Chapter 5 - The Fourier Transform

Selected Solutions

(In this solution manual, the symbol, ⊗, is used for periodic convolution because the preferred symbol which appears in the text is not in the font selection of the word processor used to create this manual.)

1. The transition from the CTFS to the CTFT is illustrated by the signal,

$$\mathrm{x}(t) = \mathrm{rect}\left(\frac{t}{w}\right) * \frac{1}{T_0}\mathrm{comb}\left(\frac{t}{T_0}\right)$$

or

$$\mathrm{x}(t) = \sum_{n=-\infty}^{\infty} \mathrm{rect}\left(\frac{t-nT_0}{w}\right) \quad .$$

The complex CTFS for this signal is given by

$$\mathrm{X}[k] = \frac{Aw}{T_0}\mathrm{sinc}\left(\frac{kw}{T_0}\right) \quad .$$

Plot the "modified" CTFS,

$$T_0\,\mathrm{X}[k] = Aw\,\mathrm{sinc}(w(kf_0)) \quad ,$$

for $w = 1$ and $f_0 = 0.5, 0.1$ and 0.02 versus kf_0 for the range $-8 < kf_0 < 8$.

2. Suppose a function, $\mathrm{m}(x)$, has units of $\frac{\mathrm{kg}}{\mathrm{m}^3}$ and is a function of spatial position, *x*, in meters. Write the mathematical expression for its CTFT, $\mathrm{M}(y)$. What are the units of M and *y*?

$$\mathrm{M}(y) = \int_{-\infty}^{\infty} \mathrm{m}(x)e^{-j2\pi yx}dx$$

The units of M are $\frac{\text{kg}}{\text{m}^2}$ because of the multiplication by *dx* in the integral and the units of *y* are m^{-1} because they are always the reciprocal of the units of the independent variable of the function transformed.

3. Using the integral definition of the Fourier transform, find the CTFT of these functions.

(a) $\text{x}(t) = \text{tri}(t)$

Substitute the definition of the triangle function into the integral and use even and odd symmetry to reduce the work.

Also, use $\sin(x)\sin(y) = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$ to put the final expression into the form of a sinc-squared function.

(b) $\text{x}(t) = \delta\left(t+\frac{1}{2}\right) - \delta\left(t-\frac{1}{2}\right)$

Use the sampling property of the impulse.

4. In Figure E4 there is one example each of a lowpass, highpass, bandpass and bandstop signal. Identify them.

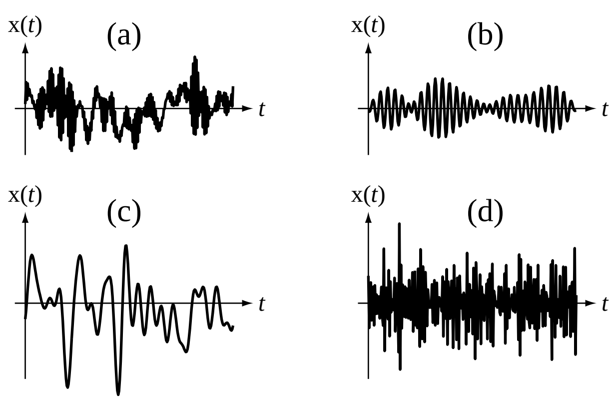


Figure E4 Signals with different frequency content

(a)	bandstop	Composed of very high and very low frequencies and nothing between
(b)	bandpass	Looks most like a sinusoid.

(c) lowpass Smoother than the others, therefore has mostly low frequencies.
(d) highpass Fast variation without any underlying low frequencies

5. Starting with the definition of the CTFT find the radian-frequency form of the generalized CTFT of a constant. Then verify that a change of variable, $\omega \to 2\pi f$, yields the correct result in cyclic-frequency form. Check your answer against the Fourier transform table in Appendix E.

Similar to the derivation in the text for cyclic frequencies. Use the scaling property of the impulse to compare with the cyclic-frequency result.

6. Starting with the definition of the CTFT, find the generalized CTFT of a sine of the form, $A\sin(\omega_0 t)$ and check your answer against the results given above. Check your answer against the Fourier transform table in Appendix E.

Similar to Exercise 5.

7. Find the CTFS and CTFT of each of these periodic signals and compare the results. After finding the transforms, formulate a general method of converting between the two forms for periodic signals.

(a) $\mathrm{x}(t) = A\cos(2\pi f_0 t)$

The CTFS is simply two impulses, $\mathrm{X}[k] = \frac{A}{2}(\delta[k-1] + \delta[k+1])$.

The CTFT is $\mathrm{X}(f) = \frac{A}{2}(\delta(f - f_0) + \delta(f + f_0)) = \mathrm{X}[1]\delta(f - f_0) + \mathrm{X}[-1]\delta(f + f_0)$.

$$\mathrm{X}(f) = \sum_{k=-\infty}^{\infty} \mathrm{X}[k]\delta(f - kf_0)$$

The CTFT is a set of continuous-frequency impulses whose weights at frequencies, kf_0, are the same as the weights of the discrete-harmonic-number impulses at harmonic number, k, in the CTFS harmonic function.

(b) $\mathrm{x}(t) = \mathrm{comb}(t)$

8. Let a signal be defined by

$$\mathrm{x}(t)=2\cos(4\pi t)+5\cos(15\pi t)\ .$$

Find the CTFT's of $\mathrm{x}\left(t-\frac{1}{40}\right)$ and $\mathrm{x}\left(t+\frac{1}{20}\right)$ and identify the resultant phase shift of each sinusoid in each case. Plot the phase of the CTFT and draw a straight line through the 4 phase points which result in each case. What is the general relationship between the slope of that line and the time delay?

The slope of the line is $-2\pi f$ times the delay.

9. Using the frequency-shifting property, find and plot versus time the inverse CTFT of

$$\mathrm{X}(f)=\mathrm{rect}\left(\frac{f-20}{2}\right)+\mathrm{rect}\left(\frac{f+20}{2}\right)\ .$$

10. Find the CTFT of

$$\mathrm{x}(t)=\mathrm{sinc}(t)\ .$$

Then make the transformation, $t\to 2t$, in $\mathrm{x}(t)$ and find the CTFT of the transformed signal.

11. Using the multiplication-convolution duality of the CTFT, find an expression for $\mathrm{y}(t)$ which does not use the convolution operator, $*$, and plot $\mathrm{y}(t)$.

(a) $\mathrm{y}(t)=\mathrm{rect}(t)*\cos(\pi t)$

$$\mathrm{y}(t)=\mathcal{F}^{-1}\left\{\mathrm{sinc}(f)\frac{1}{2}\left[\delta\left(f-\frac{1}{2}\right)+\delta\left(f+\frac{1}{2}\right)\right]\right\}$$

$$\mathrm{y}(t)=\frac{1}{2}\mathcal{F}^{-1}\left[\delta\left(f-\frac{1}{2}\right)\mathrm{sinc}\left(\frac{1}{2}\right)+\delta\left(f+\frac{1}{2}\right)\mathrm{sinc}\left(-\frac{1}{2}\right)\right]$$

Using the equivalence property of the impulse,

$$\mathrm{y}(t)=\frac{1}{2}\mathcal{F}^{-1}\left[\frac{2}{\pi}\delta\left(f-\frac{1}{2}\right)+\frac{2}{\pi}\delta\left(f+\frac{1}{2}\right)\right]=\frac{1}{\pi}\mathcal{F}^{-1}\left[\delta\left(f-\frac{1}{2}\right)+\delta\left(f+\frac{1}{2}\right)\right]$$

$$y(t) = \frac{2}{\pi}\cos(\pi t)$$

(b) Similar to (a)

(c) $y(t) = \text{sinc}(t) * \text{sinc}\left(\frac{t}{2}\right)$

This convolution would be very difficult to do directly in the time domain. But, using transform methods, it is quite easy.

$$y(t) = \mathcal{F}^{-1}\{\text{rect}(f) \times 2\,\text{rect}(2f)\} = 2\mathcal{F}^{-1}\{\text{rect}(2f)\} = \text{sinc}\left(\frac{t}{2}\right)$$

(d) Similar to (c).

(e) $y(t) = e^{-t}\,u(t) * \sin(2\pi t)$

Use the equivalence property of the impulse, then find a common denominator and simplify. Then, use

$$A\cos(x) + B\sin(x) = \sqrt{A^2 + B^2}\cos\left(x - \tan^{-1}\left(\frac{B}{A}\right)\right)$$

to get

$$y(t) = \frac{\cos(2\pi t + 2.984)}{\sqrt{1 + (2\pi)^2}}$$

12. Using the CTFT of the rectangle function and the differentiation property of the CTFT find the Fourier transform of

$$x(t) = \delta(t-1) - \delta(t+1) \;.$$

Check your answer against the CTFT found using the table and the time-shifting property.

Let $y(t) = -\text{rect}\left(\frac{t}{2}\right)$. Then $x(t) = \frac{d}{dt}(y(t))$. (This comes from the definition of a generalized derivative in Chapter 2.)

$$-\text{rect}\left(\frac{t}{2}\right) \xleftrightarrow{\mathcal{F}} -2\,\text{sinc}(2f)$$

Using the differentiation property of the CTFT,

$$\frac{d}{dt}\left(-\text{rect}\left(\frac{t}{2}\right)\right) \xleftrightarrow{\mathcal{F}} j2\pi f\left[-2\,\text{sinc}(2f)\right] = -j4\pi f\,\text{sinc}(2f)$$

$$\frac{d}{dt}\left(-\text{rect}\left(\frac{t}{2}\right)\right) \xleftrightarrow{\mathcal{F}} -j4\pi f\,\frac{\sin(2\pi f)}{2\pi f} = -j2\sin(2\pi f)$$

Use the the CTFT of the impulse and the time-shifting property to check this answer.

13. Find the CTFS and CTFT of these periodic functions and compare answers.

(a) $\mathrm{x}(t) = \text{rect}(t) * \frac{1}{2}\text{comb}\left(\frac{t}{2}\right)$

Find the CTFS harmonic function using the integral definition or Appendix E.

$$\mathrm{X}[k] = \frac{1}{2}\text{sinc}\left(\frac{k}{2}\right)$$

$$\mathrm{X}(f) = \text{sinc}(f)\,\text{comb}(2f) = \frac{1}{2}\text{sinc}(f)\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{2}\right)$$

$$\mathrm{X}(f) = \sum_{k=-\infty}^{\infty}\frac{1}{2}\text{sinc}\left(\frac{k}{2}\right)\delta\left(f-\frac{k}{2}\right) = \sum_{k=-\infty}^{\infty}\mathrm{X}[k]\,\delta(f-kf_0)$$

The CTFT impulses at kf_0 have the same strengths as the CTFT harmonic function impulses at *k*.

(b) $\mathrm{x}(t) = \text{tri}(10t) * 4\,\text{comb}(4t)$

Find the CTFS harmonic function using the integral definition or Appendix E.

$$\mathrm{X}[k] = \frac{2}{5}\text{sinc}^2\left(\frac{2k}{5}\right) = \frac{5}{4}\,\frac{\cos\left(\frac{4}{5}\pi k\right)-1}{(\pi k)^2}$$

$$\mathrm{X}(f) = \frac{1}{10}\text{sinc}^2\left(\frac{f}{10}\right)\text{comb}\left(\frac{f}{4}\right) = \frac{4}{10}\text{sinc}^2\left(\frac{f}{10}\right)\sum_{k=-\infty}^{\infty}\delta(f-4k)$$

$$X(f) = \frac{2}{5}\sum_{k=-\infty}^{\infty}\mathrm{sinc}^2\left(\frac{2k}{5}\right)\delta(f-4k) \quad \text{Checks with CTFS.}$$

14. Using Parseval's theorem, find the signal energy of these signals.

(a) $x(t) = 4\,\mathrm{sinc}\left(\frac{t}{5}\right)$

(b) $x(t) = 2\,\mathrm{sinc}^2(3t)$

$$E_x = \int_{-\infty}^{\infty}|x(t)|^2\,dt = \int_{-\infty}^{\infty}|X(f)|^2\,df = \int_{-\infty}^{\infty}\left|\frac{2}{3}\mathrm{tri}\left(\frac{f}{3}\right)\right|^2 df = \frac{4}{9}\int_{-\infty}^{\infty}\mathrm{tri}^2\left(\frac{f}{3}\right)df$$

$$E_x = \frac{8}{9}\int_0^3 \mathrm{tri}^2\left(\frac{f}{3}\right)df = \frac{8}{9}\int_0^3\left(1-\frac{f}{3}\right)^2 df = \frac{8}{9}\int_0^3\left(1-\frac{2f}{3}+\frac{f^2}{9}\right)df$$

$$E_x = \frac{8}{9}\left[f - \frac{f^2}{3}+\frac{f^3}{27}\right]_0^3 = \frac{8}{9}\left[3-\frac{9}{3}+\frac{27}{27}\right] = \frac{8}{9}$$

15. What is the total area under the function, $g(t) = 100\,\mathrm{sinc}\left(\frac{t-8}{30}\right)$?

Use $\int_{-\infty}^{\infty} g(t)\,dt = G(0)$

16. Using the integration property, find the CTFT of these functions and compare with the CTFT found using other properties.

