a) $$\mathrm{x}(t) = \frac{1}{10}\sum_{k=-30}^{30} \mathrm{sinc}\left(\frac{k}{10}\right)e^{j200\pi kt} \quad , \quad -15\mathrm{ms} < t < 15\mathrm{ms}$$

The MATLAB program can exactly follow the mathematical process in the equation.

For example:

```
f0 = 100 ; T0 = 1/f0 ; fmax = 30*f0 ;
Tmin = 1/fmax ; dt = Tmin/16 ;
t = -0.015:dt:0.015 ; x = t*0 ;
for k = -30:30,
     x = x + sinc(k/10)*exp(j*200*pi*k*t) ;
end
x = real(x/10) ;
plot(t,x,'k') ;
```

(Although the vector, x, should be real, it may have some very small, but non-zero imaginary parts because of round-off in the calculations. That is why the ‘real’ command is used in the next-to-last line.)



(b) $$\mathrm{x}(t) = \frac{j}{4}\sum_{k=-9}^{9}\left[\mathrm{sinc}\left(\frac{k+2}{2}\right) - \mathrm{sinc}\left(\frac{k-2}{2}\right)\right]e^{j10\pi kt} \quad , \quad -200\text{ms} < t < 200\text{ms}$$

2. Show by direct analytical integration that the integral of the function,

$$\mathrm{g}(t) = A\sin(2\pi t)B\sin(4\pi t)$$

is zero over the interval, $-\frac{1}{2} < t < \frac{1}{2}$.

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{g}(t)dt = AB\int_{-\frac{1}{2}}^{\frac{1}{2}} \sin(2\pi t)\sin(4\pi t)dt$$

Using $\sin(x)\sin(y) = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{g}(t)dt = \frac{AB}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} [\cos(2\pi t - 4\pi t) - \cos(2\pi t + 4\pi t)]dt$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{g}(t)dt = \frac{AB}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} [\cos(-2\pi t) - \cos(6\pi t)]dt = \frac{AB}{2}\left[\frac{\sin(-2\pi t)}{-2\pi} - \frac{\sin(6\pi t)}{6\pi}\right]_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{g}(t)dt = \frac{AB}{2}\left[\frac{\sin(-\pi)}{-2\pi} - \frac{\sin(3\pi)}{6\pi} - \left(\frac{\sin(\pi)}{-2\pi} - \frac{\sin(-3\pi)}{6\pi}\right)\right] = 0$$

3. Convert the function, $\mathrm{g}(t) = (1+j)e^{j4\pi t} + (1-j)e^{-j4\pi t}$, to an equivalent form in which "j" does not appear.

   Two very important trigonometric identities to remember are

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2} \quad , \quad \sin(x) = \frac{e^{jx} - e^{-jx}}{j2}$$

4. Using MATLAB plot these products over the time range indicated and observe in each case that the net area under the product is zero.

(a) $\mathrm{x}(t) = -3\sin(16\pi t) \times 2\cos(24\pi t) \quad , \quad 0 < t < \frac{1}{4}$

(b) $\mathrm{x}(t) = -3\sin(16\pi t) \times 2\cos(24\pi t) \quad , \quad 0 < t < 1$

(c) $\mathrm{x}(t) = -3\sin(16\pi t) \times 2\cos(24\pi t) \quad , \quad -\frac{1}{16} < t < \frac{3}{16}$

(d) $\mathrm{x}(t) = \mathrm{x}_1(t)\,\mathrm{x}_2(t)$

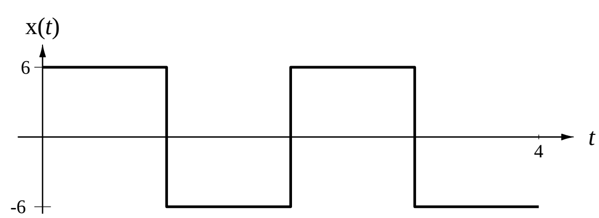
where

$\mathrm{x}_1(t)$ is an even, 50%-duty-cycle square wave with a fundamental period of

4 seconds and an average value of zero

and

$\mathrm{x}_2(t)$ is an odd, 50%-duty-cycle square wave with a fundamental period of 4 seconds and an average value of zero



(e) $\mathrm{x}(t) = \mathrm{x}_1(t)\,\mathrm{x}_2(t)$

where $\mathrm{x}_1(t) = \mathrm{rect}(2t) * \mathrm{comb}(t)$ and $\mathrm{x}_2(t) = \left[\mathrm{rect}\left(4\left(t - \frac{1}{8}\right)\right) * \frac{1}{2}\mathrm{comb}\left(\frac{t}{2}\right)\right] - \frac{1}{8}$

5. A sine function can be written as

$$\sin(2\pi f_0 t) = \frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{j2} \ .$$

This is a very simple complex CTFS in which the harmonic function is only non-zero at two harmonic numbers, +1 and –1. Verify that we can write the harmonic function directly as

$$\mathrm{X}[k] = \frac{j}{2}\left(\delta[k+1] - \delta[k-1]\right) .$$

Write the equivalent expressions for $\sin\left(2\pi(-f_0)t\right)$ and show that the harmonic function is the complex conjugate of the previous one for $\sin\left(2\pi f_0 t\right)$.

If the harmonic function is correct this equality should hold:

$$\sin\left(2\pi f_0 t\right) = \frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{j2} = \sum_{k=-\infty}^{\infty} \mathrm{X}[k] e^{j2\pi(kf_0)t} = \sum_{k=-\infty}^{\infty} \frac{j}{2}\left(\delta[k+1] - \delta[k-1]\right) e^{j2\pi(kf_0)t}$$

Finish simplifying and see whether it does.

6. For each signal, find a complex CTFS which is valid for all time, plot the magnitude and phase of the harmonic function versus harmonic number, k, then convert the answers to the trigonometric form of the harmonic function.

(a) $\mathrm{x}(t) = 4\,\mathrm{rect}(4t) * \mathrm{comb}(t)$

This form is in Appendix E. But let's do one directly from the definition.

$$\mathrm{X}[k] = \frac{1}{T_0}\int_{T_0} \mathrm{x}(t) e^{-j2\pi(kf_0)t} dt = 4\int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{rect}(4t) e^{-j2\pi kt} dt = 4\int_{-\frac{1}{8}}^{\frac{1}{8}} e^{-j2\pi kt} dt$$

Finish and check your answer against Appendix E.

(b) $\mathrm{x}(t) = 4\,\mathrm{rect}(4t) * \frac{1}{4}\mathrm{comb}\left(\frac{t}{4}\right)$

(c) A periodic signal which is described over one fundamental period by

$$\mathrm{x}(t) = \begin{cases} \mathrm{sgn}(t) & , \ |t| < 1 \\ 0 & , \ 1 < |t| < 2 \end{cases}$$

7. Using the CTFS table of transforms and the CTFS properties, find the CTFS harmonic function of each of these periodic signals using the time interval, T_F, indicated.

