Chapter 6 - Fourier Transform Analysis of Signals and Systems

Selected Solutions

(In this solution manual, the symbol, ⊗, is used for periodic convolution because the preferred symbol which appears in the text is not in the font selection of the word processor used to create this manual.)

1. A system has an impulse response,

$$h_{LP}(t) = 3e^{-10t}\,u(t) \ ,$$

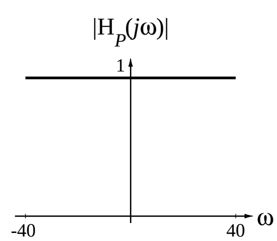
and another system has an impulse response,

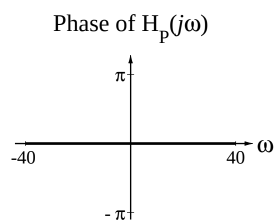
$$h_{HP}(t) = \delta(t) - 3e^{-10t}\,u(t) \ .$$

(a) Sketch the magnitude and phase of the transfer function of these two systems in a parallel connection.

$$H_{LP}(j\omega) = \frac{3}{j\omega+10} \ , \ H_{HP}(j\omega) = 1 - \frac{3}{j\omega+10}$$

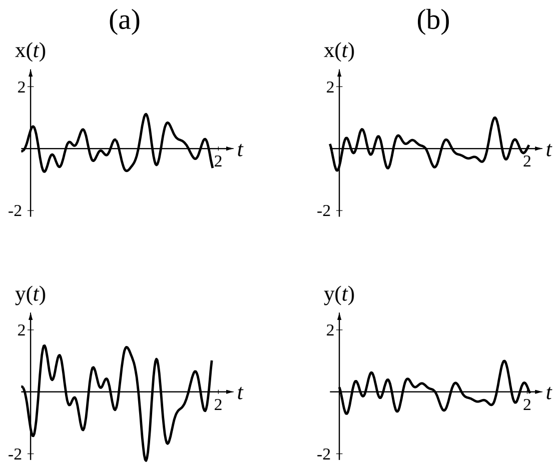
$$H_P(j\omega) = \frac{3}{j\omega+10} + 1 - \frac{3}{j\omega+10} = 1$$





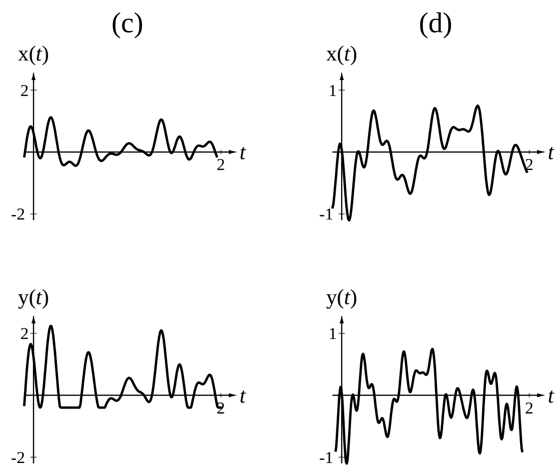
(b) Sketch the magnitude and phase of the transfer function of these two systems in a cascade connection.

2. Below are some pairs of signals, $x(t)$ and $y(t)$. In each case decide whether or not $y(t)$ is a distorted version of $x(t)$.



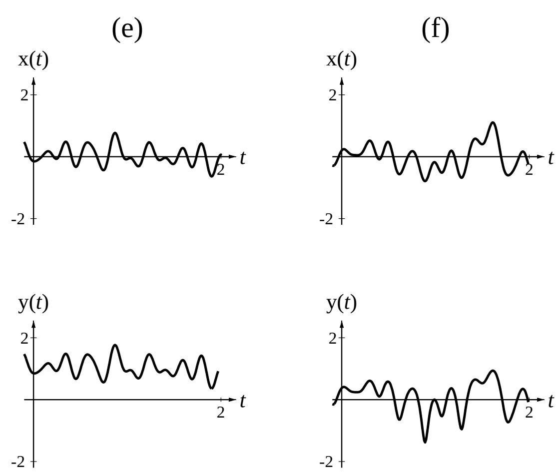
(a) Inverted and amplified, undistorted

(b) Time shifted, undistorted



(c) Clipped at a negative value, distorted

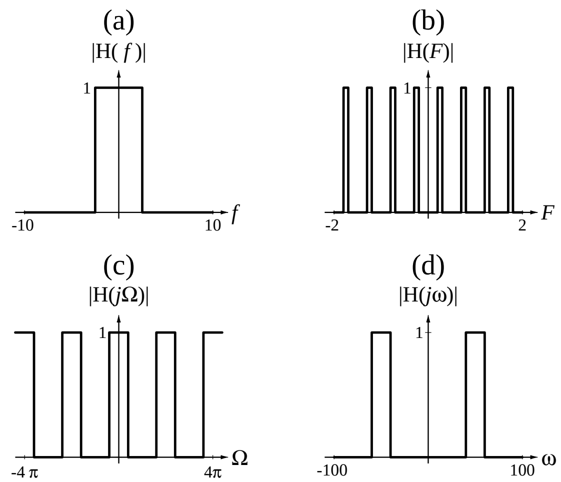
(d) Time compressed, distorted



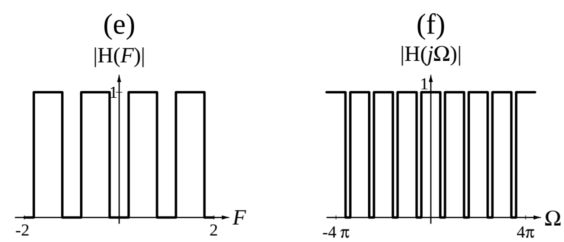
(e) Constant added, distorted

(f) Log-amplified, distorted

3. Classify each of these transfer functions as having a lowpass, highpass, bandpass or bandstop frequency response.



(a) Lowpass (b) Bandpass (c) Lowpass (d) Bandpass



(e) Highpass (f) Bandstop

4. Classify each of these transfer functions as having a lowpass, highpass, bandpass or bandstop frequency response.

(a) $\mathrm{H}(f) = 1 - \mathrm{rect}\left(\frac{|f| - 100}{10}\right)$ (b) $\mathrm{H}(F) = \mathrm{rect}(10F) * \mathrm{comb}(F)$

(c) $\mathrm{H}(j\Omega) = \left[\mathrm{rect}\left(20\pi\left(\Omega - \frac{\pi}{4}\right)\right) + \mathrm{rect}\left(20\pi\left(\Omega + \frac{\pi}{4}\right)\right)\right] * \mathrm{comb}\left(\frac{\Omega}{2\pi}\right)$

Bandpass

5. A system has an impulse response,

$$\mathrm{h}(t) = 10\,\mathrm{rect}\left(\frac{t - 0.01}{0.02}\right).$$

What is its null bandwidth?

6. A system has an impulse response,

$$\mathrm{h}[n] = \left(\frac{7}{8}\right)^n \mathrm{u}[n] \quad .$$

What is its half-power DT-frequency bandwidth?

Using

$$\alpha^n \mathrm{u}[n] \xleftrightarrow{\mathcal{F}} \frac{1}{1-\alpha e^{-j\Omega}}$$

the transfer function is

$$\mathrm{H}(j\Omega) = \frac{1}{1-\frac{7}{8}e^{-j\Omega}} = \frac{8}{8-7e^{-j\Omega}} .$$

This is a DT lowpass filter. Its maximum transfer function magnitude occurs at $\Omega = 0$. The -3 dB point must be the first frequency at which the square of the magnitude of the transfer function is one-half of its maximum value (the "half-power" bandwidth).

The low-frequency gain is

$$\mathrm{H}(0) = 8$$

The -3 dB point occurs where

$$\left|\mathrm{H}(j\Omega_{-3dB})\right|^2 = \frac{8^2}{2} = 32 \quad .$$

Solving,

$$\Omega_{hp} = 0.1337 \pm 2n\pi$$

So the -3 dB DT-frequency bandwidth in radians is 0.1337. In cycles it is 0.0213. (Notice that the bandwidths are not in radians/s or in Hz. This is because they are DT bandwidths, not CT bandwidths.)

7. Determine whether or not the CT systems with these transfer functions are causal.

(a) $\mathrm{H}(f) = \mathrm{sinc}(f)$ (b) $\mathrm{H}(f) = \mathrm{sinc}(f)e^{-j\pi f}$

(c) $\mathrm{H}(j\omega) = \mathrm{rect}(\omega)$

(d) $\mathrm{H}(j\omega) = \mathrm{rect}(\omega)e^{-j\omega}$ $\mathrm{h}(t) = \frac{1}{2\pi}\mathrm{sinc}\left(\frac{t-1}{2\pi}\right)$ Not Causal

(e) $\mathrm{H}(f) = A$

(f) $\mathrm{H}(f) = Ae^{j2\pi f}$ $\mathrm{h}(t) = A\delta(t+1)$ Not Causal

8. Determine whether or not the DT systems with these transfer functions are causal.

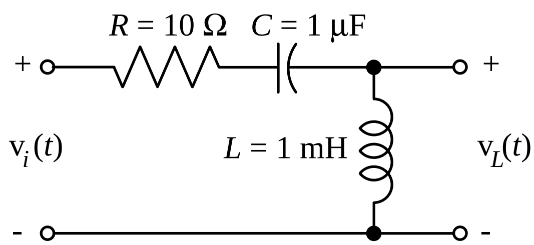
(a) $\mathrm{H}(F) = \dfrac{\sin(7\pi F)}{\sin(\pi F)}$ (b) $\mathrm{H}(F) = \dfrac{\sin(7\pi F)}{\sin(\pi F)} e^{-j2\pi F}$

(c) $\mathrm{H}(F) = \dfrac{\sin(3\pi F)}{\sin(\pi F)} e^{-j2\pi F}$ $\mathrm{h}[n] = \mathrm{rect}_1[n-1]$ Causal

(d) $\mathrm{H}(F) = \mathrm{rect}(10F) * \mathrm{comb}(F)$

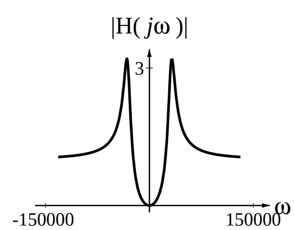
9. Find and sketch the frequency response of each of these circuits given the indicated excitation and response.

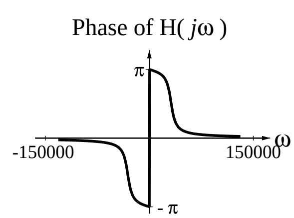
(a) Excitation, $\mathrm{v}_i(t)$ - Response, $\mathrm{v}_L(t)$



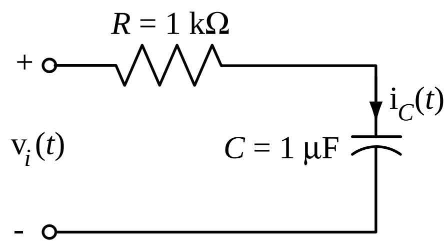
Using voltage-division principles,

$$\mathrm{H}(j\omega) = \frac{\mathrm{V}_L(j\omega)}{\mathrm{V}_i(j\omega)} = \frac{Z_L(j\omega)}{Z_L(j\omega) + Z_C(j\omega) + Z_R(j\omega)} = \frac{j\omega L}{j\omega L + \dfrac{1}{j\omega C} + R} = \frac{-\omega^2 LC}{1 - \omega^2 LC + j\omega RC}$$



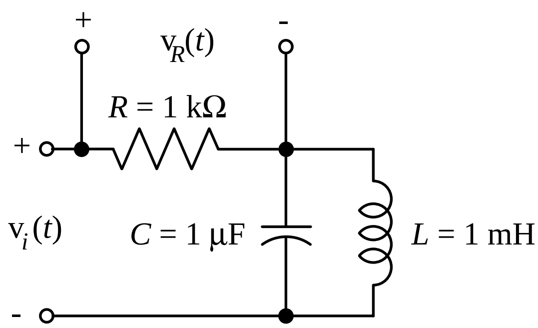


(b) Excitation, $\mathrm{v}_i(t)$ - Response, $\mathrm{i}_C(t)$

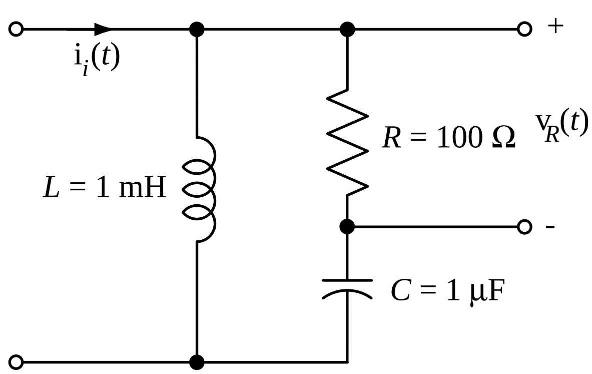


In this case the transfer function is the reciprocal of the input impedance.

(c) Excitation, $v_i(t)$ - Response, $v_R(t)$



(d) Excitation, $i_i(t)$ - Response, $v_R(t)$



Divide the excitation current between the two branches and multiply the current in the right branch by *R* to get the response voltage. Then solve for the ratio of the response voltage to the excitation current.

10. Classify each of these transfer functions as having a lowpass, highpass, bandpass or bandstop frequency response.

(a) $\mathrm{H}(f) = \dfrac{1}{1 + jf}$

(b) $\mathrm{H}(f) = \dfrac{jf}{1 + jf}$

(c) $\mathrm{H}(j\omega) = -\dfrac{j10\omega}{100 - \omega^2 + j10\omega}$

(d) $\mathrm{H}(F) = \dfrac{\sin(3\pi F)}{\sin(\pi F)}$

This case is not as "pure" as the previous ones. It is generally lowpass because the transfer function magnitude at lower frequencies is generally greater than at high frequencies. But there are nulls in the transfer function that make it look somewhat like a bandstop filter or a multiple bandstop filter.