(a) $g(t) = \begin{cases} 1 & , \ |t|<1 \\ 2-|t| & , \ 1<|t|<2 \\ 0 & , \ \text{elsewhere} \end{cases}$

Find the CTFT of the derivative of this function (which is two separated rectangles). Then use the integration property to find the CTFT of the original function.

(b) $g(t) = 8\,\mathrm{rect}\left(\frac{t}{3}\right)$

17. Sketch the magnitudes and phases of the CTFT's of these signals in the *f* form.

(a) $\mathrm{x}(t) = \delta(\mathrm{t} - 2)$

Remember, there are many alternate correct ways of plotting phase. So your phase plot may be correct even if it does not look like the answer provided in the text.

(b) $\mathrm{x}(t) = \mathrm{u}(t) - \mathrm{u}(t-1)$

This can be done directly using the two unit steps or by converting them into a shifted rectangle.

(c) $\mathrm{x}(t) = 5\,\mathrm{rect}\left(\frac{t+2}{4}\right)$ (d) $\mathrm{x}(t) = 25\,\mathrm{sinc}(10(t-2))$

(e) $\mathrm{x}(t) = 6\sin(200\pi t)$ (f) $\mathrm{x}(t) = 2e^{-3t}\,\mathrm{u}(3t)$

(g) $\mathrm{x}(t) = 4e^{-3t^2} = 4e^{-\pi\left(\sqrt{\frac{3}{\pi}}t\right)^2} \xleftrightarrow{\mathcal{F}} \mathrm{X}(f) = 4\sqrt{\frac{\pi}{3}}e^{-\frac{\pi^2}{3}f^2}$

18. Sketch the magnitudes and phases of the CTFT's of these signals in the ω form.

(a) $\mathrm{x}(t) = \frac{1}{2}\mathrm{comb}\left(\frac{t}{2}\right) \xleftrightarrow{\mathcal{F}} \mathrm{X}(j\omega) = \mathrm{comb}\left(\frac{\omega}{\pi}\right) = \pi\sum_{k=-\infty}^{\infty}\delta(\omega - k\pi)$

(b) $\mathrm{x}(t) = \mathrm{sgn}(2t)$

(c) $\mathrm{x}(t) = 10\,\mathrm{tri}\left(\frac{t-4}{20}\right)$ (d) $\mathrm{x}(t) = \frac{\mathrm{sinc}^2\left(\frac{t+1}{3}\right)}{10}$

(e) $\mathrm{x}(t) = \frac{\cos\left(200\pi t - \frac{\pi}{4}\right)}{4} = \frac{\cos\left(200\pi\left(t - \frac{1}{800}\right)\right)}{4}$

(f) $\mathrm{x}(t) = 2e^{-3t}\,\mathrm{u}(t)$ (g) $\mathrm{x}(t) = 7e^{-5|t|}$

19. Sketch the inverse CTFT's of these functions.

(a) $\mathrm{X}(f) = -15\,\mathrm{rect}\left(\frac{f}{4}\right)$ (b) $\mathrm{X}(f) = \frac{\mathrm{sinc}(-10f)}{30}$

(c) $e^{-a|t|} \xleftrightarrow{\mathcal{F}} \frac{2a}{a^2 + (2\pi f)^2}$, $f \to \frac{f}{2\pi}$, $\mathrm{x}(t) \to 2\pi\,\mathrm{x}(2\pi t)$

$$\mathrm{x}(t) = 6\pi e^{-3|2\pi t|} \xleftrightarrow{\mathcal{F}} \mathrm{X}(f) = \frac{18}{9 + f^2}$$

(d) $\mathrm{X}(f) = \frac{1}{10 + jf}$ (e) $\mathrm{X}(f) = \frac{\delta(f-3) + \delta(f+3)}{6}$

(f) $\mathrm{X}(f) = 8\delta(5f)$ (g) $\mathrm{X}(f) = -\frac{3}{j\pi f}$

20. Sketch the inverse CTFT's of these functions.

(a) $e^{-\pi t^2} \xleftrightarrow{\mathcal{F}} e^{-\frac{\omega^2}{4\pi}}$, $\omega \to 4\sqrt{\pi}\omega$, $\mathrm{x}(t) \to \frac{1}{4\sqrt{\pi}}\mathrm{x}\left(\frac{t}{4\sqrt{\pi}}\right)$

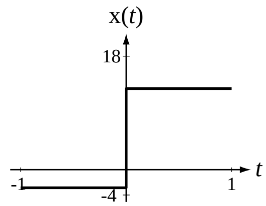
$$\mathrm{x}(t) = \frac{1}{4\sqrt{\pi}} e^{-\frac{t^2}{16}} \xleftrightarrow{\mathcal{F}} \mathrm{X}(j\omega) = e^{-4\omega^2} = e^{-16\pi\frac{\omega^2}{4\pi}} = e^{-\frac{(4\sqrt{\pi}\omega)^2}{4\pi}}$$

(b) $\mathrm{X}(j\omega) = 7\,\mathrm{sinc}^2\left(\frac{\omega}{\pi}\right)$ (c) $\mathrm{X}(j\omega) = j\pi[\delta(\omega + 10\pi) - \delta(\omega - 10\pi)]$

(d) $\mathrm{X}(j\omega) = \frac{\mathrm{comb}\left(\frac{4\omega}{\pi}\right)}{5}$

(e) $\mathrm{sgn}(t) \xleftrightarrow{\mathcal{F}} \frac{2}{j\omega}$, $1 \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega)$

$$\mathrm{x}(t) = \frac{5\pi}{2}\mathrm{sgn}(t) + 5 \xleftrightarrow{\mathcal{F}} \mathrm{X}(j\omega) = \frac{5\pi}{j\omega} + 10\pi\delta(\omega)$$



(f) $X(j\omega) = \frac{6}{3 + j\omega}$ (g) $X(j\omega) = 20\,\mathrm{tri}(8\omega)$

21. Find the CTFT's of these signals in either the f or ω form, whichever is more convenient.

(a) $x(t) = 3\cos(10t) + 4\sin(10t)$

$\omega_0 = 10$ and $f_0 = \frac{5}{\pi}$ Therefore the ω form is slightly more convenient.

$$X(j\omega) = 3\pi[\delta(\omega - 10) + \delta(\omega + 10)] + j4\pi[\delta(\omega + 10) - \delta(\omega - 10)]$$

$$X(j\omega) = (3 - j4)\pi\delta(\omega - 10) + (3 + j4)\pi\delta(\omega + 10)$$

$$X(j\omega) = (5\pi e^{-j0.927})\delta(\omega - 10) + (5\pi e^{j0.927})\delta(\omega + 10)$$

(b) $x(t) = \mathrm{comb}\left(\frac{t}{2}\right) - \mathrm{comb}\left(\frac{t-1}{2}\right)$

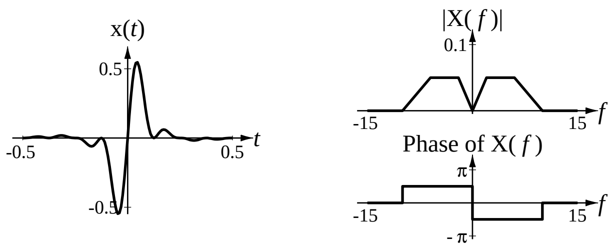
(c) $x(t) = 4\,\mathrm{sinc}(4t) - 2\,\mathrm{sinc}\left(4\left(t - \frac{1}{4}\right)\right) - 2\,\mathrm{sinc}\left(4\left(t + \frac{1}{4}\right)\right)$

(d) $x(t) = \left[2e^{(-1+j2\pi)t} + 2e^{(-1-j2\pi)t}\right]u(t)$ (e) $x(t) = 4e^{-\frac{|t|}{16}}$

22. Sketch the magnitudes and phases of these functions. Sketch the inverse CTFT's of the functions also.

(a) $X(j\omega) = \frac{10}{3 + j\omega} - \frac{4}{5 + j\omega}$ (b) $X(f) = 4\left[\mathrm{sinc}\left(\frac{f-1}{2}\right) + \mathrm{sinc}\left(\frac{f+1}{2}\right)\right]$

(c) $\mathrm{x}(t) = 1.6\,\mathrm{sinc}^2(8t)\sin(4\pi t) \xleftrightarrow{\mathcal{F}} \mathrm{X}(f) = \frac{j}{10}\left[\mathrm{tri}\left(\frac{f+2}{8}\right) - \mathrm{tri}\left(\frac{f-2}{8}\right)\right]$



(d) $\mathrm{X}(f) = \delta(f+1050) + \delta(f+950) + \delta(f-950) + \delta(f-1050)$

(e) $\mathrm{X}(f) = \begin{bmatrix} \delta(f+1050) + 2\delta(f+1000) + \delta(f+950) \\ +\delta(f-950) + 2\delta(f-1000) + \delta(f-1050) \end{bmatrix}$

23. Sketch these signals versus time. Sketch the magnitudes and phase of their CTFT's in either the f or ω form, whichever is more convenient.

(a) $\mathrm{x}(t) = \mathrm{rect}(2t) * \mathrm{comb}(t) - \mathrm{rect}(2t) * \mathrm{comb}\left(t - \frac{1}{2}\right)$

$$\mathrm{X}(f) = \frac{1}{2}\mathrm{sinc}\left(\frac{f}{2}\right)\mathrm{comb}(f)\left(1 - e^{-j\pi f}\right)$$

$$\mathrm{X}(f) = je^{-j\frac{\pi f}{2}}\,\mathrm{sinc}\left(\frac{f}{2}\right)\mathrm{comb}(f)\sin\left(\frac{\pi f}{2}\right)$$

$$\mathrm{X}(f) = \sum_{k=-\infty}^{\infty} e^{-j\frac{\pi}{2}(k-1)}\,\mathrm{sinc}\left(\frac{k}{2}\right)\sin\left(\frac{\pi k}{2}\right)\delta(f-k)$$

Non-zero only for odd values of k. At those odd values, $e^{-j\frac{\pi}{2}(k-1)}\sin\left(\frac{\pi k}{2}\right)$, always evaluates to +1. Therefore

$$\mathrm{X}(f) = \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \mathrm{sinc}\left(\frac{k}{2}\right)\delta(f-k)$$

(b)

Same as answer in part (a).

(c) $x(t) = e^{-\frac{t}{4}} u(t) * \sin(2\pi t)$

Find transform, use impulse equivalence property, get a common denominator and simplify.

$$x(t) = \frac{4\sin(2\pi t) - 32\pi\cos(2\pi t)}{1 + 64\pi^2}$$

(d) $x(t) = e^{-\pi t^2} * [\text{rect}(2t) * \text{comb}(t)]$

This is a "Gaussian" smoothing operation. The square wave is heavily smoothed. So heavily, in fact, that about all that remains is the average value (1/2) plus a small sinusoid.

(e) $x(t) = \text{rect}(t) * [\text{tri}(2t) * \text{comb}(t)]$

$x(t)$ is a constant, 1/2. Can you show that by convolving directly?

(f) $x(t) = \text{sinc}(2.01t) * \text{comb}(t)$

Parts (f) and (g) look almost identical, yet the results are quite different. Why? (Hint: The operation of convolving with a sinc function produces an effect commonly known as an ideal lowpass filter. One which makes a very fast transition in the frequency domain from passing to stopping a signal.)

(g) $x(t) = \text{sinc}(1.99t) * \text{comb}(t)$

(h) $x(t) = e^{-t^2} * e^{-t^2}$

A Gaussian convolved with Gaussian produces a Gaussian.

24. Sketch the magnitudes and phases of these functions. Sketch the inverse CTFT's of the functions also.

(a) $\mathrm{X}(f) = \mathrm{sinc}\left(\frac{f}{100}\right) * \left[\delta(f-1000) + \delta(f+1000)\right]$

The time-domain function is a "burst" of a sinusoid.

(b) $\mathrm{X}(f) = \mathrm{sinc}(10f) * \mathrm{comb}(f)$

25. Sketch these signals versus time. Sketch the magnitudes and phases of the CTFT's of these signals in either the f or ω form, whichever is more convenient. In some cases the time sketch may be conveniently done first. In other cases it may be more convenient to do the time sketch after the CTFT has been found, by finding the inverse CTFT.