(a) $\mathrm{x}(t) = 10\sin(20\pi t)$, $T_F = \dfrac{1}{10}$

(b) $\mathrm{x}(t) = 2\cos(100\pi(t - 0.005))$, $T_F = \dfrac{1}{50}$

In this case, the representation time is the same as the fundamental period. From Appendix E,

$$\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{FS}} \frac{1}{2}(\delta[k-1] + \delta[k+1]) \ , \ T_F = T_0$$

Linearity Property:

$$2\cos(100\pi t) \xleftrightarrow{\mathcal{FS}} \delta[k-1] + \delta[k+1] \ , \ T_F = T_0 = \frac{1}{50}$$

Time Shifting Property:

$$2\cos(100\pi(t - 0.005)) \xleftrightarrow{\mathcal{FS}} (\delta[k-1] + \delta[k+1])e^{-j0.5\pi k} \ , \ T_F = T_0 = \frac{1}{50}$$

$$f_0 t_0 = 50 \times \frac{1}{200} = \frac{1}{4}$$

$$2\cos(100\pi(t - 0.005)) \xleftrightarrow{\mathcal{FS}} (\delta[k-1] + \delta[k+1])e^{-j\frac{\pi k}{2}} \ , \ T_F = T_0 = \frac{1}{50}$$

Using the equivalence property of the impulse,

$$\mathrm{g}[n]A\delta[n - n_0] = \mathrm{g}[n_0]A\delta[n - n_0],$$

we can write

$$e^{-j0.5\pi k}\delta[k-1] + e^{-j0.5\pi k}\delta[k+1] = \underset{-j}{e^{-j\frac{\pi}{2}}}\delta[k-1] + \underset{j}{e^{j\frac{\pi}{2}}}\delta[k+1]$$

$$2\cos(100\pi(t - 0.005)) \xleftrightarrow{\mathcal{FS}} j(\delta[k+1] - \delta[k-1]) \ , \ T_F = T_0 = \frac{1}{50}$$

From Appendix E,

$$2\sin(100\pi t) \xleftrightarrow{\mathcal{FS}} j(\delta[k+1] - \delta[k-1]) \ , \ T_F = T_0 = \frac{1}{50}$$

(c) $\mathrm{x}(t) = -4\cos(500\pi t)$, $T_F = \dfrac{1}{50}$

$$f_0 = 250 \Rightarrow T_0 = \frac{1}{250} \Rightarrow T_F = 5T_0$$

Use the table entry,

$$T_F = mT_0$$
$$\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{FS}} \frac{1}{2}(\delta[k-m] + \delta[k+m])$$
$$(m \text{ an integer})$$

(d) $\mathrm{x}(t) = \frac{d}{dt}\left(e^{-j10\pi t}\right)$, $T_F = \frac{1}{5}$

$$T_F = T_0 = \frac{1}{5}$$

Start with

$$e^{j2\pi f_0 t} \xleftrightarrow{\mathcal{FS}} \delta[k-1] \quad , \; T_F = T_0 = \frac{1}{5}$$

from Appendix E. Then apply the derivative property of the CTFS.

(e) $\mathrm{x}(t) = \mathrm{rect}(t) * \mathrm{comb}\left(\frac{t}{4}\right)$, $T_F = 4$

(f) $\mathrm{x}(t) = \mathrm{rect}(t) * \mathrm{comb}(t)$, $T_F = 1$

Use the fact that $\mathrm{sinc}(k) = \delta[k]$.

(g) $\mathrm{x}(t) = \mathrm{tri}(t) * \mathrm{comb}(t)$, $T_F = 1$

8. If a periodic signal, $\mathrm{x}(t)$, has a fundamental period of 10 seconds and its harmonic function is

$$\mathrm{X}[k] = 4\,\mathrm{sinc}\left(\frac{k}{20}\right),$$

with a representation period of 10 seconds, what is the harmonic function of $\mathrm{z}(t) = \mathrm{x}(4t)$ using the same representation period, 10 seconds?

Using the time-scaling property, $\mathrm{Z}[k]=\begin{cases}\mathrm{X}\left[\frac{k}{a}\right] & , \frac{k}{a} \text{ an integer} \\ 0 & , \text{ otherwise}\end{cases}$,

$$\mathrm{Z}[k]=\left\{\begin{matrix}\mathrm{X}\left[\frac{k}{4}\right] & , \frac{k}{4} \text{ an integer} \\ 0 & , \text{ otherwise}\end{matrix}\right\}=\left\{\begin{matrix}4\,\mathrm{sinc}\left(\frac{k}{80}\right) & , \frac{k}{4} \text{ an integer} \\ 0 & , \text{ otherwise}\end{matrix}\right\}$$

This can be written more compactly as

$$\mathrm{Z}[k]=4\,\mathrm{sinc}\left(\frac{k}{80}\right)\mathrm{comb}_4[k] \ .$$

9. A periodic signal, $\mathrm{x}(t)$, has a fundamental period of 4 ms and its harmonic function is

$$\mathrm{X}[k]=15(\delta[k-1]+\delta[k+1]) \ ,$$

with a representation period of 4 ms. Find the integral of $\mathrm{x}(t)$.

Start with

$$\cos(2\pi f_0 t)\xleftrightarrow{\mathcal{FS}}\frac{1}{2}(\delta[k-1]+\delta[k+1]) \quad , \quad T_F=T_0=4\text{ ms}$$

Linearity

$$30\cos(500\pi t)\xleftrightarrow{\mathcal{FS}}15(\delta[k-1]+\delta[k+1]) \quad , \quad T_F=T_0=4\text{ ms}$$

Then use the integration property.

10. If $\mathrm{X}[k]$ is the harmonic function over one fundamental period of a unit-amplitude 50%-duty-cycle square wave with an average value of zero and a fundamental period of $1\,\mu$s, find an expression consisting of only real-valued functions for the signal whose harmonic function is $\mathrm{X}[k-10]+\mathrm{X}[k+10]$.

    Find the time-domain function which corresponds to the CTFS harmonic function, $\mathrm{X}[k]$, then apply the frequency-shifting (harmonic-number-shifting) property.

11. Find the harmonic function for a sine wave of the general form, $A\sin(2\pi f_0 t)$. Then, using Parseval's theorem, find its signal power and verify that it is the same as the signal power found directly from the function itself.

$$A\sin(2\pi f_0 t) \xleftrightarrow{\mathcal{FS}} j\frac{A}{2}\left(\delta[k+1]-\delta[k-1]\right) \quad , \; T_F = T_0$$

$$P_x = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \left|A\sin(2\pi f_0 t)\right|^2 dt = \frac{A^2}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \sin^2(2\pi f_0 t)\,dt = \frac{A^2}{2T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \left[1-\cos(4\pi f_0 t)\right]dt$$

$$P_x = \frac{A^2}{2}$$

By Parseval's theorem,

$$P_x = \sum_{k=-\infty}^{\infty}\left|j\frac{A}{2}\left(\delta[k+1]-\delta[k-1]\right)\right|^2 = \frac{A^2}{4}(1+1) = \frac{A^2}{2} \; . \text{ Check.}$$

12. Show for a cosine and a sine that the CTFS harmonic functions have the property,

$$\mathrm{X}[k] = \mathrm{X}^*[-k] \; .$$

For a cosine,

$$\mathrm{X}[k] = \frac{1}{2}\left(\delta[k-1]+\delta[k+1]\right)$$

$$\mathrm{X}[-k] = \frac{1}{2}\left(\delta[-k-1]+\delta[-k+1]\right) = \frac{1}{2}\left(\delta[k+1]+\delta[k-1]\right) = \mathrm{X}[k] = \mathrm{X}^*[k] \; . \text{ Check.}$$

13. Find and sketch the time functions associated with these harmonic functions assuming $T_F = 1$.

(a) $\mathrm{X}[k] = \delta[k-2]+\delta[k]+\delta[k+2]$

(b) $\mathrm{X}[k] = 10\,\mathrm{sinc}\left(\frac{k}{10}\right)$

14. Find the even and odd parts, $\mathrm{x}_e(t)$ and $\mathrm{x}_o(t)$, of

$$x(t) = 20\cos\left(40\pi t + \frac{\pi}{6}\right).$$

Then find the harmonic functions, $X_e[k]$ and $X_o[k]$, corresponding to them. Then, using the time shifting property find the harmonic function, $X[k]$ and compare it to the sum of the two harmonic functions, $X_e[k]$ and $X_o[k]$.

Use $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ to find the even and odd parts.

15. Using the direct summation formula find and sketch the DTFS harmonic function of $\text{comb}_{N_0}[n]$ with $N_F = N_0$.

$$X[k] = \frac{1}{N_0}\sum_{n=\langle N_0 \rangle} \text{comb}_{N_0}[n] e^{-j2\pi(kF_0)n}$$

One can choose any convenient period for the summation. In the period chosen below there is exactly one non-zero impulse in the comb function at $n = 0$.

$$X[k] = \frac{1}{N_0}\sum_{n=-\frac{N_0}{2}}^{\frac{N_0}{2}-1} \text{comb}_{N_0}[n] e^{-j2\pi(kF_0)n} = \frac{1}{N_0}$$

16. Using the DTFS table of transforms and the DTFS properties, find the DTFS harmonic function of each of these periodic signals using the representation period, N_F, indicated.

