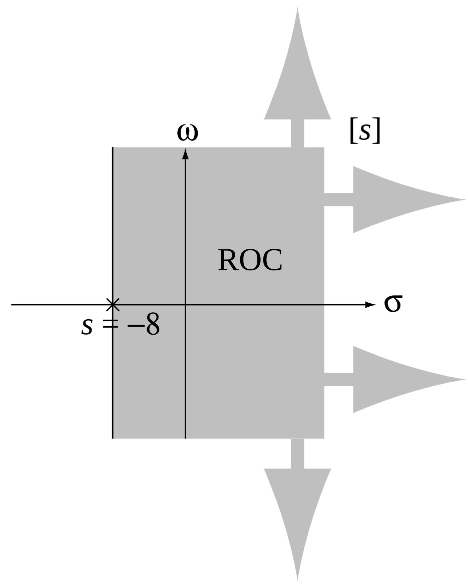
Chapter 9 - The Laplace Transform

Selected Solutions

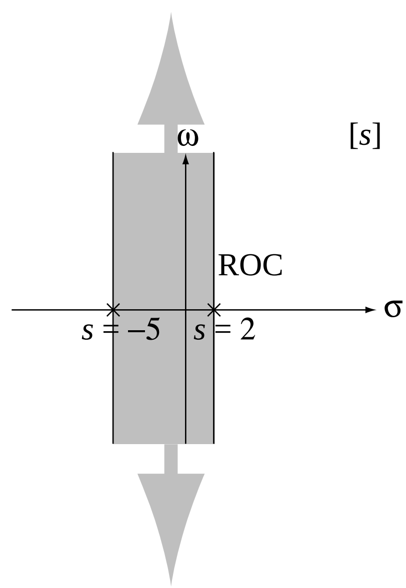
1. Sketch the pole-zero plot and region of convergence (if it exists) for these signals.

(a) $x(t) = e^{-8t} u(t)$



(b) $x(t) = e^{3t} \cos(20\pi t) u(-t)$

(c) $x(t) = e^{2t} u(-t) - e^{-5t} u(t)$



2. Starting with the definition of the Laplace transform,

$$\mathcal{L}\left(\mathrm{g}(t)\right) = \mathrm{G}(s) = \int_{0^-}^{\infty} \mathrm{g}(t) e^{-st} dt \ ,$$

find the Laplace transforms of these signals.

(a) $\mathrm{x}(t) = e^{t}\,\mathrm{u}(t)$

(b) $\mathrm{x}(t) = e^{2t}\cos(200\pi t)\mathrm{u}(t)$

(c) $\mathrm{x}(t) = \mathrm{ramp}(t)$

$$\mathrm{X}(s) = \int_{0^-}^{\infty} \mathrm{x}(t) e^{-st} dt = \int_{0^-}^{\infty} \mathrm{ramp}(t) e^{-st} dt = \int_{0^-}^{\infty} t e^{-st} dt$$

Using

$$\int x e^{ax} dx = \frac{e^{ax}}{a^2}(ax-1)$$

$$\mathrm{X}(s) = \left[\frac{e^{-st}}{(-s)^2}(-st-1)\right]_{0^-}^{\infty} = \frac{1}{s^2} \ , \ \mathrm{Re}(s) = \sigma > 0$$

(d) $\mathrm{x}(t) = t e^{t}\,\mathrm{u}(t)$

3. Using the time-shifting property, find the Laplace transform of these signals.

(a) $\mathrm{x}(t) = \mathrm{u}(t) - \mathrm{u}(t-1)$

(b) $\mathrm{x}(t) = 3e^{-3(t-2)}\,\mathrm{u}(t-2)$

(c) $\mathrm{x}(t) = 3e^{-3t}\,\mathrm{u}(t-2)$

$$3\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{3}{s}$$

Using the time shifting property,

$$3\mathrm{u}(t-2) \xleftrightarrow{\mathcal{L}} \frac{3e^{-2s}}{s}$$

$$3e^{-3t}\,\mathrm{u}(t-2) \xleftrightarrow{\mathcal{L}} \frac{3e^{-2(s+3)}}{s+3}$$

Alternate solution:

$$\mathrm{x}(t) = 3e^{-6}e^{-3(t-2)}\,\mathrm{u}(t-2)$$

Using the time shifting property,

$$\mathrm{X}(s)=\frac{3e^{-6}e^{-2s}}{s+3}=\frac{3e^{-2s-6}}{s+3}$$

(d) $\mathrm{x}(t)=5\sin(\pi(t-1))\mathrm{u}(t-1)$

4. Using the complex-frequency-shifting property, find and sketch the inverse Laplace transform of

$$\mathrm{X}(s)=\frac{1}{(s+j4)+3}+\frac{1}{(s-j4)+3}\ .$$

5. Using the time-scaling property, find the Laplace transforms of these signals.

(a) $\mathrm{x}(t)=\delta(4t)$

(b) $\mathrm{x}(t)=\mathrm{u}(4t)$

$$\mathrm{u}(t)\xleftrightarrow{\mathcal{L}}\frac{1}{s}\ ,\ \ \mathrm{Re}(s)>0$$

$$\mathrm{u}(4t)\xleftrightarrow{\mathcal{L}}\frac{1}{4}\frac{1}{\frac{s}{4}}=\frac{1}{s}\ ,\ \ \mathrm{Re}(s)>0$$

6. Using the time-differentiation property, find the Laplace transforms of these signals.

(a) $\mathrm{x}(t)=\frac{d}{dt}(\mathrm{u}(t))$

$$\frac{d}{dt}(\mathrm{g}(t))\xleftrightarrow{\mathcal{L}}s\,\mathrm{G}(s)-\mathrm{g}(0^-)$$

$$\mathrm{u}(t)\xleftrightarrow{\mathcal{L}}\frac{1}{s}\ ,\ \ \mathrm{Re}(s)>0$$

$$\frac{d}{dt}(\mathrm{u}(t))\xleftrightarrow{\mathcal{L}}s\frac{1}{s}-\underbrace{\mathrm{u}(0^-)}_{=0}=1\ ,\ \ \text{All } s$$

(b) $\mathrm{x}(t)=\frac{d}{dt}(e^{-10t}\,\mathrm{u}(t))$

(c) $\mathrm{x}(t)=\frac{d}{dt}(4\sin(10\pi t)\mathrm{u}(t))$

(d) $\mathrm{x}(t)=\frac{d}{dt}(10\cos(15\pi t)\mathrm{u}(t))$

7. Using multiplication-convolution duality, find the Laplace transforms of these signals and sketch the signals versus time.

