Chapter 8 - Correlation, Energy Spectral Density and Power Spectral Density

Selected Solutions

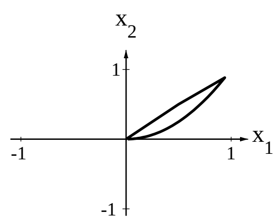
(In this solution manual, the symbol, ⊗, is used for periodic convolution because the preferred symbol which appears in the text is not in the font selection of the word processor used to create this manual.)

1. Plot correlograms of the following pairs of CT and DT signals.

(a) $\mathrm{x}_1(t) = \cos(2\pi t)$ and $\mathrm{x}_2(t) = 2\cos(4\pi t)$

(b) $\mathrm{x}_1[n] = \sin\left(\frac{2\pi n}{16}\right)$ and $\mathrm{x}_2[n] = 2\cos\left(\frac{2\pi n}{8}\right)$

(c) $\mathrm{x}_1(t) = e^{-t}\,\mathrm{u}(t)$ and $\mathrm{x}_2(t) = e^{-2t}\,\mathrm{u}(t)$



(d) $\mathrm{x}_1[n] = e^{-\frac{n}{10}}\cos\left(\frac{2\pi n}{10}\right)\mathrm{u}[n]$ and $\mathrm{x}_1[n] = e^{-\frac{n}{10}}\sin\left(\frac{2\pi n}{10}\right)\mathrm{u}[n]$

2. Plot correlograms of the following pairs of CT and DT signals.

(a) $\mathrm{x}_1(t) = \cos(2\pi t)$ and $\mathrm{x}_2(t) = \cos^2(2\pi t)$

(b) $\mathrm{x}_1[n] = n$ and $\mathrm{x}_2[n] = n^3$, $-10 < n < 10$

(c) $\mathrm{x}_1(t) = t$ and $\mathrm{x}_2(t) = 2 - t^2$, $-4 < t < 4$

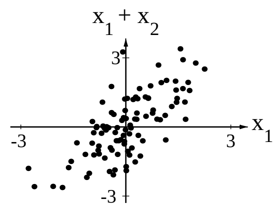
3. In MATLAB generate two vectors, `x1` and `x2`, representing DT signals using the following code fragment,

```
x1 = randn(100,1) ; x2 = randn(100,1) ; x3 = randn(100,1) ;
```

Then plot correlograms of the following pairs of DT signals.

(a) `x1` and `x2`

(b) `x1` and `x1+x2`



(c) x1+x2 and x1+x3 (d) x1+x2/10 and -x1+x3/10

4. Plot the correlation function for each of the following pairs of energy signals.

(a) $x_1(t) = 4\,\mathrm{rect}(t)$ and $x_2(t) = -3\,\mathrm{rect}(2t)$

Using

$$R_{xy}(\tau) = x(-\tau) * y(\tau),$$

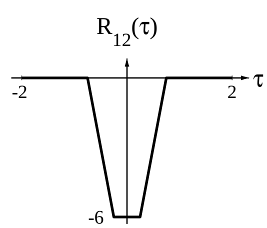
$$R_{12}(\tau) = 4\,\mathrm{rect}(-\tau) * \left(-3\,\mathrm{rect}(2\tau)\right) = -12\,\mathrm{rect}(\tau) * \mathrm{rect}(2\tau)$$

Express the wider rectangle as the sum of two narrower rectangles, each of which is the same width as the other rectangle, then do two easy convolutions instead of one hard one.

$$R_{12}(\tau) = -12\left[\mathrm{rect}\left(2\left(\tau + \frac{1}{4}\right)\right) + \mathrm{rect}\left(2\left(\tau - \frac{1}{4}\right)\right)\right] * \mathrm{rect}(2\tau)$$

$$-12\left\{\mathrm{rect}\left(2\left(\tau + \frac{1}{4}\right)\right) + \mathrm{rect}\left(2\left(\tau - \frac{1}{4}\right)\right)\right\} * \mathrm{rect}(2\tau) \xleftrightarrow{\mathcal{F}} -3\,\mathrm{sinc}^2\left(\frac{f}{2}\right)\left(e^{j\frac{\pi f}{2}} + e^{-j\frac{\pi f}{2}}\right)$$

$$R_{12}(\tau) = -6\left[\mathrm{tri}\left(2\left(\tau + \frac{1}{4}\right)\right) + \mathrm{tri}\left(2\left(\tau - \frac{1}{4}\right)\right)\right]$$



(b) $x_1[n] = 2\,\mathrm{rect}_3[n]$ and $x_2[n] = 5\,\mathrm{rect}_8[n]$

(c) $x_1(t) = 4e^{-t}\,u(t)$ and $x_2(t) = 4e^{-t}\,u(t)$

$$R_{12}(\tau) = 8\mathcal{F}^{-1}\left(\frac{2}{1+\omega^2}\right) = 8e^{-|\tau|}$$

(d) $\mathrm{x}_1[n] = 2e^{-\frac{n}{16}}\sin\left(\frac{2\pi n}{8}\right)\mathrm{u}[n]$ and $\mathrm{x}_2[n] = -3e^{-\frac{n}{16}}\sin\left(\frac{2\pi n}{8} - \frac{\pi}{4}\right)\mathrm{u}[n]$