(a) $x[n] = 6\cos\left(\frac{2\pi n}{32}\right)$, $N_F = 32$

(b) The solution is $X[k] = j5(\text{comb}_{12}[k+1] - \text{comb}_{12}[k-1])e^{-j\frac{\pi}{3}k}$

We can demonstrate that this solution is correct by reconstituting the signal using

$$x[n] = \sum_{k=\langle N_0 \rangle} X[k] e^{j2\pi\frac{kn}{N_0}} = \sum_{k=\langle N_0 \rangle} j5(\text{comb}_{12}[k+1] - \text{comb}_{12}[k-1])e^{-j\frac{\pi}{3}k} e^{j2\pi\frac{kn}{N_0}}$$

Since the summation extends only over one period, $N_0 = 12$, choose the simplest period, $-6 \le k < 6$. In that period the two comb functions are simply two impulses at $k = \pm 1$.

$$\mathrm{x}[n]=\sum_{k=-6}^{5} j5(\delta[k+1]-\delta[k-1])e^{j2\pi k\left(\frac{n}{12}-\frac{1}{6}\right)}$$

$$\mathrm{x}[n]=j5\left[e^{-j2\pi\left(\frac{n}{12}-\frac{1}{6}\right)}-e^{j2\pi\left(\frac{n}{12}-\frac{1}{6}\right)}\right]=-j5\left[e^{j2\pi\left(\frac{n}{12}-\frac{1}{6}\right)}-e^{-j2\pi\left(\frac{n}{12}-\frac{1}{6}\right)}\right]$$

$$\mathrm{x}[n]=-j5\left[j2\sin\left(2\pi\left(\frac{n}{12}-\frac{1}{6}\right)\right)\right]=10\sin\left(2\pi\left(\frac{n-2}{12}\right)\right)$$

We could have chosen a different period, for example $4\le k<16$. Then

$$\mathrm{x}[n]=\sum_{k=4}^{15} j5(\delta[k-11]-\delta[k-13])e^{j2\pi k\left(\frac{n}{12}-\frac{1}{6}\right)}$$

$$\mathrm{x}[n]=j5\left[e^{j22\pi\left(\frac{n}{12}-\frac{1}{6}\right)}-e^{j26\pi\left(\frac{n}{12}-\frac{1}{6}\right)}\right]=j5\left[e^{j2\pi\left(\frac{11n}{12}-\frac{11}{6}\right)}-e^{j2\pi\left(\frac{13n}{12}-\frac{13}{6}\right)}\right]$$

$$\mathrm{x}[n]=j5e^{j2\pi\left(\frac{12n}{12}-\frac{12}{6}\right)}\left[e^{j2\pi\left(-\frac{n}{12}+\frac{1}{6}\right)}-e^{j2\pi\left(\frac{n}{12}-\frac{1}{6}\right)}\right]=j5\underbrace{e^{j2\pi n}}_{=1}\underbrace{e^{-j4\pi}}_{=1}\left[e^{-j2\pi\left(\frac{n}{12}-\frac{1}{6}\right)}-e^{j2\pi\left(\frac{n}{12}-\frac{1}{6}\right)}\right]$$

$$\mathrm{x}[n]=j5\left[-j2\sin\left(2\pi\left(\frac{n}{12}-\frac{1}{6}\right)\right)\right]=10\sin\left(2\pi\left(\frac{n-2}{12}\right)\right)$$

which is the same answer as before (with somewhat more effort).

(c) $$\mathrm{x}[n]=\begin{cases}\mathrm{x}_1\left[\frac{n}{8}\right] & , \frac{n}{8} \text{ an integer} \\ 0 & , \text{otherwise}\end{cases} \quad , \; N_F=48$$

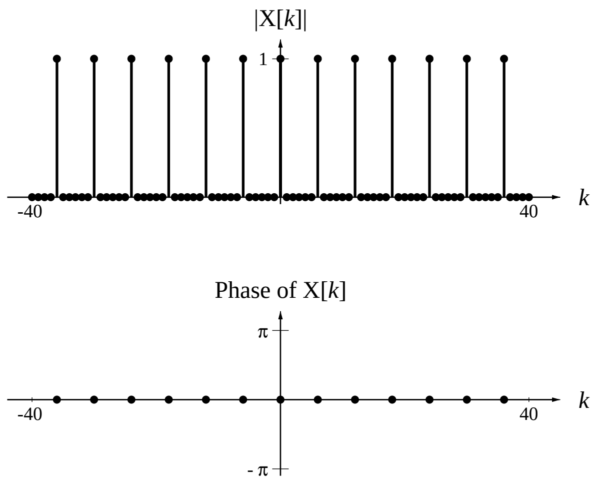
where $\mathrm{x}_1[n]=\sin\left(\frac{2\pi n}{6}\right)$

Use the time-scaling property of the DTFS.

(d) $\mathrm{x}[n]=e^{j2\pi n}$, $N_F=6$

Notice that this function, although written as $x[n] = e^{j2\pi n}$ could be more simply written as $x[n] = 1$ because for any integer value of discrete time, *n*, the result is always the same, 1. Therefore

$$e^{j2\pi n} = 1 \xleftrightarrow{\mathcal{FS}} \text{comb}_6[k]$$



If we take a more straightforward approach to finding the DTFS harmonic function using

$$N_F = mN_0$$

$$e^{j2\pi \frac{n}{N_0}} \xleftrightarrow{\mathcal{FS}} \text{comb}_{N_F}[k-m]$$

we get

$$e^{j2\pi n} \xleftrightarrow{\mathcal{FS}} \text{comb}_6[k-6]$$

Observe that this answer and the previous answer are identical. That is,

$$\text{comb}_6[k] = \text{comb}_6[k-6],$$

as must be true.

(e) $\quad x[n] = \cos\left(\frac{2\pi n}{16}\right) - \cos\left(\frac{2\pi(n-1)}{16}\right)$, $N_F = 16$

(f) $\quad x[n] = -\sin\left(\frac{33\pi n}{32}\right) = -\sin\left(\frac{33 \times 2\pi n}{64}\right)$, $N_F = 64$

The period of this signal is 64.

This form does not appear directly in Appendix E. Therefore we must either use what is in Appendix E with some properties to get to this form or simply apply the definition of the DTFS harmonic function directly.

From Appendix E,

$$e^{j2\pi\frac{n}{N_0}} \xleftrightarrow{\mathcal{FS}} \text{comb}_{N_0}[k-1]$$

Using the frequency-shifting property,

$$e^{j2\pi\frac{mn}{N_0}} e^{j2\pi\frac{n}{N_0}} \xleftrightarrow{\mathcal{FS}} \text{comb}_{N_0}[k-m-1]$$

$$e^{j2\pi\frac{n}{N_0}(m+1)} \xleftrightarrow{\mathcal{FS}} \text{comb}_{N_0}[k-(m+1)]$$

or

$$e^{j2\pi\frac{mn}{N_0}} \xleftrightarrow{\mathcal{FS}} \text{comb}_{N_0}[k-m]$$

Then

$$\frac{e^{j2\pi\frac{mn}{N_0}} + e^{-j2\pi\frac{mn}{N_0}}}{2} = \cos\left(\frac{2\pi mn}{N_0}\right) \xleftrightarrow{\mathcal{FS}} \frac{1}{2}\left(\text{comb}_{N_0}[k-m] + \text{comb}_{N_0}[k+m]\right)$$

and

$$\frac{e^{j2\pi\frac{mn}{N_0}} - e^{-j2\pi\frac{mn}{N_0}}}{j2} = \sin\left(\frac{2\pi mn}{N_0}\right) \xleftrightarrow{\mathcal{FS}} \frac{j}{2}\left(\text{comb}_{N_0}[k+m] - \text{comb}_{N_0}[k-m]\right)$$

$$\text{X}[k] = -\frac{j}{2}\left(\text{comb}_{64}[k+33] - \text{comb}_{64}[k-33]\right)$$