(a) $\mathrm{x}(t) = e^{-\pi t^2}\sin(20\pi t)$

(b) $\mathrm{x}(t) = \cos(400\pi t)\mathrm{comb}(100t) = \frac{1}{100}\sum_{n=-\infty}^{\infty}\cos(4\pi n)\delta\left(t - \frac{n}{100}\right)$

A graph of this function looks just like a comb function, even though the comb is multiplied by a cosine. Why?

Given that the time-domain function looks like a comb function you should expect its CTFT to look like the CTFT of a comb, which is another comb.

$$\mathrm{X}(f) = \frac{1}{2}\left[\delta(f-200) + \delta(f+200)\right] * \frac{1}{100}\mathrm{comb}\left(\frac{f}{100}\right)$$

$$\mathrm{X}(f) = \frac{1}{200}\left[\underbrace{\mathrm{comb}\left(\frac{f-200}{100}\right)}_{=\mathrm{comb}\left(\frac{f}{100}\right)} + \underbrace{\mathrm{comb}\left(\frac{f+200}{100}\right)}_{=\mathrm{comb}\left(\frac{f}{100}\right)}\right]$$

$$\mathrm{X}(f) = \frac{1}{100}\mathrm{comb}\left(\frac{f}{100}\right) = \frac{1}{100}\sum_{k=-\infty}^{\infty}\delta\left(\frac{f}{100} - k\right) = \sum_{k=-\infty}^{\infty}\delta(f - 100k)$$

(c) $\mathrm{x}(t) = \left[1 + \cos(400\pi t)\right]\cos(4000\pi t)$

(d) $\mathrm{x}(t) = \left[1 + \mathrm{rect}(100t) * 50\,\mathrm{comb}(50t)\right]\cos(500\pi t)$

(e) $\mathrm{x}(t) = \mathrm{rect}\left(\frac{t}{7}\right)\mathrm{comb}(t)$

$$\mathrm{X}(f) = 7\,\mathrm{sinc}(7f) * \mathrm{comb}(f) = 7\,\mathrm{sinc}(7f) * \sum_{k=-\infty}^{\infty} \delta(f-k)$$

$$\mathrm{X}(f) = 7\sum_{k=-\infty}^{\infty} \mathrm{sinc}(7(f-k))$$

This periodically-repeated sinc function is equivalent to a Dirichlet function.

26. Sketch the magnitudes and phases of these functions. Sketch the inverse CTFT's of the functions also.

(a) $\mathrm{X}(f) = \mathrm{sinc}\left(\frac{f}{4}\right)\mathrm{comb}(f)$

(b) $\mathrm{X}(f) = \left[\mathrm{sinc}\left(\frac{f-1}{4}\right) + \mathrm{sinc}\left(\frac{f+1}{4}\right)\right]\mathrm{comb}(f)$

(c) $\mathrm{X}(f) = \mathrm{sinc}(f)\mathrm{sinc}(2f)$

$$\mathrm{x}(t) = \mathrm{rect}(t) * \frac{1}{2}\mathrm{rect}\left(\frac{t}{2}\right) = \frac{1}{2}\mathrm{rect}(t) * \mathrm{rect}\left(\frac{t}{2}\right)$$

The result of the graphical convolution can be expressed in the form,

$\mathrm{x}(t) = \frac{1}{4}\left[3\mathrm{tri}\left(\frac{2t}{3}\right) - \mathrm{tri}(2t)\right]$. A generalization of this result his leads to the pair,

$$\frac{a+b}{2}\mathrm{tri}\left(\frac{2t}{a+b}\right) - \frac{a-b}{2}\mathrm{tri}\left(\frac{2t}{a-b}\right) \xleftrightarrow{\mathcal{F}} |ab|\,\mathrm{sinc}(af)\,\mathrm{sinc}(bf)$$

$$a > b > 0$$

in Appendix E.

27. Sketch these signals versus time and the magnitudes and phases of their CTFT's.

(a) $\mathrm{x}(t) = \frac{d}{dt}[\mathrm{sinc}(t)]$

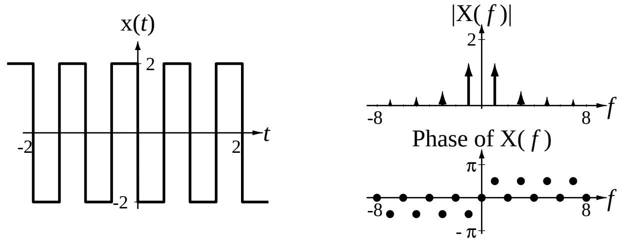
(b) $\mathrm{x}(t) = \frac{d}{dt}\left[4\,\mathrm{rect}\left(\frac{t}{6}\right)\right]$

(c) $\mathrm{x}(t) = \frac{d}{dt}\left[\mathrm{tri}(2t) * \mathrm{comb}(t)\right]$

Using the convolution property that the derivative of a convolution is the convolution of either of the two functions with the derivative of the other,

$$\mathrm{x}(t) = 2\left\{\left[\mathrm{rect}\left(2\left(t+\frac{1}{4}\right)\right) - \mathrm{rect}\left(2\left(t-\frac{1}{4}\right)\right)\right] * \mathrm{comb}(t)\right\}$$

$$\mathrm{X}(f) = j2\pi f\,\frac{1}{2}\mathrm{sinc}^2\left(\frac{f}{2}\right)\mathrm{comb}(f) = j\pi\sum_{k=-\infty}^{\infty} k\,\mathrm{sinc}^2\left(\frac{k}{2}\right)\delta(f-k)$$



28. Sketch these signals versus time and the magnitudes and phases of their CTFT's.

(a) $\mathrm{x}(t) = \int_{-\infty}^{t} \sin(2\pi\lambda)\,d\lambda$

$$\mathrm{X}(f) = \frac{1}{j2\pi f} \times \frac{j}{2}\left[\delta(f+1) - \delta(f-1)\right] + \frac{1}{2}\left[\mathcal{F}\left(\sin(2\pi t)\right)\right]_{f=0}\delta(f) = \frac{\delta(f+1)-\delta(f-1)}{4\pi f}$$

$$\mathrm{X}(f) = -\frac{\delta(f+1)}{4\pi} - \frac{\delta(f-1)}{4\pi} = -\frac{1}{2\pi} \times \frac{1}{2}\left[\delta(f+1) + \delta(f-1)\right]$$

(b) $\mathrm{x}(t) = \int_{-\infty}^{t} \mathrm{rect}(\lambda)\,d\lambda$

(c) $\mathrm{x}(t) = \int_{-\infty}^{t} 3\mathrm{sinc}(2\lambda)\,d\lambda$

Let $u = 2\pi\lambda$. Then $\mathrm{x}(t) = \frac{3}{2\pi}\int_{-\infty}^{2\pi t} \mathrm{sinc}\left(\frac{u}{\pi}\right)du = \frac{3}{2\pi}\int_{-\infty}^{2\pi t} \frac{\sin(u)}{u}\,du$

For $t \le 0$:

$$\mathrm{x}(t) = \frac{3}{2\pi}\left[\int_{-\infty}^{0} \frac{\sin(u)}{u}\,du - \int_{2\pi t}^{0} \frac{\sin(u)}{u}\,du\right] = \frac{3}{2\pi}\left[-\int_{0}^{-\infty} \frac{\sin(u)}{u}\,du + \int_{0}^{2\pi t} \frac{\sin(u)}{u}\,du\right]$$

$$\mathrm{x}(t) = \frac{3}{2\pi}\left[\underbrace{\int_{0}^{\infty} \frac{\sin(\lambda)}{\lambda}\,d\lambda}_{=\mathrm{Si}(\infty)=\frac{\pi}{2}} + \underbrace{\int_{0}^{2\pi t} \frac{\sin(u)}{u}\,du}_{=\mathrm{Si}(2\pi t)}\right] = \frac{3}{2\pi}\left[\frac{\pi}{2} + \mathrm{Si}(2\pi t)\right] = \frac{3}{2\pi}\left[\frac{\pi}{2} - \mathrm{Si}(-2\pi t)\right]$$

"Si" is called the sine integral. It is a special function of calculus and can be computed by the MATLAB function, `sinint`.

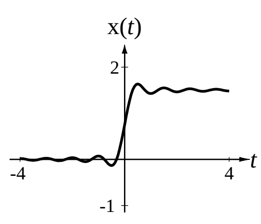
For $t \geq 0$:

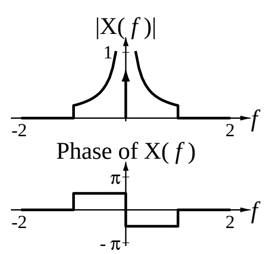
$$\mathrm{x}(t) = \frac{3}{2\pi}\left[\int_{-\infty}^{0} \frac{\sin(u)}{u}\,du + \int_{0}^{2\pi t} \frac{\sin(u)}{u}\,du\right] = \frac{3}{2\pi}\left[-\int_{0}^{-\infty} \frac{\sin(u)}{u}\,du + \int_{0}^{2\pi t} \frac{\sin(u)}{u}\,du\right]$$

$$\mathrm{x}(t) = \frac{3}{2\pi}\left[\frac{\pi}{2} + \mathrm{Si}(2\pi t)\right]$$

Therefore, for any t, $\mathrm{x}(t) = \frac{3}{2\pi}\left[\frac{\pi}{2} + \mathrm{Si}(2\pi t)\right]$.

$$\mathrm{X}(f) = \frac{1}{j2\pi f}\frac{3}{2}\mathrm{rect}\left(\frac{f}{2}\right) + \frac{1}{2}\mathcal{F}\left[3\,\mathrm{sinc}(2t)\right]_{f=0}\delta(f) = \frac{3}{j4\pi f}\mathrm{rect}\left(\frac{f}{2}\right) + \frac{3}{4}\delta(f)$$





29. From the definition, find the DTFT of

$$\mathrm{x}[n] = 10\,\mathrm{rect}_4[n]\ .$$

and compare with the Fourier transform table in Appendix E.

Apply the definition and put into closed form by using the formula for the summation of a geometric series,

$$\sum_{n=0}^{N-1} r^n = \begin{cases} 1 & , \quad |r| = 1 \\ \dfrac{1-r^N}{1-r} & , \quad \text{otherwise} \end{cases} .$$

Then convert to a sine function by factoring out the proper complex exponential and recognize the ratio of two sine functions as a Dirichlet function. Check your answer against Appendix E.

30. From the definition, derive a general expression for the F and Ω forms of the DTFT of functions of the form,

$$\mathrm{x}[n] = A\sin(2\pi F_0 n) = A\sin(\Omega_0 n) .$$

(It should remind you of the CTFT of $\mathrm{x}(t) = A\sin(2\pi f_0 t) = A\sin(\omega_0 t)$.) Compare with the Fourier transform table in Appendix E.

$$\mathrm{X}(F) = \sum_{n=-\infty}^{\infty} \mathrm{x}[n] e^{-j2\pi Fn} = \sum_{n=-\infty}^{\infty} A\sin(2\pi F_0 n) e^{-j2\pi Fn} = A \sum_{n=-\infty}^{\infty} \frac{e^{j2\pi F_0 n} - e^{-j2\pi F_0 n}}{j2} e^{-j2\pi Fn}$$

$$\mathrm{X}(F) = \frac{A}{j2} \sum_{n=-\infty}^{\infty} \left[e^{j2\pi(F_0 - F)n} - e^{-j2\pi(F_0 + F)n} \right]$$

Then, using

$$\sum_{n=-\infty}^{\infty} e^{j2\pi xn} = \mathrm{comb}(x)$$

we get

$$\mathrm{X}(F) = \frac{A}{j2} \left[\mathrm{comb}(F_0 - F) - \mathrm{comb}(-F_0 - F) \right] = A\frac{j}{2} \left[-\mathrm{comb}(F_0 - F) + \mathrm{comb}(-F_0 - F) \right]$$

$$\mathrm{X}(F) = A\frac{j}{2} \left[\mathrm{comb}(F + F_0) - \mathrm{comb}(F - F_0) \right]$$

The Ω form can be found by the transformation, $F \rightarrow \dfrac{\Omega}{2\pi}$.

$$\mathrm{X}(j\Omega) = A\frac{j}{2}\left[\mathrm{comb}\left(\frac{\Omega}{2\pi}+\frac{\Omega_0}{2\pi}\right)-\mathrm{comb}\left(\frac{\Omega}{2\pi}-\frac{\Omega_0}{2\pi}\right)\right]$$

$$\mathrm{X}(j\Omega) = j\pi A\left[\mathrm{comb}(\Omega+\Omega_0)-\mathrm{comb}(\Omega-\Omega_0)\right]$$

31. A DT signal is defined by

$$\mathrm{x}[n] = \mathrm{sinc}\left(\frac{n}{8}\right) .$$

Sketch the magnitude and phase of the DTFT of $\mathrm{x}[n-2]$.

