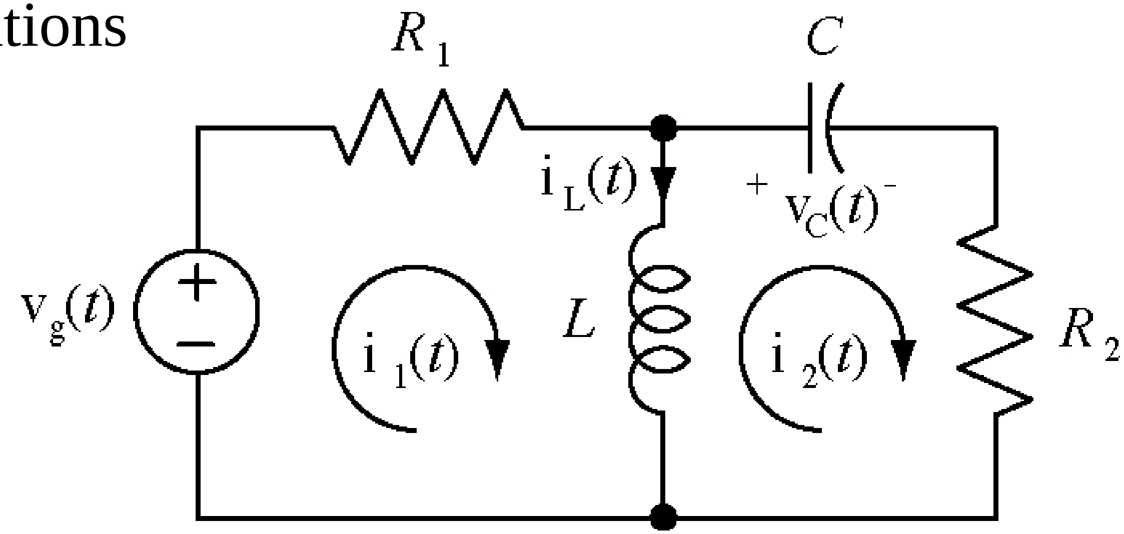
Laplace Transform Analysis of Signals and Systems

Transfer Functions

- Transfer functions of CT systems can be found from analysis of
  - Differential Equations
  - Block Diagrams
  - Circuit Diagrams

Transfer Functions

A circuit can be described by a system of differential equations

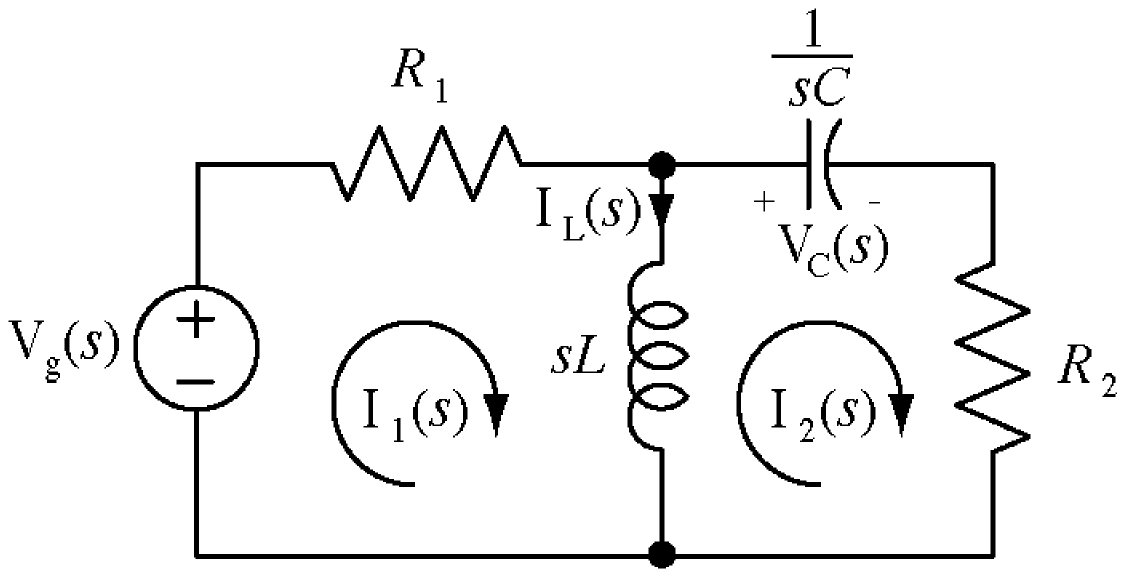


$$R_1\, \mathrm{i}_1(t) + L\left[\frac{d}{dt}\left(\mathrm{i}_1(t)\right) - \frac{d}{dt}\left(\mathrm{i}_2(t)\right)\right] = \mathrm{v}_g(t)$$