(e) $\mathrm{H}(j\Omega) = j[\sin(\Omega) + \sin(2\Omega)]$

This case is also not perfectly clear. The response at zero frequency is zero and the response at $\Omega = \pi$ is also zero. These criteria fit a bandpass filter. But the response is also zero at $\Omega = \dfrac{2\pi}{3}$. So it might again look like a bandstop in some ways.

11. Plot the magnitude frequency responses, both on a linear-magnitude and on a log-magnitude scale, of the systems with these transfer functions, over the frequency range specified.

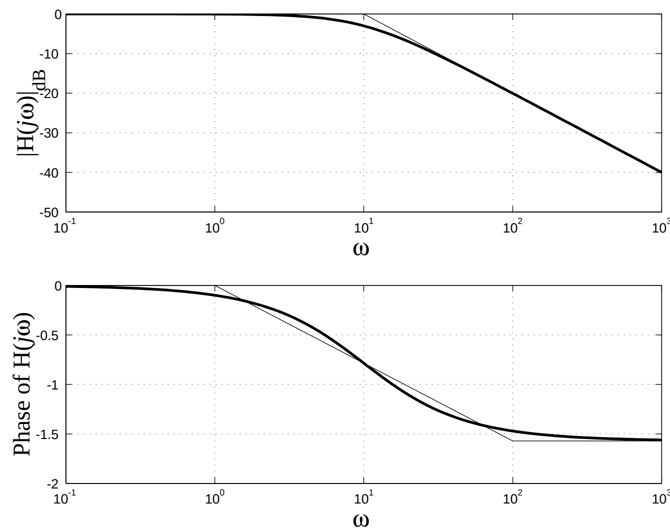
(a) $\mathrm{H}(f) = \dfrac{20}{20 - 4\pi^2 f^2 + j42\pi f}$, $-100 < f < 100$

(b) $\mathrm{H}(j\omega) = \dfrac{2\times 10^5}{(100 + j\omega)(1700 - \omega^2 + j20\omega)}$, $-500 < \omega < 500$

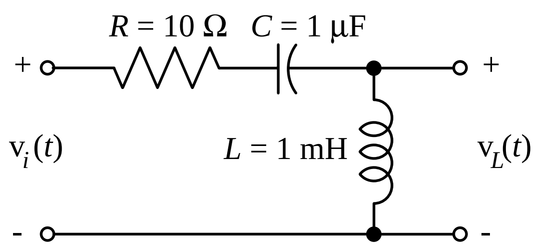
12. Draw asymptotic and exact magnitude and phase Bode diagrams for the frequency responses of the following circuits and systems.

(a) An RC lowpass filter with $R = 1\ \mathrm{M\Omega}$ and $C = 0.1\ \mu\mathrm{F}$.

$$\mathrm{H}(j\omega) = \frac{\dfrac{1}{j\omega C}}{\dfrac{1}{j\omega C} + R} = \frac{1}{j\omega RC + 1} = \frac{1}{j\omega 10^6 10^{-7} + 1} = \frac{1}{j0.1\omega + 1}$$

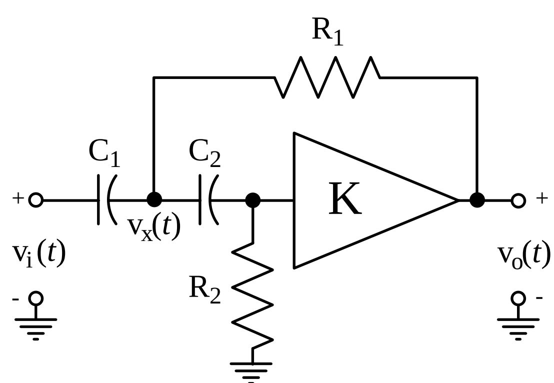


(b)



13. Find the transfer functions, $H(f) = \frac{V_o(f)}{V_i(f)}$, of these active filters and identify them as lowpass, highpass, bandpass or bandstop.

(a)



The triangle with the "K" inside is an ideal voltage amplifier of gain K (not an operational amplifier). It is, in circuit theory parlance, a "voltage-dependent voltage source". "Ideal" means its input impedance is infinite so no input current flows and its output impedance is zero so the output voltage is independent of the output current (and therefore any load connected to the output).

Writing Kirchhoff's current law at the $v_X(t)$ node and then writing the relationship between $V_x(f)$ and $V_o(f)$ using voltage division and the voltage amplifier gain, K,

$$V_x(f)\left(j2\pi fC_1+\frac{1}{\frac{1}{j2\pi fC_2}+R_2}+G_1\right)-V_i(f)j2\pi fC_1-V_o(f)G_1=0$$

$$V_x(f)\frac{R_2}{\frac{1}{j2\pi fC_2}+R_2}K=V_o(f)$$

(G_1 is $\frac{1}{R_1}$. That is, it is the conductance of the resistor, R_1.)

$$V_x(f)\left(j2\pi fC_1+\frac{j2\pi fC_2}{1+j2\pi fR_2C_2}+G_1\right)-V_i(f)j2\pi fC_1-V_o(f)G_1=0$$

$$V_x(f)\frac{j2\pi fR_2C_2}{1+j2\pi fR_2C_2}K=V_o(f)$$

Writing the two equations as one matrix equation,

$$\begin{bmatrix}\left(j2\pi fC_1+\frac{j2\pi fC_2}{1+j2\pi fR_2C_2}+G_1\right) & -G_1\\ j2\pi fKR_2C_2 & -(1+j2\pi fR_2C_2)\end{bmatrix}\begin{bmatrix}V_x(f)\\ V_o(f)\end{bmatrix}=\begin{bmatrix}j2\pi fC_1\\ 0\end{bmatrix}V_i(f)$$

Solving by Cramer's rule with the excitation voltage as a forcing function,

$$\Delta=-\left(j2\pi fC_1+\frac{j2\pi fC_2}{1+j2\pi fR_2C_2}+G_1\right)(1+j2\pi fR_2C_2)+j2\pi fKG_1R_2C_2$$

$$\Delta=(2\pi f)^2R_2C_1C_2-j2\pi f\left[C_1+C_2+G_1R_2C_2(1-K)\right]-G_1$$

$$V_o(f)=\frac{1}{\Delta}\begin{vmatrix}\left(j2\pi fC_1+\frac{j2\pi fC_2}{1+j2\pi fR_2C_2}+G_1\right) & j2\pi fC_1\\ j2\pi fKR_2C_2 & 0\end{vmatrix}V_i(f)$$

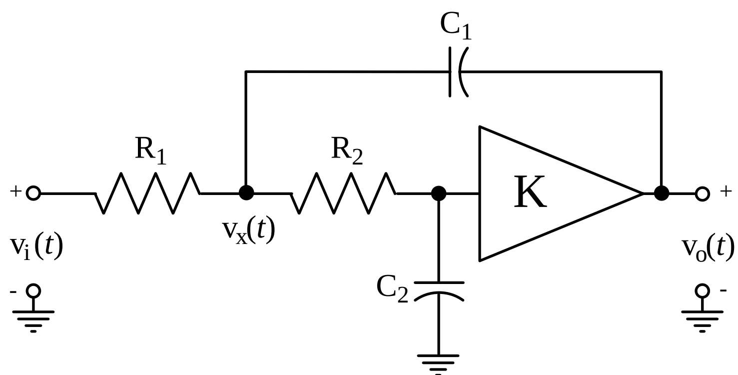
$$H(f)=\frac{V_o(f)}{V_i(f)}=\frac{(2\pi f)^2KR_2C_1C_2}{(2\pi f)^2R_2C_1C_2-j2\pi f\left[C_1+C_2+G_1R_2C_2(1-K)\right]-G_1}$$

$$H(f)=\frac{(2\pi f)^2KR_1R_2C_1C_2}{(2\pi f)^2R_1R_2C_1C_2-j2\pi f\left(R_1(C_1+C_2)+R_2C_2(1-K)\right)-1}$$

Highpass

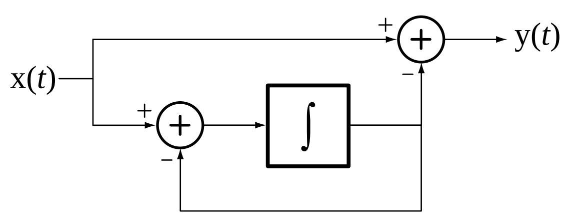
The circuit is a highpass filter because at low frequencies the response approaches zero and at high frequencies it approaches K. This can be seen both in the transfer function formula and in the physical nature of the circuit connection itself. At zero frequency no current flows through the capacitors and therefore not current flows through resistor, R_2. Therefore the voltage at the input of the amplifier is zero making the response voltage zero. At high frequencies the capacitor impedances become practically zero, making the excitation voltage and the voltage at the amplifier input equal. Therefore the transfer function must be K at high frequencies.

(b)



Similar to (a)

14. Show that this system has a highpass frequency response.



Write the differential equation from the block diagram. You should get

$$\frac{d}{dt}\mathrm{y}(t) + \mathrm{y}(t) = \frac{d}{dt}\mathrm{x}(t)$$

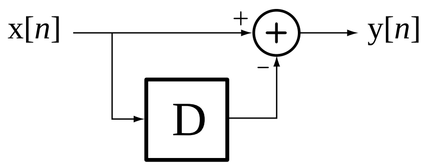
Fourier transform both sides and solve for the ratio of Y to X.

15. Draw the block diagram of a system with a bandpass frequency response using two integrators as functional blocks. Then find its transfer function and verify that it has a bandpass frequency response.

Lowpass cascaded with highpass. Find the transfer function of both stages (lowpass and highpass) and multiply the transfer functions.

16. Find the transfer function, $H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)}$, and sketch the frequency response of each of these DT filters over the range, $-4\pi < \Omega < 4\pi$.

(a)

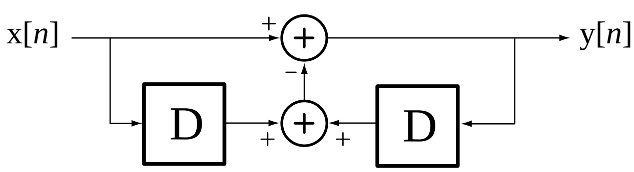


$$y[n] = x[n] - x[n-1]$$

$$H(j\Omega) = 1 - e^{-j\Omega}$$

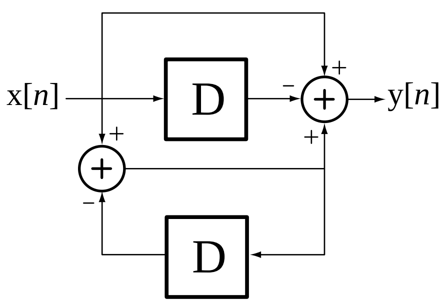
(b) Similar to (a)

(c)



$$H(j\Omega) = \frac{1 - e^{-j\Omega}}{1 + e^{-j\Omega}}$$

(d)



Let z be the output of the left-hand summer. Then

$$y[n] = x[n] - x[n-1] + z[n]$$
$$z[n] = x[n] - z[n-1]$$

Take the DTFT of both equations, eliminate Z and solve for the ratio of Y to X.

$$H(j\Omega) = \frac{2 - e^{-j2\Omega}}{1 + e^{-j\Omega}}$$

17. Find the minimum stop band attenuation of a moving-average filter with $N = 3$. Define the stop band as the frequency region, $F_c < F < \frac{1}{2}$, where F_c is the DT frequency of the first null in the frequency response.

From the text, for a moving-average filter

$$\mathrm{H}(F) = \frac{e^{-j\pi NF}}{N+1}\frac{\sin(\pi(N+1)F)}{\sin(\pi F)}$$

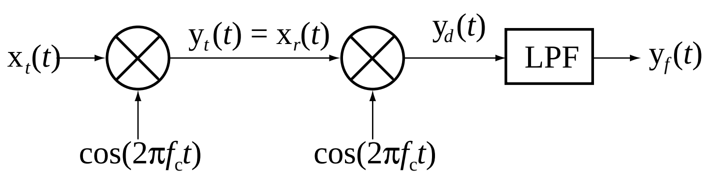
The first null in the frequency response occurs at

$$\pi(N+1)F = \pi \Rightarrow F = \frac{1}{N+1} = \frac{1}{4} .$$

The phrase, "minimum stop band attenuation" refers to the point in the stop band at which the reduction in magnitude is the smallest. That is, the point in the stop band in which the transfer function is the largest. The biggest magnitude response after the null frequency is at the next maximum of $\mathrm{H}(F)$ which occurs at

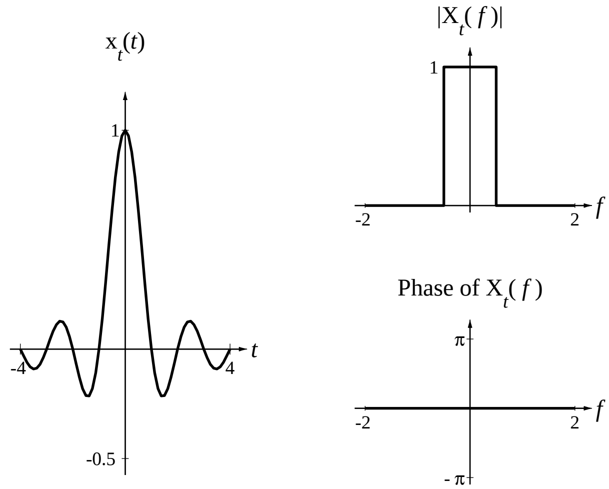
$$\pi(N+1)F = \frac{3\pi}{2} \Rightarrow F = \frac{3}{2(N+1)} = \frac{3}{8} .$$

18. In the system below, $\mathrm{x}_t(t) = \mathrm{sinc}(t)$, $f_c = 10$ and the cutoff frequency of the lowpass filter is 1 Hz. Plot the signals, $\mathrm{x}_t(t)$, $\mathrm{y}_t(t)$, $\mathrm{y}_d(t)$ and $\mathrm{y}_f(t)$ and the magnitudes and phases of their CTFT's.