32. A DT signal is defined by

$$\mathrm{x}[n] = \sin\left(\frac{\pi n}{6}\right) .$$

Sketch the magnitude and phase of the DTFT of $\mathrm{x}[n-3]$ and $\mathrm{x}[n+12]$.

The DTFT of $\mathrm{x}[n+12]$ should be exactly the same as the DTFT of $\mathrm{x}[n]$. Why?

33. The DTFT of a DT signal is defined by

$$\mathrm{X}(j\Omega) = 4\left[\mathrm{rect}\left(\frac{2}{\pi}\left(\Omega-\frac{\pi}{2}\right)\right)+\mathrm{rect}\left(\frac{2}{\pi}\left(\Omega+\frac{\pi}{2}\right)\right)\right]*\mathrm{comb}\left(\frac{\Omega}{2\pi}\right) .$$

Sketch $\mathrm{x}[n]$.

Start with

$$\mathrm{sinc}\left(\frac{n}{w}\right) \xleftrightarrow{\mathcal{F}} w\,\mathrm{rect}\left(\frac{w\Omega}{2\pi}\right)*\mathrm{comb}\left(\frac{\Omega}{2\pi}\right)$$

and apply the frequency-shifting and linearity properties to produce (after simplification)

$$2\,\mathrm{sinc}\left(\frac{n}{4}\right)\cos\left(\frac{\pi}{2}n\right) \xleftrightarrow{\mathcal{F}} 4\left[\mathrm{rect}\left(\frac{2}{\pi}\left(\Omega-\frac{\pi}{2}\right)\right)+\mathrm{rect}\left(\frac{2}{\pi}\left(\Omega+\frac{\pi}{2}\right)\right)\right]*\mathrm{comb}\left(\frac{\Omega}{2\pi}\right)$$

Remember in applying the frequency-shifting property, if either (but not both) of two functions being convolved shifts, the result of the convolution shifts by the same amount.

34. Sketch the magnitude and phase of the DTFT of

$$\mathrm{x}[n] = \mathrm{rect}_4[n] * \cos\left(\frac{2\pi n}{6}\right).$$

Then sketch $\mathrm{x}[n]$.

From the table,

$$\mathrm{rect}_{N_w}[n] \xleftrightarrow{\mathcal{F}} (2N_w + 1)\,\mathrm{drcl}(F, 2N_w + 1)$$

and

$$\cos(2\pi F_0 n) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\mathrm{comb}(F - F_0) + \mathrm{comb}(F + F_0)\right]$$

$$\mathrm{X}(F) = 9\,\mathrm{drcl}(F, 9) \times \frac{1}{2}\left[\mathrm{comb}\left(F - \frac{1}{6}\right) + \mathrm{comb}\left(F + \frac{1}{6}\right)\right]$$

Since both functions are periodic with period, one, at every impulse in the comb function the value of the Dirichlet function will be the same.

$$\mathrm{X}(F) = \frac{9}{2}\left[\mathrm{drcl}\left(\frac{1}{6}, 9\right)\mathrm{comb}\left(F - \frac{1}{6}\right) + \mathrm{drcl}\left(-\frac{1}{6}, 9\right)\mathrm{comb}\left(F + \frac{1}{6}\right)\right]$$

$$\mathrm{X}(F) = \frac{9}{2}\underbrace{\mathrm{drcl}\left(\frac{1}{6}, 9\right)}_{\frac{\sin\left(\frac{3\pi}{2}\right)}{9\sin\left(\frac{\pi}{6}\right)}}\left[\mathrm{comb}\left(F - \frac{1}{6}\right) + \mathrm{comb}\left(F + \frac{1}{6}\right)\right]$$

$$\mathrm{X}(F) = -\left[\mathrm{comb}\left(F - \frac{1}{6}\right) + \mathrm{comb}\left(F + \frac{1}{6}\right)\right]$$

Then, using

$$\cos(2\pi F_0 n) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\mathrm{comb}(F - F_0) + \mathrm{comb}(F + F_0)\right]$$

$$-2\cos\left(\frac{2\pi n}{6}\right) \xleftrightarrow{\mathcal{F}} -\left[\mathrm{comb}\left(F - \frac{1}{6}\right) + \mathrm{comb}\left(F + \frac{1}{6}\right)\right]$$

and, therefore,

$$\mathrm{x}[n] = -2\cos\left(\frac{2\pi n}{6}\right)$$

35. Sketch the inverse DTFT of

$$\mathrm{X}(F) = [\mathrm{rect}(4F) * \mathrm{comb}(F)] \otimes \mathrm{comb}(2F) \ .$$

Find the individual inverse DTFT's and multiply in the time domain.

36. Using the differencing property of the DTFT and the transform pair,

$$\mathrm{tri}\left(\frac{n}{2}\right) \xleftrightarrow{\mathcal{F}} 1 + \cos(2\pi F) \ ,$$

find the DTFT of $\frac{1}{2}(\delta[n+1] + \delta[n] - \delta[n-1] - \delta(n-2))$. Compare it with Fourier transform found using the table in Appendix E.

The first backward difference of $\mathrm{tri}\left(\frac{n}{2}\right)$ is $\frac{1}{2}(\delta[n+1] + \delta[n] - \delta[n-1] - \delta(n-2))$.

Apply the differencing property and simplify.

Other route to the DTFT:

$$\frac{1}{2}(\delta[n+1] + \delta[n] - \delta[n-1] - \delta[n-2]) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left(e^{j2\pi F} + 1 - e^{-j2\pi F} - e^{-j4\pi F}\right)$$

37. Using Parseval's theorem, find the signal energy of

$$\mathrm{x}[n] = \mathrm{sinc}\left(\frac{n}{10}\right)\sin\left(\frac{2\pi n}{4}\right) \ .$$

$$E_x = \sum_{n=-\infty}^{\infty} |\mathrm{x}[n]|^2 = \int_1 |\mathrm{X}(F)|^2 dF$$

Find the individual DTFT's, periodically convolve them in F and integrate the square of the magnitude of that result over one period (one). Remember, periodic convolution of two periodic functions is the same as the aperiodic convolution of one period of either function with the entire other function.

$$E_x = \int_1 \left| j5\left[\text{rect}(10F) * \text{comb}\left(F + \frac{1}{4}\right) - \text{rect}(10F) * \text{comb}\left(F - \frac{1}{4}\right) \right] \right|^2 dF$$

Since we are integrating only over a range of one, only one impulse in each comb is significant.

$$E_x = 25\int_1 \left| \left[\text{rect}\left(10\left(F + \frac{1}{4}\right)\right) - \text{rect}\left(10\left(F - \frac{1}{4}\right)\right) \right] \right|^2 dF$$

The square of the sum equals the sum of the squares because there is no cross product; the two rectangles do not overlap.

$$E_x = 5$$

38. Sketch the magnitude and phase of the CTFT of

$$x_1(t) = \text{rect}(t)$$

and of the CTFS of

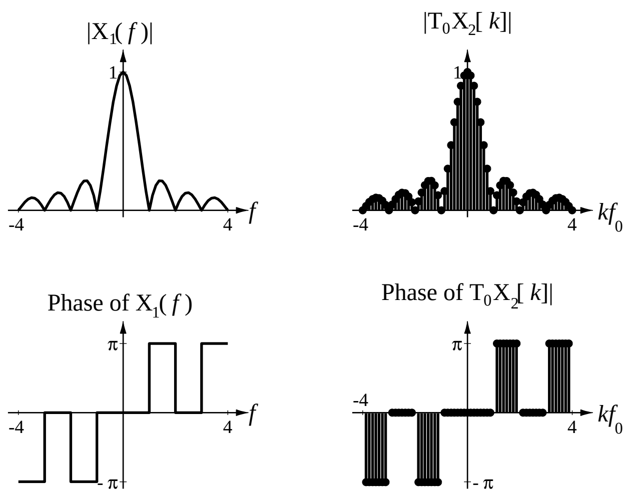
$$x_2(t) = \text{rect}(t) * \frac{1}{8}\text{comb}\left(\frac{t}{8}\right).$$

For comparison purposes, sketch $X_1(f)$ versus f and $T_0 X_2[k]$ versus kf_0 on the same set of axes. (T_0 is the period of $x_2(t)$ and $T_0 = \frac{1}{f_0}$.)

$$X_1(f) = \text{sinc}(f)$$

Using the relationship between an the CTFT of an aperiodic signal and the CTFS of a periodic extension of that signal,

$$X_2[k] = f_s X_1(f_s k) = \frac{1}{8}\text{sinc}\left(\frac{k}{8}\right)$$



39. Sketch the magnitude and phase of the CTFT of

$$x_1(t) = 4\cos(4\pi t)$$

and of the DTFT of

$$x_2[n] = x_1(nT_s)$$

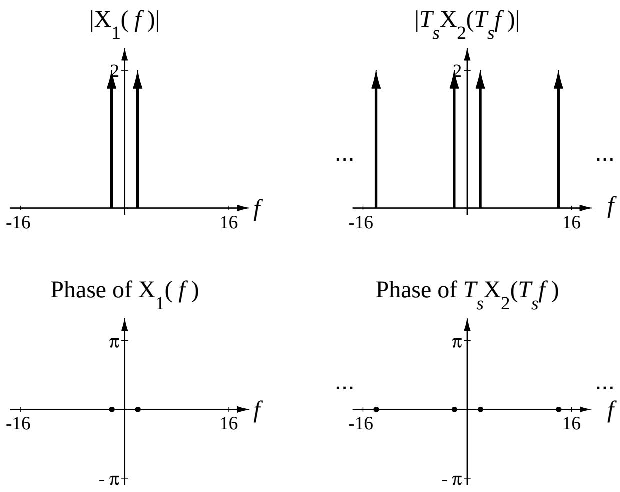
where $T_s = \dfrac{1}{16}$. For comparison purposes sketch $X_1(f)$ and $T_s X_2(T_s f)$ versus f on the same set of axes.

$$X_1(f) = 2[\delta(f-2) + \delta(f+2)]$$

$$X_2(F) = 2\left[\text{comb}\left(F - \frac{1}{8}\right) + \text{comb}\left(F + \frac{1}{8}\right)\right]$$

$$X_2(T_s f) = 2\sum_{k=-\infty}^{\infty}\left[\delta\left(\frac{f}{16} - \frac{1}{8} - k\right) + \delta\left(\frac{f}{16} + \frac{1}{8} - k\right)\right]$$

$$T_s X_2(T_s f) = 2\sum_{k=-\infty}^{\infty}\left[\delta(f - 2 - 16k) + \delta(f + 2 - 16k)\right]$$



40. Sketch the magnitude and phase of the DTFT of

$$x_1[n] = \frac{\operatorname{sinc}\left(\frac{n}{16}\right)}{4}$$

and of the DTFS of

$$x_2[n] = \frac{\operatorname{sinc}\left(\frac{n}{16}\right)}{4} * \operatorname{comb}_{32}[n] \ .$$

For comparison purposes sketch $X_1(F)$ versus F and $N_0 X_2[k]$ versus kF_0 on the same set of axes.

Similar to Exercise 38.

41. A system is excited by a signal,

$$x(t) = 4\operatorname{rect}\left(\frac{t}{2}\right)$$

$$y(t) = 4\left(1 - e^{-(t+1)}\right)u(t+1) - 4\left(1 - e^{-(t-1)}\right)u(t-1)$$

and its response is

$$y(t) = 10\left[\left(1 - e^{-(t+1)}\right)u(t+1) - \left(1 - e^{-(t-1)}\right)u(t-1)\right] \ .$$

What is its impulse response?

Find the CTFT of both x and y. Take their ratio which is the transfer function, H. Find the inverse transform of H which is h, the impulse response.

$$\mathrm{h}(t) = \frac{5}{2} e^{-t} \mathrm{u}(t)$$

42. Sketch the magnitudes and phases of the CTFT's of the following functions.

(a) $\mathrm{g}(t) = 5\delta(4t)$

(b) $\mathrm{g}(t) = \mathrm{comb}\left(\frac{t+1}{4}\right) - \mathrm{comb}\left(\frac{t-3}{4}\right)$

(c) $\mathrm{g}(t) = \mathrm{u}(2t) + \mathrm{u}(t-1)$

Since $\mathrm{u}(2t)$ has the same value as $\mathrm{u}(t)$ for any t, $\mathrm{u}(2t) = \mathrm{u}(t)$ and their transforms must also be equal when using the time scaling property.