$$L\left[\frac{d}{dt}\left(\mathrm{i}_2(t)\right) - \frac{d}{dt}\left(\mathrm{i}_1(t)\right)\right] + \frac{1}{C}\int_{0^-}^{t} \mathrm{i}_2(\lambda)\, d\lambda + \mathrm{v}_c\left(0^-\right) + R_2\, \mathrm{i}_2(t) = 0$$

Transfer Functions

Using the Laplace transform, a circuit can be described by a system of algebraic equations

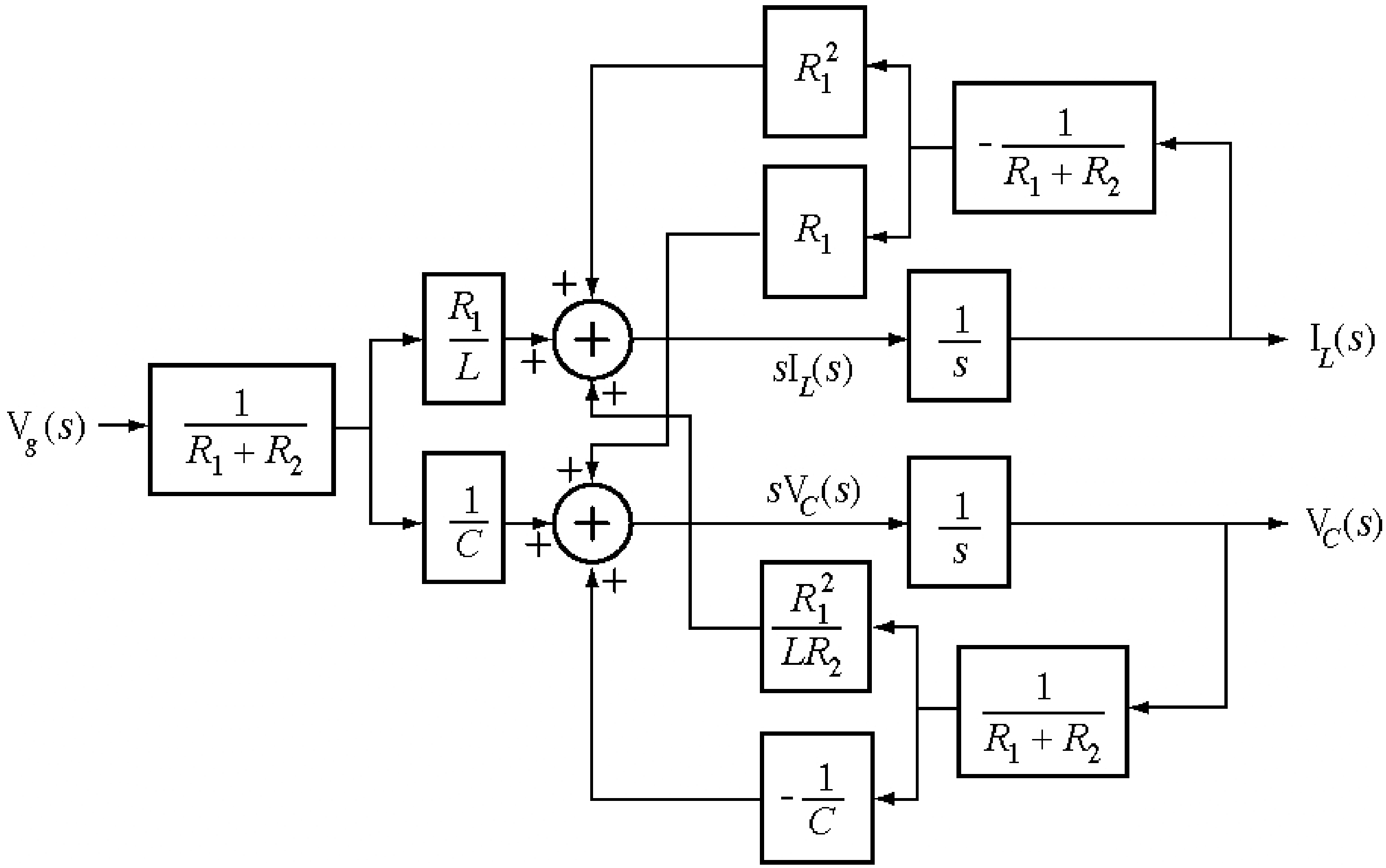


$$R_1 I_1(s) + sL\, I_1(s) - sL\, I_2(s) = V_g(s)$$

$$sL\, I_2(s) - sL\, I_1(s) + \frac{1}{sC} I_2(s) + R_2\, I_2(s) = 0$$

Transfer Functions

A circuit can even be described by a block diagram.



Transfer Functions

A mechanical system can be described by a system of differential equations