For $m \geq 0$,

$$\mathrm{R}_{12}[m] = -3e^{-\frac{m}{16}}\sum_{n=-\infty}^{0} e^{\frac{n}{8}}\left(\cos\left(-\frac{2\pi m}{8} + \frac{\pi}{4}\right) - \cos\left(-\frac{4\pi n}{8} + \frac{2\pi m}{8} - \frac{\pi}{4}\right)\right)$$

For $m < 0$,

$$\mathrm{R}_{12}[m] = -3e^{-\frac{m}{16}}\sum_{n=-\infty}^{m} e^{\frac{n}{8}}\left(\cos\left(-\frac{2\pi m}{8} + \frac{\pi}{4}\right) - \cos\left(-\frac{4\pi n}{8} + \frac{2\pi m}{8} - \frac{\pi}{4}\right)\right)$$

5. Plot the correlation function for each of the following pairs of power signals.

(a) $\mathrm{x}_1(t) = 6\sin(12\pi t)$ and $\mathrm{x}_2(t) = 5\cos(12\pi t)$

LCM of the two periods is $\frac{1}{6}$. Therefore both signals are at the fundamental of the CTFS representation.

$$\mathrm{R}_{12}(\tau) \xleftrightarrow{\mathcal{FS}} \mathrm{X}_1^*[k]\,\mathrm{X}_2[k]$$

$$\mathrm{X}_1[k] = j3(\delta[k+1] - \delta[k-1]) \quad \text{and} \quad \mathrm{X}_2[k] = \frac{5}{2}(\delta[k-1] + \delta[k+1])$$

$$\mathrm{R}_{12}(\tau) \xleftrightarrow{\mathcal{FS}} -j\frac{15}{2}(\delta[k+1] - \delta[k-1])(\delta[k-1] + \delta[k+1])$$

$$\mathrm{R}_{12}(\tau) \xleftrightarrow{\mathcal{FS}} -j\frac{15}{2}(\delta[k+1] - \delta[k-1])$$

$$\mathrm{R}_{12}(\tau) = -15\sin(12\pi\tau)$$

Alternate Solution:

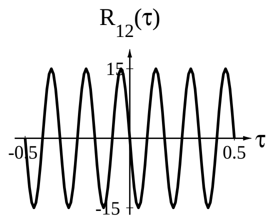
$$\mathrm{R}_{12}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_T 6\sin(12\pi t)5\cos(12\pi(t+\tau))\,dt = \lim_{T\to\infty}\frac{30}{T}\int_T \sin(12\pi t)\cos(12\pi(t+\tau))\,dt$$

$$\mathrm{R}_{12}(\tau) = \lim_{T\to\infty}\frac{15}{T}\int_T [\sin(-12\pi\tau) + \sin(24\pi t + 12\pi\tau)]\,dt = \lim_{T\to\infty}\frac{15}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} [\sin(-12\pi\tau) + \sin(24\pi t + 12\pi\tau)]\,dt$$

$$\mathrm{R}_{12}(\tau) = \lim_{T\to\infty}\frac{15}{T}\left[t\sin(-12\pi\tau) - \frac{\cos(24\pi t + 12\pi\tau)}{24\pi}\right]_{-\frac{T}{2}}^{\frac{T}{2}}$$

$$\mathrm{R}_{12}(\tau) = \lim_{T\to\infty}\frac{15}{T}\left[\frac{T}{2}\sin(-12\pi\tau) - \frac{\cos\left(\frac{24\pi T}{2} + 12\pi\tau\right)}{24\pi} + \frac{T}{2}\sin(-12\pi\tau) + \frac{\cos\left(-\frac{24\pi T}{2} + 12\pi\tau\right)}{24\pi}\right]$$

$$\mathrm{R}_{12}(\tau) = 15\sin(-12\pi\tau) = -15\sin(12\pi\tau)$$



(b) $\mathrm{x}_1[n] = 6\sin\left(\frac{2\pi n}{12}\right)$ and $\mathrm{x}_2[n] = 5\sin\left(\frac{2\pi n}{12}\right)$

(c) $\mathrm{x}_1(t) = 6\sin(12\pi t)$ and $\mathrm{x}_2(t) = 5\sin\left(12\pi t - \frac{\pi}{4}\right)$

6. Find the autocorrelations of the following CT and DT energy and power signals and show that, at zero shift, the value of the autocorrelation is the signal energy or power and that all the properties of autocorrelation functions are satisfied.

(a) $\mathrm{x}(t) = e^{-3t}\,\mathrm{u}(t)$

(b) $\mathrm{x}[n] = \mathrm{rect}_5[n-5]$

$$\mathrm{R}_\mathrm{x}[m] = \mathrm{rect}_5[-m-5] * \mathrm{rect}_5[m-5] = \mathrm{rect}_5[m+5] * \mathrm{rect}_5[m-5]$$

$$\mathrm{R}_\mathrm{x}[m] \xleftrightarrow{\mathcal{F}} 11\,\mathrm{drcl}(F,11)e^{j10\pi F}\,11\,\mathrm{drcl}(F,11)e^{-j10\pi F} = 121\,\mathrm{drcl}^2(F,11)$$
$$\mathrm{R}_\mathrm{x}[m] = \mathrm{rect}_5[m] * \mathrm{rect}_5[m] = 11\,\mathrm{tri}\left(\frac{m}{11}\right)$$