$\mathrm{x}_t(t) = \mathrm{sinc}(t)$ $\quad\quad\quad$ $\mathrm{X}_t(f) = \mathrm{rect}(f)$

Modulation

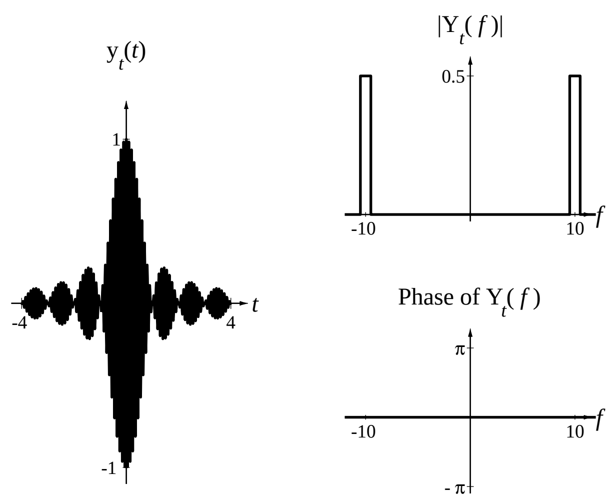


$$y_t(t) = \text{sinc}(t)\cos(20\pi t)$$

Fourier transforming,

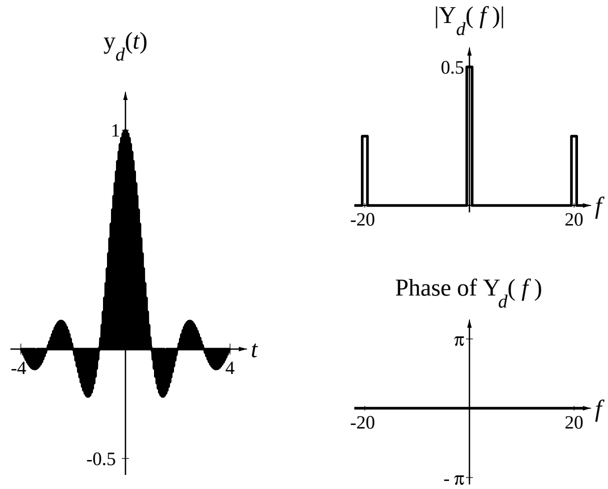
$$Y_t(f) = \frac{1}{2}\left[\text{rect}(f-10) + \text{rect}(f+10)\right]$$

Modulated Carrier



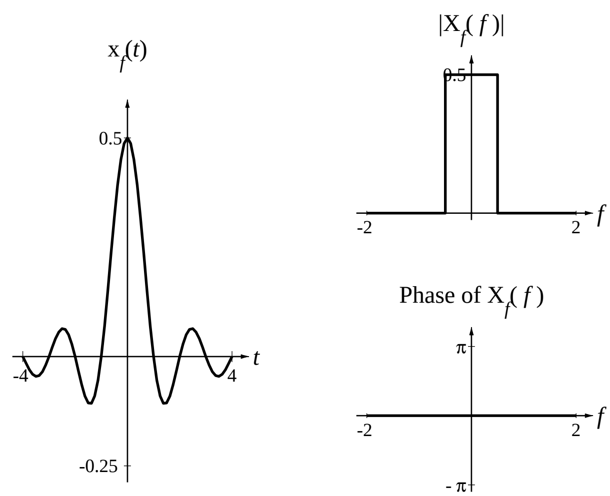
$$y_d(t) = \text{sinc}(t)\cos^2(20\pi t) \qquad Y_d(f) = \frac{1}{4}\left[\text{rect}(f-20) + 2\,\text{rect}(f) + \text{rect}(f+20)\right]$$

Demodulated Carrier

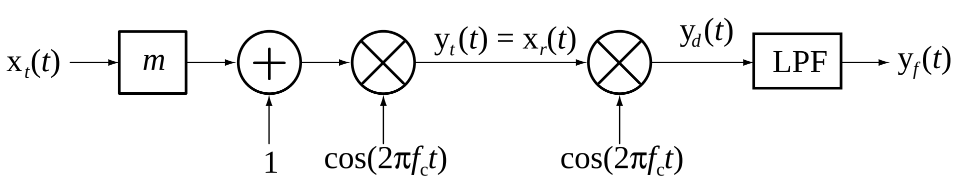


$$y_f(t) = \frac{1}{2}\mathrm{sinc}(t) \qquad \mathrm{Y}_f(f) = \frac{2\mathrm{rect}(f)}{4}$$

Demodulated and Filtered Carrier



19. In the system below, $x_t(t) = \mathrm{sinc}(10t) * \mathrm{comb}(t)$, $m = 1$, $f_c = 100$ and the cutoff frequency of the lowpass filter is 10 Hz. Plot the signals, $x_t(t)$, $y_t(t)$, $y_d(t)$ and $y_f(t)$ and the magnitudes and phases of their CTFT's.



Similar to Exercise 18.

20. An RC lowpass filter with a time constant of 16 ms is excited by a DSBSC signal,

$$x(t) = \sin(2\pi t)\cos(20\pi t) \ .$$

Find the phase and group delays at a the carrier frequency.

The transfer function of the RC lowpass filter is

$$H(j\omega) = \frac{A}{1 + j\omega\tau} = \frac{A}{1 + j0.016\omega} \ .$$

The phase of the transfer function is

$$\phi(j\omega) = -\tan^{-1}(\omega\tau) = -\tan^{-1}(0.016\omega) \ .$$

The carrier frequency is 10 Hz. Therefore

$$\phi(j20\pi) = -\tan^{-1}(0.016 \times 20\pi) = 0.788$$

and the phase delay is $-\frac{\phi(j20\pi)}{\omega_c} = \frac{0.788}{20\pi} = 0.01254 \text{ or } 12.54 \text{ ms}$. The derivative of the phase shift function is

$$\frac{d}{d\omega}(\phi(j\omega)) = -\frac{\tau}{1 + (\omega\tau)^2} \ .$$

Evaluating this derivative at the carrier frequency we get

$$-\left[\frac{d}{d\omega}(\phi(j\omega))\right]_{\omega=\omega_c} = \frac{\tau}{1 + (\omega_c\tau)^2} = \frac{0.016}{1 + (0.016 \times 20\pi)^2} = 7.95 \text{ ms} \ .$$

21. A pulse train,

$$p(t) = \text{rect}(100t) * 10\,\text{comb}(10t)$$

is modulated by a signal,

$$x(t) = \sin(4\pi t) \ .$$

Plot the response of the modulator, $y(t)$, and the CTFT's of the excitation and response.

Similar to Exercise 18.

$$y(t) = \sin(4\pi t)\sum_{n=-\infty}^{\infty} \text{rect}\left(100\left(t - \frac{n}{10}\right)\right)$$

$$Y(f) = \frac{j}{20}\left[\sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{k}{10}\right)\delta(f + 2 - 10k) - \sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{k}{10}\right)\delta(f - 2 - 10k)\right]$$

22. In the system below, let the excitation be

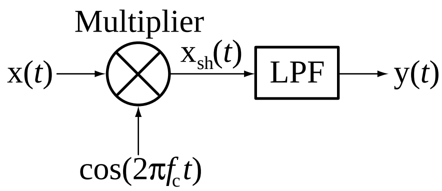
$$x(t) = \text{rect}(1000t) * 250\,\text{comb}(250t)$$

and let the filter be ideal, with unity passband gain. Plot the signal power of the response, $y(t)$, of this system versus the sweep frequency, f_c, over the range, $0 < f_c < 2000$ for a LPF bandwidth of

(a) 5 Hz

(b) 50 Hz

and (c) 500 Hz.



$$x_{sh}(t) = [\text{rect}(1000t) * 250\,\text{comb}(250t)]\cos(2\pi f_c t)$$

$$x_{sh}(t) = \cos(2\pi f_c t)\sum_{n=-\infty}^{\infty}\text{rect}\left(1000\left(t - \frac{n}{250}\right)\right)$$

$$X_{sh}(f) = \left[\frac{1}{1000}\text{sinc}\left(\frac{f}{1000}\right)\text{comb}\left(\frac{f}{250}\right)\right] * \frac{1}{2}[\delta(f - f_c) + \delta(f + f_c)]$$

$$X_{sh}(f) = \frac{1}{8}\left[\text{sinc}\left(\frac{f}{1000}\right)\sum_{k=-\infty}^{\infty}\delta(f - 250k)\right] * [\delta(f - f_c) + \delta(f + f_c)]$$

$$X_{sh}(f) = \frac{1}{8}\sum_{k=-\infty}^{\infty}\text{sinc}\left(\frac{k}{4}\right)[\delta(f - f_c - 250k) + \delta(f + f_c - 250k)]$$

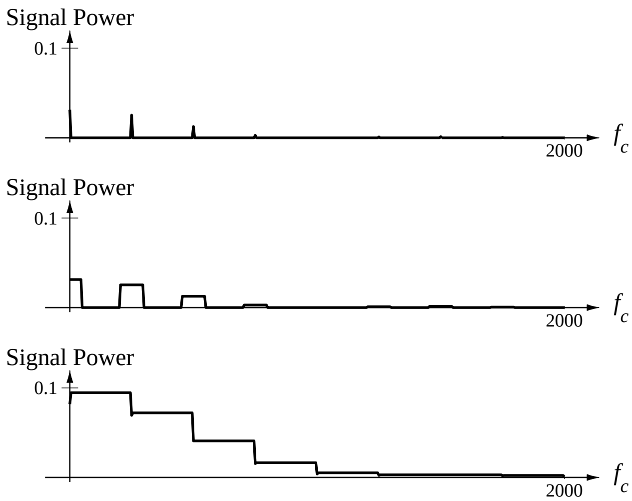
$$Y(f) = \left\{\frac{1}{8}\sum_{k=-\infty}^{\infty}\text{sinc}\left(\frac{k}{4}\right)[\delta(f - f_c - 250k) + \delta(f + f_c - 250k)]\right\}\text{rect}\left(\frac{f}{2B}\right)$$

where "B" is the bandwidth of the LPF.

$$Y(f) = \frac{1}{8}\left\{\begin{array}{l}\text{rect}\left(\frac{f}{2B}\right)\sum_{k=-\infty}^{\infty}\text{sinc}\left(\frac{k}{4}\right)\delta(f - f_c - 250k) \\ +\text{rect}\left(\frac{f}{2B}\right)\sum_{k=-\infty}^{\infty}\text{sinc}\left(\frac{k}{4}\right)\delta(f + f_c - 250k)\end{array}\right\}$$

$$Y(f) = \frac{1}{8}\left\{\sum_{|f_c + 250k| < B}\text{sinc}\left(\frac{k}{4}\right)\delta(f - f_c - 250k) + \sum_{|f_c - 250k| < B}\text{sinc}\left(\frac{k}{4}\right)\delta(f + f_c - 250k)\right\}$$

$$P_y = \frac{1}{64}\left\{ \sum_{|f_c+250k|<B} \mathrm{sinc}^2\left(\frac{k}{4}\right) + \sum_{|f_c-250k|<B} \mathrm{sinc}^2\left(\frac{k}{4}\right) \right\}$$

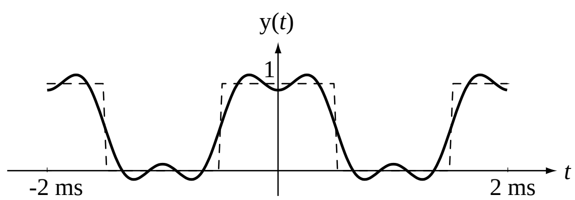


23. A signal, x(t) is described by

$$\mathrm{x}(t) = 500\,\mathrm{rect}(1000t) * \mathrm{comb}(500t)$$

(a) If x(t), is the excitation of an ideal lowpass filter with a cutoff frequency of 3 kHz, plot the excitation , x(t) and the response, y(t) on the same scale and compare.

Fourier transform the excitation, to yield X. Write the transfer function, H, of the ideal filter as a rectangle function. Form the transform of the response, Y, from X times H. Recognize it as a finite summation of impulses. Inverse transform the impulses in pairs to form y, and graph y.



This looks like the partial sums in the discussion of convergence of the CTFS because, mathematically, the same thing is happening.

(b) Similar to (a)

24. Determine whether or not the CT systems with these transfer functions are causal.

The test for causality is that a causal system has an impulse response that is zero fall time, $t < 0$.