$$\mathrm{f}(t) - K_d\,\mathrm{x}_1'(t) - K_{s1}\left[\mathrm{x}_1(t) - \mathrm{x}_2(t)\right] = m_1\,\mathrm{x}_1''(t)$$

$$K_{s1}\left[\mathrm{x}_1(t) - \mathrm{x}_2(t)\right] - K_{s2}\,\mathrm{x}_2(t) = m_2\,\mathrm{x}_2''(t)$$

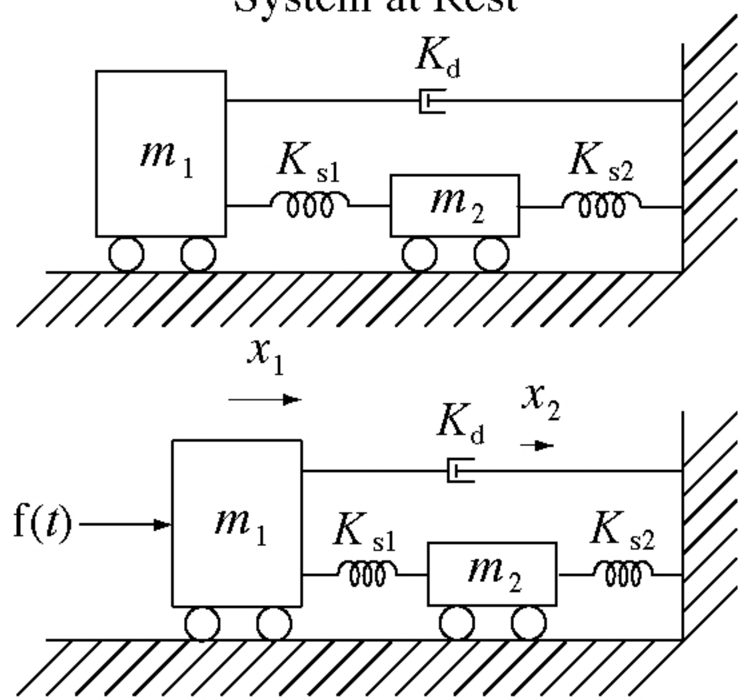
or a system of algebraic equations.

$$\mathrm{F}(s) - K_d s\,\mathrm{X}_1(s) - K_{s1}\left[\mathrm{X}_1(s) - \mathrm{X}_2(s)\right] = m_1 s^2\,\mathrm{X}_1(s)$$

$$K_{s1}\left[\mathrm{X}_1(s) - \mathrm{X}_2(s)\right] - K_{s2}\,\mathrm{X}_2(s) = m_2 s^2\,\mathrm{X}_2(s)$$

f(t) is the system excitation signal and the velocity of mass, m_2, $\mathrm{x}_2'(t)$, is the system response signal, y(t)