(d) $\mathrm{g}(t) = \mathrm{sgn}(t) - \mathrm{sgn}(-t)$

(e) $\mathrm{g}(t) = \mathrm{rect}\left(\frac{t+1}{2}\right) + \mathrm{rect}\left(\frac{t-1}{2}\right)$

$$\mathrm{rect}\left(\frac{t+1}{2}\right) + \mathrm{rect}\left(\frac{t-1}{2}\right) \xleftrightarrow{\mathcal{F}} 2\,\mathrm{sinc}(2f)e^{j2\pi f} + 2\,\mathrm{sinc}(2f)e^{-j2\pi f}$$

$$\mathrm{rect}\left(\frac{t+1}{2}\right) + \mathrm{rect}\left(\frac{t-1}{2}\right) \xleftrightarrow{\mathcal{F}} 4\,\mathrm{sinc}(2f)\cos(2\pi f)$$

Using the definition of the sinc function,

$$\mathrm{rect}\left(\frac{t+1}{2}\right) + \mathrm{rect}\left(\frac{t-1}{2}\right) \xleftrightarrow{\mathcal{F}} 4\,\frac{\sin(2\pi f)\cos(2\pi f)}{2\pi f}$$

Using $\sin(x)\cos(y) = \frac{1}{2}\left[\sin(x-y) + \sin(x+y)\right]$

$$\mathrm{rect}\left(\frac{t+1}{2}\right) + \mathrm{rect}\left(\frac{t-1}{2}\right) \xleftrightarrow{\mathcal{F}} 4\,\frac{\frac{1}{2}\left[\sin(0) + \sin(4\pi f)\right]}{2\pi f} = \frac{\sin(4\pi f)}{\pi f} = 4\,\mathrm{sinc}(4f)$$

(f) $\mathrm{g}(t)=\mathrm{rect}\left(\frac{t}{4}\right)$

Same answer as in (e) because the function, $\mathrm{g}(t)=\mathrm{rect}\left(\frac{t}{4}\right)$ is the same as the

function, $\mathrm{g}(t)=\mathrm{rect}\left(\frac{t+1}{2}\right)+\mathrm{rect}\left(\frac{t-1}{2}\right)$.

(g) $\mathrm{g}(t)=5\,\mathrm{tri}\left(\frac{t}{5}\right)-2\,\mathrm{tri}\left(\frac{t}{2}\right)$

(h) $\mathrm{g}(t)=\frac{3}{2}\mathrm{rect}\left(\frac{t}{8}\right)*\mathrm{rect}\left(\frac{t}{2}\right)$

43. Sketch the magnitudes and phases of the CTFT's of the following functions.

(a) $\mathrm{rect}(4t)$

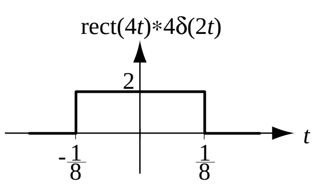
(b) $\mathrm{rect}(4t)*4\delta(t)$

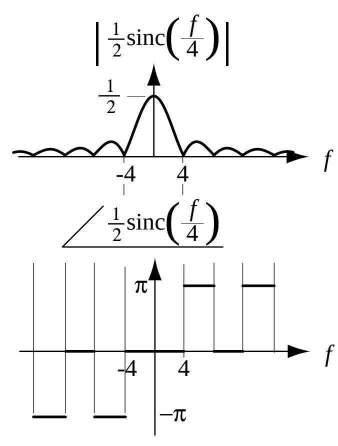
(c) $\mathrm{rect}(4t)*4\delta(t-2)$

(d) $\mathrm{rect}(4t)*4\delta(2t)$

$$\mathrm{rect}(4t)*4\delta(2t)\xleftrightarrow{\mathcal{F}}\frac{1}{4}\mathrm{sinc}\left(\frac{f}{4}\right)\times 2=\frac{1}{2}\mathrm{sinc}\left(\frac{f}{4}\right)$$

$$\mathrm{rect}(4t)*4\delta(2t)\xleftrightarrow{\mathcal{F}}=\frac{1}{2}\mathrm{sinc}\left(\frac{\omega}{8\pi}\right)$$



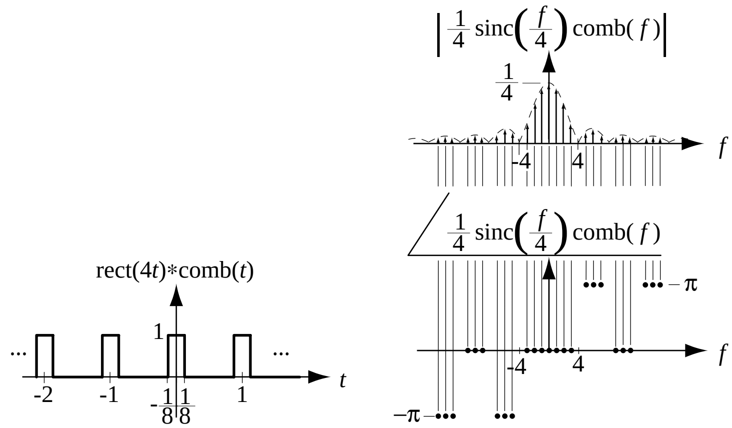


(e) $\mathrm{rect}(4t)*\mathrm{comb}(t)$

$$\mathrm{rect}(4t)*\mathrm{comb}(t)\xleftrightarrow{\mathcal{F}}\frac{1}{4}\mathrm{sinc}\left(\frac{f}{4}\right)\mathrm{comb}(f)=\frac{1}{4}\mathrm{sinc}\left(\frac{f}{4}\right)\sum_{k=-\infty}^{\infty}\delta(f-k)$$

$$\text{rect}(4t) * \text{comb}(t) \xleftrightarrow{\mathcal{F}} \frac{1}{4}\sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{k}{4}\right)\delta(f-k)$$

$$\text{rect}(4t) * \text{comb}(t) \xleftrightarrow{\mathcal{F}} \frac{1}{4}\sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{k}{4}\right)\delta\left(\frac{\omega}{2\pi}-k\right) = \frac{\pi}{2}\sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{k}{4}\right)\delta(\omega-2\pi k)$$



(f) $\text{rect}(4t) * \text{comb}(t-1)$

Same as part (e).

(g) $\text{rect}(4t) * \text{comb}(2t)$

(h) $\text{rect}(t) * \text{comb}(2t)$

CTFT is $\delta(f)$.

44. Plot these signals over two periods centered at $t = 0$.

(a) $\quad \text{x}(t) = 2\cos(20\pi t) + 4\sin(10\pi t) + 3\cos(-20\pi t) - 3\sin(-10\pi t)$

(b) $\quad \text{x}(t) = 5\cos(20\pi t) + 7\sin(10\pi t)$

Compare the results of parts (a) and (b).

This Exercise is intended to show the equivalence between positive and negative frequencies.

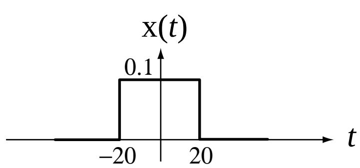
45. A periodic signal has a period of four seconds.

(a) What is the lowest positive frequency at which its CTFT could be non-zero?

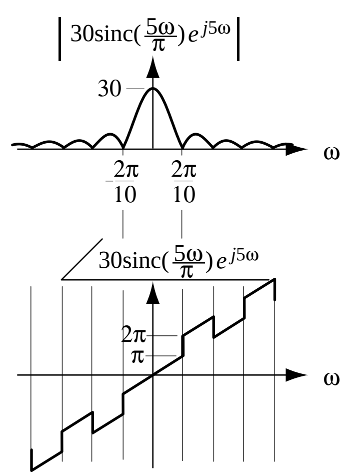
(b) What is the next-lowest positive frequency at which its CTFT could be non-zero?

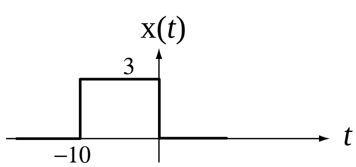
46. Sketch the magnitude and phase of the CTFT of each of the following signals (ω form):

(a)

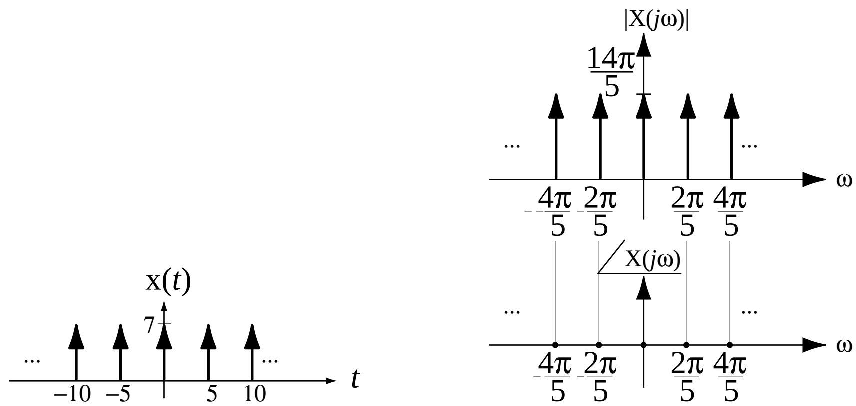


(b) $\mathrm{x}(t) = 3\,\mathrm{rect}\left(\frac{t+5}{10}\right)$





(c) $\mathrm{x}(t) = \frac{7}{5}\,\mathrm{comb}\left(\frac{t}{5}\right)$



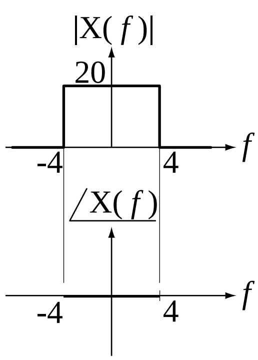
(d) $\mathrm{x}(t) = \frac{7}{5}\mathrm{comb}\left(\frac{t-2}{5}\right)$

Compare this CTFT with the CTFT of $\frac{7}{5}\mathrm{comb}\left(\frac{t+3}{5}\right)$. Since the two time-

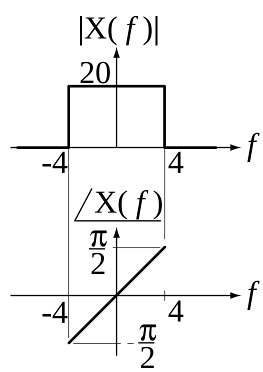
domain

signals are the same, the two CTFT's must be the same also. Are they?

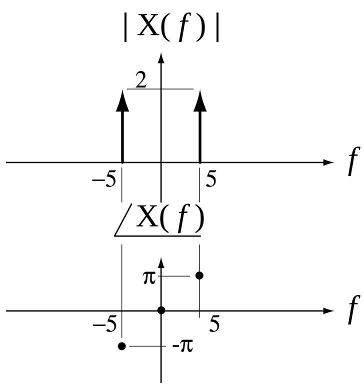
47. Sketch the inverse CTFT's of the following functions:



(a)

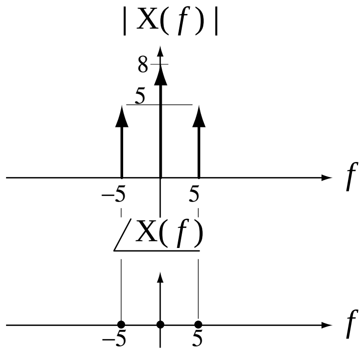


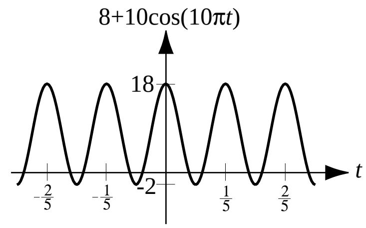
(b)



(c)

(d) $\mathrm{X}(f) = 8\delta(f) + 5\delta(f-5) + 5\delta(f+5)$





48. Find the inverse CTFT of this real, frequency-domain function (Figure E48) and sketch it. (Let $A = 1$, $f_1 = 95\,\mathrm{kHz}$ and $f_2 = 105\,\mathrm{kHz}$.)