(a) $\mathrm{x}(t) = e^{-t}\,\mathrm{u}(t) * \mathrm{u}(t)$

(b) $\mathrm{x}(t) = e^{-t}\sin(20\pi t)\mathrm{u}(t) * \mathrm{u}(t)$

$$e^{-t}\sin(20\pi t)\mathrm{u}(t) * \mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{20\pi}{(s+1)^2 + (20\pi)^2}\frac{1}{s}$$

$$\frac{20\pi}{(s+1)^2 + (20\pi)^2}\frac{1}{s} = \frac{20\pi}{1+(20\pi)^2}\frac{1}{s} + \frac{As+B}{(s+1)^2 + (20\pi)^2}$$

Multiply through by *s*, let *s* approach infinity and solve for *A*. After finding *A*, let $s = 1$ and solve for *B*,

$$\mathrm{X}(s) = \frac{20\pi}{1+(20\pi)^2}\left[\frac{1}{s} - \frac{s+1}{(s+1)^2 + (20\pi)^2} - \frac{1}{20\pi}\frac{20\pi}{(s+1)^2 + (20\pi)^2}\right]$$

$$\mathrm{x}(t) = \frac{20\pi}{1+(20\pi)^2}\left\{1 - e^{-t}\left[\cos(20\pi t) + \frac{\sin(20\pi t)}{20\pi}\right]\right\}\mathrm{u}(t)$$

(c) $\mathrm{x}(t) = 8\cos\left(\frac{\pi t}{2}\right)\mathrm{u}(t) * [\mathrm{u}(t) - \mathrm{u}(t-1)]$

(d) $\mathrm{x}(t) = 8\cos(2\pi t)\mathrm{u}(t) * [\mathrm{u}(t) - \mathrm{u}(t-1)]$

After time, $t = 1$, the solution is zero.

8. Using the initial and final value theorems, find the initial and final values (if possible) of the signals whose Laplace transforms are these functions.

Initial Value Theorem

$$\mathrm{g}(0^+) = \lim_{s\to\infty} s\mathrm{G}(s)$$

Final Value Theorem

$$\lim_{t\to\infty}\mathrm{g}(t) = \lim_{s\to 0} s\mathrm{G}(s) \quad , \text{ if } \lim_{t\to\infty}\mathrm{g}(t) \text{ exists}$$

(a) $\mathrm{X}(s) = \frac{10}{s+8}$, One pole in open LHP

$$\mathrm{x}(0^+) = \lim_{s\to\infty} s\frac{10}{s+8} = 10$$

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\frac{10}{s+8} = 0$$

and the limit exists because the only pole of $X(s) = \frac{10}{s+8}$ is in the open LHP

(b) $X(s) = \frac{s+3}{(s+3)^2+4}$, Poles at $-3 \pm j2$

(c) $X(s) = \frac{s}{s^2+4}$, Poles at $\pm j2$

$$x(0^+) = \lim_{s\to\infty} s\frac{s}{s^2+4} = 1$$

Final-value theorem does not apply because there are two poles on the ω axis

(d) $X(s) = \frac{10s}{s^2+10s+300}$, Poles at $-5 \pm j16.583$

(e) $X(s) = \frac{8}{s(s+20)}$, Poles at 0 and -20

(f) $X(s) = \frac{8}{s^2(s+20)}$, Double pole at zero.

9. Find the inverse Laplace transforms of these functions.

(a) $X(s) = \frac{24}{s(s+8)}$

$$X(s) = \frac{3}{s} - \frac{3}{s+8}$$

$$x(t) = 3\left(1-e^{-8t}\right)u(t)$$

(b) $X(s) = \frac{20}{s^2+4s+3}$

(c) $X(s) = \frac{5}{s^2+6s+73}$

(d) $X(s) = \frac{10}{s\left(s^2+6s+73\right)}$

$$X(s) = \frac{10}{73}\left[\frac{1}{s} - \frac{s+6}{(s+3)^2+64}\right] = \frac{10}{73}\left[\frac{1}{s} - \frac{s+3}{(s+3)^2+64} - \frac{3}{8}\frac{8}{(s+3)^2+64}\right]$$

(e) $$\mathrm{X}(s)=\frac{4}{s^2(s^2+6s+73)}$$

$$\mathrm{X}(s)=\frac{4}{s^2(s^2+6s+73)}=\frac{A}{s^2}+\frac{B}{s}+\frac{Cs+D}{s^2+6s+73}$$

Using the "cover up" method, $A=\frac{4}{73}\cong 0.0548$. Using

$$K_{qk}=\frac{1}{(m-k)!}\frac{d^{m-k}}{ds^{m-k}}\left[(s-p_q)^m \mathrm{H}(s)\right]_{s\to p_q} \quad , \quad k=1,2,\quad ,m$$

$$B=\frac{1}{1!}\frac{d}{ds}\left[\frac{4}{(s^2+6s+73)}\right]_{s\to 0}=\left[-4(s^2+6s+73)^{-2}(2s^2+6)\right]_{s\to 0}=-\frac{24}{(73)^2}\cong -0.0045$$

$$\mathrm{X}(s)=\frac{4}{s^2(s^2+6s+73)}=\frac{\frac{4}{73}}{s^2}-\frac{\frac{24}{(73)^2}}{s}+\frac{Cs+D}{s^2+6s+73}$$

Multiply through by s and let s approach infinity,

$$0=-\frac{24}{(73)^2}+C\Rightarrow C=\frac{24}{(73)^2}\cong 0.0045$$

$$\mathrm{X}(s)=\frac{4}{s^2(s^2+6s+73)}=\frac{\frac{4}{73}}{s^2}-\frac{\frac{24}{(73)^2}}{s}+\frac{\frac{24}{(73)^2}s+D}{(s+3)^2+64}$$

Then let $s=1$.

$$\frac{4}{80}=\frac{4}{73}-\frac{24}{(73)^2}+\frac{\frac{24}{(73)^2}+D}{80}\Rightarrow D=4-\frac{21464}{(73)^2}=-\frac{148}{(73)^2}=-0.0278$$

$$\mathrm{X}(s)=\frac{\frac{4}{73}}{s^2}-\frac{\frac{24}{(73)^2}}{s}+\frac{\frac{24}{(73)^2}s-\frac{148}{(73)^2}}{s^2+6s+73}=\frac{1}{(73)^2}\left[\frac{292}{s^2}-\frac{24}{s}+24\frac{s-\frac{37}{6}}{(s+3)^2+64}\right]$$

$$\mathrm{X}(s)=\frac{1}{(73)^2}\left[\frac{292}{s^2}-\frac{24}{s}+24\left(\frac{s+3}{(s+3)^2+64}-\frac{55}{48}\frac{8}{(s+3)^2+64}\right)\right]$$

$$x(t) = \frac{1}{(73)^2}\left[292t - 24 + 24e^{-3t}\left(\cos(8t) - \frac{55}{48}\sin(8t)\right)\right]u(t)$$