System at Rest

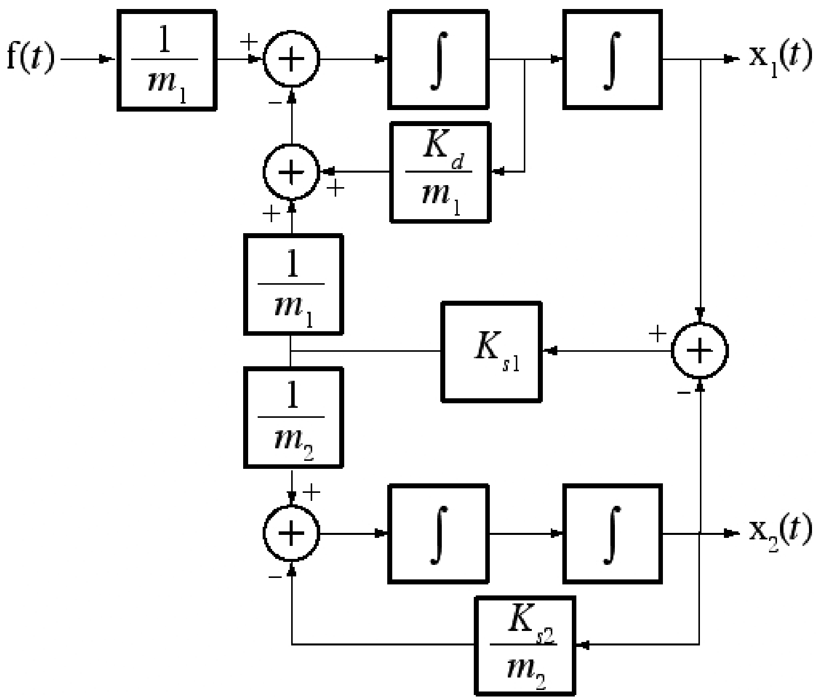


System in Motion

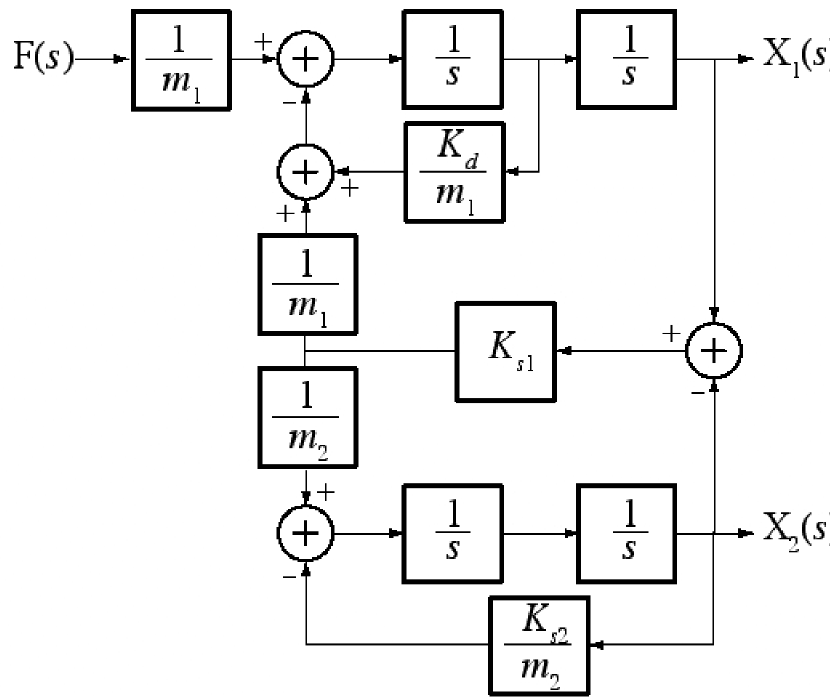
Transfer Functions

The mechanical system can also be described by a block diagram.