We can demonstrate that this solution is correct by reconstituting the signal using

$$\text{x}[n] = \sum_{k=\langle N_0 \rangle} \text{X}[k] e^{j2\pi\frac{kn}{N_0}} = \sum_{k=\langle N_0 \rangle} -\frac{j}{2}\left(\text{comb}_{64}[k+33] - \text{comb}_{64}[k-33]\right) e^{j2\pi\frac{kn}{N_0}}$$

Since the summation extends only over one period, $N_0 = 64$, choose the simplest period, $-32 \le k < 32$.

$$\text{x}[n] = \sum_{k=-32}^{31} -\frac{j}{2}\left(\text{comb}_{64}[k+33] - \text{comb}_{64}[k-33]\right) e^{j2\pi\frac{kn}{N_0}}$$

We must now determine for which values of k, $-32 \le k < 32$, the comb functions are not zero. Take the first comb function,

$$\mathrm{comb}_{64}[k+33] \ .$$

Its impulses occur whenever $k+33$ is an integer multiple of 64. The only value of k in the range, $-32 \le k < 32$, for which that is true is $k = 31$. Similarly, for

$$\mathrm{comb}_{64}[k-33]$$

the only value for which it is non-zero is $k = -31$. Therefore we can write the summation as

$$\mathrm{x}[n] = \sum_{k=-32}^{31} -\frac{j}{2}\left(\delta[k-31] - \delta[k+31]\right)e^{j2\pi\frac{kn}{N_0}} = -\frac{j}{2}\left(e^{j2\pi\frac{31n}{64}} - e^{j2\pi\frac{-31n}{64}}\right)$$

$$\mathrm{x}[n] = \frac{j}{2}\left(e^{-j2\pi\frac{31n}{64}} - e^{j2\pi\frac{31n}{64}}\right) = \frac{j}{2}\left(-j2\sin\left(2\pi\frac{31n}{64}\right)\right) = \sin\left(2\pi\frac{31n}{64}\right) = \sin\left(\frac{31\pi n}{32}\right)$$

But this can also be written as

$$\mathrm{x}[n] = \frac{j}{2}\left(\underbrace{e^{-j2\pi\frac{64n}{64}}}_{=1}e^{j2\pi\frac{33n}{64}} - \underbrace{e^{j2\pi\frac{64n}{64}}}_{=1}e^{-j2\pi\frac{33n}{64}}\right) = \frac{j}{2}\left(e^{j2\pi\frac{33n}{64}} - e^{-j2\pi\frac{33n}{64}}\right)$$

$$\mathrm{x}[n] = \frac{j}{2}\left(j2\sin\left(2\pi\frac{33n}{64}\right)\right) = -\sin\left(\frac{33\pi n}{32}\right)$$

If these two results are to both be correct

$$\sin\left(\frac{31\pi n}{32}\right) = -\sin\left(\frac{33\pi n}{32}\right)$$

for any integer value of n. We can write

$$\sin\left(\frac{31\pi n}{32}\right) = \sin\left(\frac{31\pi n}{32} - 2\pi n\right) = \sin\left(\frac{31\pi n - 64\pi n}{32}\right) = \sin\left(-\frac{33\pi n}{32}\right) = -\sin\left(\frac{33\pi n}{32}\right)$$

proving that the two expressions are indeed equivalent for integer values of *n*.

(g) $\mathrm{x}[n] = \mathrm{rect}_5[n] * \mathrm{comb}_{11}[n]$

Using $\mathrm{rect}_W[n] * \mathrm{comb}_{N_0}[n] \xleftrightarrow{\mathcal{FS}} \dfrac{2W+1}{N_0}\mathrm{drcl}\left(\dfrac{k}{N_0}, 2W+1\right)$

And, from Appendix A, $\mathrm{drcl}\left(\dfrac{n}{2m+1}, 2m+1\right) = \mathrm{comb}_{2m+1}[n]$ and the fact that $2W+1 = N_0$,

$$\mathrm{X}[k] = \mathrm{comb}_{11}[k]$$

Agrees with $1 \xleftrightarrow{\mathcal{FS}} \mathrm{comb}_{N_0}[k]$ because $\mathrm{x}[n] = \mathrm{rect}_5[n] * \mathrm{comb}_{11}[n] = 1$ and in $1 \xleftrightarrow{\mathcal{FS}} \mathrm{comb}_{N_0}[k]$, N_0 can be arbitrarily chosen. The meaning of the result is the same regardless of the choice of N_0.

This result was obtained for a period of $N_0 = 11$. But when the function is a constant, we can choose any period we like and get an equivalent result (because a constant repeats exactly in any "period" you choose). For example, if we let $N_0 = 4$ the transform pair, $1 \xleftrightarrow{\mathcal{FS}} \mathrm{comb}_{N_0}[k]$, yields $\mathrm{X}[k] = \mathrm{comb}_4[k]$. Then, reconstituting the signal from its DTFS,

$$\mathrm{x}[n] = \sum_{k=\langle N_0 \rangle} \mathrm{X}[k] e^{j2\pi \frac{kn}{N_0}} = \sum_{k=\langle N_0 \rangle} \mathrm{comb}_4[k] e^{j2\pi \frac{kn}{N_0}}$$

Summing over the period, $-2 \le k < 2$, yields

$$\mathrm{x}[n] = \sum_{k=-2}^{1} \mathrm{comb}_4[k] e^{j2\pi \frac{kn}{4}} = e^{j2\pi \frac{(0)n}{4}} = 1 \ .$$

If we chose the period, $2 \le k < 6$ we would get

$$\mathrm{x}[n] = \sum_{k=2}^{5} \mathrm{comb}_4[k] e^{j2\pi \frac{kn}{4}} = e^{j2\pi \frac{(4)n}{4}} = 1 \ ,$$

which is exactly the same result. In general, for any choice of period and any range of *k* covering one period,

$$\mathrm{x}[n] = \sum_{k=k_0}^{k_0+N_0-1} \mathrm{comb}_{N_0}[k] e^{j2\pi \frac{kn}{N_0}} = e^{j2\pi \frac{qN_0 n}{N_0}} = 1$$

where the integer, q, lies in the range, $k_0 \le q < k_0 + N_0$.

(h) $\mathrm{x}[n] = \mathrm{rect}_2[n] * \mathrm{comb}_{21}[n-3]$, $N_F = 21$

17. Find the DTFS harmonic function of

$$\mathrm{x}[n] = \sum_{m=-\infty}^{n} \mathrm{comb}_3[m] - \mathrm{comb}_3[m-1]$$

with $N_F = N_0 = 3$.

18. Find the average signal power of

$$\mathrm{x}[n] = \mathrm{rect}_4[n] * \mathrm{comb}_{20}[n]$$

directly in the DT domain and then find its harmonic function, $\mathrm{X}[k]$, and find the signal power in the "k" domain and show that they are the same.

In the DT domain:

$$P_x = \frac{9}{20}$$

In the "k" domain:
Do the summation numerically in MATLAB to get $P_x = \frac{9}{20}$.

19. Using the frequency shifting property of the DTFS find the DT-domain signal, $\mathrm{x}[n]$, corresponding to the harmonic function,

$$\mathrm{X}[k] = \frac{7}{32}\mathrm{drcl}\left(\frac{k-16}{32}, 7\right).$$

This frequency shifting causes the sign of the DT-domain function to alternate.

20. Find the DTFS harmonic function for

$$\mathrm{x}[n] = \mathrm{rect}_3[n] * \mathrm{comb}_8[n]$$

with the representation period, $N_F = 8$. Then, using MATLAB, plot the DTFS representation,

$$x_F[n] = \sum_{k=0}^{7} X[k] e^{j2\pi \frac{kn}{8}}$$

over the DT range, $-8 \le n < 8$. For comparison, plot the function,

$$x_{F2}[n] = \sum_{k=13}^{20} X[k] e^{j2\pi \frac{kn}{8}}$$

over the same range. The plots should be identical.