Even function, maximum at $m = 0$. Energy is

$$E_x = \sum_{n=-\infty}^{\infty}\left|\mathrm{rect}_5[n-5]\right|^2 = 11 = \mathrm{R}_\mathrm{x}[0].\ \text{Check.}$$

(c) $\mathrm{x}(t) = \mathrm{rect}\left(2\left(t - \frac{1}{4}\right)\right) - \mathrm{rect}\left(2\left(t - \frac{3}{4}\right)\right)$

7. Find the autocorrelation functions of the following power signals.

(a) $\mathrm{x}(t) = 5\sin(24\pi t) - 2\cos(18\pi t)$

The period of this signal is $\frac{1}{3}$ second. Therefore $f_0 = 3$ and its CTFS is

$$\mathrm{X}[k] = j\frac{5}{2}(\delta[k+4]-\delta[k-4]) - (\delta[k-3]+\delta[k+3])$$

The autocorrelation can be found from

$$\mathrm{R}_x(\tau) \xleftrightarrow{\mathcal{FS}} \mathrm{X}^*[k]\,\mathrm{X}[k] = \left| j\frac{5}{2}(\delta[k+4]-\delta[k-4]) - (\delta[k-3]+\delta[k+3]) \right|^2$$

$$\mathrm{R}_x(\tau) \xleftrightarrow{\mathcal{FS}} \frac{25}{4}(\delta[k+4]+\delta[k-4]) + (\delta[k-3]+\delta[k+3])$$

$$\mathrm{R}_x(\tau) = \frac{25}{2}\cos(24\pi t) + 2\cos(18\pi t)$$

(b) $\mathrm{x}[n] = -4\sin\left(\frac{2\pi n}{36}\right) - 2\cos\left(\frac{2\pi n}{40}\right)$

8. A signal is sent from a transmitter to a receiver and is corrupted by noise along the way. The signal shape is of the functional form,

$$\mathrm{x}(t) = A\sin(2\pi f_0 t)\,\mathrm{rect}\left(\frac{f_0}{4}\left(t - \frac{1}{2f_0}\right)\right)$$

What is the transfer function of a matched filter for this signal?

$$\mathrm{h}(t) = K\sin(2\pi f_0(t_0 - t))\,\mathrm{rect}\left(\frac{f_0}{4}\left(t_0 - t - \frac{1}{2f_0}\right)\right)$$

The sine is odd and the rect is even. Therefore

$$\mathrm{h}(t) = -K\sin(2\pi f_0(t - t_0))\,\mathrm{rect}\left(\frac{f_0}{4}\left(t - t_0 + \frac{1}{2f_0}\right)\right)$$

$$\mathrm{H}(f) = j\frac{2K}{f_0}e^{-j2\pi f\left(t_0 - \frac{1}{2f_0}\right)}\left[\mathrm{sinc}\left(\frac{4(f+f_0)}{f_0}\right) - \mathrm{sinc}\left(\frac{4(f-f_0)}{f_0}\right)\right]$$

9. Find the signal power of the following sums or differences of signals and compare it to the power in the individual signals. How does the comparison relate to the correlation between the two signals that are summed or differenced?

(a) $\mathrm{x}(t) = \sin(2\pi t) + \cos(2\pi t)$

$$P_x = \frac{1}{T_0}\int_{T_0} |\sin(2\pi t) + \cos(2\pi t)|^2 dt = \frac{1}{T_0}\int_{T_0} \left(\underbrace{\sin^2(2\pi t) + \cos^2(2\pi t)}_{=1} + 2\sin(2\pi t)\cos(2\pi t) \right) dt$$

$$P_x = \int_1 dt + 2\underbrace{\int_1 \sin(2\pi t)\cos(2\pi t)\,dt}_{=0} = 1$$

The power of the sin is $\frac{1}{2}$ and the power of the cosine is $\frac{1}{2}$. The power of the sum equals the sum of the powers because these two signals are uncorrelated at zero shift.

(b) $\quad \mathrm{x}(t) = \sin(2\pi t) + \cos\left(2\pi t - \frac{\pi}{4}\right)$

(c) $\quad \mathrm{x}[n] = \mathrm{rect}_2[n] * \mathrm{comb}_{10}[n] - \mathrm{tri}\left(\frac{n}{2}\right) * \mathrm{comb}_{10}[n]$

The signal power of the sum is less than sum of the signal powers because the two signals are negatively correlated at zero shift.

(d) $\quad \mathrm{x}[n] = \mathrm{rect}_2[n] * \mathrm{comb}_{10}[n] + \mathrm{tri}\left(\frac{n-5}{2}\right) * \mathrm{comb}_{10}[n]$

The signal power of the sum equals the sum of the signal powers because the two signals are uncorrelated at zero shift.