Time Domain



Frequency Domain

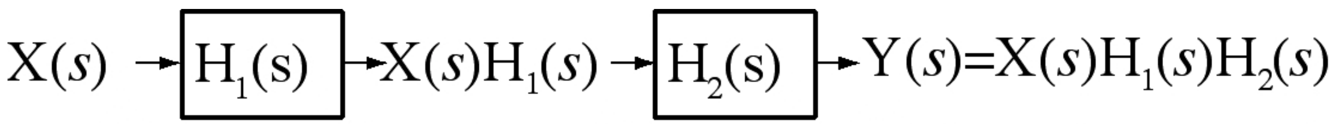


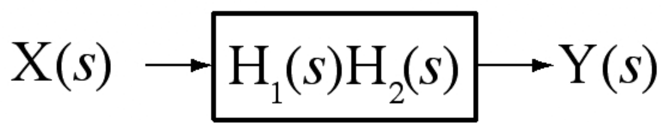
System Stability

- System stability is very important
- A continuous-time LTI system is stable if its impulse response is absolutely integrable
- This translates into the frequency domain as the requirement that all the poles of the system transfer function must lie in the <u>open</u> left half-plane of the *s* plane (pp. 675-676)
- "Open left half-plane" means *not* including the ω axis

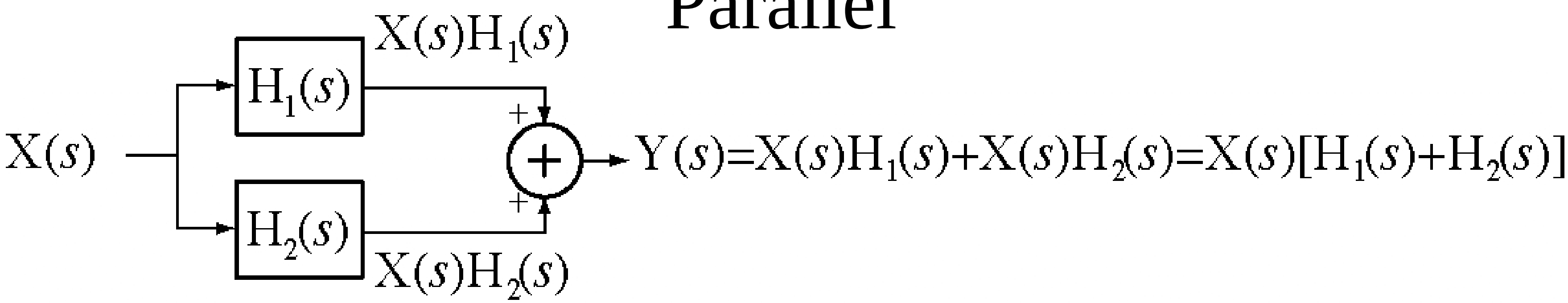
System Interconnections

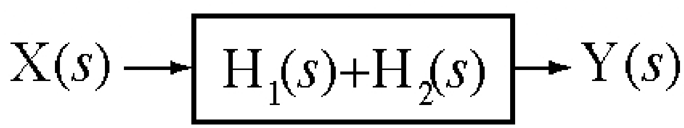
Cascade





Parallel





System Interconnections

Feedback

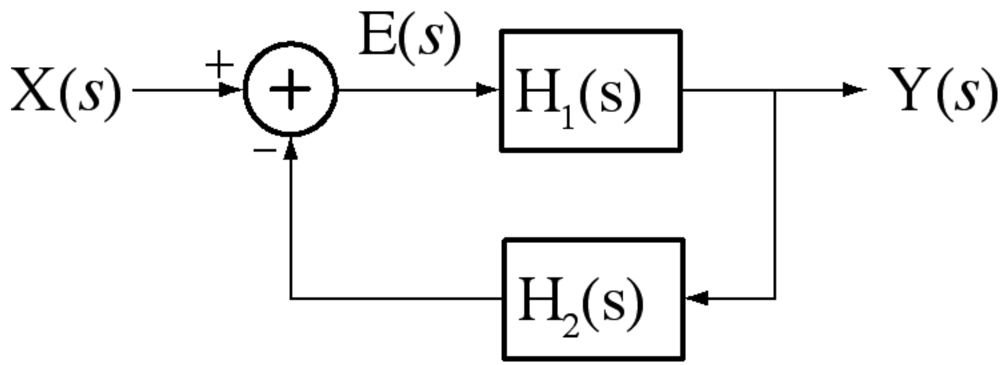
$\mathrm{E}(s)$ Error signal

$\mathrm{H}_1(s)$ Forward path transfer function or the "plant"

$\mathrm{H}_2(s)$ Feedback path transfer function or the "sensor".