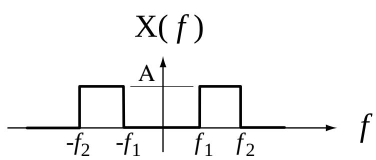
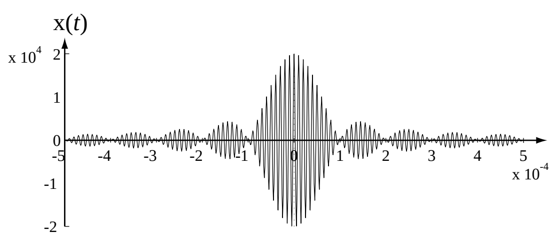


Figure E48 A real frequency-domain function



49. Find the CTFT (either form) of this signal (Figure E49) and sketch its magnitude and phase versus frequency on separate graphs. (Let $A = -B = 1$ and let $t_1 = 1$ and $t_2 = 2$.) Hint: Express this signal as the sum of two functions and use the linearity property.

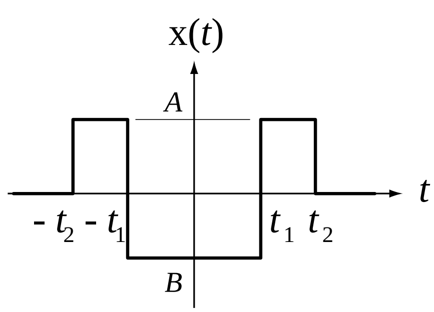
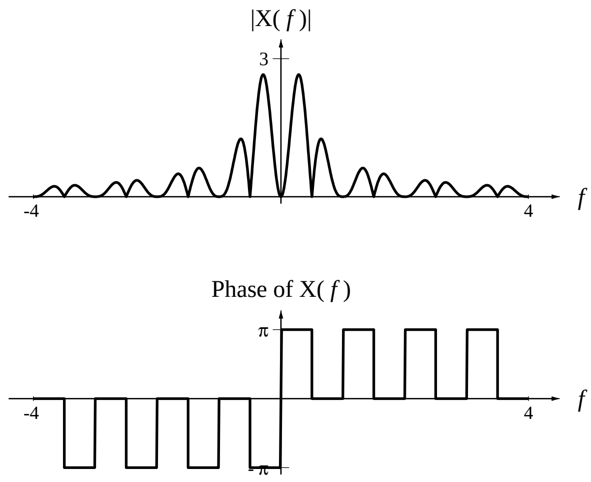


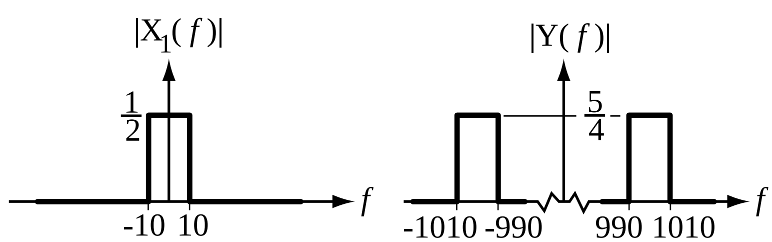
Figure E49 A CT function



50. In many communication systems a device called a "mixer" is used. In its simplest form a mixer is simply an analog multiplier. That is, its response signal, y(*t*), is the product of its two excitation signals. If the two excitation signals are

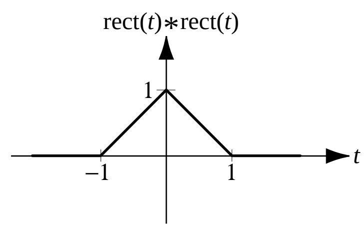
$$x_1(t) = 10\,\mathrm{sinc}(20t) \quad \text{and} \quad x_2(t) = 5\cos(2000\pi t)$$

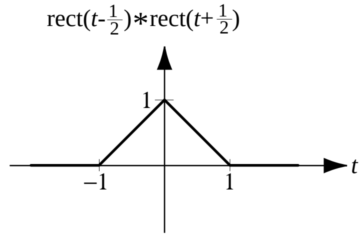
plot the magnitude of the CTFT of y(*t*), $Y(f)$, and compare it to the magnitude of the CTFT of $x_1(t)$. In simple terms what does a mixer do?



51. Sketch a graph of the convolution of the two functions in each case:

(a) $\mathrm{rect}(t) * \mathrm{rect}(t)$

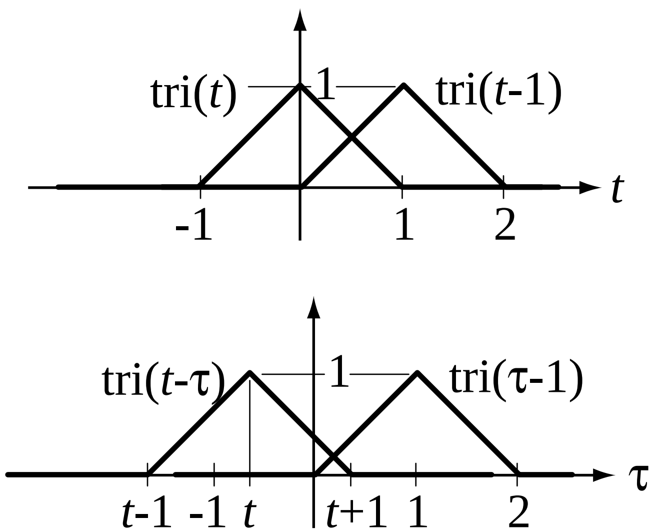




(b) $\text{rect}\left(t-\frac{1}{2}\right)*\text{rect}\left(t+\frac{1}{2}\right)$

(c) $\text{tri}(t)*\text{tri}(t-1)$

This is a very challenging problem. It cannot be done using the transforms and tables in Appendix E but must be done in the time domain.



For t < -1, the non-zero portions of the two functions do not overlap and the convolution is zero.

For t > 3, the non-zero portions of the two functions do not overlap and the convolution is zero.

For -1 < t < 0:

The non-zero portions overlap for 0 < τ < t+1 and, in that range of τ,

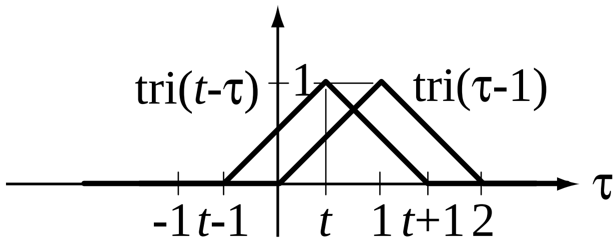
$$\text{tri}(t-\tau)=t+1-\tau \quad \text{and} \quad \text{tri}(\tau-1)=\tau$$

Therefore, for -1 < t < 0,

$$\text{tri}(t) * \text{tri}(t-1) = \int_0^{t+1} (t+1-\tau)\tau d\tau = \int_0^{t+1} \left[(t+1)\tau - \tau^2\right] d\tau$$

$$\text{tri}(t) * \text{tri}(t-1) = \left[(t+1)\frac{\tau^2}{2} - \frac{\tau^3}{3}\right]_0^{t+1} = \frac{(t+1)^3}{2} - \frac{(t+1)^3}{3} = \frac{(t+1)^3}{6}$$

For 0 < t < 1:



$$\text{tri}(t) * \text{tri}(t-1) = \int_{-\infty}^{\infty} \text{tri}(t-\tau)\text{tri}(\tau) d\tau$$

The non-zero portions overlap for $0 < \tau < t+2$ and, in that range of τ, there are three cases to consider, $0 < \tau < t$, $t < \tau < 1$ and $1 < \tau < t+1$. Therefore

$$\text{tri}(t) * \text{tri}(t-1) = \int_0^{t} \text{tri}(t-\tau)\text{tri}(\tau) d\tau + \int_t^{1} \text{tri}(t-\tau)\text{tri}(\tau) d\tau + \int_1^{t+1} \text{tri}(t-\tau)\text{tri}(\tau) d\tau$$

Case 1: $0 < \tau < t$

$$\text{tri}(t-\tau) = 1 - t + \tau \quad \text{and} \quad \text{tri}(\tau - 1) = \tau$$

Case 2: $t < \tau < 1$

$$\text{tri}(t-\tau) = 1 + t - \tau \quad \text{and} \quad \text{tri}(\tau - 1) = \tau$$

Case 3: $1 < \tau < t+1$

$$\text{tri}(t-\tau) = 1 + t - \tau \quad \text{and} \quad \text{tri}(\tau - 1) = 2 - \tau$$

Therefore

$$\text{tri}(t) * \text{tri}(t-1) = \int_0^{t} (1 - t + \tau)\tau d\tau + \int_t^{1} (1 + t - \tau)\tau d\tau + \int_1^{t+1} (1 + t - \tau)(2 - \tau) d\tau$$

$$\text{tri}(t) * \text{tri}(t-1) = \int_0^{t} \left[(1-t)\tau + \tau^2\right] d\tau + \int_t^{1} \left[(1+t)\tau - \tau^2\right] d\tau + \int_1^{t+1} \left[2(1+t) - 2\tau - (1+t)\tau + \tau^2\right] d\tau$$

$$\text{tri}(t)*\text{tri}(t-1)=\left[(1-t)\frac{\tau^2}{2}+\frac{\tau^3}{3}\right]_0^t+\left[(1+t)\frac{\tau^2}{2}-\frac{\tau^3}{3}\right]_t^1+\left[2(1+t)\tau-\tau^2-(1+t)\frac{\tau^2}{2}+\frac{\tau^3}{3}\right]_1^{t+1}$$

$$\text{tri}(t)*\text{tri}(t-1)=\left[(1-t)\frac{t^2}{2}+\frac{t^3}{3}\right]+\left[(1+t)\frac{1}{2}-\frac{1}{3}-(1+t)\frac{t^2}{2}+\frac{t^3}{3}\right]$$
$$+\left[2(1+t)^2-(t+1)^2-(1+t)\frac{(t+1)^2}{2}+\frac{(t+1)^3}{3}-2(1+t)+1+(1+t)\frac{1}{2}-\frac{1}{3}\right]$$

$$\text{tri}(t)*\text{tri}(t-1)=(1-t)\frac{t^2}{2}+\frac{2t^3}{3}-(1+t)\left(1+\frac{t^2}{2}\right)+\frac{1}{3}+(1+t)^2-\frac{(t+1)^3}{6}$$

$$\text{tri}(t)*\text{tri}(t-1)=-\frac{t^3}{2}+\frac{t^2}{2}+\frac{t}{2}+\frac{1}{6}$$

For the remaining regions of t, the convolution simply repeats with even symmetry about the point, t = 1. The analytical solutions can be found by the following successive changes of variable:

$$t\to t+1\,,\ t\to -t\,,\ t\to t-1$$

These three successive changes of variable can be condensed into one,

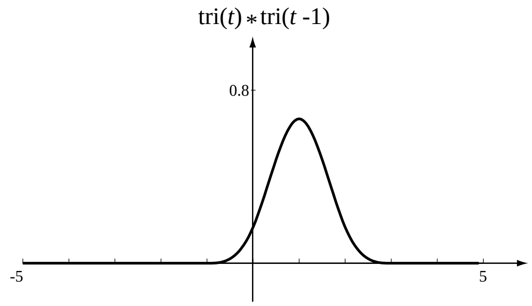
$$t\to -t+2$$

Then, for 1 < t < 2,

$$\text{tri}(t)*\text{tri}(t-1)=\left[-\frac{t^3}{2}+\frac{t^2}{2}+\frac{t}{2}+\frac{1}{6}\right]_{t\to -t+2}=\left[-\frac{(2-t)^3}{2}+\frac{(2-t)^2}{2}+\frac{(2-t)}{2}+\frac{1}{6}\right]$$

and, for 2 < t < 3,

$$\text{tri}(t)*\text{tri}(t-1)=\left[\frac{(t+1)^3}{6}\right]_{t\to -t+2}=\frac{[(-t+2)+1]^3}{6}$$



(d) $3\delta(t) * 10\cos(t)$

(e) $10\,\text{comb}(t) * \text{rect}(t)$ (f) $5\,\text{comb}(t) * \text{tri}(t)$

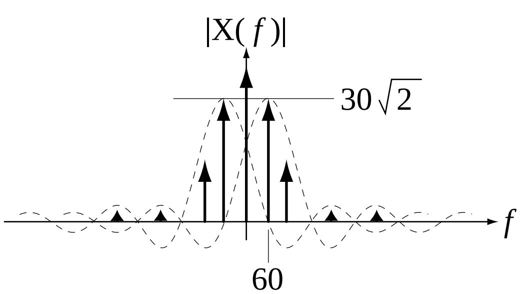
52. In electronics, one of the first circuits studied is the rectifier. There are two forms, the half-wave rectifier and the full-wave rectifier. The half-wave rectifier cuts off half of an excitation sinusoid and leaves the other half intact. The full-wave rectifier reverses the polarity of half of the excitation sinusoid and leaves the other half intact. Let the excitation sinusoid be a typical household voltage, 120 Vrms at 60 Hz, and let both types of rectifiers alter the negative half of the sinusoid while leaving the positive half unchanged. Find and plot the magnitudes of the CTFT's of the responses of both types of rectifiers (either form).