21. A periodic signal, x(*t*), with a period of 4 seconds is described over one period by

$$x(t) = 3 - t \ \ , \ \ 0 < t < 4 \ .$$

Plot the signal and find its trigonometric CTFS description. Then plot on the same scale approximations to the signal, $x_N(t)$, given by

$$x_N(t) = X_c[0] + \sum_{k=1}^{N} X_c[k]\cos(2\pi(kf_F)t) + X_s[k]\sin(2\pi(kf_F)t)$$

for $N = 1, 2$ and 3. (In each case the time scale of the plot should cover at least two periods of the original signal.)

Find the trigonometric harmonics series by direct integration using

$$X_c[0] = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\,dt \ ,$$

$$X_c[k] = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(2\pi(kf_0)t)\,dt$$

$$X_s[k] = \frac{2}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin(2\pi(kf_0)t)\,dt$$

Use

$$\int x^n \cos(x)\,dx = x^n \sin(x) - n\int x^{n-1}\sin(x)\,dx \ ,$$

or

$$\int x^n \sin(x)\,dx = -x^n \cos(x) + n\int x^{n-1}\cos(x)\,dx$$

from Appendix A, and make the change of variable

$$\lambda = \frac{\pi k}{2}t$$

to finish the integrals.

22. A periodic signal, x(*t*), with a period of 2 seconds is described over one period by

$$\mathrm{x}(t) = \begin{cases} \sin(2\pi t) & , \ |t| < \frac{1}{2} \\ 0 & , \ \frac{1}{2} < |t| < 1 \end{cases} .$$

Plot the signal and find its complex CTFS description. Then plot on the same scale approximations to the signal, $\mathrm{x}_N(t)$, given by

$$\mathrm{x}_N(t) = \sum_{k=-N}^{N} \mathrm{X}[k] e^{j2\pi(kf_F)t}$$

for $N = 1, 2$ and 3. (In each case the time scale of the plot should cover at least two periods of the original signal.)

23. Find and plot two periods of the complex CTFS description of $\cos(2\pi t)$

(a) Over the interval, $0 < t < 1$,

and (b) Over the interval, $0 < t < 1.5$.

(a) The period of $\cos(2\pi t)$ is 1, therefore the complex Fourier series description is simply the exponential description of the cosine function

$$\cos(2\pi t) = \frac{e^{j2\pi t} + e^{-j2\pi t}}{2} .$$

That is,

$$\mathrm{X}[1] = \mathrm{X}[-1] = \frac{1}{2} \ \text{ and } \ \mathrm{X}[k] = 0 \, , |k| \neq 1 \ .$$

The plot is simply the plot of two periods of the cosine because the complex Fourier series description is exact everywhere.

(b)

$$T_F = \frac{3}{2} \text{ and } f_F = \frac{2}{3}$$

$$\mathrm{X}[k] = \frac{1}{T_F} \int_{t_0}^{t_0+T_F} \mathrm{x}(t) e^{-j2\pi(kf_F)t} dt$$

$$\mathrm{X}[k] = \frac{1}{3} \left\{ \begin{array}{l} \dfrac{\sin\left[3\pi\left(\frac{2k}{3}-1\right)\right]}{2\pi\left(\frac{2k}{3}-1\right)} + \dfrac{\sin\left[3\pi\left(\frac{2k}{3}+1\right)\right]}{2\pi\left(\frac{2k}{3}+1\right)} \\ +j\dfrac{\cos\left[3\pi\left(\frac{2k}{3}-1\right)\right]-1}{2\pi\left(\frac{2k}{3}-1\right)} + j\dfrac{\cos\left[3\pi\left(\frac{2k}{3}+1\right)\right]-1}{2\pi\left(\frac{2k}{3}+1\right)} \end{array} \right\}$$

24. Using MATLAB, plot the following signals over the time range, $-3 < t < 3$.

(a) $\mathrm{x}_0(t) = 1$

(b) $\mathrm{x}_1(t) = \mathrm{x}_0(t) + 2\cos(2\pi t)$

(c) $\mathrm{x}_2(t) = \mathrm{x}_1(t) + 2\cos(4\pi t)$

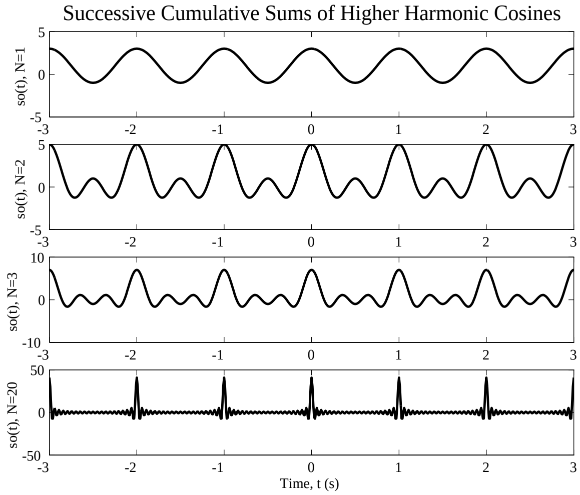
(d) $\mathrm{x}_{20}(t) = \mathrm{x}_{19}(t) + 2\cos(40\pi t)$

For each part, (a) through (d), numerically evaluate the area of the signal over the time range, $-\frac{1}{2} < t < \frac{1}{2}$.

The "*N*th" member of this sequence of functions can be represented by the general form

$$\mathrm{x}_N(t) = \mathrm{x}_{N-1}(t) + 2\cos(2\pi N t).$$

Based on what you observed in parts (a) through (d) what is the limit as "N" approaches infinity? Find the trigonometric Fourier series expression for the unit comb function and compare it to this result.



25. Using the CTFS table of transforms and the CTFS properties, find the CTFS harmonic function of each of these periodic signals using the time interval, T_F, indicated.

(a) $$\mathrm{x}(t) = 3\,\mathrm{rect}\left(2\left(t - \frac{1}{4}\right)\right) * \mathrm{comb}(t) \ , \ T_F = 1$$

Remember:

If $\mathrm{g}(t) = \mathrm{g}_0(t) * \delta(t)$

Then $\mathrm{g}(t - t_0) = \mathrm{g}_0(t - t_0) * \delta(t) = \mathrm{g}_0(t) * \delta(t - t_0)$

And $\mathrm{g}(t - t_0) \neq \mathrm{g}_0(t - t_0) * \delta(t - t_0) = \mathrm{g}(t - 2t_0)$

(b) $$\mathrm{x}(t) = 5[\mathrm{tri}(t-1) - \mathrm{tri}(t+1)] * \frac{1}{4}\mathrm{comb}\left(\frac{t}{4}\right) \ , \ T_F = 4$$

Using $\mathrm{tri}\left(\frac{t}{w}\right) * \frac{1}{T_0}\mathrm{comb}\left(\frac{t}{T_0}\right) \xleftrightarrow{\mathcal{FS}} \frac{w}{T_0}\mathrm{sinc}^2\left(\frac{w}{T_0}k\right)$ **,** with $w = 1$ and $T_0 = 4$,

$$\mathrm{tri}(t) * \frac{1}{4}\mathrm{comb}\left(\frac{t}{4}\right) \xleftrightarrow{\mathcal{FS}} \frac{1}{4}\mathrm{sinc}^2\left(\frac{k}{4}\right)$$

Then using the time-shifting property, $\mathrm{x}(t - t_0) \xleftrightarrow{\mathcal{FS}} e^{-j2\pi(kf_0)t_0}\,\mathrm{X}[k]$,

$$\mathrm{tri}(t-1) * \frac{1}{4}\mathrm{comb}\left(\frac{t}{4}\right) \xleftrightarrow{\mathcal{FS}} \frac{1}{4}\mathrm{sinc}^2\left(\frac{k}{4}\right)e^{-j\frac{\pi k}{2}}$$

and

$$\mathrm{tri}(t+1) * \frac{1}{4}\mathrm{comb}\left(\frac{t}{4}\right) \xleftrightarrow{\mathcal{FS}} \frac{1}{4}\mathrm{sinc}^2\left(\frac{k}{4}\right) e^{j\frac{\pi k}{2}}$$