$\mathrm{T}(s) = \mathrm{H}_1(s)\mathrm{H}_2(s)$

Loop transfer function



$$\mathrm{E}(s) = \mathrm{X}(s) - \mathrm{H}_2(s)\mathrm{Y}(s)$$

$$\mathrm{Y}(s) = \mathrm{H}_1(s)\mathrm{E}(s)$$

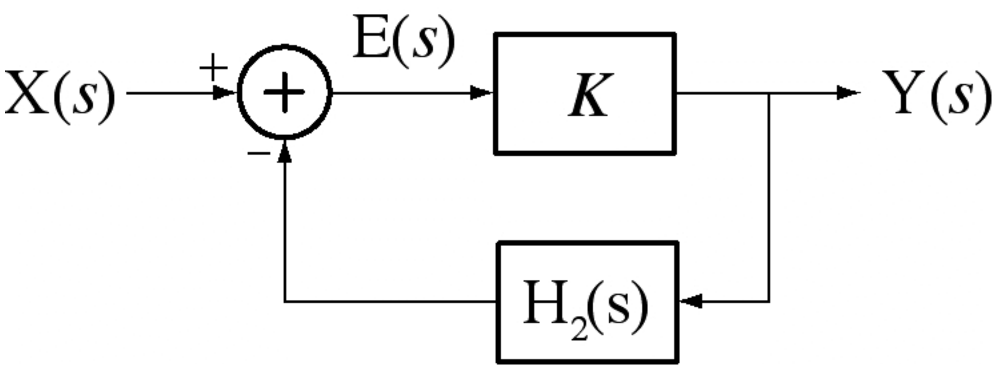
$$\mathrm{H}(s) = \frac{\mathrm{Y}(s)}{\mathrm{X}(s)} = \frac{\mathrm{H}_1(s)}{1 + \mathrm{H}_1(s)\mathrm{H}_2(s)}$$

$$\mathrm{T}(s) = \mathrm{H}_1(s)\mathrm{H}_2(s)$$

$$\mathrm{H}(s) = \frac{\mathrm{H}_1(s)}{1 + \mathrm{T}(s)}$$

Analysis of Feedback Systems

Beneficial Effects



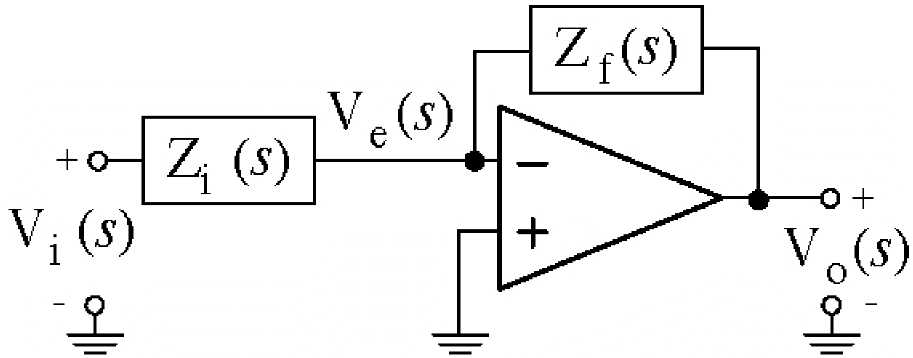
$$\mathrm{H}(s) = \frac{K}{1 + K\,\mathrm{H}_2(s)}$$

If K is large enough that $K\,\mathrm{H}_2(s) >> 1$ then $\mathrm{H}(s) \approx \dfrac{1}{\mathrm{H}_2(s)}$. This

means that the overall system is the approximate inverse of the system in the feedback path. This kind of system can be useful for reversing the effects of another system.

Analysis of Feedback Systems

A very important example of feedback systems is an electronic amplifier based on an operational amplifier