Half-Wave Case:

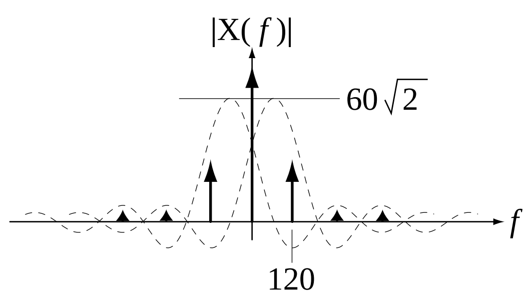
$$\text{x}(t) = 120\sqrt{2}\cos(120\pi t)\left[\text{rect}(120t) * 60\,\text{comb}(60t)\right]$$

Full-Wave Case:

$$\text{x}(t) = 120\sqrt{2}\cos(120\pi t)\left[2\,\text{rect}(120t) * 60\,\text{comb}(60t) - 1\right]$$



Half-Wave



Full-Wave

53. Find the DTFT of each of these signals:

(a) $\mathrm{x}[n]=\left(\frac{1}{3}\right)^n \mathrm{u}[n-1]$

$$\mathrm{X}(j\Omega)=\sum_{n=-\infty}^{\infty}\mathrm{x}[n]e^{-j\Omega n}=\sum_{n=-\infty}^{\infty}\left(\frac{1}{3}\right)^n \mathrm{u}[n-1]e^{-j\Omega n}=\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n e^{-j\Omega n}$$

$$\mathrm{X}(j\Omega)=\sum_{m=0}^{\infty}\left(\frac{1}{3}\right)^{m+1} e^{-j\Omega(m+1)}=\sum_{m=0}^{\infty}\left(\frac{e^{-j\Omega}}{3}\right)^{m+1}$$

$$\mathrm{X}(j\Omega)=\frac{e^{-j\Omega}}{3}\sum_{m=0}^{\infty}\left(\frac{e^{-j\Omega}}{3}\right)^{m}=\frac{e^{-j\Omega}}{3}\frac{1}{1-\frac{e^{-j\Omega}}{3}}=\frac{e^{-j\Omega}}{3-e^{-j\Omega}}$$

Alternate Solution:

$$\mathrm{x}[n]=\left(\frac{1}{3}\right)^n \mathrm{u}[n-1]=\left(\frac{1}{3}\right)^n \mathrm{u}[n]-\delta[n]$$

Using

$$\alpha^n \mathrm{u}[n] \xleftrightarrow{\mathcal{F}} \frac{1}{1-\alpha e^{-j\Omega}} \qquad \text{and} \qquad \delta[n] \xleftrightarrow{\mathcal{F}} 1$$

$$\mathrm{x}[n]=\left(\frac{1}{3}\right)^n \mathrm{u}[n]-\frac{1}{3}\delta[n]=\frac{1}{1-\frac{e^{-j\Omega}}{3}}-1$$

$$\mathrm{x}[n]=\frac{1-\left(1-\frac{e^{-j\Omega}}{3}\right)}{1-\frac{e^{-j\Omega}}{3}}=\frac{\frac{e^{-j\Omega}}{3}}{1-\frac{e^{-j\Omega}}{3}}=\frac{e^{-j\Omega}}{3-e^{-j\Omega}}$$

Second Alternate Solution:

$$\mathrm{x}[n]=\frac{1}{3}\left(\frac{1}{3}\right)^{n-1} \mathrm{u}[n-1]$$

$$\mathrm{X}(j\Omega)=\frac{1}{3}\frac{1}{1-\frac{1}{3}e^{-j\Omega}}e^{-j\Omega}=\frac{e^{-j\Omega}}{3-e^{-j\Omega}}$$

(b) $\mathrm{x}[n]=\sin\left(\frac{\pi}{4}n\right)\left(\frac{1}{4}\right)^n \mathrm{u}[n-2]$

$$\sin\left(\frac{\pi n}{4}\right) = \sin\left(\frac{2\pi n}{8}\right) = \cos\left(\frac{2\pi(n-2)}{8}\right)$$

$$\mathrm{x}[n] = \left(\frac{1}{4}\right)^2 \cos\left(\frac{2\pi(n-2)}{8}\right)\left(\frac{1}{4}\right)^{n-2} \mathrm{u}[n-2]$$

$$\alpha^n \cos(\Omega_0 n)\mathrm{u}[n] \xleftrightarrow{\mathcal{F}} \frac{1-\alpha\cos(\Omega_0)e^{-j\Omega}}{1-2\alpha\cos(\Omega_0)e^{-j\Omega}+\alpha^2 e^{-j2\Omega}} \quad , \quad |\alpha|<1$$

$$\mathrm{X}(j\Omega) = \left(\frac{1}{4}\right)^2 e^{-j2\Omega} \frac{1-\frac{1}{4}\cos\left(\frac{\pi}{4}\right)e^{-j\Omega}}{1-\frac{1}{2}\cos\left(\frac{\pi}{4}\right)e^{-j\Omega}+\frac{1}{16}e^{-j2\Omega}} = \frac{e^{-j2\Omega}}{16} \frac{1-\frac{\sqrt{2}}{8}e^{-j\Omega}}{1-\frac{\sqrt{2}}{4}e^{-j\Omega}+\frac{1}{16}e^{-j2\Omega}}$$

Alternate Solution:

$$\mathrm{x}[n] = \frac{e^{j\frac{\pi}{4}n}-e^{-j\frac{\pi}{4}n}}{j2}\left(\frac{1}{4}\right)^n \mathrm{u}[n-2] = \frac{1}{j2}\left(\left(\frac{e^{j\frac{\pi}{4}}}{4}\right)^n - \left(\frac{e^{-j\frac{\pi}{4}}}{4}\right)^n\right)\mathrm{u}[n-2]$$

$$\mathrm{x}[n] = \frac{1}{j2}\left(\left(\frac{e^{j\frac{\pi}{4}}}{4}\right)^2\left(\frac{e^{j\frac{\pi}{4}}}{4}\right)^{n-2} - \left(\frac{e^{-j\frac{\pi}{4}}}{4}\right)^2\left(\frac{e^{-j\frac{\pi}{4}}}{4}\right)^{n-2}\right)\mathrm{u}[n-2]$$

$$\left(\frac{e^{j\frac{\pi}{4}}}{4}\right)^n \mathrm{u}[n] \xleftrightarrow{\mathcal{F}} \frac{1}{1-\frac{e^{j\frac{\pi}{4}}}{4}e^{-j\Omega}}$$

$$\left(\frac{e^{j\frac{\pi}{4}}}{4}\right)^{n-2} \mathrm{u}[n-2] \xleftrightarrow{\mathcal{F}} \frac{e^{-j2\Omega}}{1-\frac{e^{j\frac{\pi}{4}}}{4}e^{-j\Omega}}$$

$$\mathrm{X}(j\Omega) = \frac{1}{j2}\left(\left(\frac{e^{j\frac{\pi}{4}}}{4}\right)^2 \frac{e^{-j2\Omega}}{1-\frac{e^{j\frac{\pi}{4}}}{4}e^{-j\Omega}} - \left(\frac{e^{-j\frac{\pi}{4}}}{4}\right)^2 \frac{e^{-j2\Omega}}{1-\frac{e^{-j\frac{\pi}{4}}}{4}e^{-j\Omega}}\right)$$

$$\mathrm{X}(j\Omega)=\frac{e^{-j2\Omega}}{j2}\frac{\left(\frac{e^{j\frac{\pi}{4}}}{4}\right)^2\left(1-\frac{e^{-j\frac{\pi}{4}}}{4}e^{-j\Omega}\right)-\left(\frac{e^{-j\frac{\pi}{4}}}{4}\right)^2\left(1-\frac{e^{j\frac{\pi}{4}}}{4}e^{-j\Omega}\right)}{\left(1-\frac{e^{j\frac{\pi}{4}}}{4}e^{-j\Omega}\right)\left(1-\frac{e^{-j\frac{\pi}{4}}}{4}e^{-j\Omega}\right)}$$

$$\mathrm{X}(j\Omega)=\frac{e^{-j2\Omega}}{j32}\frac{\left(e^{j\frac{\pi}{4}}\right)^2-\left(e^{j\frac{\pi}{4}}\right)^2\frac{e^{-j\frac{\pi}{4}}}{4}e^{-j\Omega}-\left(e^{-j\frac{\pi}{4}}\right)^2+\left(e^{-j\frac{\pi}{4}}\right)^2\frac{e^{j\frac{\pi}{4}}}{4}e^{-j\Omega}}{1-\left(\frac{e^{j\frac{\pi}{4}}}{4}+\frac{e^{-j\frac{\pi}{4}}}{4}\right)e^{-j\Omega}+\frac{e^{-j2\Omega}}{16}}$$

$$\mathrm{X}(j\Omega)=\frac{e^{-j2\Omega}}{j32}\frac{e^{j\frac{\pi}{2}}-e^{-j\frac{\pi}{2}}-\frac{e^{j\frac{\pi}{4}}}{4}e^{-j\Omega}+\frac{e^{-j\frac{\pi}{4}}}{4}e^{-j\Omega}}{1-\frac{1}{2}\cos\left(\frac{\pi}{4}\right)e^{-j\Omega}+\frac{e^{-j2\Omega}}{16}}$$

$$\mathrm{X}(j\Omega)=\frac{e^{-j2\Omega}}{j32}\frac{j2-\left(\frac{e^{j\frac{\pi}{4}}-e^{-j\frac{\pi}{4}}}{4}\right)e^{-j\Omega}}{1-\frac{1}{2}\cos\left(\frac{\pi}{4}\right)e^{-j\Omega}+\frac{e^{-j2\Omega}}{16}}$$

$$\mathrm{X}(j\Omega)=\frac{e^{-j2\Omega}}{16}\frac{1-\frac{1}{4}\sin\left(\frac{\pi}{4}\right)e^{-j\Omega}}{1-\frac{1}{2}\cos\left(\frac{\pi}{4}\right)e^{-j\Omega}+\frac{e^{-j2\Omega}}{16}}=\frac{e^{-j2\Omega}}{16}\frac{1-\frac{\sqrt{2}}{8}e^{-j\Omega}}{1-\frac{\sqrt{2}}{4}e^{-j\Omega}+\frac{e^{-j2\Omega}}{16}}$$

(c) $\mathrm{x}[n]=\mathrm{sinc}\left(\frac{2\pi n}{8}\right)*\mathrm{sinc}\left(\frac{2\pi(n-4)}{8}\right)$

(d) $\mathrm{x}[n]=\mathrm{sinc}^2\left(\frac{2\pi n}{8}\right)$

Using

$$\text{sinc}\left(\frac{n}{w}\right) \xleftrightarrow{\mathcal{F}} w\,\text{rect}(wF) * \text{comb}(F)$$

Find the DTFT of $\text{sinc}\left(\frac{n}{w}\right)$. Then periodically convolve it with itself.

$$X(F) = \frac{8}{2\pi}\text{tri}\left(\frac{8}{2\pi}F\right) * \text{comb}(F)$$

54. Sketch the magnitudes and phases of the DTFT's of the following functions:

(a) $\text{rect}_2[n]$ (b) $\text{rect}_2[n] * (-5\delta[n])$

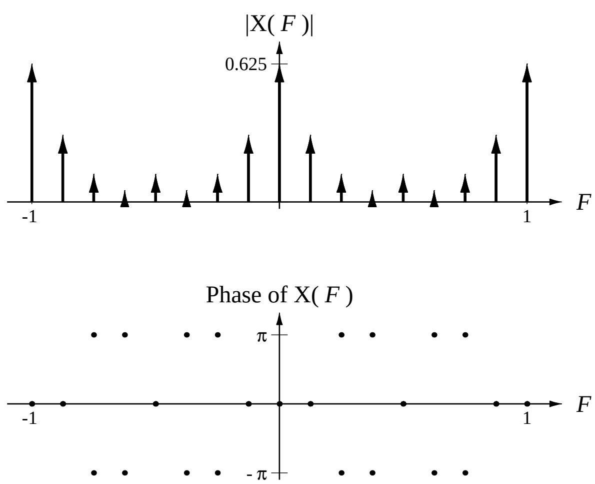
(c) $\text{rect}_2[n] * 3\delta[n+3]$

(d) $\text{rect}_2[n] * (-5\delta[4n]) = \text{rect}_2[n] * (-5\delta[n])$

Remember, there is no scaling property for the discrete-time impulse.