(a) $\mathrm{H}(j\omega) = \dfrac{2}{j\omega}$ (b) $\mathrm{H}(j\omega) = \dfrac{10}{6 + j4\omega}$

(c) $\mathrm{H}(j\omega) = \dfrac{4}{25-\omega^2+j6\omega} = \dfrac{4}{(j\omega+3)^2+16}$

Using $e^{-at}\sin(\omega_0 t)\mathrm{u}(t) \xleftrightarrow{\mathcal{F}} \dfrac{\omega_0}{(j\omega+a)^2+\omega_0^2}$

$\mathrm{h}(t) = e^{-3t}\sin(4t)\mathrm{u}(t)$ Causal

(d) $\mathrm{H}(j\omega) = \dfrac{4}{25-\omega^2+j6\omega}e^{j\omega}$ (e) $\mathrm{H}(j\omega) = \dfrac{4}{25-\omega^2+j6\omega}e^{-j\omega}$

(f) $\mathrm{H}(j\omega) = \dfrac{j\omega+9}{45-\omega^2+j6\omega}$ (g) $\mathrm{H}(j\omega) = \dfrac{49}{49+\omega^2}$

25. Determine whether or not the DT systems with these transfer functions are causal.

(a) $\mathrm{H}(F) = [\mathrm{rect}(10F) * \mathrm{comb}(F)]e^{-j20\pi F}$

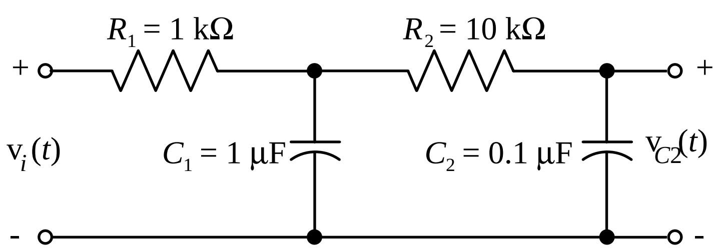
(b) $\mathrm{H}(F) = j\sin(2\pi F)$ (c) $\mathrm{H}(F) = 1 - e^{-j4\pi F}$

(d) $\mathrm{H}(j\Omega) = \dfrac{8e^{j\Omega}}{8-5e^{-j\Omega}}$

Similar to Exercise 24, except for discrete time.

26. Find and sketch the frequency response of each of these circuits given the indicated excitation and response.

(a) Excitation, $\mathrm{v}_i(t)$ - Response, $\mathrm{v}_{C2}(t)$



The transfer function can be found in multiple ways. One way is to think of this circuit as two voltage dividers. The first voltage division is from the excitation, $\mathrm{v}_i(t)$, to the voltage across the first capacitor. The second voltage division is from that voltage to the response voltage, $\mathrm{v}_{C2}(t)$.

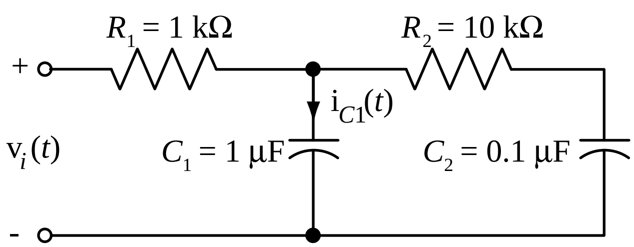
$$\mathrm{H}(j\omega) = \frac{\mathrm{V}_{C2}(j\omega)}{\mathrm{V}_i(j\omega)} = \underbrace{\frac{Z_\pi(j\omega)}{R_1 + Z_\pi(j\omega)}}_{\text{first voltage division}} \underbrace{\frac{\frac{1}{j\omega C_2}}{R_2 + \frac{1}{j\omega C_2}}}_{\text{second voltage division}} = \frac{Z_\pi(j\omega)}{R_1 + Z_\pi(j\omega)} \frac{1}{j\omega R_2 C_2 + 1}$$

$$Z_\pi(j\omega) = \frac{\frac{1}{j\omega C_1}\left(R_2 + \frac{1}{j\omega C_2}\right)}{\frac{1}{j\omega C_1} + R_2 + \frac{1}{j\omega C_2}}$$

Substitute $Z_\pi(j\omega)$ into the expression for the transfer function and simplify.

$$\mathrm{H}(j\omega) = \frac{1}{1 - \omega^2 R_1 R_2 C_1 C_2 + j\omega[(C_1 + C_2)R_1 + R_2 C_2]}$$

(b) Excitation, $\mathrm{v}_i(t)$ - Response, $\mathrm{i}_{C1}(t)$



Think of the transfer function as the transfer function from the excitation to the current in R_1 times the transfer function from the current in R_1 to $\mathrm{i}_{C1}(t)$.

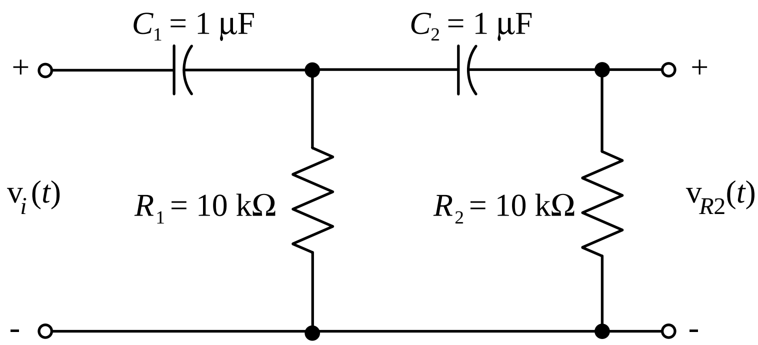
$$\mathrm{H}(j\omega) = \frac{\mathrm{I}_{C1}(j\omega)}{\mathrm{V}_i(j\omega)} = \underbrace{\frac{\mathrm{I}_{R1}(j\omega)}{\mathrm{V}_i(j\omega)}}_{\text{first transfer function}} \underbrace{\frac{R_2 + \frac{1}{j\omega C_2}}{\frac{1}{j\omega C_1} + R_2 + \frac{1}{j\omega C_2}}}_{\text{second transfer function}} = \underbrace{\frac{1}{Z_i(j\omega)}}_{\text{input impedance}} \underbrace{\frac{j\omega R_2 C_2 + 1}{j\omega R_2 C_2 + 1 + \frac{C_2}{C_1}}}_{\text{second transfer function}}$$

$$Z_i(j\omega) = R_1 + Z_\pi(j\omega) = R_1 + \frac{j\omega R_2 C_2 + 1}{j\omega(C_1 + C_2) - \omega^2 R_2 C_1 C_2}$$

Combine expressions and simplify to yield

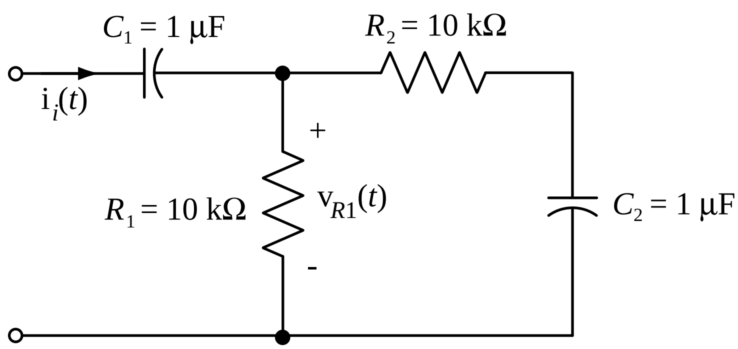
$$\mathrm{H}(j\omega) = \frac{j\omega C_1(j\omega R_2 C_2 + 1)}{1 - \omega^2 R_1 R_2 C_1 C_2 + j\omega[R_1(C_1 + C_2) + R_2 C_2]}$$

(c) Excitation, $\mathrm{v}_i(t)$ - Response, $\mathrm{v}_{R2}(t)$

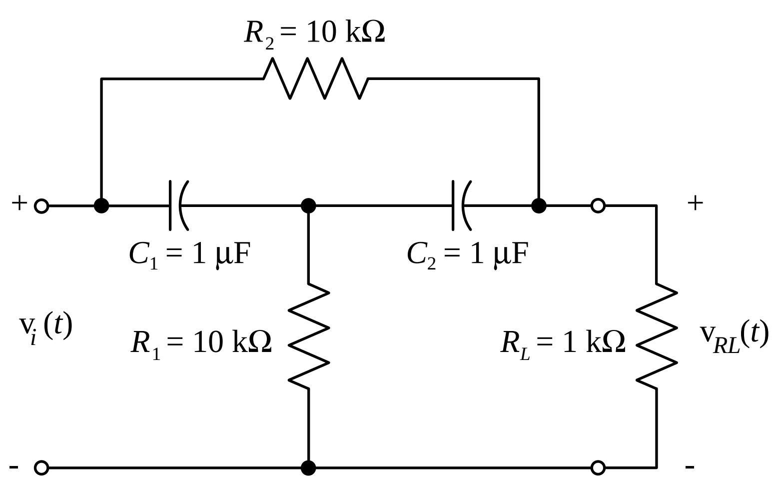


Similar to (a).

(d) Excitation, $i_i(t)$ - Response, $v_{R1}(t)$



(e) Excitation, $v_i(t)$ - Response, $v_{RL}(t)$



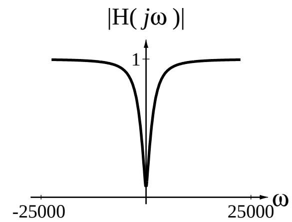
Write two nodal equations and solve for the transfer function.

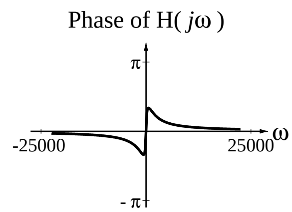
Summing currents to zero at the middle node and the right-hand node,

$$V_{R_1}(j\omega)[j\omega C_1 + j\omega C_2 + G_1] - V_i(j\omega)j\omega C_1 - V_{RL}(j\omega)j\omega C_2 = 0$$
$$V_{RL}(j\omega)[j\omega C_2 + G_L + G_2] - V_i(j\omega)G_2 - V_{R_1}(j\omega)j\omega C_2 = 0$$

Solve for the transfer function,

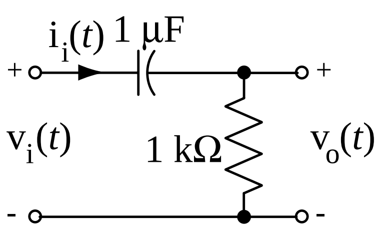
$$\mathrm{H}(j\omega)=\frac{-\omega^2R_1R_2C_1C_2+j\omega R_1(C_1+C_2)+1}{-\omega^2R_1R_2C_1C_2+j\omega\left[(C_1+C_2)\left(1+\frac{R_2}{R_L}\right)R_1+R_2C_2\right]+\frac{R_L+R_2}{R_L}}$$



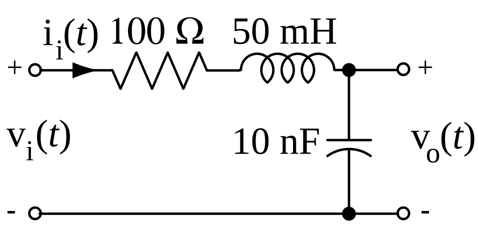


27. Find and sketch versus frequency the magnitude and phase of the input impedance, $\mathrm{Z}_{in}(f)=\frac{\mathrm{V}_i(f)}{\mathrm{I}_i(f)}$ and transfer function, $\mathrm{H}(f)=\frac{\mathrm{V}_o(f)}{\mathrm{V}_i(f)}$, for each of these filters.

(a)



(b)



28. The signal, x(*t*), in Exercise 23 is the excitation of an RC lowpass filter with $R=1\mathrm{k}\Omega$ and $C=0.3\mu\mathrm{F}$. Sketch the excitation and response voltages versus time on the same scale.

From Exercise 23,

$$\mathrm{x}(t)=500\,\mathrm{rect}(1000t)*\mathrm{comb}(500t)$$

$$X(f) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{n}{2}\right)\delta(f - 500n)$$

The transfer function is

$$H(f) = \frac{1}{j2\pi fRC + 1}$$

Therefore the output is

$$Y(f) = \frac{1}{2}\frac{1}{j2\pi fRC + 1}\sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{n}{2}\right)\delta(f - 500n)$$

$$Y(f) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \text{sinc}\left(\frac{n}{2}\right)\frac{1}{j1000\pi nRC + 1}\delta(f - 500n)$$

Converting to the time domain,

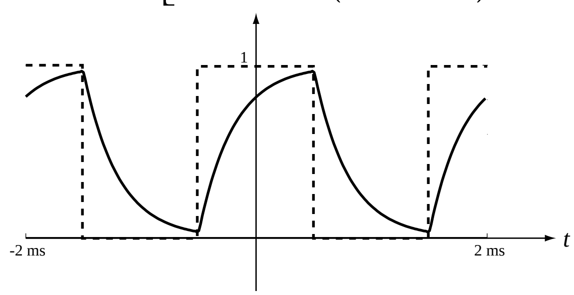
$$y(t) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \frac{\text{sinc}\left(\frac{n}{2}\right)}{j1000\pi nRC + 1}e^{j1000\pi nt}$$

or

$$y(t) = \frac{1}{2}\left\{1 + \sum_{n=1}^{\infty} \text{sinc}\left(\frac{n}{2}\right)\left[\frac{e^{j1000\pi nt}}{j1000\pi nRC + 1} + \frac{e^{-j1000\pi nt}}{-j1000\pi nRC + 1}\right]\right\}$$

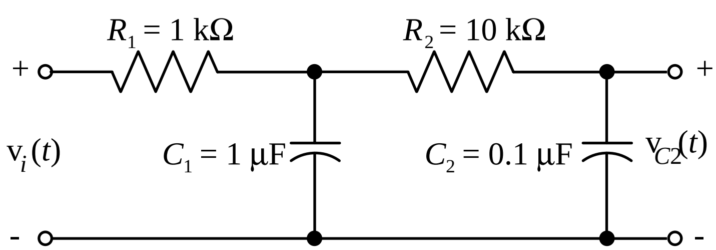
$$y(t) = \frac{1}{2}\left\{1 + \sum_{n=1}^{\infty} \text{sinc}\left(\frac{n}{2}\right)\left[\frac{j1000\pi nRC\left(e^{-j1000\pi nt} - e^{j1000\pi nt}\right) + e^{j1000\pi nt} + e^{-j1000\pi nt}}{(1000\pi nRC)^2 + 1}\right]\right\}$$

$$y(t) = \frac{1}{2}\left\{1 + \sum_{n=1}^{\infty} \text{sinc}\left(\frac{n}{2}\right)\left[\frac{2000\pi nRC\sin(1000\pi nt) + 2\cos(1000\pi nt)}{(1000\pi nRC)^2 + 1}\right]\right\}$$



29. Draw asymptotic and exact magnitude and phase Bode diagrams for the frequency responses of the following circuits and systems.