(f) $X(s) = \dfrac{2s}{s^2 + 2s + 13}$

(g) $X(s) = \dfrac{s}{s+3}$

(h) $X(s) = \dfrac{s}{s^2 + 4s + 4}$

(i) $X(s) = \dfrac{s^2}{s^2 - 4s + 4}$

(j) $X(s) = \dfrac{10s}{s^4 + 4s^2 + 4}$

$$X(s) = \frac{\frac{j5\sqrt{2}}{4}}{\left(s + j\sqrt{2}\right)^2} - \frac{\frac{j5\sqrt{2}}{4}}{\left(s - j\sqrt{2}\right)^2} \Rightarrow x(t) = \frac{j5\sqrt{2}}{4}\left(te^{-j\sqrt{2}t} - te^{j\sqrt{2}t}\right)u(t)$$

10. Using a table of Laplace transforms, find the CTFT's of these signals.

(a) $x(t) = 10e^{-100t}\,u(t)$

(b) $x(t) = 3e^{-50t}\cos(100\pi t)\,u(t)$

11. Using the Laplace transform, solve these differential equations for $t \geq 0$.

(a) $x'(t) + 10\,x(t) = u(t)$, $x(0^-) = 1$

$$s\,X(s) - x(0^-) + 10\,X(s) = \frac{1}{s}$$

$$X(s) = \frac{\frac{1}{s} + 1}{s + 10} = \frac{s+1}{s(s+10)}$$

$$X(s) = \frac{\frac{1}{10}}{s} + \frac{\frac{9}{10}}{s+10} \Rightarrow x(t) = \frac{1 + 9e^{-10t}}{10}u(t)$$

Checking initial conditions,

$$x(0^+) = 1$$

which agrees with the initial condition, $x(0^-) = 1$. For this system and this excitation the response cannot change instantaneously.

(b) $x''(t) - 2x'(t) + 4x(t) = u(t)$, $x(0^-) = 0$, $\left[\frac{d}{dt}x(t)\right]_{t=0^-} = 4$

(c) $x'(t) + 2x(t) = \sin(2\pi t)u(t)$, $x(0^-) = -4$

12. Using the Laplace transform, find and sketch the time-domain response, $y(t)$, of the systems with these transfer functions to the sinusoidal excitation, $x(t) = A\cos(10\pi t)u(t)$.

(a) $H(s) = \frac{1}{s+1}$

$$Y(s) = \frac{A}{1+(10\pi)^2}\left[-\frac{1}{s+1} + \frac{s+(10\pi)^2}{s^2+(10\pi)^2}\right] = \frac{A}{1+(10\pi)^2}\left[-\frac{1}{s+1} + \frac{s}{s^2+(10\pi)^2} + 10\pi\frac{10\pi}{s^2+(10\pi)^2}\right]$$

(b) $H(s) = \frac{s-2}{(s-2)^2+16}$

$$Y(s) = \frac{s-2}{(s-2)^2+16}\frac{As}{s^2+(10\pi)^2} = A\frac{s^2-2s}{s^2(s-2)^2+16s^2+(10\pi)^2(s-2)^2+16(10\pi)^2}$$

$$Y(s) = \frac{s^2-2s}{s^4-4s^3+s^2\left(20+(10\pi)^2\right)-4(10\pi)^2 s+20(10\pi)^2}$$

$$Y(s) = \frac{s^2-2s}{s^4-4s^3+1006.97s^2-3947.84s+19739.2}$$

Using MATLAB,

```
»X

Transfer function:
    s
---------
s^2 + 987

»H
```

```
Transfer function:
    s - 2
--------------
s^2 - 4 s + 20

»Y

Transfer function:
                s^2 - 2 s
----------------------------------------
s^4 - 4 s^3 + 1007 s^2 - 3948 s + 1.974e04

»[z,p,k] = zpkdata(Y,'v') ;
»z

z =

     0
     2

»p

p =

 -0.00000000000000 +31.41592653589795i
 -0.00000000000000 -31.41592653589795i
  2.00000000000000 + 4.00000000000000i
  2.00000000000000 - 4.00000000000000i

r =

 -0.00105906221326 - 0.01610704734047i
 -0.00105906221326 + 0.01610704734047i
  0.00105906221326 + 0.00203398509622i
  0.00105906221326 - 0.00203398509622i
```

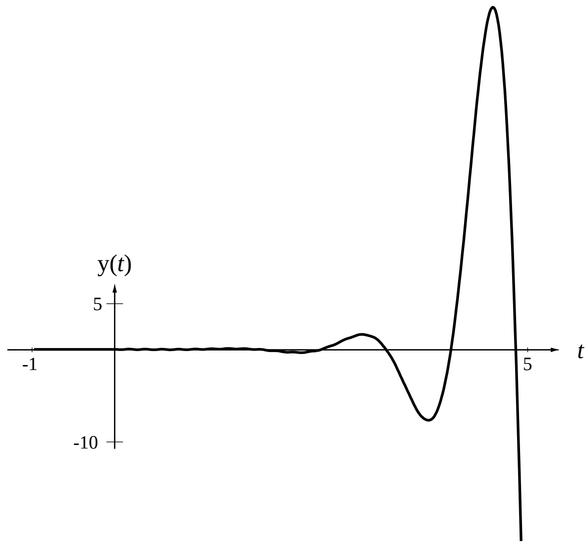
$$\mathrm{Y}(s) = A\left[\begin{array}{l}\dfrac{-0.00106 - j0.01611}{s - j10\pi} + \dfrac{-0.00106 + j0.01611}{s + j10\pi} \\ + \dfrac{0.00106 + j0.00203}{s - 2 - j4} + \dfrac{0.00106 - j0.00203}{s - 2 + j4}\end{array}\right]$$

$$\mathrm{Y}(s) = A\left[\frac{-0.00212s + 1.0122}{s^2 + (10\pi)^2} + \frac{0.00212s - 0.02048}{(s-2)^2 + 16}\right]$$

$$\mathrm{Y}(s) = 0.00212A\left[\frac{s - 9.66}{(s-2)^2 + 16} - \frac{s + 477.45}{s^2 + (10\pi)^2}\right]$$