10. Find the crosscorrelation functions of the following pairs of periodic signals.

(a) $\quad \mathrm{x}_1(t) = \mathrm{rect}\left(\frac{t}{6}\right) * \mathrm{comb}\left(\frac{t}{24}\right)$ and $\quad \mathrm{x}_2(t) = \mathrm{rect}\left(\frac{t-3}{6}\right) * \mathrm{comb}\left(\frac{t}{24}\right)$

$$\mathrm{R}_{12}(\tau) = 6\,\mathrm{tri}\left(\frac{\tau-3}{6}\right) * \mathrm{comb}\left(\frac{\tau}{24}\right) = 144\,\mathrm{tri}\left(\frac{\tau-3}{6}\right) * \sum_{n=-\infty}^{\infty} \delta(\tau - 24n)$$

(b) $\quad \mathrm{x}_1[n] = \sin^2\left(\frac{2\pi n}{8}\right)$ and $\quad \mathrm{x}_2[n] = \sin^2\left(\frac{2\pi n}{10}\right)$

(c) $\quad \mathrm{x}_1(t) = e^{-j10\pi t}$ and $\quad \mathrm{x}_2(t) = \cos(10\pi t)$

11. Find the ESD's of the following energy signals.

(a) $\quad \mathrm{x}[n] = A\delta[n - n_0]$

(b) $\quad \mathrm{x}(t) = e^{-100t}\,\mathrm{u}(t)$

$$\mathrm{X}(j\omega)=\frac{1}{100+j\omega}\Rightarrow\Psi_x(j\omega)=\frac{1}{10^4+\omega^2}$$

(c) $\mathrm{x}[n]=10\left(\frac{7}{8}\right)^n\sin\left(\frac{2\pi n}{12}\right)\mathrm{u}[n]$ (d) $\mathrm{x}(t)=A\,\mathrm{tri}\left(\frac{t-t_0}{w}\right)$

12. Find the ESD of the response, $\mathrm{y}(t)$ or $\mathrm{y}[n]$, of each system with impulse response, $\mathrm{h}(t)$ or $\mathrm{h}[n]$, to the excitation, $\mathrm{x}(t)$ or $\mathrm{x}[n]$.

(a) $\mathrm{x}[n]=\delta[n]$, $\mathrm{h}[n]=\left(-\frac{9}{10}\right)^n\mathrm{u}[n]$

$$\mathrm{X}(j\Omega)=1\Rightarrow\Psi_x(j\Omega)=1 \quad \text{and} \quad \mathrm{H}(j\Omega)=\frac{1}{1+\frac{9}{10}e^{-j\Omega}}=\frac{10}{10+9e^{-j\Omega}}$$

$$\Psi_y(j\Omega)=\Psi_x(j\Omega)\left|\mathrm{H}(j\Omega)\right|^2=\frac{100}{(10+9\cos(\Omega))^2+81\sin^2(\Omega)}=\frac{100}{181+180\cos(\Omega)}$$

(b) $\mathrm{x}(t)=e^{-100t}\mathrm{u}(t)$, $\mathrm{h}(t)=e^{-100t}\mathrm{u}(t)$

(c) $\mathrm{x}[n]=\mathrm{rect}_3[n]$, $\mathrm{h}[n]=\mathrm{rect}_2[n-2]$

(d) $\mathrm{x}(t)=4e^{-t}\cos(2\pi t)\mathrm{u}(t)$, $\mathrm{h}(t)=\mathrm{rect}\left(t-\frac{1}{2}\right)$

13. Find the PSD's of these signals.

(a) $\mathrm{x}(t)=A\cos(2\pi f_0 t+\theta)$

$$\mathrm{R}_x(\tau)=\frac{A^2}{2}\cos(2\pi f_0\tau)\Rightarrow G_x(f)=\frac{A^2}{4}\left[\delta(f-f_0)+\delta(f+f_0)\right]$$

In finding PSD's, it is usually easier to find the autocorrelation first and then Fourier transform it to find the PSD rather than taking the direct route using the definition.

(b) $\mathrm{x}(t)=3\,\mathrm{rect}(100t)*\mathrm{comb}(25t)$

$$\mathrm{G}_x(f)=3.6\times10^{-5}\,\mathrm{sinc}^2\left(\frac{f}{100}\right)\mathrm{comb}\left(\frac{f}{25}\right)$$

(c) $\mathrm{x}[n]=8\sin\left(\frac{2\pi n}{12}\right)$

(d) $\mathrm{x}[n]=3\,\mathrm{rect}_4[n]*\mathrm{comb}_{20}[n]$

$$\mathrm{G_x}(F) = \frac{9}{20}\frac{\sin^2(9\pi F)}{\sin^2(\pi F)}\mathrm{comb}(20F)$$

14. Find the PSD of the response, $\mathrm{y}(t)$ or $\mathrm{y}[n]$, of each system with impulse response, $\mathrm{h}(t)$ or $\mathrm{h}[n]$, to the excitation, $\mathrm{x}(t)$ or $\mathrm{x}[n]$.