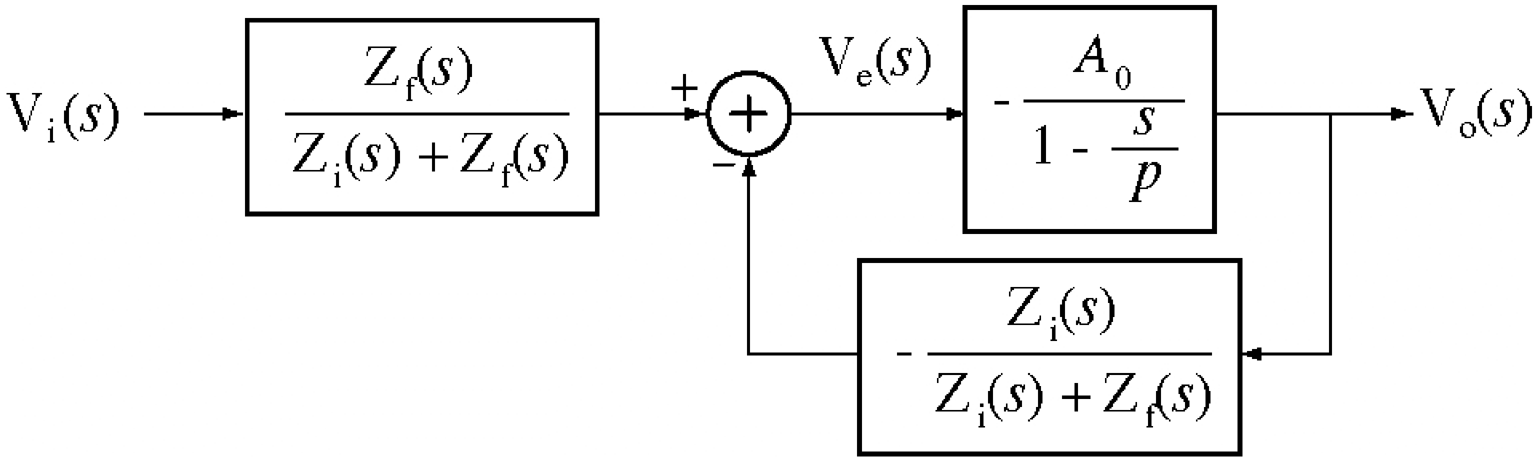


Let the operational amplifier gain be

$$\mathrm{H}_1(s) = \frac{\mathrm{V}_o(s)}{\mathrm{V}_e(s)} = -\frac{A_0}{1 - \dfrac{s}{p}}$$

Analysis of Feedback Systems

The amplifier can be modeled as a feedback system with this block diagram.



The overall gain can be written as

$$\frac{V_0(s)}{V_i(s)} = \frac{-A_0 Z_f(s)}{\left(1-\frac{s}{p}+A_0\right)Z_i(s)+\left(1-\frac{s}{p}\right)Z_f(s)}$$

Analysis of Feedback Systems

If the operational amplifier low-frequency gain, A_0 , is very large (which it usually is) then the overall amplifier gain reduces at low-frequencies to

$$\frac{V_0(s)}{V_i(s)} \cong -\frac{Z_f(s)}{Z_i(s)}$$

the gain formula based on an ideal operational amplifier.

Analysis of Feedback Systems

If $A_0 = 10^7$ and $p = -100$ then $\mathrm{H}(-j100) = -9.999989 + j0.000011$

If $A_0 = 10^6$ and $p = -100$ then $\mathrm{H}(-j100) = -9.99989 + j0.00011$

The change in overall system gain is about 0.001% for a change in open-loop gain of a factor of 10.

The half-power bandwidth of the operational amplifier itself is 15.9 Hz ($100/2\pi$). The half-power bandwidth of the overall amplifier is approximately 14.5 MHz, an increase in bandwidth of a factor of approximately 910,000.

Analysis of Feedback Systems

Feedback can stabilize an unstable system. Let a forward-path transfer function be

$$\mathrm{H}_1(s) = \frac{1}{s - p} \quad , \quad p > 0$$

This system is unstable because it has a pole in the right half-plane. If we then connect feedback with a transfer function, K, a constant, the overall system gain becomes

$$\mathrm{H}(s) = \frac{1}{s - p + K}$$

and, if $K > p$, the overall system is now stable.