Then, using linearity,

$$5\left[\mathrm{tri}(t-1) - \mathrm{tri}(t+1)\right] * \frac{1}{4}\mathrm{comb}\left(\frac{t}{4}\right) \xleftrightarrow{\mathcal{FS}} 5\left(e^{-j\frac{\pi k}{2}} - e^{j\frac{\pi k}{2}}\right)\frac{1}{4}\mathrm{sinc}^2\left(\frac{k}{4}\right)$$

or

$$5\left[\mathrm{tri}(t-1) - \mathrm{tri}(t+1)\right] * \frac{1}{4}\mathrm{comb}\left(\frac{t}{4}\right) \xleftrightarrow{\mathcal{FS}} -j\frac{5}{2}\sin\left(\frac{\pi k}{2}\right)\mathrm{sinc}^2\left(\frac{k}{4}\right)$$

(c) $\mathrm{x}(t) = 3\sin(6\pi t) + 4\cos(8\pi t)$, $T_F = 1$

(d) $\mathrm{x}(t) = 2\cos(24\pi t) - 8\cos(30\pi t) + 6\sin(36\pi t)$, $T_F = 2$

(e) $\mathrm{x}(t) = \int_{-\infty}^{t}\left[\mathrm{comb}(\lambda) - \mathrm{comb}\left(\lambda - \frac{1}{2}\right)\right]d\lambda$, $T_F = 1$

(f) $\mathrm{x}(t) = 4\cos(100\pi t)\sin(1000\pi t)$, $T_F = \frac{1}{50}$

(g) $\mathrm{x}(t) = \left[14\,\mathrm{rect}\left(\frac{t}{8}\right) * \mathrm{comb}\left(\frac{t}{12}\right)\right] \otimes \left[7\,\mathrm{rect}\left(\frac{t}{5}\right) * \mathrm{comb}\left(\frac{t}{8}\right)\right]$, $T_F = 24$

Let

$$\mathrm{x}_1(t) = \mathrm{rect}\left(\frac{t}{8}\right) * \mathrm{comb}\left(\frac{t}{12}\right) \quad \text{and} \quad \mathrm{x}_2(t) = \mathrm{rect}\left(\frac{t}{5}\right) * \mathrm{comb}\left(\frac{t}{8}\right)$$

Then, using

$$\mathrm{rect}\left(\frac{t}{w}\right) * \frac{1}{T_0}\mathrm{comb}\left(\frac{t}{T_0}\right) \xleftrightarrow{\mathcal{FS}} \frac{w}{T_0}\mathrm{sinc}\left(\frac{w}{T_0}k\right) ,$$

$$\mathrm{x}(t) = 98\,\mathrm{x}_1(t) \otimes \mathrm{x}_2(t)$$

$$\mathrm{X}_1[k] = 8\,\mathrm{sinc}\left(\frac{2k}{3}\right) \quad , \; T_F = 12$$

$$\mathrm{X}_1[k]=\begin{cases}8\,\mathrm{sinc}\left(\frac{k}{3}\right) & , \ \frac{k}{2} \text{ an integer} \\ 0 & , \ \text{otherwise}\end{cases} \quad , \ T_F=24$$

$$\mathrm{X}_2[k]=5\,\mathrm{sinc}\left(\frac{5k}{8}\right) \quad , \ T_F=8$$

$$\mathrm{X}_2[k]=\begin{cases}5\,\mathrm{sinc}\left(\frac{5k}{24}\right) & , \ \frac{k}{3} \text{ an integer} \\ 0 & , \ \text{otherwise}\end{cases} \quad , \ T_F=24$$

$$\mathrm{X}[k]=98\times 24\begin{cases}40\,\mathrm{sinc}\left(\frac{k}{3}\right)\mathrm{sinc}\left(\frac{5k}{24}\right) & , \ \frac{k}{6} \text{ an integer} \\ 0 & , \ \text{otherwise}\end{cases} \quad , \ T_F=24$$

All the values of this function are zero *except* when $k=0$. Then

$$\mathrm{X}[k]=94080\delta[k] \quad , \ T_F=24 \quad .$$

(This result indicates that the periodic convolution of the two signals is a constant, 94080. That can be confirmed by graphically periodically convolving the two signals. That is by finding the area under the product over one cycle and observing that, as one signal is shifted, the area does not change. Signal one is a periodic sequence of pulses of height 168, width 8 and period 12. Signal two is a periodic sequence of pulses of height 56, width 5 and period 8. The overall overlap width is a constant 10. That product is $168\times 56\times 10=94080$.)

(h) $$\mathrm{x}(t)=\left[8\,\mathrm{rect}\left(\frac{t}{2}\right)*\mathrm{comb}\left(\frac{t}{5}\right)\right]\otimes\left[-2\,\mathrm{rect}\left(\frac{t}{6}\right)*\mathrm{comb}\left(\frac{t}{20}\right)\right] \ , \ T_F=20$$

$$\mathrm{X}[k]=-192\begin{cases}\mathrm{sinc}\left(\frac{k}{10}\right)\mathrm{sinc}\left(\frac{3k}{20}\right) & , \ \frac{k}{4} \text{ an integer} \\ 0 & , \ \text{otherwise}\end{cases} \quad , \ T_F=20$$

26. A signal, $\mathrm{x}(t)$, is described over one period by

$$\mathrm{x}(t) = \begin{cases} -A\,,\ -\dfrac{T_0}{2} < t < 0 \\ A\,,\ 0 < t < \dfrac{T_0}{2} \end{cases} .$$

Find its complex CTFS and then, using the integration property find the CTFS of its integral and plot the resulting CTFS representation of the integral.

Find its harmonic function using the integral definition. You should get

$$\mathrm{X}[k] = jA\frac{\cos(k\pi) - 1}{k\pi}$$

Then apply the integration property. Verify that the harmonic is correct at $k = 0$. (You will need to apply L'Hôpital's rule.)

27. In some types of communication systems binary data are transmitted using a technique called binary phase-shift keying (BPSK) in which a "1" is represented by a "burst" of a CT sine wave and a "0" is represented by a burst which is the exact negative of the burst that represents a "1". Let the sine frequency be 1 MHz and let the burst width be 10 periods of the sine wave. Find and plot the CTFS harmonic function for a periodic binary signal consisting of alternating "1's" and "0's" using its fundamental period as the representation period.

    The signal can be represented by

$$\mathrm{x}(t) = \sin\left(2\times 10^6 \pi t\right)\left[2\,\mathrm{rect}\left(10^5 t\right) * 5\times 10^4\,\mathrm{comb}\left(5\times 10^4 t\right) - 1\right]$$

Using mulitplication-convolution duality,

$$\mathrm{X}[k] = \frac{j}{2}\left(\mathrm{comb}[k+20] - \mathrm{comb}[k-20]\right) \otimes \left(10^5 \times 10^{-5}\,\mathrm{sinc}\left(\frac{k}{2}\right) - \delta[k]\right)$$

Remember that a periodic convolution is the same as the aperiodic convolution of one period of one of the periodic signals with the entire other signal. Choose one period of the difference of two combs.

28. Using the DTFS table of transforms and the DTFS properties, find the harmonic function of each of these periodic signals using the representation period, N_F, indicated.