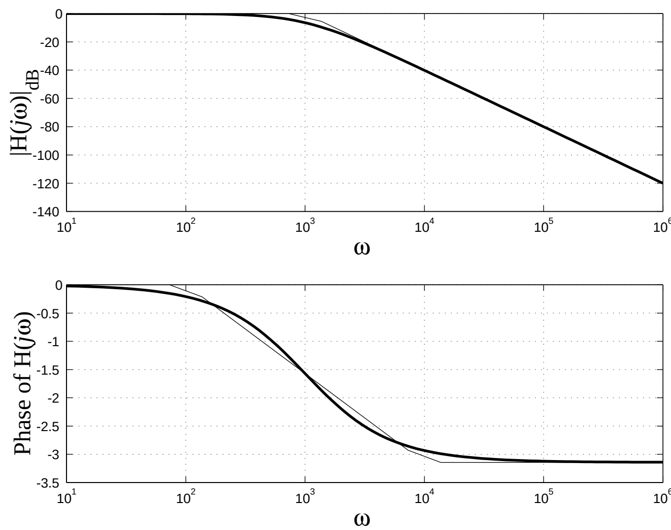
(a)



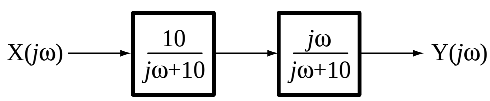
From Exercise 26(b)

$$\mathrm{H}(j\omega)=\frac{1}{1-\omega^2 R_1R_2C_1C_2+j\omega\left[(C_1+C_2)R_1+R_2C_2\right]}$$

$$\mathrm{H}(j\omega)=\frac{1}{1-10^{-6}\omega^2+j2.1\times10^{-3}\omega}$$



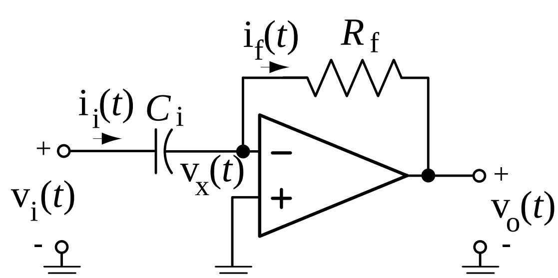
(b)



(c) A system whose transfer function is $\mathrm{H}(j\omega)=\dfrac{j20\omega}{10{,}000-\omega^2+j20\omega}$

$$\mathrm{H}(j\omega)=\frac{j20\omega}{(j\omega+10-j99.5)(j\omega+10+j99.5)}=\frac{j20\omega}{10000\left[1+j\dfrac{\omega}{500}-\dfrac{\omega^2}{10000}\right]}$$

30. Find the transfer function for the following circuit. What function does it perform?

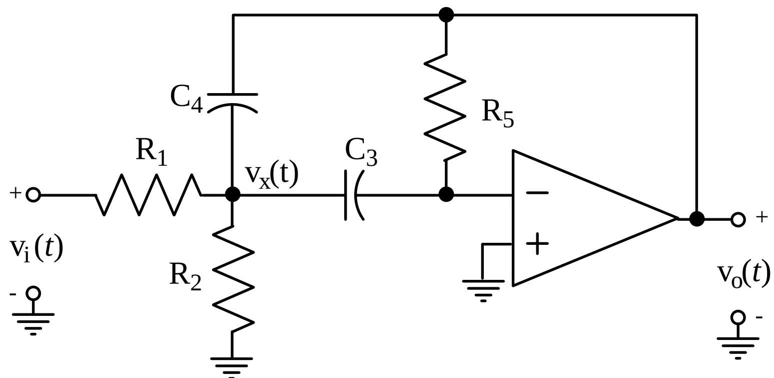


31. Design an active highpass filter using an ideal operational amplifier, two resistors and one capacitor and derive its transfer function to verify that it is high pass.

Use an inverting amplifier configuration. Let the feedback impedance be a simple resistor. Choose an input impedance that is high at low frequencies and approaches a constant at high frequencies so that the transfer function, $-\frac{Z_f(j\omega)}{Z_i(j\omega)}$ approaches zero at low frequencies and approaches a constant at high frequencies.

32. Find the transfer functions, $\mathrm{H}(f)=\frac{\mathrm{V}_o(f)}{\mathrm{V}_i(f)}$, of these active filters and identify them as lowpass, highpass, bandpass or bandstop.

(a)



Sum currents to zero at node, $\mathrm{v}_x(t)$, and at the input node of the operational amplifier, which must be at zero volts because the ideal operational amplifier gain is infinite. Remember the input impedance of the operational amplifier is infinite so no current flows into its input terminals.

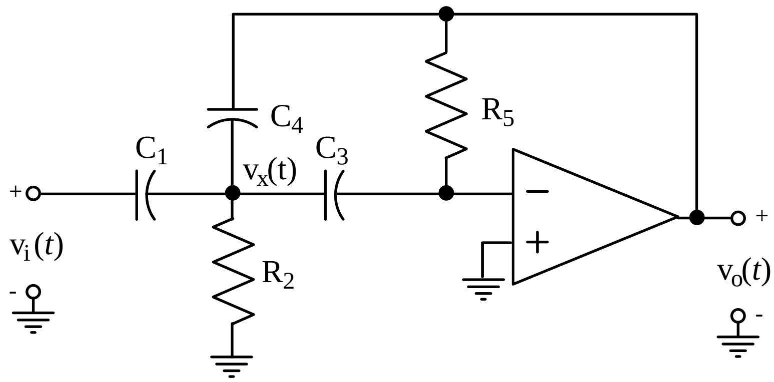
$$\mathrm{V}_x(f)(G_1+G_2+j2\pi fC_3+j2\pi fC_4)-\mathrm{V}_i(f)G_1-\mathrm{V}_o(f)j2\pi fC_4=0$$
$$-\mathrm{V}_x(f)j2\pi fC_3-\mathrm{V}_o(f)G_5=0$$

Solve for the transfer function.

$$\mathrm{H}(f)=\frac{j2\pi fR_5C_3}{(2\pi f)^2R_1R_5C_3C_4-j2\pi fR_1(C_4+C_3)-\left(1+\frac{R_1}{R_2}\right)}$$

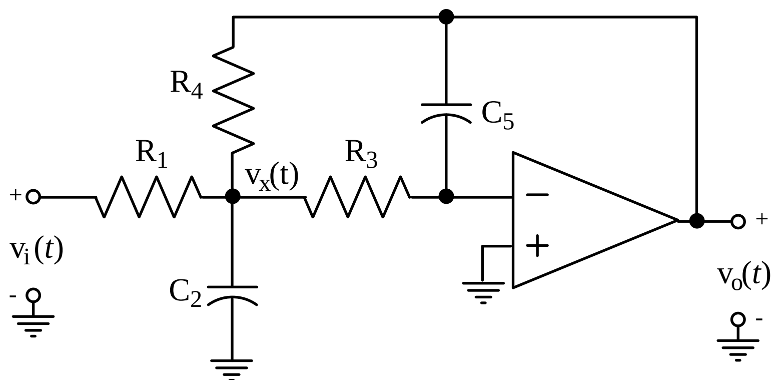
What kind of filter is this?

(b)



Similar to (a).

(c)



Similar to (a)

33. When music is recorded on analog magnetic tape and later played back, a high-frequency noise component, called tape "hiss" is added to the music. For purposes of analysis assume that the spectrum of the music is flat at –30 dB across the audio spectrum from 20 Hz to 20 kHz. Also assume that the spectrum of the signal played back on the tape deck has an added component making the playback signal have a Bode diagram as illustrated in Figure E33.

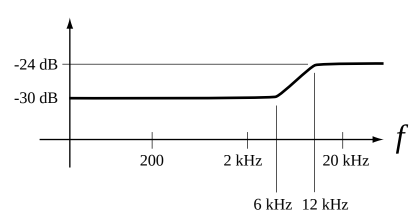


Figure E33 Bode diagram of playback signal

The extra high-frequency noise could be attenuated by a lowpass filter but that would also attenuate the high-frequency components of the music, reducing its fidelity. One solution to the problem is to "pre-emphasize" the high-frequency part of the music during the recording process so that when the lowpass filter is applied to the playback the net effect on the music is zero but the "hiss" has been attenuated. Design an active filter which could be used during the recording process to do the pre-emphasis.

The pre-emphasis active filter should have a low-frequency gain of one and a high-frequency gain of 6 dB and should transition between those gains between 6 and 12 kHz with a slope (asymptotic slope) of 6 dB/octave. So the pre-emphasis filter should have one real zero to create the first corner and one real pole to create the second corner. If we use the inverting amplifier configuration with an op-amp and make the feedback impedance a resistor then the source impedance can create the zero and pole at the required locations. The design idea is to make the source impedance have the required low-frequency value and then to make its high-frequency value lower, increasing the overall gain at high frequencies, as illustrated in Figure S33.

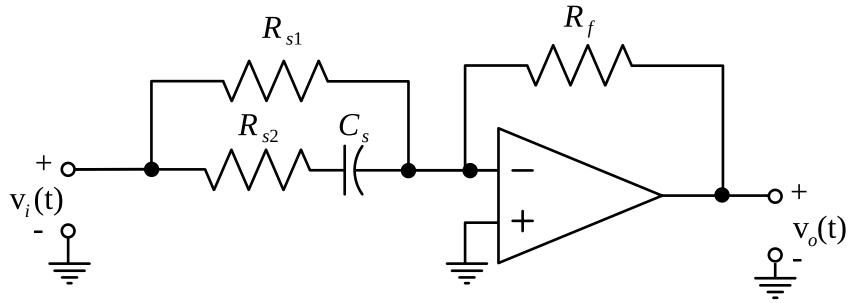


Figure S33 Pre-emphasis filter

The source impedance is

$$Z_s(f) = \frac{R_{s1}\left(R_{s2} + \frac{1}{j2\pi f C_s}\right)}{R_{s1} + R_{s2} + \frac{1}{j2\pi f C_s}} = \frac{j2\pi f C_s R_{s1} R_{s2} + R_{s1}}{j2\pi f C_s\left(R_{s1} + R_{s2}\right) + 1} = \frac{R_{s1} R_{s2}}{R_{s1} + R_{s2}} \frac{j2\pi f + \frac{1}{C_s R_{s2}}}{j2\pi f + \frac{1}{C_s\left(R_{s1} + R_{s2}\right)}}$$

The numerator provides the zero of the source impedance (the pole of the overall gain) and the denominator provides the pole of the source impedance (the zero of the overall gain). So we want the gain-pole location to be set by

$$\frac{1}{C_s R_{s2}} = 2\pi \times 12000$$

and the gain-zero location to be set by

$$\frac{1}{C_s\left(R_{s1} + R_{s2}\right)} = 2\pi \times 6000 \quad .$$

There is no unique solution so let's arbitrarily set $R_{s2} = 10\ \text{k}\Omega$. Then it follows that $C_s = 1.33\ \text{nF}$ and $R_{s1} = 10\ \text{k}\Omega$. The overall gain is

$$\mathrm{H}(f) = -\frac{R_f}{\dfrac{R_{s1}R_{s2}}{R_{s1}+R_{s2}} \dfrac{j2\pi f + \dfrac{1}{C_s R_{s2}}}{j2\pi f + \dfrac{1}{C_s(R_{s1}+R_{s2})}}} = -R_f \frac{R_{s1}+R_{s2}}{R_{s1}R_{s2}} \frac{j2\pi f + \dfrac{1}{C_s(R_{s1}+R_{s2})}}{j2\pi f + \dfrac{1}{C_s R_{s2}}}$$

At low frequencies,

$$\mathrm{H}(f) = -\frac{R_f}{R_{s1}} .$$

To make the low-frequency gain one, set $R_f = R_{s1} = 10\ \mathrm{k\Omega}$.

34. One problem with causal CT filters is that the response of the filter always lags the excitation. This problem cannot be eliminated if the filtering is done in real time but if the signal is recorded for later "off-line" filtering one simple way of eliminating the lag effect is to filter the signal, record the response and then filter that recorded response with the same filter but playing the signal back through the system *backward*. Suppose the filter is a single-pole filter with a transfer function of the form,

$$\mathrm{H}(j\omega) = \frac{1}{1 + j\dfrac{\omega}{\omega_c}} ,$$

where ω_c is the cutoff frequency (half-power frequency) of the filter.

(a) What is the effective transfer function of the entire process of filtering the signal forward, then backward?

(b) What is the effective impulse response?