(e) $\text{rect}_2[n] * \text{comb}_8[n]$

Since this function is periodic, its DTFT must contain only impulses.



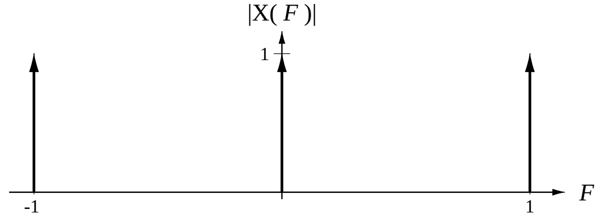
(f) $\text{rect}_2[n] * \text{comb}_8[n-3]$

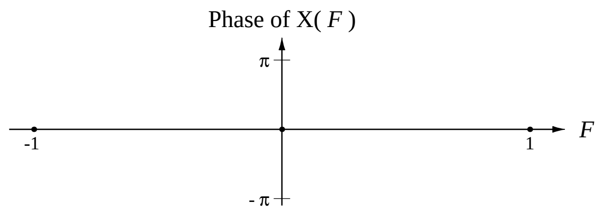
Similar to (e).

(g) $\text{rect}_2[n] * \text{comb}_8[2n] = \text{rect}_2[n] * \sum_{m=-\infty}^{\infty} \delta[2n - 8m] = \text{rect}_2[n] * \sum_{m=-\infty}^{\infty} \delta[2(n - 4m)]$

$$\text{rect}_2[n] * \text{comb}_8[2n] = \text{rect}_2[n] * \sum_{m=-\infty}^{\infty} \delta[n - 4m] = \text{rect}_2[n] * \text{comb}_4[n]$$

(h) $\text{rect}_2[n] * \text{comb}_5[n]$





55. Sketch the inverse DTFT's of these functions.

(a) $\text{X}(F) = \text{comb}(F) - \text{comb}\left(F - \frac{1}{2}\right)$

Using $1 \xleftrightarrow{\mathcal{F}} \text{comb}(F)$ and $e^{j2\pi F_0 n}\,\text{x}[n] \longleftrightarrow \text{X}(F - F_0)$

$$1 - e^{j\pi n} \xleftrightarrow{\mathcal{F}} \text{comb}(F) - \text{comb}\left(F - \frac{1}{2}\right)$$

$$e^{j\frac{\pi n}{2}}\left(e^{-j\frac{\pi n}{2}} - e^{+j\frac{\pi n}{2}}\right) \xleftrightarrow{\mathcal{F}} \text{comb}(F) - \text{comb}\left(F - \frac{1}{2}\right)$$

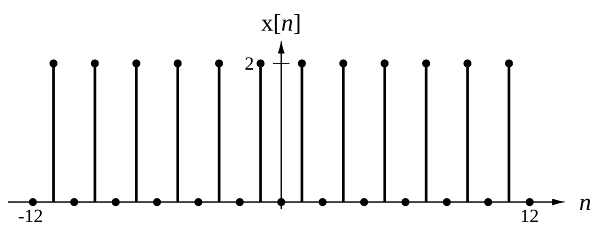
$$-j2e^{j\frac{\pi n}{2}} \sin\left(\frac{\pi n}{2}\right) \xleftrightarrow{\mathcal{F}} \text{comb}(F) - \text{comb}\left(F - \frac{1}{2}\right)$$

$$-2e^{j\frac{\pi}{2}(n+1)} \sin\left(\frac{\pi n}{2}\right) \xleftrightarrow{\mathcal{F}} \text{comb}(F) - \text{comb}\left(F - \frac{1}{2}\right)$$

$$-2\left[\cos\left(\frac{\pi}{2}(n+1)\right)+j\sin\left(\frac{\pi}{2}(n+1)\right)\right]\sin\left(\frac{\pi n}{2}\right)\xleftrightarrow{\mathcal{F}}\text{comb}(F)-\text{comb}\left(F-\frac{1}{2}\right)$$

$$-2\left[\underbrace{\cos\left(\frac{\pi}{2}(n+1)\right)}_{-\sin\left(\frac{\pi n}{2}\right)}\sin\left(\frac{\pi n}{2}\right)+\underbrace{j\sin\left(\frac{\pi}{2}(n+1)\right)\sin\left(\frac{\pi n}{2}\right)}_{=0}\right]\xleftrightarrow{\mathcal{F}}\text{comb}(F)-\text{comb}\left(F-\frac{1}{2}\right)$$

$$2\sin^2\left(\frac{\pi n}{2}\right)\xleftrightarrow{\mathcal{F}}\text{comb}(F)-\text{comb}\left(F-\frac{1}{2}\right)$$



(b) $\mathrm{X}(F)=j\,\text{comb}\left(F+\frac{1}{8}\right)-j\,\text{comb}\left(F-\frac{1}{8}\right)$

(c) $\mathrm{X}(F)=\left[\text{sinc}\left(10\left(F-\frac{1}{4}\right)\right)+\text{sinc}\left(10\left(F+\frac{1}{4}\right)\right)\right]*\text{comb}(F)$

Use

$$\text{sinc}\left(\frac{t}{w}\right)*f_0\,\text{comb}(f_0 t)=wf_0\left[\cos\left(\frac{\pi t}{w}\right)+\left(\frac{T_0}{w}-1\right)\text{drcl}\left(f_0 t,\frac{T_0}{w}-1\right)\right]$$

from Appendix A (because $\frac{T_0}{2w}$ is an integer).

$$\mathrm{X}(F)=\frac{1}{10}\left\{\cos\left(10\pi\left(F-\frac{1}{4}\right)\right)+9\,\text{drcl}\left(F-\frac{1}{4},9\right)+\cos\left(10\pi\left(F+\frac{1}{4}\right)\right)+9\,\text{drcl}\left(F+\frac{1}{4},9\right)\right\}$$

$$\int_1 \cos(10\pi F)e^{j2\pi Fn}dF\xleftrightarrow{\mathcal{F}}\cos(10\pi F)$$

$$\frac{1}{2}\int_1 \left(e^{j10\pi F} + e^{-j10\pi F}\right) e^{j2\pi Fn} dF \xleftrightarrow{\mathcal{F}} \cos(10\pi F)$$

$$\frac{1}{2}\left[\int_1 e^{j2\pi F(n+5)} dF + \int_1 e^{j2\pi F(n-5)} dF\right] \xleftrightarrow{\mathcal{F}} \cos(10\pi F)$$

$$\frac{1}{2}\left[\begin{array}{l}\int_1 \left[\cos\left(2\pi F(n+5)\right) + j\sin\left(2\pi F(n+5)\right)\right] dF \\ +\int_1 \left[\cos\left(2\pi F(n-5)\right) + j\sin\left(2\pi F(n-5)\right)\right] dF\end{array}\right] \xleftrightarrow{\mathcal{F}} \cos(10\pi F)$$

These integrals are zero unless $n = \pm 5$. Therefore

$$\frac{1}{2}\left(\delta[n+5] + \delta[n-5]\right) \xleftrightarrow{\mathcal{F}} \cos(10\pi F) \quad .$$

Then using $e^{j2\pi F_0 n}\,\mathrm{x}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(F - F_0)$,

$$\mathrm{rect}_4[n] \xleftrightarrow{\mathcal{F}} 9\,\mathrm{drcl}(F,9)$$

Combining inverse transforms,

$$\frac{1}{2}\left(\delta[n+5] + \delta[n-5]\right) + \mathrm{rect}_4[n] \xleftrightarrow{\mathcal{F}} \cos(10\pi F) + 9\,\mathrm{drcl}(F,9) \quad .$$

Then, using $e^{j2\pi F_0 n}\,\mathrm{x}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(F - F_0)$

$$e^{j\frac{\pi n}{2}}\left\{\frac{1}{2}\left(\delta[n+5] + \delta[n-5]\right) + \mathrm{rect}_4[n]\right\} \xleftrightarrow{\mathcal{F}} \cos\left(10\pi\left(F - \frac{1}{4}\right)\right) + 9\,\mathrm{drcl}\left(F - \frac{1}{4}, 9\right)$$

and

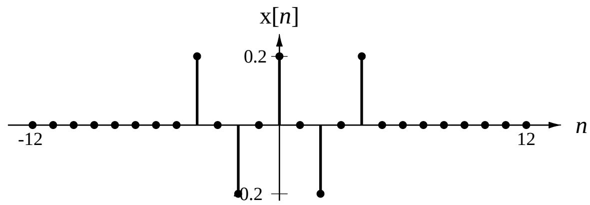
$$e^{-j\frac{\pi n}{2}}\left\{\frac{1}{2}\left(\delta[n+5] + \delta[n-5]\right) + \mathrm{rect}_4[n]\right\} \xleftrightarrow{\mathcal{F}} \cos\left(10\pi\left(F + \frac{1}{4}\right)\right) + 9\,\mathrm{drcl}\left(F + \frac{1}{4}, 9\right) \quad .$$

Then, finally

$$\frac{1}{10}\left(\begin{array}{l} e^{j\frac{\pi n}{2}}\left\{\frac{1}{2}\left(\delta[n+5]+\delta[n-5]\right)+\mathrm{rect}_4[n]\right\} \\ +e^{-j\frac{\pi n}{2}}\left\{\frac{1}{2}\left(\delta[n+5]+\delta[n-5]\right)+\mathrm{rect}_4[n]\right\} \end{array}\right) \xleftrightarrow{\mathcal{F}} \frac{1}{10}\left\{\begin{array}{l} \cos\left(10\pi\left(F-\frac{1}{4}\right)\right)+9\,\mathrm{drcl}\left(F-\frac{1}{4},9\right) \\ +\cos\left(10\pi\left(F+\frac{1}{4}\right)\right)+9\,\mathrm{drcl}\left(F+\frac{1}{4},9\right) \end{array}\right\}$$

The impulses on the left side cancel and we get

$$\frac{\mathrm{rect}_4[n]}{5}\cos\left(\frac{\pi n}{2}\right) \xleftrightarrow{\mathcal{F}} \frac{1}{10}\left\{\begin{array}{l} \cos\left(10\pi\left(F-\frac{1}{4}\right)\right)+9\,\mathrm{drcl}\left(F-\frac{1}{4},9\right) \\ +\cos\left(10\pi\left(F+\frac{1}{4}\right)\right)+9\,\mathrm{drcl}\left(F+\frac{1}{4},9\right) \end{array}\right\}$$



(d) $\mathrm{X}(F)=\left[\delta\left(F-\frac{1}{4}\right)+\delta\left(F-\frac{3}{16}\right)+\delta\left(F-\frac{5}{16}\right)\right]*\mathrm{comb}(2F)$

Express $\mathrm{comb}(2F)$ as $\frac{1}{2}\left[\mathrm{comb}(F)+\mathrm{comb}\left(F-\frac{1}{2}\right)\right]$. Then do the convolution.

Then do the inverse transform and simplify.

$$\mathrm{X}(F)=\frac{1}{2}\left\{\begin{array}{l} \left[\mathrm{comb}\left(F-\frac{1}{4}\right)+\mathrm{comb}\left(F-\frac{3}{16}\right)+\mathrm{comb}\left(F-\frac{5}{16}\right)\right] \\ +\left[\mathrm{comb}\left(F-\frac{1}{4}-\frac{1}{2}\right)+\mathrm{comb}\left(F-\frac{3}{16}-\frac{1}{2}\right)+\mathrm{comb}\left(F-\frac{5}{16}-\frac{1}{2}\right)\right] \end{array}\right\}$$

$$\cos\left(\frac{\pi n}{2}\right)+\cos\left(\frac{3\pi n}{8}\right)+\cos\left(\frac{5\pi n}{8}\right) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left\{\begin{array}{l} \mathrm{comb}\left(F-\frac{1}{4}\right)+\mathrm{comb}\left(F-\frac{3}{16}\right)+\mathrm{comb}\left(F-\frac{5}{16}\right) \\ +\mathrm{comb}\left(F-\frac{3}{4}\right)+\mathrm{comb}\left(F-\frac{11}{16}\right)+\mathrm{comb}\left(F-\frac{13}{16}\right) \end{array}\right\}$$

56. Using the relationship between the CTFT of a signal and the CTFS of a periodic extension of that signal, find the CTFS of

$$x(t) = \text{rect}\left(\frac{t}{w}\right) * \frac{1}{T_0}\text{comb}\left(\frac{t}{T_0}\right)$$

and compare it with the table entry.

57. Using the relationship between the DTFT of a signal and the DTFS of a periodic extension of that signal, find the DTFS of

$$\text{rect}_{N_w}[n] * \text{comb}_{N_0}[n]$$

and compare it with the table entry.