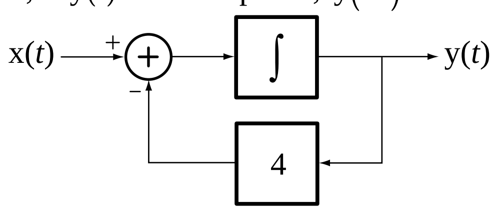
$$\mathrm{Y}(s) = 0.00212A\left[\frac{s-2}{(s-2)^2 + 16} - \frac{7.66}{4}\frac{4}{(s-2)^2 + 16} - \frac{s}{s^2 + (10\pi)^2} - \frac{477.45}{10\pi}\frac{10\pi}{s^2 + (10\pi)^2}\right]$$

$$\mathrm{y}(t) = 0.00212A\left\{e^{2t}\left[\cos(4t) - 1.915\sin(4t)\right] - \cos(10\pi t) - 15.2\sin(10\pi t)\right\}\mathrm{u}(t)$$



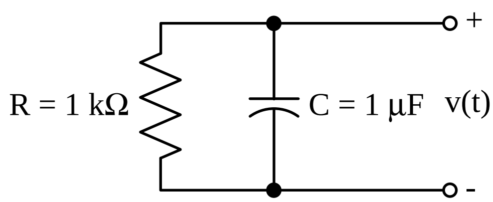
13. Write the differential equations describing these systems and find and sketch the indicated responses.

(a) $\mathrm{x}(t)=\mathrm{u}(t)$, $\mathrm{y}(t)$ is the response, $\mathrm{y}(0^-)=0$



The solution is continuous at $t=0$ because, if it were not the discontinuity would cause an impulse on the left-hand side of the equation which could not be equated to the step excitation on the right-hand side.

(b) $\mathrm{v}(0^-)=10$, $\mathrm{v}(t)$ is the response



$$C\mathrm{v}'(t)+\frac{\mathrm{v}(t)}{R}=0$$

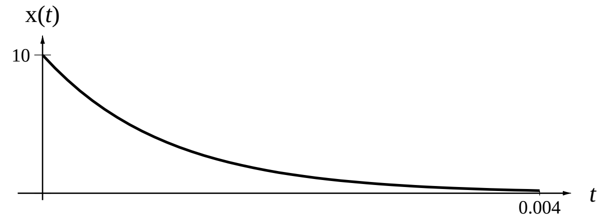
$$C\left[s\mathrm{V}(s)-\mathrm{v}(0^-)\right]+\frac{\mathrm{V}(s)}{R}=0$$

$$\mathrm{V}(s)=\frac{10C}{sC+\frac{1}{R}}=10\frac{1}{s+\frac{1}{RC}}$$

$$\mathrm{v}(t) = 10e^{-\frac{t}{RC}}\,\mathrm{u}(t) = 10e^{-1000t}\,\mathrm{u}(t) \ \ , \ \ t > 0$$

$$\mathrm{v}(0^+) = 10 = \mathrm{v}(0^-)\ .\ \text{Check.}$$

The solution is continuous at $t = 0$ because the capacitor voltage cannot change instantaneously.



14. Find the three parts, $\mathrm{x}_{ac}(t)$, $\mathrm{x}_0(t)$ and $\mathrm{x}_c(t)$, of the following signals.

(a) $\mathrm{x}(t) = e^{-10t}\,\mathrm{u}(t) - e^{2t}\,\mathrm{u}(-t)$

$$\mathrm{x}_{ac}(t) = -e^{2t}\,\mathrm{u}(-t) \ \ , \ \ \mathrm{x}_0(t) = 0 \ \ , \ \ \mathrm{x}_c(t) = e^{-10t}\,\mathrm{u}(t)$$

(b) $\mathrm{x}(t) = K$

(c) $\mathrm{x}(t) = \mathrm{u}(t)$

(d) $\mathrm{x}(t) = \dfrac{d}{dt}(\mathrm{u}(t))$

15. Find the bilateral Laplace transforms of these signals.

(a) $\mathrm{x}(t) = 3e^{-7t}\,\mathrm{u}(t) - 12e^{4t}\,\mathrm{u}(-t)$

$$\mathrm{x}_c(t) = 3e^{-7t}\,\mathrm{u}(t) \Rightarrow \mathrm{X}_c(s) = \frac{3}{s+7} \ \ , \ \ \mathrm{Re}(s) > -7$$

$$\mathrm{x}_0(t) = 0 \Rightarrow \mathrm{X}_0(s) = 0$$

$$\mathrm{x}_{ac}(t) = -12e^{4t}\,\mathrm{u}(-t) \Rightarrow \mathrm{x}_{ac}(-t) = -12e^{-4t}\,\mathrm{u}(t) \Rightarrow \mathrm{X}_{ac}(-s) = -\frac{12}{s+4} \ \ , \ \ \mathrm{Re}(s) > -4$$

$$\mathrm{X}_{ac}(s) = \frac{12}{s-4} \ \ , \ \ \mathrm{Re}(s) < 4$$

$$\mathrm{X}(s) = \frac{3}{s+7} + \frac{12}{s-4} = 3\frac{5s+24}{s^2+3s-28} \ \ , \ \ -7 < \mathrm{Re}(s) < 4$$

(b) $\mathrm{x}(t) = 50e^{-10|t|}$

16. Find the responses, $\mathrm{y}(t)$, of these systems to these excitations.

(a) $\mathrm{h}(t) = e^{-5t}\,\mathrm{u}(t)$, $\mathrm{x}(t) = 3e^{-7t}\,\mathrm{u}(t) - 12e^{4t}\,\mathrm{u}(-t)$

(b) $\mathrm{h}(t) = \mathrm{tri}(t)$, $\mathrm{x}(t) = e^{-t}\,\mathrm{u}(t)$

Using

$$\mathrm{tri}(t) \xleftrightarrow{\mathcal{L}} \left(\frac{e^{\frac{s}{2}} - e^{-\frac{s}{2}}}{s}\right)^2 \quad , \quad \text{All } s$$

$$\mathrm{H}(s) = \left(\frac{e^{\frac{s}{2}} - e^{-\frac{s}{2}}}{s}\right)^2 \quad , \text{ All } s \quad \text{and} \quad \mathrm{X}(s) = \frac{1}{s+1} \quad , \ \mathrm{Re}(s) > -1$$