(a) The impulse response of the filter is $\mathrm{h}(t) = \mathcal{F}^{-1}[\mathrm{H}(j\omega)]$. The response of the filter on the first pass through the filter forward is $\mathrm{y}_1(t) = \mathrm{x}(t) * \mathrm{h}(t)$ in the time domain or $\mathrm{Y}_1(j\omega) = \mathrm{X}(j\omega)\mathrm{H}(j\omega)$ in the frequency domain. The response of the filter on the second pass through the filter backward is $\mathrm{y}_2(t) = \mathrm{y}_1(-t) * \mathrm{h}(t)$ in the time domain or $\mathrm{Y}_2(j\omega) = \mathrm{Y}_1^*(j\omega)\mathrm{H}(j\omega)$ in the frequency domain. (This uses the CTFT property, $\mathrm{g}(-t) \xleftrightarrow{\mathcal{F}} \mathrm{G}^*(j\omega)$.) The final signal is $\mathrm{y}_2(-t)$ in the time domain and $\mathrm{Y}_2^*(j\omega)$ in the frequency domain. Therefore the final signal in the frequency domain is

$$\mathrm{Y}_2^*(j\omega) = \left[\mathrm{Y}_1^*(j\omega)\mathrm{H}(j\omega)\right]^* = \mathrm{Y}_1(j\omega)\mathrm{H}^*(j\omega) = \mathrm{X}(j\omega)\mathrm{H}(j\omega)\mathrm{H}^*(j\omega)$$

and in the time domain this is $\mathrm{y}_2(t) = \mathrm{x}(t) * \mathrm{h}(t) * \mathrm{h}(-t)$. So the effective transfer function is

$$\mathrm{H}(j\omega)\mathrm{H}^*(j\omega) = \frac{1}{1+j\dfrac{\omega}{\omega_c}}\frac{1}{1-j\dfrac{\omega}{\omega_c}} = \frac{1}{1+\left(\dfrac{\omega}{\omega_c}\right)^2}$$

and, using $e^{-a|t|} \xleftrightarrow{\mathcal{F}} \dfrac{2a}{a^2+\omega^2}$, the effective impulse response is

$$\mathrm{h}(t) * \mathrm{h}(-t) = \omega_c e^{-\omega_c t}\,\mathrm{u}(t) * \omega_c e^{\omega_c t}\,\mathrm{u}(-t) = \frac{\omega_c}{2} e^{-\omega_c |t|}.$$

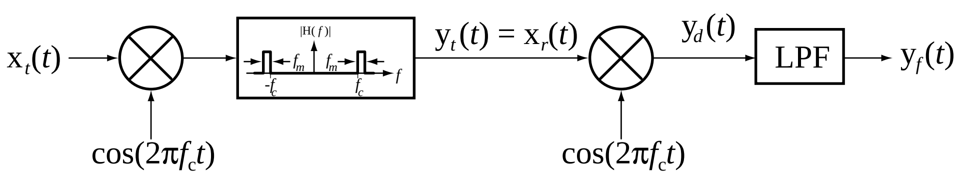
Just as there is no phase shift in the frequency domain, the effective impulse response, $\mathrm{h}(-t) * \mathrm{h}(t)$ is symmetrical about $t = 0$ which also means it creates no time delay when it is convolved with the excitation.

35. Repeat Exercise 18 but with the second $\cos(2\pi f_c t)$ replaced by $\sin(2\pi f_c t)$.

Similar to Exercise 18.

36. In the system below, $\mathrm{x}_t(t) = \mathrm{sinc}(t)$, $f_c = 10$ and the cutoff frequency of the lowpass filter is 1 Hz. Plot the signals, $\mathrm{x}_t(t)$, $\mathrm{y}_t(t)$, $\mathrm{y}_d(t)$ and $\mathrm{y}_f(t)$ and the magnitudes and phases of their CTFT's.

This is a single sideband system. Analysis is similar to several previous communication system exercises. The biggest difference is the addition of the filter which removes a sideband before transmission.



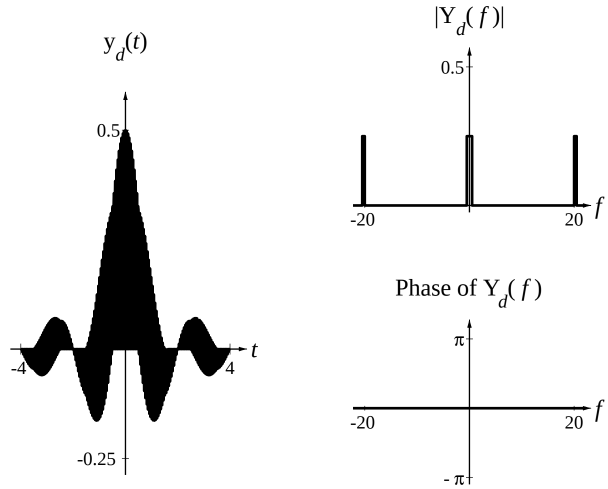
$$\mathrm{Y}_t(f) = \frac{1}{2}\left[\mathrm{rect}\left(\frac{f-10.25}{0.5}\right) + \mathrm{rect}\left(\frac{f+10.25}{0.5}\right)\right]$$

$$\mathrm{y}_t(t) = \frac{1}{2}\mathrm{sinc}\left(\frac{t}{2}\right)\cos(20.5\pi t)$$

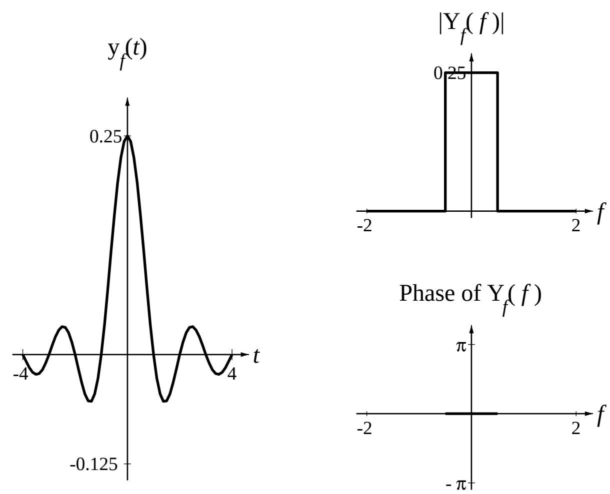
$$\mathrm{y}_d(t) = \frac{1}{2}\mathrm{sinc}\left(\frac{t}{2}\right)\cos(20.5\pi t)\cos(20\pi t)$$

$$\mathrm{Y}_d(f) = \frac{1}{4}\begin{bmatrix} \mathrm{rect}(2(f-20.25)) + \mathrm{rect}(2(f+0.25)) \\ +\mathrm{rect}(2(f-0.25)) + \mathrm{rect}(2(f+20.25)) \end{bmatrix}$$

Demodulated Carrier



Demodulated and Filtered Carrier



37. A quadrature modulator modulates a sine carrier, $\sin(20\pi t)$, with a signal, $x_1(t) = \text{sinc}(t)$, and a cosine carrier, $\cos(20\pi t)$, with a signal, $x_2(t) = \text{rect}(t)$. The quadrature demodulator has a phase error making its local oscillators be $\sin\left(20\pi t - \frac{\pi}{6}\right)$ and $\cos\left(20\pi t - \frac{\pi}{6}\right)$. Plot the two demodulated and filtered signals, $x_{1f}(t)$ and $x_{2f}(t)$.

$$y(t) = \text{sinc}(t)\sin(20\pi t) + \text{rect}(t)\cos(20\pi t)$$

$$X_{1d}(f) = \mathcal{F}\left\{\left[\text{sinc}(t)\sin(20\pi t) + \text{rect}(t)\cos(20\pi t)\right]\sin\left(20\pi t - \frac{\pi}{6}\right)\right\}$$

$$\mathrm{X}_{1d}(f)=\begin{bmatrix}\mathrm{rect}(f)*\dfrac{j}{2}[\delta(f+10)-\delta(f-10)]\\+\mathrm{sinc}(f)*\dfrac{1}{2}[\delta(f-10)+\delta(f+10)]\end{bmatrix}*\frac{j}{2}[\delta(f+10)-\delta(f-10)]e^{-j\frac{\pi f}{60}}$$

$$\mathrm{X}_{1d}(f)=\frac{j}{4}\begin{bmatrix}j[\mathrm{rect}(f+10)-\mathrm{rect}(f-10)]\\+[\mathrm{sinc}(f-10)+\mathrm{sinc}(f+10)]\end{bmatrix}*[\delta(f+10)-\delta(f-10)]e^{-j\frac{\pi f}{60}}$$

$$\mathrm{X}_{1d}(f)=\frac{j}{4}\begin{bmatrix}j\begin{bmatrix}\mathrm{rect}(f+10)*\left[\delta(f+10)e^{-j\frac{\pi f}{60}}-\delta(f-10)e^{-j\frac{\pi f}{60}}\right]\\-\mathrm{rect}(f-10)*\left[\delta(f+10)e^{-j\frac{\pi f}{60}}-\delta(f-10)e^{-j\frac{\pi f}{60}}\right]\end{bmatrix}\\+\begin{bmatrix}\mathrm{sinc}(f-10)*\left[\delta(f+10)e^{-j\frac{\pi f}{60}}-\delta(f-10)e^{-j\frac{\pi f}{60}}\right]\\+\mathrm{sinc}(f+10)*\left[\delta(f+10)e^{-j\frac{\pi f}{60}}-\delta(f-10)e^{-j\frac{\pi f}{60}}\right]\end{bmatrix}\end{bmatrix}$$

$$\mathrm{X}_{1d}(f)=\frac{j}{4}\begin{bmatrix}j\begin{bmatrix}\left[\mathrm{rect}(f+10)*\delta(f+10)e^{j\frac{\pi}{6}}-\mathrm{rect}(f+10)*\delta(f-10)e^{-j\frac{\pi}{6}}\right]\\-\left[\mathrm{rect}(f-10)*\delta(f+10)e^{j\frac{\pi}{6}}-\mathrm{rect}(f-10)*\delta(f-10)e^{-j\frac{\pi}{6}}\right]\end{bmatrix}\\+\begin{bmatrix}\left[\mathrm{sinc}(f-10)*\delta(f+10)e^{j\frac{\pi}{6}}-\mathrm{sinc}(f-10)*\delta(f-10)e^{-j\frac{\pi}{6}}\right]\\+\left[\mathrm{sinc}(f+10)*\delta(f+10)e^{j\frac{\pi}{6}}-\mathrm{sinc}(f+10)*\delta(f-10)e^{-j\frac{\pi}{6}}\right]\end{bmatrix}\end{bmatrix}$$

$$\mathrm{X}_{1d}(f)=\frac{j}{4}\begin{bmatrix}j\left[\mathrm{rect}(f+20)e^{j\frac{\pi}{6}}-\mathrm{rect}(f)e^{-j\frac{\pi}{6}}-\mathrm{rect}(f)e^{j\frac{\pi}{6}}+\mathrm{rect}(f-20)e^{-j\frac{\pi}{6}}\right]\\+\left[\mathrm{sinc}(f)e^{j\frac{\pi}{6}}-\mathrm{sinc}(f-20)e^{-j\frac{\pi}{6}}+\mathrm{sinc}(f+20)e^{j\frac{\pi}{6}}-\mathrm{sinc}(f)e^{-j\frac{\pi}{6}}\right]\end{bmatrix}$$

$$\mathrm{X}_{1d}(f)=\frac{j}{4}\begin{bmatrix}j\left[\mathrm{rect}(f+20)e^{j\frac{\pi}{6}}-2\cos\left(\dfrac{\pi}{6}\right)\mathrm{rect}(f)+\mathrm{rect}(f-20)e^{-j\frac{\pi}{6}}\right]\\+\left[j2\sin\left(\dfrac{\pi}{6}\right)\mathrm{sinc}(f)-\mathrm{sinc}(f-20)e^{-j\frac{\pi}{6}}+\mathrm{sinc}(f+20)e^{j\frac{\pi}{6}}\right]\end{bmatrix}$$

Now apply the lowpass filter,

$$X_{1f}(f) = \frac{j}{4}\left[-j2\cos\left(\frac{\pi}{6}\right)\text{rect}(f) + j2\sin\left(\frac{\pi}{6}\right)\text{sinc}(f)\right]$$

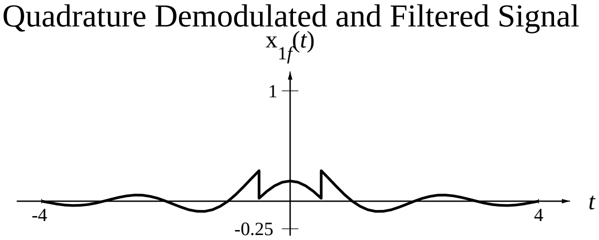
$$X_{1f}(f) = \frac{1}{2}\left[\cos\left(\frac{\pi}{6}\right)\text{rect}(f) - \sin\left(\frac{\pi}{6}\right)\text{sinc}(f)\right]$$

$$x_{1f}(t) = \frac{1}{2}\left[\cos\left(\frac{\pi}{6}\right)\text{sinc}(t) - \sin\left(\frac{\pi}{6}\right)\text{rect}(t)\right]$$

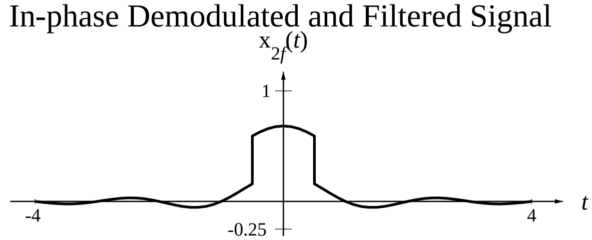
Similarly,

$$x_{2f}(t) = \frac{1}{2}\left[\cos\left(\frac{\pi}{6}\right)\text{rect}(t) + \sin\left(\frac{\pi}{6}\right)\text{sinc}(t)\right]$$

Quadrature Demodulated and Filtered Signal



In-phase Demodulated and Filtered Signal



38. A pulse train,

$$p(t) = \frac{1}{w}\text{rect}\left(\frac{t}{w}\right) * 4\,\text{comb}(4t)$$

is modulated by a signal,

$$x(t) = \text{sinc}(t) \ .$$

Plot the response of the modulator, $y(t)$, and the CTFT's of the excitation and response for

(a) $w = 10$ ms

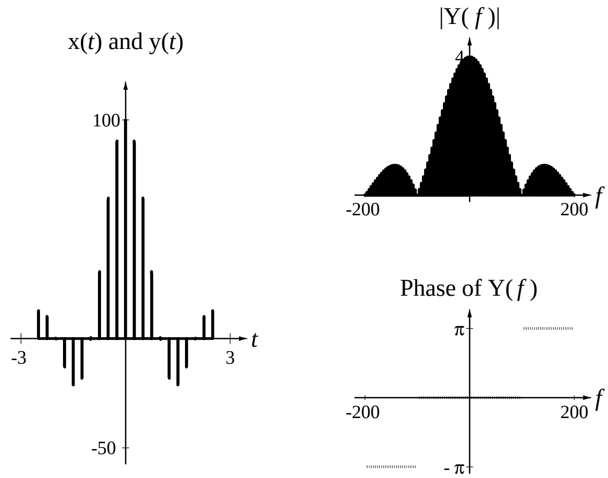
and (b) $w = 1$ ms .