(a) $\mathrm{x}(t) = 4\cos\left(32\pi t - \frac{\pi}{4}\right)$, $\mathrm{h}(t) = e^{-\frac{t}{10}}\mathrm{u}(t)$

$$\mathrm{G}_x(f) = 4[\delta(f-16) + \delta(f+16)]$$

and $\mathrm{H}(f) = \dfrac{1}{\frac{1}{10} + j2\pi f} = \dfrac{10}{1 + j20\pi f}$

$$\mathrm{G_y}(f) = 4[\delta(f-16) + \delta(f+16)]\frac{100}{1+(20\pi f)^2} = 400\left[\frac{\delta(f-16)}{1+(20\pi f)^2} + \frac{\delta(f+16)}{1+(20\pi f)^2}\right]$$

$$\mathrm{G_y}(f) = 400\frac{\delta(f-16)+\delta(f+16)}{1+(320\pi)^2}$$

(b) $\mathrm{x}(t) = 4\,\mathrm{comb}(2t)$, $\mathrm{h}(t) = \mathrm{rect}(t-1)$

(c) $\mathrm{x}[n] = 2\,\mathrm{comb}_8[n]$, $\mathrm{h}[n] = \left(\frac{11}{12}\right)^n \mathrm{u}[n-1]$

$$\mathrm{G_y}(F) = \frac{0.42\,\mathrm{comb}(8F)}{1 - 1.8333\cos(2\pi F) + 0.8403}$$

(d) $\mathrm{x}[n] = (-0.9)^n\mathrm{u}[n]$, $\mathrm{h}[n] = (0.5)^n\mathrm{u}[n]$

15. Plot correlograms of the following pairs of CT and DT signals.

(a) $\mathrm{x_1}(t) = \left[\mathrm{tri}\left(4\left(t - \frac{1}{4}\right)\right) - \mathrm{tri}\left(4\left(t - \frac{3}{4}\right)\right)\right] * \mathrm{comb}(t)$ and $\mathrm{x_2}(t) = \sin(2\pi t)$

(b) $\mathrm{x_1}[n] = \left[\mathrm{tri}\left(\frac{n-8}{8}\right) - \mathrm{tri}\left(\frac{n-24}{8}\right)\right] * \mathrm{comb}_{32}[n]$ and
$\mathrm{x_2}[n] = \cos\left(\frac{2\pi n}{32}\right)$

(c) $\mathrm{x}_1(t) = \left[\mathrm{tri}\left(4\left(t-\frac{1}{4}\right)\right) - \mathrm{tri}\left(4\left(t-\frac{3}{4}\right)\right)\right] * \mathrm{comb}(t)$ and

$\mathrm{x}_2(t) = \left[\mathrm{tri}(4t) - \mathrm{tri}\left(4\left(t-\frac{1}{2}\right)\right)\right] * \mathrm{comb}(t)$

(d) $\mathrm{x}_1[n] = \left[\mathrm{rect}\left(\frac{n-8}{16}\right) - \mathrm{rect}\left(\frac{n-24}{16}\right)\right] * \mathrm{comb}_{32}[n]$ and

$\mathrm{x}_2[n] = \sin\left(\frac{2\pi n}{32}\right)$

16. Plot a correlogram for the following sets of samples from two signals, x and y. In each case, from the nature of the correlogram what relationship, if any, exists between the two sets of data?

(a) x = {6,5,8,-2,3,-10,9,-2,-4,3,-2,6,0,-5,-7,1,9,9,4,-6}
y = {-1,-10,-4,4,5,-2,-3,-5,-9,2,6,-5,-1,-10,-9,0,4,-10,9,-1}

(b)
x = {4,6,0,0,5,-6,8,-9,0,8,7,2,-5,-3,-4,-4,8,0,4,7}
y = {-11,-13,3,-1,-8,10,-16,16,1,-17,-14,-3,9,7,12,9,-17,1,-8,-17}

(c)
x = {0,6,11,16,19,20,19,16,11,6,-0,-7,-12,-17,-20,-20,-20,-17,-12,-7}
y = {19,15,10,8,3,-9,-12,-19,-19,-25,-19,-17,-12,-5,-1,5,8,12,17,20}

17. Plot the correlation function for each of the following pairs of energy signals.

(a) $\mathrm{x}_1(t) = \mathrm{rect}(t)\sin(10\pi t)$ and $\mathrm{x}_2(t) = \mathrm{rect}(t)\cos(10\pi t)$

(b) $\mathrm{x}_1[n] = \delta[n-1] - \delta[n+1]$ and $\mathrm{x}_2[n] = -\delta[n-1] + \delta[n+1]$

(c) $\mathrm{x}_1(t) = e^{-t^2}$ and $\mathrm{x}_2(t) = e^{-2t^2}$

$$\mathrm{R}_{12}(\tau) = \sqrt{\frac{\pi}{3}} e^{-\frac{2}{3}\tau^2}$$

18. Plot the correlation function for each of the following pairs of power signals.

(a) $\mathrm{x}_1[n] = -3\sin\left(\frac{2\pi n}{20}\right)$ and $\mathrm{x}_2[n] = 8\sin\left(\frac{2\pi n}{10}\right)$

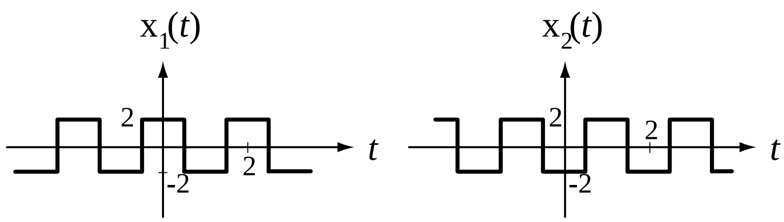
(b) $\mathrm{x}_1(t) = \mathrm{rect}(4t) * \mathrm{comb}(t)$ and $\mathrm{x}_2(t) = \mathrm{rect}(4t) * \mathrm{comb}(t)$

(c)
$\mathrm{x}_1(t) = 4\,\mathrm{rect}(t) * \frac{1}{2}\mathrm{comb}\left(\frac{t}{2}\right) - 2$ and $\mathrm{x}_2(t) = 4\,\mathrm{rect}(t-1) * \frac{1}{2}\mathrm{comb}\left(\frac{t}{2}\right) - 2$.