(a) $\mathrm{x}[n] = e^{-j\frac{2\pi n}{16}} \otimes \mathrm{comb}_{24}[n] \quad , \quad N_F = 48$

From the table,

$$e^{j\frac{2\pi n}{16}} \xleftrightarrow{\mathcal{FS}} \mathrm{comb}_{16}[k-1] \quad , \quad N_F = 16$$

Therefore, using the change of period property,

$$e^{-j\frac{2\pi n}{16}} \xleftrightarrow{\mathcal{FS}} \begin{cases} \mathrm{comb}_{16}\left[\frac{k}{3}-1\right] & , \ \frac{k}{3} \text{ an integer} \\ 0 & , \text{otherwise} \end{cases} \quad , \quad N_F = 48$$

Also, from the table,

$$N_F = mN_0$$

$$e^{j\frac{2\pi n}{N_0}} \xleftrightarrow{\mathcal{FS}} \mathrm{comb}_{N_F}[k-m]$$

$$48 = 3 \times 16$$

$$e^{j\frac{2\pi n}{16}} \xleftrightarrow{\mathcal{FS}} \mathrm{comb}_{48}[k-3]$$

The function, $\mathrm{comb}_{16}\left[\frac{k}{3}-1\right]$, is a sequence of impulses. The impulses occur wherever $\frac{k}{3}-1 = 16m$, m any integer. Rearranging, the impulses occur where $k-3 = 48m$. This describes a comb of the form, $\mathrm{comb}_{48}[k-3]$. Therefore the two answers agree.

From the table,

$$\mathrm{comb}_{N_0}[n] \xleftrightarrow{\mathcal{FS}} \frac{1}{N_0} \quad , \quad N_F = 24$$

Therefore, using the time scaling property,

$$\text{comb}_{N_0}[n] \xleftrightarrow{\mathcal{FS}} \begin{Bmatrix} \frac{1}{24} & , & \frac{k}{2} \text{ an integer} \\ 0 & , & \text{otherwise} \end{Bmatrix} \quad , \quad N_F = 48$$

Then, since the two DTFS's are both done with reference to the same representation period, using the multiplication-convolution duality property,

$$X[k] = 2\begin{Bmatrix} \text{comb}_{48}[k-3] & , & \frac{k}{3} \text{ an integer} \\ 0 & , & \text{otherwise} \end{Bmatrix} \begin{Bmatrix} \frac{1}{24} & , & \frac{k}{2} \text{ an integer} \\ 0 & , & \text{otherwise} \end{Bmatrix}$$

The non-zero impulses in the first harmonic function occur at values of *k* for which $k+3$ is an integer multiple of 48. Therefore all these *k*'s must be odd. The values of *k* for which the second harmonic function is non-zero are all even. Therefore

$$X[k] = 0 \quad , \quad \text{for all } k.$$

This result implies that the original DT function, $x[n] = e^{-j\frac{2\pi n}{16}} \otimes \text{comb}_{24}[n]$, is zero. Show that to be true by actually doing the convolution directly.

(b) $x[n] = \left(\text{rect}_5[n] * \text{comb}_{24}[n]\right)\sin\left(\frac{2\pi n}{6}\right) \quad , \; N_F = 24$

(c) $x[n] = x_1[n] - x_1[n-1]$ where $x_1[n] = \text{tri}\left(\frac{n}{8}\right) * \text{comb}_{20}[n] \quad , \; N_F = 20$

29. Find the signal power of

$$x[n] = 5\sin\left(\frac{14\pi n}{15}\right) - 8\cos\left(\frac{26\pi n}{30}\right) .$$

Use Parseval's theorem to find the signal power from the harmonic function.

30. Find the DTFS harmonic function, $X[k]$, of $x[n] = \left(\text{rect}_1[n-1] - \text{rect}_1[n-4]\right) * \text{comb}_6[n]$. Then plot the partial sum, $x_N[n] = \sum_{k=-N}^{N} X[k] e^{j\frac{\pi n k}{3}}$, for $N = 0,1,2$ and then the total sum, $x[n] = \sum_{k=0}^{5} X[k] e^{j\frac{\pi n k}{3}}$.

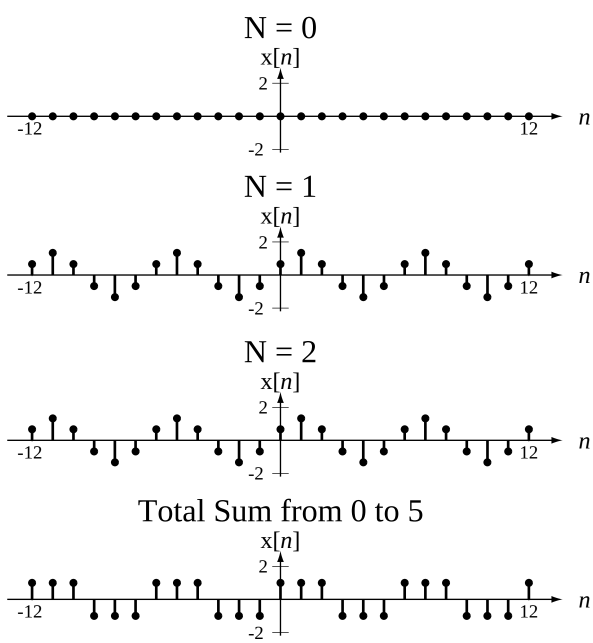
$$x[n] = \text{rect}_1[n-1] * \text{comb}_6[n] - \text{rect}_1[n-4] * \text{comb}_6[n]$$

Using $\text{rect}_{N_W}[n] * \text{comb}_{N_0}[n] \xleftrightarrow{\mathcal{FS}} \frac{1}{N_0} \frac{\sin\left((2N_W+1)\frac{k\pi}{N_0}\right)}{\sin\left(\frac{k\pi}{N_0}\right)}$

$$\mathrm{X}[k] = \frac{1}{6}\frac{\sin\left(\frac{k\pi}{2}\right)}{\sin\left(\frac{k\pi}{6}\right)} e^{-j\frac{\pi k}{3}} - \frac{1}{6}\frac{\sin\left(\frac{k\pi}{2}\right)}{\sin\left(\frac{k\pi}{6}\right)} e^{-j\frac{4\pi k}{3}} = \frac{1}{6}\frac{\sin\left(\frac{k\pi}{2}\right)}{\sin\left(\frac{k\pi}{6}\right)}\left(e^{-j\frac{\pi k}{3}} - e^{-j\frac{4\pi k}{3}}\right)$$

$$\mathrm{X}[k] = \frac{1}{6} e^{-j\frac{5\pi k}{6}} \frac{\sin\left(\frac{k\pi}{2}\right)}{\sin\left(\frac{k\pi}{6}\right)}\left(e^{j\frac{\pi k}{2}} - e^{-j\frac{\pi k}{2}}\right) = \frac{j}{3} e^{-j\frac{5\pi k}{6}} \frac{\sin^2\left(\frac{k\pi}{2}\right)}{\sin\left(\frac{k\pi}{6}\right)}$$

$$\mathrm{x}_N[n] = \frac{j}{3}\sum_{k=-N}^{N} \frac{\sin^2\left(\frac{k\pi}{2}\right)}{\sin\left(\frac{k\pi}{6}\right)} e^{-j\frac{5\pi k}{6}} e^{j\frac{\pi nk}{3}} = \frac{j}{3}\sum_{k=-N}^{N} \frac{\sin^2\left(\frac{k\pi}{2}\right)}{\sin\left(\frac{k\pi}{6}\right)} e^{j\frac{\pi k}{3}\left(n-\frac{5}{2}\right)}$$