Therefore

$$\mathrm{Y}(s) = \frac{1}{s+1}\left(\frac{e^{\frac{s}{2}} - e^{-\frac{s}{2}}}{s}\right)^2 = \frac{e^{s} - 2 + e^{-s}}{s^2(s+1)} \quad , \ \mathrm{Re}(s) > -1$$

$$\mathrm{Y}(s) = \left(e^{s} - 2 + e^{-s}\right)\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right) \quad , \ \mathrm{Re}(s) > -1$$

$$\mathrm{y}(t) = \begin{bmatrix} \mathrm{ramp}(t+1) - 2\,\mathrm{ramp}(t) + \mathrm{ramp}(t-1) \\ -\mathrm{u}(t+1) + 2\,\mathrm{u}(t) - \mathrm{u}(t-1) \\ +\mathrm{e}^{-(t+1)}\,\mathrm{u}(t+1) - 2e^{-t}\,\mathrm{u}(t) + e^{-(t-1)}\,\mathrm{u}(t-1) \end{bmatrix}$$

$$\mathrm{y}(t) = \begin{Bmatrix} \left[\mathrm{ramp}(t+1) - 1 + \mathrm{e}^{-(t+1)}\right]\mathrm{u}(t+1) \\ -2\left[\mathrm{ramp}(t) - 1 + e^{-t}\right]\mathrm{u}(t) \\ +\left[\mathrm{ramp}(t-1) - 1 + e^{-(t-1)}\right]\mathrm{u}(t-1) \end{Bmatrix}$$

(c) $\mathrm{h}(t) = e^{-10t}\,\mathrm{u}(t)$, $\mathrm{x}(t) = 50e^{-10|t|}$

17. Sketch the pole-zero plot and region of convergence (if it exists) for these signals.

(a) $\mathrm{x}(t) = e^{-t}\,\mathrm{u}(-t) - e^{-4t}\,\mathrm{u}(t)$

(b) $\mathrm{x}(t) = e^{-2t}\,\mathrm{u}(-t) - e^{t}\,\mathrm{u}(t)$

18. Using the integral definition find the the unilateral Laplace transform of these time functions.

(a) $\mathrm{g}(t) = e^{-at}\,\mathrm{u}(t)$

(b) $\mathrm{g}(t) = e^{-a(t-\tau)}\,\mathrm{u}(t-\tau)\ ,\ \tau > 0$

(c) $\mathrm{g}(t) = e^{-a(t+\tau)}\,\mathrm{u}(t+\tau)\ ,\ \tau > 0$

$$\mathrm{G}(s) = e^{-a\tau}\left[-\frac{1}{s+a}e^{-(s+a)t}\right]_{0^-}^{\infty} = \frac{e^{-a\tau}}{s+a}\ ,\ \tau > 0$$

(d) $\mathrm{g}(t) = \sin(\omega_0 t)\,\mathrm{u}(t)$

(e) $\mathrm{g}(t) = \mathrm{rect}(t)$

(f) $\mathrm{g}(t) = \mathrm{rect}\left(t - \frac{1}{2}\right)$

19. Using MATLAB (or any other appropriate computer mathematics tool) do the inversion integral of

$$\mathrm{G}(s) = \frac{1}{s+10}$$

numerically. That is, approximate the inversion integral with a summation of the form

$$\mathrm{g}(t) \quad \mathcal{L}^{1}(G(s)) = \frac{1}{j2\pi}\sum_{n=-N}^{N}\frac{e^{s_n t}}{s_n+10}\Delta s_n = \frac{1}{j2\pi}\sum_{n=-N}^{N}\frac{e^{(\sigma + jn\Delta\omega)t}}{\sigma + jn\Delta\omega + 10}\,j\Delta\omega\ ,\ \sigma > 0\ .$$

Choose the combination of large N and small $\Delta\omega$ so that the summation will range over a contour from well below to well above the real axis. Plot g(t) versus t by computing the value of g(t) at every value of t from the above summation approximation to the inversion integral. Compare to the analytical result. Try at least three different values of σ to see the effect on the result. (Ideally there is no effect of changing σ as long as it is greater than -10, but actually, in this numerical approximation, there will be some small effects.)

```
%      Program to demonstrate the inverse Laplace transform numerically.

close all ;
tau = 0.1 ; dw = 1 ; p = -1/tau ;
w = dw*[-2000:2000]' ; ds = j*dw*ones(length(w),1) ;
allint = [] ;
for sigma = 0:5,
      s = sigma + j*w ;
      int = [] ; tv = [] ;
      for t = -tau*2:tau/20:tau*4,
```

```
            f   =   exp(s.*t)./(s   -   p)   ;   tv   =   [tv;t]   ;   int   =
[int;sum(f.*ds)/(j*2*pi)] ;
      end
      int = real(int) ; allint = [allint,int] ;
end
subplot(3,2,1) ; h = plot(tv, allint(:,1), 'k') ; set(h, 'LineWidth',2)
;
xlabel('Time, t (s)') ; ylabel('h(t)') ; title('sigma = 0') ;
grid ; axis([-0.2, 0.4, -0.1, 1]) ;
subplot(3,2,2) ; h = plot(tv, allint(:,2), 'k') ; set(h, 'LineWidth',2)
;
xlabel('Time, t (s)') ; ylabel('h(t)') ; title('sigma = 1') ;
grid ; axis([-0.2, 0.4, -0.1, 1]) ;
subplot(3,2,3) ; h = plot(tv, allint(:,3), 'k') ; set(h, 'LineWidth',2)
;
xlabel('Time, t (s)') ; ylabel('h(t)') ; title('sigma = 2') ;
grid ; axis([-0.2, 0.4, -0.1, 1]) ;
subplot(3,2,4) ; h = plot(tv, allint(:,4), 'k') ; set(h, 'LineWidth',2)
;
xlabel('Time, t (s)') ; ylabel('h(t)') ; title('sigma = 3') ;
grid ; axis([-0.2, 0.4, -0.1, 1]) ;
subplot(3,2,5) ; h = plot(tv, allint(:,5), 'k') ; set(h, 'LineWidth',2)
;
xlabel('Time, t (s)') ; ylabel('h(t)') ; title('sigma = 4') ;
grid ; axis([-0.2, 0.4, -0.1, 1]) ;
subplot(3,2,6) ; h = plot(tv, allint(:,6), 'k') ; set(h, 'LineWidth',2)
;
xlabel('Time, t (s)') ; ylabel('h(t)') ; title('sigma = 5') ;
grid ; axis([-0.2, 0.4, -0.1, 1]) ;
```

20. Using a table of unilateral Laplace transforms and the properties find the unilateral Laplace transforms of the following functions.