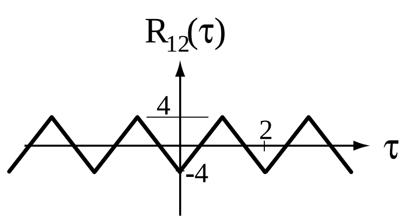
Graphical Solution:

$$\mathrm{x}_1(t) = 4\,\mathrm{rect}(t) * \frac{1}{2}\mathrm{comb}\left(\frac{t}{2}\right) - 2 \qquad \text{and} \qquad \mathrm{x}_2(t) = 4\,\mathrm{rect}(t-1) * \frac{1}{2}\mathrm{comb}\left(\frac{t}{2}\right) - 2 \ .$$

The easiest way to find the correlation function for these two signals is to first graph them, then find the correlation graphically.



For periodic functions, the correlation function is the average value of the product of the two functions over one period of the product of the two functions. In this case the periods of the two functions are the same, 2, and the period of the product is also 2. With zero shift of $\mathrm{x}_2(t)$ the product is -4 everywhere and therefore the average of the product is also -4. If $\mathrm{x}_2(t)$ is shifted by exactly half a period, 1, the product is +4 everywhere and the average value is +4. For any shift between these two shifts, the average value of the product varies linearly with the shift amount between these two extreme values of the correlation. Also the correlation function is periodic with the same period as these two functions. Therefore the correlation function is all illustrated below.



$$\mathrm{R}_{12}(\tau) = -4 + 8\sum_{n=-\infty}^{\infty} \mathrm{tri}(t - 1 - 2n)$$

19. Find the autocorrelations of the following CT and DT energy and power signals and show that, at zero shift, the value of the autocorrelation is the signal energy or power and that all the properties of autocorrelation functions are satisfied.

(a) $\mathrm{x}[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]$

$$\mathrm{R}_x[m] = \mathrm{x}[-m] * \mathrm{x}[m] = \begin{pmatrix} \delta[-m] + \delta[-m-1] \\ +\delta[-m-2] + \delta[-m-3] \end{pmatrix} * \begin{pmatrix} \delta[m] + \delta[m-1] \\ +\delta[m-2] + \delta[m-3] \end{pmatrix}$$

$$\mathrm{R}_x[m]=\begin{Bmatrix}\delta[-m]+\delta[-m+1]+\delta[-m+2]+\delta[-m+3]\\+\delta[-m-1]+\delta[-m]+\delta[-m+1]+\delta[-m+2]\\+\delta[-m-2]+\delta[-m-1]+\delta[-m]+\delta[-m+1]\\+\delta[-m-3]+\delta[-m-2]+\delta[-m-1]+\delta[-m]\end{Bmatrix}$$

$$\mathrm{R}_x[m]=\begin{Bmatrix}4\delta[m]+3\delta[-m+1]+2\delta[-m+2]+\delta[-m+3]\\+3\delta[-m-1]+2\delta[-m-2]+\delta[-m-3]\end{Bmatrix}$$

$$\mathrm{R}_x[m]=\delta[m+3]+2\delta[m+2]+3\delta[m+1]+4\delta[m]+3\delta[m-1]+2\delta[m-2]+\delta[m-3]$$

$$E_x=\sum_{n=-\infty}^{\infty}|\mathrm{x}[n]|^2=\sum_{n=-\infty}^{\infty}|\delta[n]+\delta[n-1]+\delta[n-2]+\delta[n-3]|^2=4=\mathrm{R}_x[0].\ \text{ Check.}$$

(b) $\mathrm{x}(t)=A\cos(2\pi f_0 t+\theta)$

$$\mathrm{R}_x(\tau)\xleftrightarrow{\mathcal{FS}}\mathrm{X}^*[k]\mathrm{X}[k]$$

$$\mathrm{R}_x(\tau)=\frac{A^2}{2}\cos(2\pi f_0\tau)$$

Alternate Solution:

$$\mathrm{R}_x(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\mathrm{x}(t)\,\mathrm{x}(t+\tau)dt=\lim_{T\to\infty}\frac{A^2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\cos(2\pi f_0 t+\theta)\cos(2\pi f_0(t+\tau)+\theta)dt$$

$$\mathrm{R}_x(\tau)=\lim_{T\to\infty}\frac{A^2}{2T}\int_{-\frac{T}{2}}^{\frac{T}{2}}\left[\cos(-2\pi f_0\tau)+\cos(4\pi f_0 t+2\pi f_o\tau+2\theta)\right]dt$$

$$\mathrm{R}_x(\tau)=\lim_{T\to\infty}\frac{A^2}{2T}\left[t\cos(-2\pi f_0\tau)+\frac{\sin(4\pi f_0 t+2\pi f_o\tau+2\theta)}{4\pi f_0}\right]_{-\frac{T}{2}}^{\frac{T}{2}}$$

$$\mathrm{R}_x(\tau)=\frac{A^2}{2}\cos(2\pi f_0\tau)$$

$\mathrm{R}_x(0)=\dfrac{A^2}{2}$ which is the average signal power of any sinusoid of amplitude A