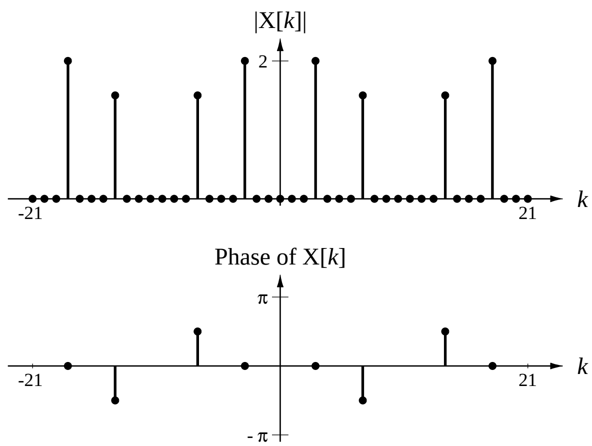
31. Find and sketch the magnitude and phase of the DTFS harmonic function of

$$x[n] = 4\cos\left(\frac{2\pi}{7}n\right) + 3\sin\left(\frac{2\pi}{3}n\right)$$

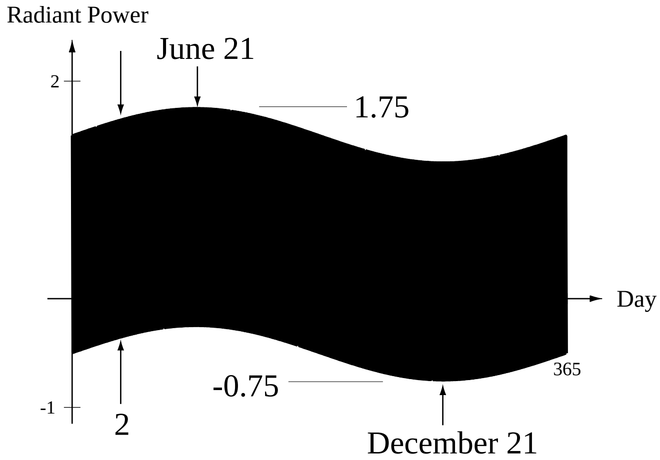
which is valid for all discrete-time.

The least common period of these two signals is $N_0 = 21$. Use the tables and the change-of-period property of the DTFS, to find the DTFS harmonic function.

$$X[k] = 2(\text{comb}_{21}[k-3] + \text{comb}_{21}[k+3]) + j\frac{3}{2}(\text{comb}_{21}[k+7] - \text{comb}_{21}[k-7])$$



32. The sun shining on the earth is a system in which the radiant power from the sun is the excitation and the atmospheric temperature (among many other things) is a response. A simplified model of the radiant power falling on a typical mid-latitude location in North America is that it is periodic with a fundamental period of one year and that every day the radiant power of the sunlight rises linearly from the time the sun rises until the sun is at its zenith then falls linearly until the sun sets. The earth absorbs and stores the radiant energy and re-radiates some of the energy into space every night. To keep the model of the excitation as simple as possible assume that the energy loss every night can be modeled as a continuation of the daily linear radiant power pattern except negative at night. There is also a variation with the seasons caused by the tilt of the earth's axis of rotation. This causes this linear rise-and-fall pattern to rise and fall sinusoidally on a much longer time scale as illustrated below.



(a) Write a mathematical description of the radiant power from the sun.

$$\mathrm{p}(t) = \left[2\,\mathrm{tri}(2t) * \mathrm{comb}(t) - \frac{1}{2}\right] + \frac{1}{4}\sin\left(\frac{2\pi t}{365}\right)$$

(b) Assume that the earth is a first-order system with a time constant of 0.16 years. What day of the year should be the hottest according to this simplified model?

The differential equation for the rate of heat flow into the earth is

$$\frac{d}{dt}\left(K\,\mathrm{T}(t)\right) = \mathrm{p}(t) - \frac{K\,\mathrm{T}(t)}{58.4}$$

where K is a proportionality constant relating the temperature of the earth to its stored energy (t in days). Since the radiant power striking the earth is periodic the temperature is also periodic and both can be expressed in a CTFS with a representation period of 365 days. Find the harmonic function for the radiant power, P[k]. The temperature must also be periodic with the same fundamental period as the radiant power. Solve for the harmonic function of the temperature, T[k], by substituting P[k] into the differential equation. Then simplify it by realizing that the higher order harmonic content in T[k] is negligible compared to the fundamental. Then find the time-domain temperature expression, T(t), and find the time at which it reaches a maximum.

$$\mathrm{T}[k] = \frac{1}{K}\frac{\begin{cases}\mathrm{sinc}^2\left(\frac{k}{2}\right) - \frac{1}{2}\delta[k] & , \ \frac{k}{365} \text{ an integer} \\ 0 & , \ \text{otherwise}\end{cases} + \frac{j}{8}\left(\delta[k+1] - \delta[k-1]\right)}{j\frac{2\pi k}{365} + 0.01712}$$

$$\mathrm{T}(t) \cong \frac{1}{8K} \frac{0.0344\cos\left(\frac{2\pi t}{365}\right) + 0.03424\sin\left(\frac{2\pi t}{365}\right)}{\left(\frac{2\pi}{365}\right)^2 + (0.01712)^2} - \frac{1}{0.03424K}$$

The maximum temperature occurs at about

$$t = 45.625 \ .$$

So the hottest day should occur about 46 days after the summer solstice or about August 6.

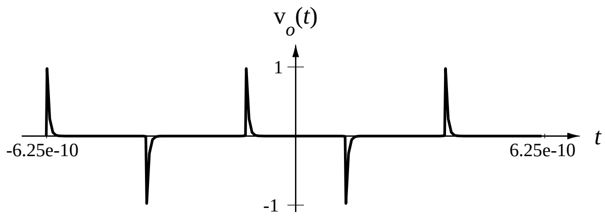
33. The speed and timing of computer computations are controlled by a clock. The clock is a periodic sequence of rectangular pulses, typically with 50% duty cycle. One problem in the design of computer circuit boards is that the clock signal can interfere with other signals on the board by being coupled into adjacent circuits through stray capacitance. Let the computer clock be modeled by a square-wave voltage source alternating between 0.4 and 1.6 V at a frequency of 2 GHz and let the coupling into an adjacent circuit be modeled by a series combination of a 0.1 pF capacitor and a 50 Ω resistance. Find and plot over two fundamental periods the voltage across the 50 Ω resistance.

Describe the excitation as a rectangle convolved with a comb, plus a constant. Find its harmonic function. Put that into the differential equation for the response voltage,

$$RC\left[\mathrm{v}_i'(t) - \mathrm{v}_o'(t)\right] = \mathrm{v}_o(t)$$

and solve for the harmonic function of the response, $\mathrm{V}_o[k] = \dfrac{j2.4\times 10^9 k\pi \,\mathrm{sinc}\left(\frac{k}{2}\right)}{\frac{1}{RC} + j4\times 10^9 k\pi}$.

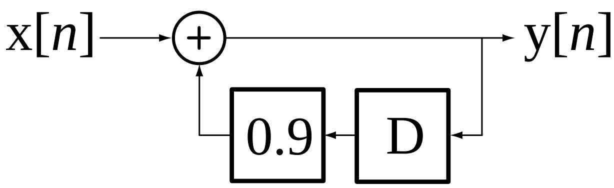
Then plot the response voltage by computing its CTFS. It should look like the graph below.



34. Find and plot versus F the magnitude of the response, $y[n]$, to the periodic excitation,

$$x[n] = \cos(2\pi F n),$$

in the system below.

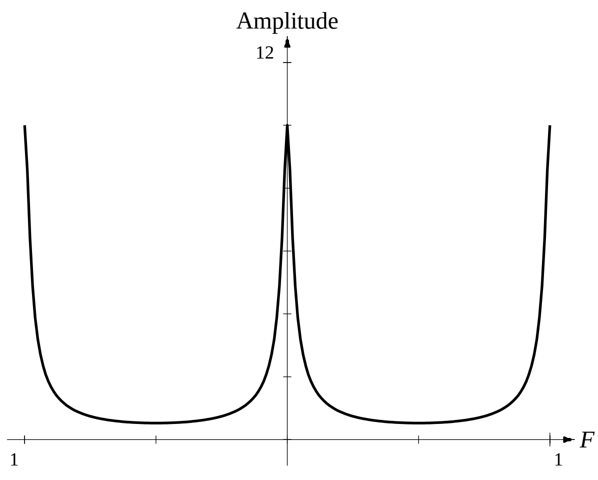


$$y[n] = x[n] + 0.9\,y[n-1]$$

or

$$y[n] - 0.9\,y[n-1] = x[n] \quad .$$

The final graph should look like the graph below.



This is a lowpass DT filter.