(a) $\mathrm{g}(t) = 5\sin(2\pi(t-1))\mathrm{u}(t-1)$

(b) $\mathrm{g}(t) = 5\sin(2\pi t)\mathrm{u}(t-1)$

$$\sin(2\pi t) = \sin(2\pi(t-1))$$

Therefore

$$5\sin(2\pi t)\mathrm{u}(t-1) \xleftrightarrow{\mathcal{L}} \frac{10\pi e^{-s}}{s^2 + (2\pi)^2}$$

(c) $\mathrm{g}(t) = 2\cos(10\pi t)\cos(100\pi t)\mathrm{u}(t)$

Use the trigonometric identity,

$$\cos(10\pi t)\cos(100\pi t) = \frac{1}{2}[\cos(10\pi t - 100\pi t) + \cos(10\pi t + 100\pi t)],$$

then complete the solution as usual.

(d) $\mathrm{g}(t) = \frac{d}{dt}\left(\mathrm{u}(t-2)\right)$

(e) $\mathrm{g}(t) = \int_{0^-}^{t} \mathrm{u}(\tau)\,d\tau$

(f) $\mathrm{g}(t) = \frac{d}{dt}\left(5e^{-\frac{t-\tau}{2}}\mathrm{u}(t-\tau)\right) \;,\; \tau > 0$

Use these properties:

Frequency Shifting
Time Shifting
Linearity
Time Differentiation Once

(g) $\mathrm{g}(t) = 2e^{-5t}\cos(10\pi t)\mathrm{u}(t)$

(h) $\mathrm{x}(t) = 5\sin\left(\pi t - \frac{\pi}{8}\right)\mathrm{u}(t)$

$$\mathrm{X}(s) = 5\frac{\pi\cos\left(\frac{\pi}{8}\right) - \sin\left(\frac{\pi}{8}\right)s}{s^2 + \pi^2}$$

21. Given

$$\mathrm{g}(t) \xleftrightarrow{\mathcal{L}} \frac{s+1}{s(s+4)}$$

find the Laplace transforms of

(a) $\mathrm{g}(2t)$

(b) $\frac{d}{dt}\left(\mathrm{g}(t)\right)$

Time Differentiation Once $\frac{d}{dt}\left(\mathrm{g}(t)\right) \xleftrightarrow{\mathcal{L}} s\frac{s+1}{s(s+4)} - \mathrm{g}(0^-)$

$$\frac{d}{dt}\left(\mathrm{g}(t)\right) \xleftrightarrow{\mathcal{L}} \frac{s+1}{s+4} - \mathrm{g}(0^-)$$

Initial Value Theorem $\mathrm{g}(0^+) = \lim_{s\to\infty} s\mathrm{G}(s) = 1$

$$\frac{d}{dt}\left(\mathrm{g}(t)\right) \xleftrightarrow{\mathcal{L}} \frac{s+1}{s+4} - 1$$
$$\frac{d}{dt}\left(\mathrm{g}(t)\right) \xleftrightarrow{\mathcal{L}} -\frac{3}{s+4}$$

(This is correct if $\mathrm{g}(0^-) = \mathrm{g}(0^+)$. That is, if g is continuous at time, $t = 0$.)

(c) $\mathrm{g}(t-4)$ (d) $\mathrm{g}(t) * \mathrm{g}(t)$

22. Find the time-domain functions which are the inverse Laplace transforms of these functions. Then, using the initial and final value theorems verify that they agree with the time-domain functions.

(a) $\mathrm{G}(s) = \dfrac{4s}{(s+3)(s+8)}$

$$\mathrm{g}(t) = \left(-\frac{12}{5}e^{-3t} + \frac{32}{5}e^{-8t}\right)\mathrm{u}(t)$$

$$\lim_{t\to\infty}\mathrm{g}(t) = \lim_{t\to\infty}\left[\left(-\frac{12}{5}e^{-3t} + \frac{32}{5}e^{-8t}\right)\mathrm{u}(t)\right] = 0$$

$$\lim_{s\to 0^+} s\mathrm{G}(s) = \lim_{s\to 0^+}\frac{4s^2}{(s+3)(s+8)} = 0 \text{ , Check.}$$

$$\lim_{t\to 0^+}\mathrm{g}(t) = \lim_{t\to 0^+}\left[\left(-\frac{12}{5}e^{-3t} + \frac{32}{5}e^{-8t}\right)\mathrm{u}(t)\right] = 4$$

$$\lim_{s\to\infty} s\mathrm{G}(s) = \lim_{s\to\infty}\frac{4s^2}{(s+3)(s+8)} = 4 \text{ , Check.}$$

(b) $\mathrm{G}(s) = \dfrac{4}{(s+3)(s+8)}$ (c) $\mathrm{G}(s) = \dfrac{s}{s^2+2s+2}$

(d) $\mathrm{G}(s) = \dfrac{e^{-2s}}{s^2+2s+2}$

23. Given

$$e^{-4t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \mathrm{G}(s)$$

find the inverse Laplace transforms of

(a) $\mathrm{G}\left(\dfrac{s}{3}\right)$

Frequency Scaling $3e^{-12t}\,\mathrm{u}(3t) \xleftrightarrow{\mathcal{L}} \mathrm{G}\left(\dfrac{s}{3}\right)$

$$3e^{-12t}\,\mathrm{u}(3t) = 3e^{-12t}\,\mathrm{u}(t)$$

(b) $\mathrm{G}(s-2) + \mathrm{G}(s+2)$ (c) $\dfrac{\mathrm{G}(s)}{s}$

24. The CTFT of

$$\mathrm{x}(t) = e^{-|t|}$$

exists but the (unilateral) Laplace transform does not. Why?

25. Compare the CTFT and the Laplace transform of a unit step. Why can the CTFT not be found from the Laplace transform?

$$\mathrm{u}(t) \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega} + \pi\delta(\omega) \qquad \text{and} \qquad \mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}$$

26. Show that the common Laplace transform pairs

$$\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \;,\; e^{-\alpha t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+\alpha} \;,\; te^{-\alpha t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+\alpha)^2}$$

$$\sin(\omega_0 t)\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{\omega_0}{s^2+\omega_0^2} \;,\; \cos(\omega_0 t)\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{s}{s^2+\omega_0^2}$$

$$e^{-\alpha t}\sin(\omega_0 t)\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{\omega_0}{(s+\alpha)^2+\omega_0^2} \;,\; e^{-\alpha t}\cos(\omega_0 t)\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$$

can be derived from only the impulse transformation, $\delta(t) \xleftrightarrow{\mathcal{L}} 1$, and the properties of the Laplace transform.