(c) $\mathrm{x}[n]=\mathrm{comb}_{12}[n]$

$$\mathrm{R}_x[m]\xleftrightarrow{\mathcal{FS}}\mathrm{X}^*[k]\mathrm{X}[k]$$

Using $\text{comb}_{N_0}[n] \xleftrightarrow{\mathcal{FS}} \dfrac{1}{N_0}$

$$\text{R}_x[m] \xleftrightarrow{\mathcal{FS}} \left(\frac{1}{N_0}\right)^* \frac{1}{N_0} = \frac{1}{N_0^2}$$

$$\text{R}_x[m] = \frac{\text{comb}_{12}[m]}{12}$$

$$\text{R}_x[0] = \frac{1}{12} \text{ . Check.}$$

Alternate Solution:

$$\text{R}_\text{x}[m] = \lim_{N\to\infty} \frac{1}{N} \sum_{n=\langle N\rangle} \text{x}[n]\text{x}[n+m] = \lim_{N\to\infty} \frac{1}{N} \sum_{n=\langle N\rangle} \text{comb}_{12}[n]\text{comb}_{12}[n+m]$$

The product, $\text{comb}_{12}[n]\text{comb}_{12}[n+m]$, is non-zero only if m is a integer multiple of 12. The summation is over a period of length N. There are two cases.

Case I. N even

Let the summation be from $-\dfrac{N}{2}$ to $\dfrac{N}{2}-1$. Then

$$\text{R}_\text{x}[m] = \lim_{N\to\infty} \frac{1}{N} \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} \text{comb}_{12}[n]\text{comb}_{12}[n+m] \text{ .}$$

In that range, for m an integer multiple of 12, the number of impulses in the product is the greatest integer in $\dfrac{N}{12}$. Therefore

$$\text{R}_\text{x}[m] = \lim_{N\to\infty} \frac{1}{N} \begin{Bmatrix} \left[\text{Greatest Integer in } \dfrac{N}{12}\right] \text{ , } m \text{ an integer multiple of 12} \\ 0 \text{ , otherwise} \end{Bmatrix}$$

$$\text{R}_\text{x}[m] = \begin{Bmatrix} \dfrac{1}{12} \text{ , } m \text{ an integer multiple of 12} \\ 0 \text{ , otherwise} \end{Bmatrix} = \frac{\text{comb}_{12}[m]}{12}$$

Case II. N odd

Let the summation be from $-\dfrac{N-1}{2}$ to $\dfrac{N-1}{2}$. Then

$$\mathrm{R_x}[m] = \lim_{N\to\infty} \frac{1}{N} \sum_{n=-\frac{N-1}{2}}^{\frac{N-1}{2}} \mathrm{comb}_{12}[n]\,\mathrm{comb}_{12}[n+m] \ .$$

The rest of the analysis is the same and so is the answer (in the limit).

$$\mathrm{R_x}[m] = \begin{Bmatrix} \frac{1}{12} & , \ m \text{ an integer multiple of } 12 \\ 0 & , \ \text{otherwise} \end{Bmatrix} = \frac{\mathrm{comb}_{12}[m]}{12} \ .$$

20. Find and sketch the autocorrelation function of

$$\mathrm{x}(t) = 10\,\mathrm{rect}(2t) * \frac{1}{4}\mathrm{comb}\left(\frac{t}{4}\right) \ .$$

$$\mathrm{R}_{12}(\tau) \xleftrightarrow{\mathcal{FS}} \mathrm{X}_1^*[k]\,\mathrm{X}_2[k]$$

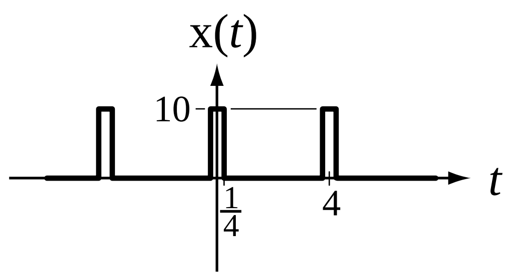
$$\mathrm{R}_{12}(\tau) = \frac{25}{2}\mathrm{tri}(2t) * \frac{1}{4}\mathrm{comb}\left(\frac{t}{4}\right)$$

$$\mathrm{R}_{12}(0) = \frac{25}{2} \ \text{. Check.}$$

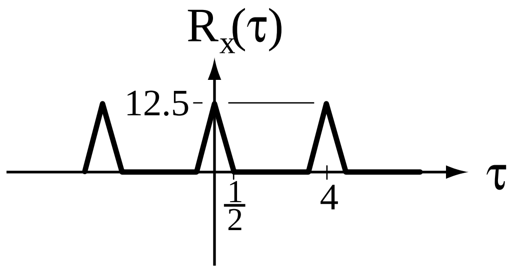
Alternate Solution:

Check to be sure that its value at a shift of zero is the same as its average signal power.

The signal is graphed below.