This is similar to previous modulation exercises.

$$\mathrm{y}(t) = \frac{1}{w}\mathrm{sinc}(t)\sum_{n=-\infty}^{\infty}\mathrm{rect}\left(\frac{t-\frac{n}{4}}{w}\right)$$

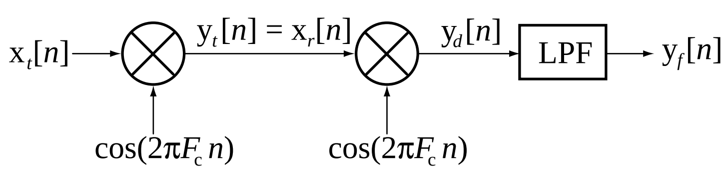
$$\mathrm{Y}(f) = 4\sum_{k=-\infty}^{\infty}\mathrm{sinc}(4wk)\mathrm{rect}(f-4k)$$

PAM Modulator Response (a)



Because of the scale, it is difficult to see what is really happening in the magnitude plot of the transform of the response. It consists of a large number of closely-spaced impulses.

39. In the system below, $\mathrm{x}_t[n] = \mathrm{sinc}\left(\frac{n}{20}\right)$, $F_c = \frac{1}{4}$ and the cutoff DT frequency of the lowpass filter is $\frac{1}{20}$. Plot the signals, $\mathrm{x}_t[n]$, $\mathrm{y}_t[n]$, $\mathrm{y}_d[n]$ and $\mathrm{y}_f[n]$ and the magnitudes and phases of their DTFT's.



This is a DT modulation system. The analysis is very similar to that used for CT modulation systems.

40. Repeat Exercise 22 but with an excitation,

$$\mathrm{x}(t) = \mathrm{rect}(1000t) * 20\,\mathrm{comb}(20t)\ .$$

$$\mathrm{x}_{sh}(t) = [\mathrm{rect}(1000t) * 20\,\mathrm{comb}(20t)]\cos(2\pi f_c t)$$

$$x_{sh}(t) = \cos(2\pi f_c t) \sum_{n=-\infty}^{\infty} \text{rect}\left(1000\left(t - \frac{n}{20}\right)\right)$$

$$X_{sh}(f) = \left[\frac{1}{1000}\text{sinc}\left(\frac{f}{1000}\right)\text{comb}\left(\frac{f}{20}\right)\right] * \frac{1}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right]$$

$$X_{sh}(f) = \frac{1}{8}\left[\text{sinc}\left(\frac{f}{1000}\right)\sum_{k=-\infty}^{\infty}\delta(f - 20k)\right] * \left[\delta(f - f_c) + \delta(f + f_c)\right]$$

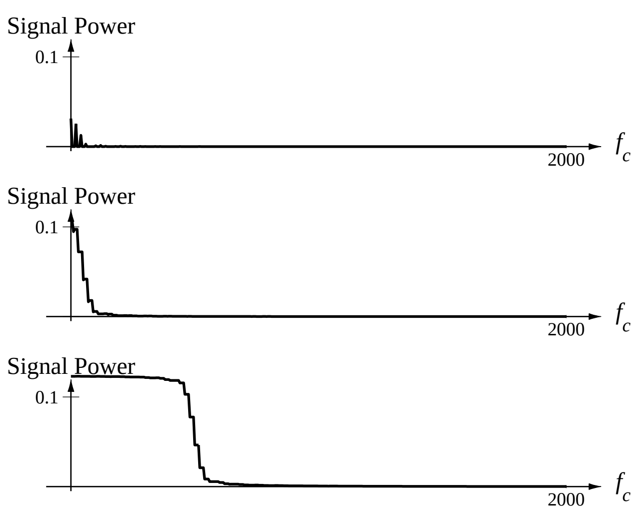
$$X_{sh}(f) = \frac{1}{8}\sum_{k=-\infty}^{\infty}\text{sinc}\left(\frac{k}{4}\right)\left[\delta(f - f_c - 20k) + \delta(f + f_c - 20k)\right]$$

$$Y(f) = \left\{\frac{1}{8}\sum_{k=-\infty}^{\infty}\text{sinc}\left(\frac{k}{4}\right)\left[\delta(f - f_c - 20k) + \delta(f + f_c - 20k)\right]\right\}\text{rect}\left(\frac{f}{2B}\right)$$

$$Y(f) = \frac{1}{8}\left\{\begin{array}{l}\text{rect}\left(\frac{f}{2B}\right)\sum_{k=-\infty}^{\infty}\text{sinc}\left(\frac{k}{4}\right)\delta(f - f_c - 20k) \\ +\text{rect}\left(\frac{f}{2B}\right)\sum_{k=-\infty}^{\infty}\text{sinc}\left(\frac{k}{4}\right)\delta(f + f_c - 20k)\end{array}\right\}$$

$$Y(f) = \frac{1}{8}\left\{\sum_{|f_c + 20k| < B}\text{sinc}\left(\frac{k}{4}\right)\delta(f - f_c - 20k) + \sum_{|f_c - 20k| < B}\text{sinc}\left(\frac{k}{4}\right)\delta(f + f_c - 20k)\right\}$$

$$P_y = \frac{1}{64}\left\{\sum_{|f_c + 20k| < B}\text{sinc}^2\left(\frac{k}{4}\right) + \sum_{|f_c - 20k| < B}\text{sinc}^2\left(\frac{k}{4}\right)\right\}$$



41. The diffraction of light can be approximately described through the use of the Fourier transform. Consider an opaque screen with a small slit being illuminated from the left by a normally-incident uniform plane light wave (Figure E41).

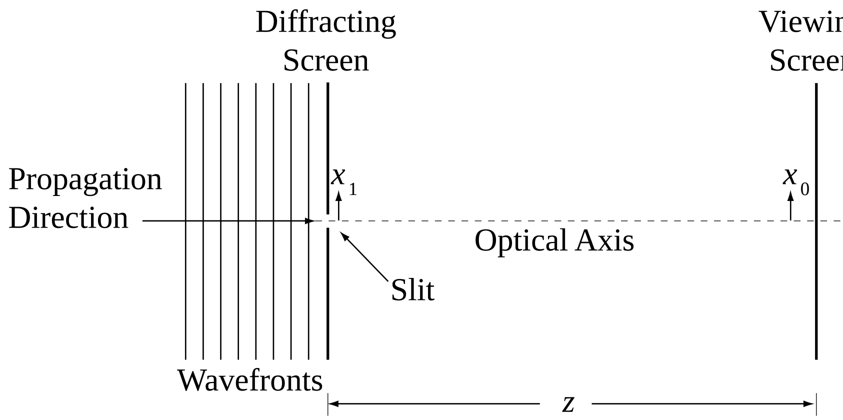


Figure E41 One-dimensional diffraction of light through a slit

If $z >> \frac{\pi x_1^2}{\lambda}$ is a good approximation for any x_1 in the slit, then the electric field strength of the light striking the viewing screen can be accurately described by

$$\mathrm{E}_0(x_0) = K \frac{e^{j\frac{2\pi z}{\lambda}}}{j\lambda z} e^{j\frac{\pi}{\lambda z}x_0^2} \int_{-\infty}^{\infty} \mathrm{E}_1(x_1) e^{-j\frac{2\pi}{\lambda z}x_0 x_1} dx_1$$

where E_1 is field strength at the diffracting screen, E_0 is field strength at the viewing screen, K is a constant of proportionality and λ is the wavelength of the light. The integral is a Fourier transform with different notation. The field strength at the viewing screen can be written as

$$\mathrm{E}_0(x_0) = K \frac{e^{j\frac{2\pi z}{\lambda}}}{j\lambda z} e^{j\frac{\pi}{\lambda z}x_0^2} \mathcal{F}\left[\mathrm{E}_1(t)\right]_{f \to \frac{x_0}{\lambda z}} .$$

The intensity, $\mathrm{I}_0(x_0)$, of the light at the viewing screen is the square of the magnitude of the field strength,

$$\mathrm{I}(x_0) = \left|\mathrm{E}_0(x_0)\right|^2 .$$

(a) Plot the intensity of light at the viewing screen if the slit width is 1 mm, the wavelength of light is 500 nm, the distance, z, is 100 m, the constant of proportionality is 10^{-3} and the electric field strength at the diffraction screen is $1\ \frac{\mathrm{V}}{\mathrm{m}}$.

The electric field exiting the diffraction slit is

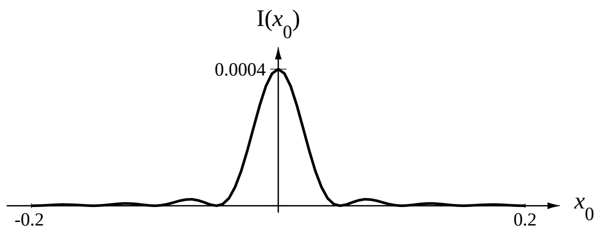
$$\mathrm{E}(x_1) = \mathrm{rect}\left(\frac{x_1}{0.001}\right)$$

Finding the screen electric field using the formula above,

$$E_0(x_0) = \frac{e^{j\frac{2\pi z}{\lambda}}}{j50} e^{j\frac{\pi}{\lambda z}x_0^2} \operatorname{sinc}\left(\frac{x_0}{5\times10^{-2}}\right)$$

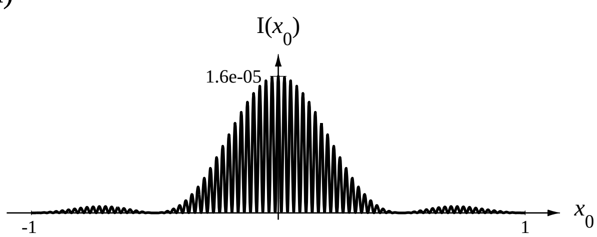
The intensity is the square of the magnitude of the electric field,

$$I_0(x_0) = \frac{\operatorname{sinc}^2\left(\frac{x_0}{5\times10^{-2}}\right)}{2500} = 4\times10^{-4}\operatorname{sinc}^2(20x_0)$$



(b) Now let the slit be replaced by two slits each 0.1 mm in width, separated by 1 mm (center-to-center) and centered on the optical axis. Plot the intensity of light at the viewing screen if the other parameters are the same as in part (a).

Similar to (a)



42. In Figure 42-1 is a circuit diagram of a half-wave rectifier followed by a capacitor to smooth the response voltage. Model the diode as ideal and let the excitation be a cosine at 60 Hz with an amplitude of $120\sqrt{2}$ volts. Let the *RC* time constant be 0.1 seconds. Then the response voltage will look as illustrated in Figure E42-2 . Find and plot the magnitude of the CTFT of the response voltage.

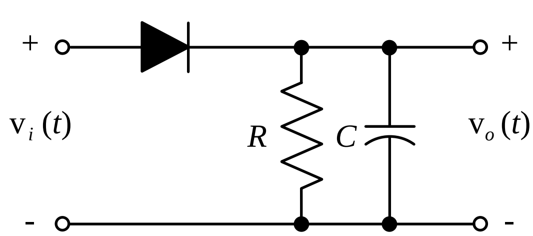


Figure E42-1 A half-wave rectifier with a capacitive smoothing filter

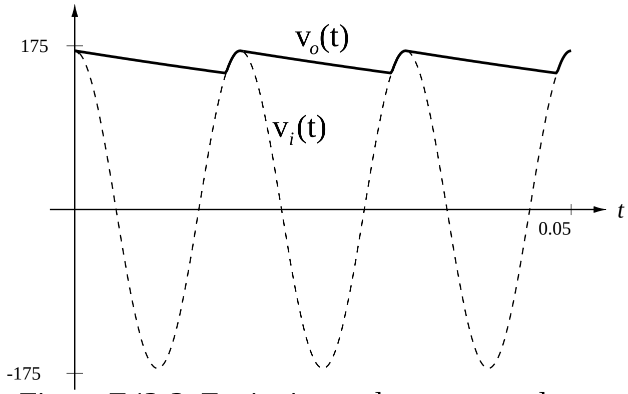


Figure E42-2 Excitation and response voltages

The response voltage has two parts, the exponential decay time and the cosinusoidal charging time. The dividing time, t_d, between these two parts is set by the intersection of the cosine and the exponential decay. The peak of the cosine is $120\sqrt{2}$. The decay time constant is 0.1 seconds. Therefore the dividing time is the solution of

$$120\sqrt{2}\cos(120\pi t_d) = 120\sqrt{2}e^{-\frac{t_d}{0.1}}$$

or

$$\cos(120\pi t_d) = e^{-\frac{t_d}{0.1}}$$

This is a transcendental equation best solved numerically. This equation is simple enough that a trial-and-error method converges very quickly to a solution. That solution is

$$t_d = 15.23906 \text{ ms} \quad .$$

Therefore the description of the response voltage over one period is

$$\mathrm{v}_{o1}(t) = 120\sqrt{2}\begin{cases} e^{-\frac{t}{0.1}} & , \ 0 < t < 15.23906 \text{ ms} \\ \cos(120\pi t) & , \ 15.23906 < t < 16\frac{2}{3} \text{ ms} \end{cases}$$

or

$$\mathrm{v}_{o1}(t) = 120\sqrt{2}\left[e^{-\frac{t}{0.1}}\mathrm{rect}\left(\frac{t-0.00761953}{0.01523906}\right) + \cos(120\pi t)\,\mathrm{rect}\left(\frac{t-0.01595286}{0.001427607}\right)\right]$$