$\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}$:

Integration $\qquad \displaystyle\int_{0^-}^{t} \mathrm{g}(\tau)\,d\tau \xleftrightarrow{\mathcal{L}} \frac{\mathrm{G}(s)}{s}$

$$\int_{0^-}^{t} \delta(\lambda)\,d\lambda \xleftrightarrow{\mathcal{L}} \frac{1}{s}$$

$$\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s}$$

$e^{-\alpha t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+\alpha}$:

Frequency Shifting $\qquad e^{-\alpha t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \dfrac{1}{s+\alpha}$

$te^{-\alpha t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+\alpha)^2}$:

$$\sin(\omega_0 t)\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{\omega_0}{s^2+\omega_0^2} \ :$$

$$\cos(\omega_0 t)\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{s}{s^2+\omega_0^2} \ :$$

$$e^{-\alpha t}\sin(\omega_0 t)\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{\omega_0}{(s+\alpha)^2+\omega_0^2} :$$

$$e^{-\alpha t}\cos(\omega_0 t)\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2} :$$

27. Given an LTI system transfer function, $\mathrm{H}(s)$, find the time-domain response, $\mathrm{y}(t)$ to the excitation, $\mathrm{x}(t)$.

(a) $\mathrm{x}(t) = \sin(2\pi t)\mathrm{u}(t) \ , \ \mathrm{H}(s) = \dfrac{1}{s+1}$

(b) $\mathrm{x}(t) = \mathrm{u}(t) \ , \ \mathrm{H}(s) = \dfrac{3}{s+2}$

(c) $\mathrm{x}(t) = \mathrm{u}(t) \ , \ \mathrm{H}(s) = \dfrac{3s}{s+2}$

(d) $\mathrm{x}(t) = \mathrm{u}(t) \ , \ \mathrm{H}(s) = \dfrac{5s}{s^2+2s+2}$

(e) $\mathrm{x}(t) = \sin(2\pi t)\mathrm{u}(t) \ , \ \mathrm{H}(s) = \dfrac{5s}{s^2+2s+2}$

$$\mathrm{X}(s) = \frac{2\pi}{s^2+(2\pi)^2}$$

$$\mathrm{Y}(s) = \frac{2\pi}{s^2+(2\pi)^2}\frac{5s}{s^2+2s+2} = 10\pi\frac{s}{\left[s^2+(2\pi)^2\right]\left(s^2+2s+2\right)}$$

$$\mathrm{Y}(s) = 10\pi\frac{s}{\left[s^2+(2\pi)^2\right]\left(s^2+2s+2\right)} = \frac{As+B}{s^2+(2\pi)^2} + \frac{Cs+D}{s^2+2s+2}$$

Letting *s* be zero,

$$0 = \frac{B}{(2\pi)^2} + \frac{D}{2}$$

Multiplying through by *s* and letting *s* approach infinity,

$$0 = A + C$$

Letting $s = 1$,

$$\frac{2\pi}{1+(2\pi)^2} = \frac{A+B}{1+(2\pi)^2} + \frac{C+D}{5} \; .$$

Letting $s = -1$,

$$\frac{-10\pi}{1+(2\pi)^2} = \frac{-A+B}{1+(2\pi)^2} - C + D \; .$$

Arranging the equations in matrix form,

$$\begin{bmatrix} 0 & \frac{1}{(2\pi)^2} & 0 & \frac{1}{2} \\ 1 & 0 & 1 & 0 \\ \frac{1}{1+(2\pi)^2} & \frac{1}{1+(2\pi)^2} & \frac{1}{5} & \frac{1}{5} \\ -\frac{1}{1+(2\pi)^2} & \frac{1}{1+(2\pi)^2} & -1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{2\pi}{1+(2\pi)^2} \\ -\frac{10\pi}{1+(2\pi)^2} \end{bmatrix}$$

Solving,

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} -0.7535 \\ 1.5875 \\ 0.7535 \\ -0.0804 \end{bmatrix}$$

where

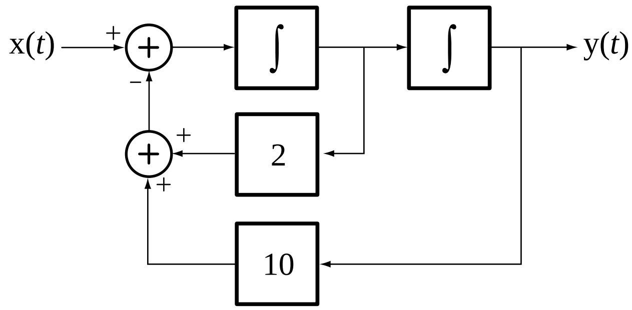
$$\mathrm{Y}(s) = \frac{-0.7535s + 1.5875}{s^2 + (2\pi)^2} + \frac{0.7535s - 0.0804}{s^2 + 2s + 2}$$

$$\mathrm{Y}(s) = -0.7535\left[\frac{s}{s^2+(2\pi)^2} - \frac{2.107}{2\pi}\frac{2\pi}{s^2+(2\pi)^2}\right] + 0.7535\left[\frac{s+1}{(s+1)^2+1} - 1.1067\frac{1}{(s+1)^2+1}\right]$$

$$\mathrm{y}(t) = \left\{-0.7535\left[\cos(2\pi t) - 0.3353\sin(2\pi t)\right] + 0.7535e^{-t}\left[\cos(t) - 1.1067\sin(t)\right]\right\}$$

28. Write the differential equations describing these systems and find and sketch the indicated responses.

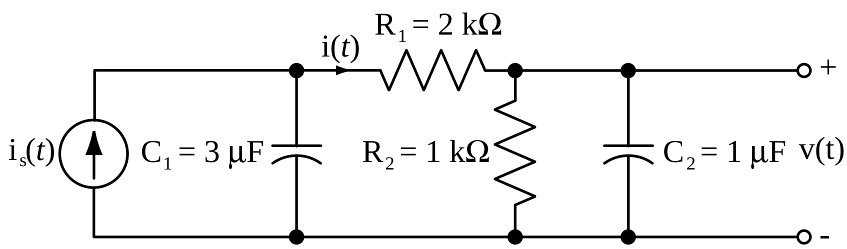
    (a) $\mathrm{x}(t) = \mathrm{u}(t)$, $\mathrm{y}(t)$ is the response , $\mathrm{y}(0^-) = -5$, $\left[\frac{d}{dt}(\mathrm{y}(t))\right]_{t=0^-} = 10$



$$\mathrm{Y}(s)=\frac{1}{10}\left(\frac{1}{s}-51\frac{s+1}{(s+1)^2+9}+\frac{49}{3}\frac{3}{(s+1)^2+9}\right)$$

Both the response and its first derivative must be continuous in response to a step excitation because of the double integration between excitation and response.