This is a periodic signal. Therefore its autocorrelation function is also periodic with the same period. The maximum autocorrelation value occurs at zero shift and that value is the average of the square of the signal (its average power), 12.5. The autocorrelation varies linearly with shift up to a shift of one-half at which time it is zero. This is true for positive or negative shifts of up to one-half. The autocorrelation for shifts, $\frac{1}{2} < |\tau| < 2$, is zero. This pattern repeats periodically. Therefore the autocorrelation function is as illustrated below.



$$R_{12}(\tau) = \frac{25}{2}\mathrm{tri}(2t) * \frac{1}{4}\mathrm{comb}\left(\frac{t}{4}\right)$$

21. Find all cross correlation and autocorrelation functions for these three signals:

$$x_1(t) = \cos(2\pi t) \;\;,\;\; x_2(t) = \sin(2\pi t) \;\;,\;\; x_3(t) = \cos(4\pi t)$$

Check your autocorrelation answers by finding the average power of each signal.

22. Find and sketch the crosscorrelation between a unit-amplitude, one Hz cosine and a 50% duty-cycle square wave which has a peak-to-peak amplitude of two, a period of one, an average value of zero and is an even function.

$$R_{12}(\tau) \xleftrightarrow{\mathcal{FS}} X_1^*[k]X_2[k]$$

$$x_1(t) = \cos(2\pi t) \qquad \text{and} \qquad x_2(t) = 2\,\mathrm{rect}(2t) * \mathrm{comb}(t) - 1$$

$$R_{12}(\tau) = \mathrm{sinc}\left(\frac{1}{2}\right)\cos(2\pi\tau)$$

Old Solution:

$$X_c[k] = \frac{4}{T_0}\int_0^{\frac{T_0}{4}} \cos(2\pi(kf_0)t)\,dt = \frac{4}{T_0}\left[\frac{\sin(2\pi(kf_0)t)}{2\pi(kf_0)}\right]_0^{\frac{T_0}{4}} = \frac{2}{k\pi}\sin\left(\frac{k\pi}{2}\right) = \mathrm{sinc}\left(\frac{k}{2}\right)$$

$$X_s[k] = 0 \;\;,\;\; \text{because the function is even}$$

All the cosines in the trigonometric expression for the square wave, except the one at the fundamental frequency of one Hz, are orthogonal to the one-Hz cosine over one period. Therefore the crosscorrelation is simply

$$R(\tau) = \frac{1}{1}\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos(2\pi t)\,\mathrm{sinc}\left(\frac{1}{2}\right)\cos(2\pi(t+\tau))\,dt = 2\,\mathrm{sinc}\left(\frac{1}{2}\right)\int_0^{\frac{1}{2}}\left[\cos(-2\pi\tau) + \cos(4\pi t + 2\pi\tau)\right]dt$$

$$R(\tau) = 2\,\mathrm{sinc}\left(\frac{1}{2}\right)\left[t\cos(2\pi\tau) + \frac{\sin(4\pi t + 2\pi\tau)}{4\pi}\right]_0^{\frac{1}{2}} = \mathrm{sinc}\left(\frac{1}{2}\right)\cos(2\pi\tau)$$

23. Find and sketch the ESD of each of these signals:

(a) $\mathrm{x}(t) = A\,\mathrm{rect}\left(\dfrac{t}{w}\right)$

$$\Psi_x(f) = |\mathrm{X}(f)|^2 = |Aw\,\mathrm{sinc}(wf)|^2 = A^2 w^2\,\mathrm{sinc}^2(wf)$$

(b) $\mathrm{x}(t) = A\,\mathrm{rect}\left(\dfrac{t+1}{w}\right)$

(c) $\mathrm{x}(t) = A\,\mathrm{sinc}\left(\dfrac{t}{w}\right)$

$$\Psi_x(f) = |\mathrm{X}(f)|^2 = |Aw\,\mathrm{rect}(wf)|^2 = A^2 w^2\,\mathrm{rect}(wf)$$

(d) $\mathrm{x}(t) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$

$$\Psi_x(f) = e^{-(2\pi f)^2}$$

24. Find the PSD's of

(a) $\mathrm{x}(t) = A$

$$\mathrm{R}_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \mathrm{x}(t)\mathrm{x}(t+\tau)\,dt = \lim_{T\to\infty} \frac{A^2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} dt = A^2$$

$$\mathrm{G}_x(f) = \mathcal{F}\left[\mathrm{R}_x(\tau)\right] = A^2 \delta(f)$$

(b) $\mathrm{x}(t) = A\cos(2\pi f_0 t)$ (c) $\mathrm{x}(t) = A\sin(2\pi f_0 t)$

25. Which of the following functions could not be the autocorrelation function of a real signal and why?

An autocorrelation function has the properties,

1. It is an even function,

2. Its maximum value occurs at zero shift,

and 3. Its Fourier transform is strictly non-negative for all f (or ω).

(a) $\mathrm{R}(\tau) = \mathrm{tri}(\tau)$

(b) $\mathrm{R}(\tau) = A\sin(2\pi f_0 \tau)$

(c) $\mathrm{R}(\tau) = \mathrm{rect}(\tau)$

(d) $\mathrm{R}(\tau) = A\,\mathrm{sinc}(B\tau)$