The CTFT of the response is the CTFT of this voltage convolved with a comb to make it periodically repeat. The CTFT of one period is

$$\mathrm{V}_{o1}(f) = 120\sqrt{2}\left[\begin{array}{l} \displaystyle\int_0^{0.01523906} e^{-\frac{t}{0.1}} e^{-j2\pi f t}\,dt \\ +\dfrac{1}{2}\left[\delta(f-60)+\delta(f+60)\right] * 0.001427607\,\mathrm{sinc}(0.001427607 f)e^{-j2\pi f(0.01595286)} \end{array}\right]$$

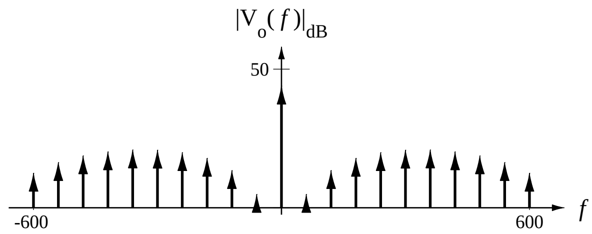
$$V_{o1}(f) = 120\sqrt{2}\left\{\begin{array}{l} \left[\dfrac{e^{-\frac{t}{0.1}}e^{-j2\pi ft}}{-10-j2\pi f}\right]_0^{0.01523906} \\ +\dfrac{1}{2}0.001427607\,\text{sinc}\big(0.001427607(f-60)\big)e^{-j2\pi(f-60)(0.01595286)} \\ +\dfrac{1}{2}0.001427607\,\text{sinc}\big(0.001427607(f+60)\big)e^{-j2\pi(f+60)(0.01595286)} \end{array}\right\}$$

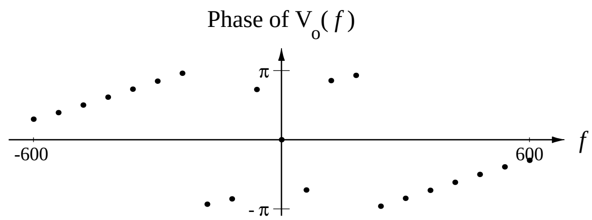
$$V_{o1}(f) = 120\sqrt{2}\left\{\begin{array}{l} \left[\dfrac{e^{-\frac{0.01523906}{0.1}}e^{-j2\pi f\,0.01523906}}{-10-j2\pi f}\right] - \left[\dfrac{1}{-10-j2\pi f}\right] \\ +\dfrac{1}{2}0.001427607\,\text{sinc}\big(0.001427607(f-60)\big)e^{-j2\pi(f-60)(0.01595286)} \\ +\dfrac{1}{2}0.001427607\,\text{sinc}\big(0.001427607(f+60)\big)e^{-j2\pi(f+60)(0.01595286)} \end{array}\right\}$$

$$V_{o1}(f) = 120\sqrt{2}\left\{\begin{array}{l} \dfrac{1-e^{-\frac{0.01523906}{0.1}}e^{-j2\pi f\,0.01523906}}{10+j2\pi f} \\ +\dfrac{1}{2}0.001427607\,\text{sinc}\big(0.001427607(f-60)\big)e^{-j2\pi(f-60)(0.01595286)} \\ +\dfrac{1}{2}0.001427607\,\text{sinc}\big(0.001427607(f+60)\big)e^{-j2\pi(f+60)(0.01595286)} \end{array}\right\}$$

The CTFT of the actual periodic response is the product of this CTFT with the CTFT of $60\,\text{comb}(60t)$ which is $\text{comb}\left(\frac{f}{60}\right)$. Therefore

$$V_{o}(f) = 120\sqrt{2}\left\{\begin{array}{l} \dfrac{1-e^{-\frac{0.01523906}{0.1}}e^{-j2\pi f\,0.01523906}}{10+j2\pi f} \\ +\dfrac{1}{2}0.001427607\,\text{sinc}\big(0.001427607(f-60)\big)e^{-j2\pi(f-60)(0.01595286)} \\ +\dfrac{1}{2}0.001427607\,\text{sinc}\big(0.001427607(f+60)\big)e^{-j2\pi(f+60)(0.01595286)} \end{array}\right\}\text{comb}\left(\frac{f}{60}\right)$$





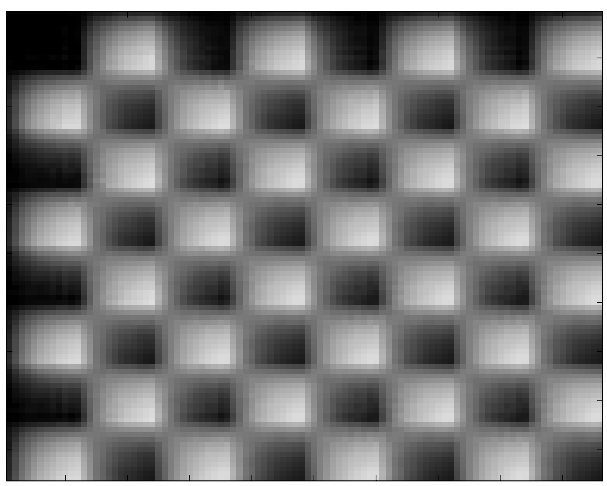
43. Create a discrete-space image consisting of 96 by 96 pixels. Let the image be a "checkerboard" consisting of 8 by 8 alternating black-and-white squares.

(a) Filter the image row-by-row and then column-by-column with a DT filter whose impulse response is

$$h[n] = 0.2(0.8)^n \, u[n]$$

and display the image on the screen using the `imagesc` command in MATLAB.

After defining the checkerboard we can filter it by convolving it with the impulse response using the MATLAB `conv` function.

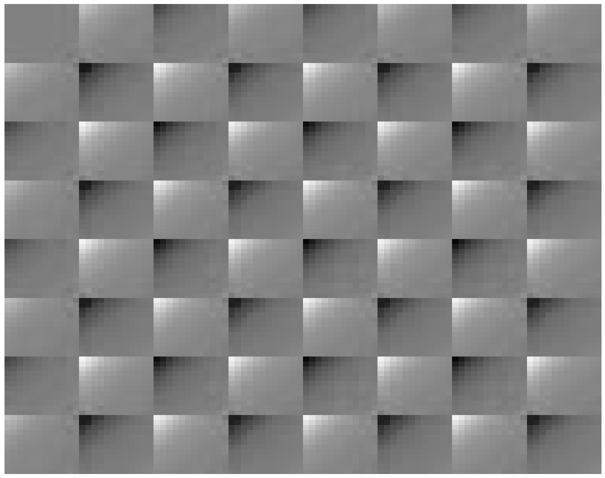


Notice how this lowpass spatial filter blurs the edges. A lowpass filter does not allow any fast transitions to occur.

(b) Filter the image row-by-row and then column-by-column with a DT filter whose impulse response is

$$h[n] = \delta[n] - 0.2(0.8)^n \, u[n]$$

and display the image on the screen using the `imagesc` command in MATLAB.



Notice how this highpass spatial filter empahsize the edges and de-emphasizes the constant regions between the edges. A highpass filter does allow fast transitions to occur.

44. In the system of Figure E0 let the CTFT of the the excitation be $X(f) = \text{tri}\left(\frac{f}{f_c}\right)$. This system is sometimes called a scrambler because it moves the frequency components of a signal to new locations making it unintelligible.

(a) Using only an analog multiplier and an ideal filter, design a "descrambler" which would recover the original signal.

(b) Sketch the magnitude spectrum of each of the signals in the scrambler-descrambler system.

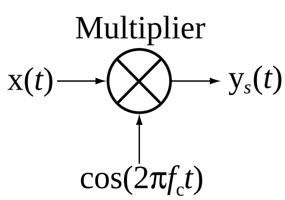


Figure E0 A "scrambler"

No help here. Left as a challenge for the student.

45. Electronic amplifiers that handle very-low-frequency signals are difficult to design because thermal drifts of offset voltages cannot be distinguished from the signals. For this reason a popular technique for designing low-frequency amplifiers is the so called "chopper-stabilized" amplifier (Figure E45).

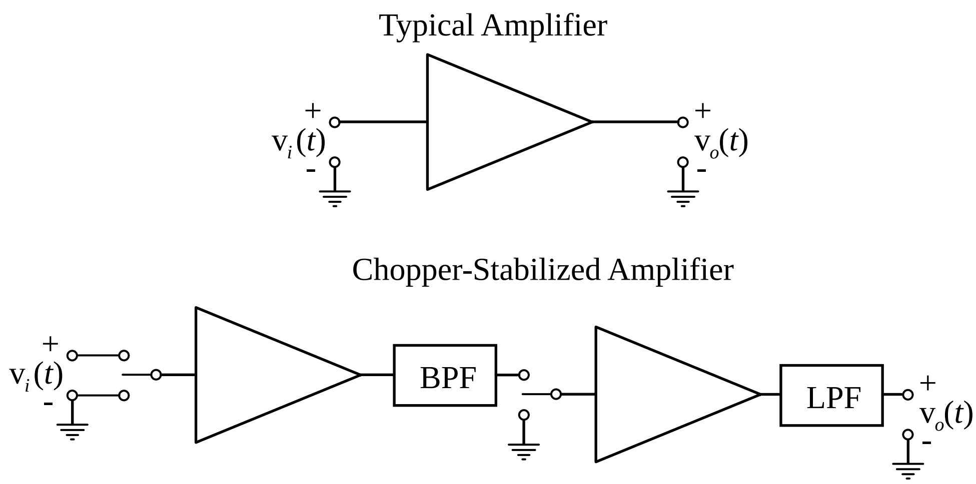


Figure E45 A chopper-stabilized amplifier

A chopper-stabilized amplifier “chops” the excitation signal by switching it on and off periodically. This action is equivalent to a pulse amplitude modulation in which the pulse train being modulated by the excitation is a 50% duty-cycle square wave which alternates between zero and one. Then the “chopped” signal is bandpass filtered to remove any slow thermal drift signals from the first amplifier. Then the amplified signal is “chopped” again at exactly the same rate and in phase with the chopping signal used at the input of the first amplifier. Then this signal may be further amplified. The last step is to lowpass filter the signal out of the last amplifier to recover an amplified version of the original signal. (This is a simplified model but it illustrates the essential features of a chopper-stabilized amplifier.)

Let the following be the parameters of the chopper-stabilized amplifier:

Chopping frequency	500 Hz
Gain of the first amplifier	100 V/V
Bandpass filter	Unity-gain, ideal, zero-phase. Passband $250 < \lvert f \rvert < 750$
Gain of the second amplifier	10 V/V
Lowpass filter	Unity-gain, ideal, zero-phase. Bandwidth 100 Hz

Let the excitation signal have a 100 Hz bandwidth. What is the effective DC gain of this chopper-stabilized amplifier?

Let the excitation be $\mathrm{v}_i(t)$. Then the signal after the first amplifier is

$$\mathrm{v}_1(t) = 100\,\mathrm{v}_i(t)\left[\mathrm{rect}(1000t) * 500\,\mathrm{comb}(500t)\right]$$

or, in the frequency domain,

$$\mathrm{V}_1(f) = 50\sum_{m=-\infty}^{\infty} \mathrm{sinc}\left(\frac{m}{2}\right)\mathrm{V}_i(f - 500m)$$

The response signal from the bandpass filter is that part of the spectrum lying between 250 and 750 Hz which is

$$\mathrm{V}_{bpf}(f) = 50\,\mathrm{sinc}\left(\frac{1}{2}\right)\left[\mathrm{V}_i(f - 500) + \mathrm{V}_i(f + 500)\right] .$$

The response of the second amplifier is the response of the BPF pulse-amplitude modulated by the synchronous switching. In the frequency domain,

$$V_2(f) = 25\,\text{sinc}\left(\frac{1}{2}\right)\sum_{m=-\infty}^{\infty}\text{sinc}\left(\frac{m}{2}\right)\left[V_i(f-500m-500)+V_i(f-500m+500)\right]$$

If the excitation is dc, then $V_i(f) = A\delta(f)$ and

$$V_2(f) = 25A\,\text{sinc}\left(\frac{1}{2}\right)\sum_{m=-\infty}^{\infty}\text{sinc}\left(\frac{m}{2}\right)\left[\delta(f-500m-500)+\delta(f-500m+500)\right] .$$

The response of the lowpass filter is that part of the signal below 100 Hz, which is

$$V_2(f) = 25A\,\text{sinc}\left(\frac{1}{2}\right)\left[\text{sinc}\left(\frac{1}{2}\right)\left[\delta(f)+\delta(f)\right]\right] = 50A\,\text{sinc}^2\left(\frac{1}{2}\right)\delta(f)$$

and the effective dc gain is

$$\text{DC Gain} = \frac{50A\,\text{sinc}^2\left(\frac{1}{2}\right)}{A} = 20.264 .$$

46. A common problem in over-the-air television signal transmission is "multipath" distortion of the received signal due to the transmitted signal bouncing off structures. Typically a strong "main" signal arrives at some time and a weaker "ghost" signal arrives later. So if the transmitted signal is $x_t(t)$, the received signal is

$$x_r(t) = K_m\,x_t(t-t_m) + K_g\,x_t(t-t_g)$$

where $K_m >> K_g$ and $t_g > t_m$.

(a) What is the transfer function of this communication channel?

(b) What would be the transfer function of an "equalization" system that would compensate for the effects of multipath?

The impulse response of the system is

$$h(t) = K_m\delta(t-t_m) + K_g\delta(t-t_g) .$$

Therefore its transfer function is

$$H(f) = K_m e^{-j2\pi f t_m} + K_g e^{-j2\pi f t_g} .$$

An equalization system would then have a transfer function of the form,

$$H_{eq}(f) = \frac{Ae^{-j2\pi f t_0}}{K_m e^{-j2\pi f t_m} + K_g e^{-j2\pi f t_g}}$$