(b) $\mathrm{i}_s(t)=\mathrm{u}(t),\ \mathrm{v}(t)$ is the response , No initial energy storage



$$\mathrm{i}_s(t)-\underbrace{\left[C_2\,\mathrm{v}'(t)+\frac{\mathrm{v}(t)}{R_2}\right]}_{\mathrm{i}(t)}=C_1\underbrace{\mathrm{v}_s'(t)}_{\substack{\text{voltage}\\ \text{across}\\ \text{current}\\ \text{source}}}$$

$$\mathrm{v}_s(t)=\mathrm{v}(t)+\underbrace{\left[C_2\,\mathrm{v}'(t)+\frac{\mathrm{v}(t)}{R_2}\right]R_1}_{\text{voltage across } R_1}$$

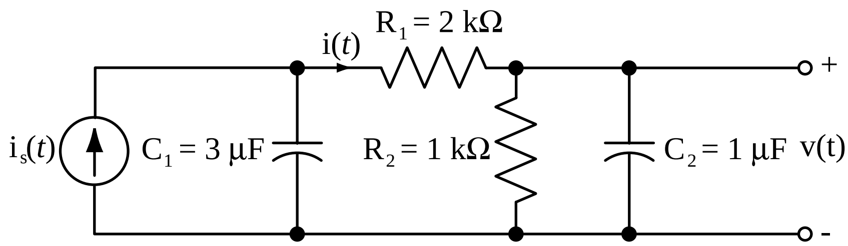
Combining equations,

$$R_1R_2C_1C_2\,\mathrm{v}''(t)+\left(R_2C_2+\left(R_2+R_1\right)C_1\right)\mathrm{v}'(t)+\mathrm{v}(t)=R_2\,\mathrm{i}_s(t)$$

$$R_1R_2C_1C_2s^2\,\mathrm{V}(s)+\left(R_2C_2+\left(R_2+R_1\right)C_1\right)s\mathrm{V}(s)+\mathrm{V}(s)=\frac{R_2}{s}$$

$$\mathrm{V}(s)=\frac{1.667\times10^8}{s(s+1560)(s+106.8)}=\frac{1000}{s}+\frac{73.52}{s+1560}-\frac{1073.52}{s+106.8}$$

(c) $\mathrm{i}_s(t) = \cos(2000\pi t)\mathrm{u}(t)$, $\mathrm{v}(t)$ is the response , No initial energy storage



From part (b)

$$R_1 R_2 C_1 C_2 \mathrm{v}''(t) + \left(R_2 C_2 + (R_2 + R_1)C_1\right)\mathrm{v}'(t) + \mathrm{v}(t) = R_2 \mathrm{i}_s(t)$$

$$R_1 R_2 C_1 C_2 s^2 \mathrm{V}(s) + \left(R_2 C_2 + (R_2 + R_1)C_1\right) s\mathrm{V}(s) + \mathrm{V}(s) = R_2 \frac{s}{s^2 + (2000\pi)^2}$$

$$\mathrm{V}(s) = -3.96 \frac{s}{s^2 + (2000\pi)^2} + \frac{6626}{2000\pi} \frac{2000\pi}{s^2 + (2000\pi)^2} + \frac{4.269}{s + 1560} - \frac{0.3104}{s + 106.84}$$

29. Find the three parts, $\mathrm{x}_{ac}(t)$, $\mathrm{x}_0(t)$ and $\mathrm{x}_c(t)$, of the following signals.

(a) $\mathrm{x}(t) = t$

(b) $\mathrm{x}(t) = \sin(\omega t)$

(c) $\mathrm{x}(t) = \frac{d}{dt}(\mathrm{sgn}(t))$

$$\mathrm{x}(t) = 2\delta(t)$$

$$\mathrm{x}_{ac}(t) = 0 \ , \ \mathrm{x}_0(t) = 2\delta(t) \ , \ \mathrm{x}_c(t) = 0$$

30. Find the bilateral Laplace transforms of these signals.

(a) $\mathrm{x}(t) = \mathrm{rect}(t)$

$$\mathrm{x}_c(t) = \mathrm{rect}(t)\mathrm{u}(t)$$

$$\mathrm{X}_c(s) = \int_{0^+}^{\frac{1}{2}} e^{-st} dt = \left[\frac{e^{-st}}{-s}\right]_{0^+}^{\frac{1}{2}} = \frac{e^{-\frac{s}{2}} - 1}{-s} = \frac{1 - e^{-\frac{s}{2}}}{s} \ , \ \text{Any } s$$

$$\mathrm{x}_0(t) = 0 \Rightarrow \mathrm{X}_0(s) = 0$$

$$\mathrm{x}_{ac}(t)=\mathrm{rect}(t)\mathrm{u}(-t)\Rightarrow \mathrm{x}_{ac}(-t)=\mathrm{rect}(-t)\mathrm{u}(t)$$

$$\mathrm{X}_{ac}(-s)=\int_{0^+}^{\frac{1}{2}} e^{-st}dt=\left[\frac{e^{-st}}{-s}\right]_{0^-}^{\frac{1}{2}}=\frac{e^{-\frac{s}{2}}-1}{-s}=\frac{1-e^{-\frac{s}{2}}}{s} \quad , \text{ Any } s$$

$$\mathrm{X}_{ac}(s)=\frac{1-e^{\frac{s}{2}}}{-s}=\frac{e^{\frac{s}{2}}-1}{s} \quad , \text{ Any } s$$

$$\mathrm{X}(s)=\frac{1-e^{-\frac{s}{2}}}{s}+\frac{e^{\frac{s}{2}}-1}{s}=\frac{e^{\frac{s}{2}}-e^{-\frac{s}{2}}}{s}=\frac{e^{-j\frac{js}{2}}-e^{j\frac{js}{2}}}{s}=\frac{\sin\left(-\frac{js}{2}\right)}{-j\frac{s}{2}}=\mathrm{sinc}\left(-\frac{js}{2\pi}\right) \quad , \text{ Any } s$$

Notice that if we make the change of variable, $s=j\omega$, we get $\mathrm{X}(j\omega)=\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$ which is the CTFT of $\mathrm{x}(t)=\mathrm{rect}(t)$ and this is allowed because the region of convergence is the entire s plane.

(b) $\mathrm{x}(t)=\mathrm{rect}(t)\sin(20\pi t)$

(c) $\mathrm{x}(t)=\left[e^{-2t}\,\mathrm{u}(t)-e^{2t}\,\mathrm{u}(-t)\right]\sin(2\pi t)$