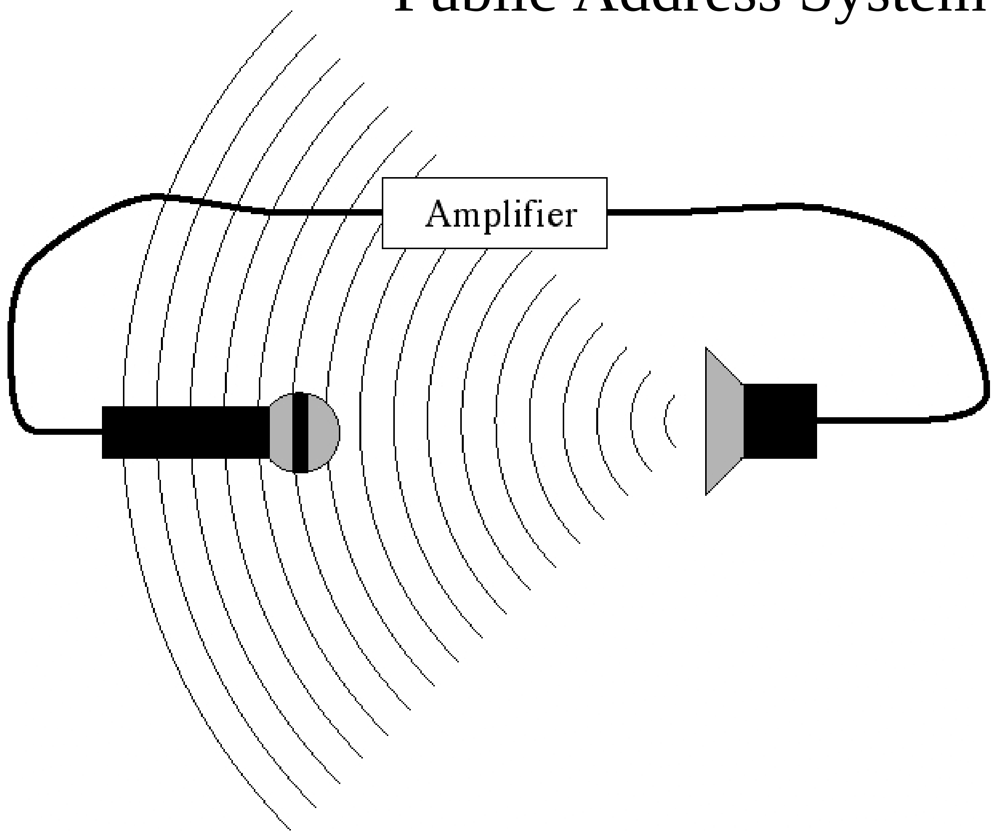
Analysis of Feedback Systems

Feedback can make an unstable system stable but it can also make a stable system unstable. Even though all the poles of the forward and feedback systems may be in the open left half-plane, the poles of the overall feedback system can be in the right half-plane.

A familiar example of this kind of instability caused by feedback is a public address system. If the amplifier gain is set too high the system will go unstable and oscillate, usually with a very annoying high-pitched tone.

Analysis of Feedback Systems

Public Address System

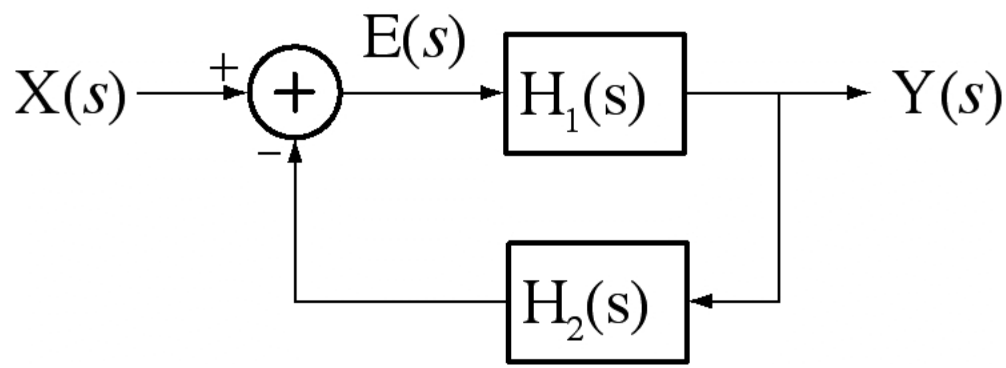


As the amplifier gain is increased, any sound entering the microphone makes a stronger sound from the speaker until, at some gain level, the returned sound from the speaker is a large as the originating sound into the microphone. At that point the system goes unstable (pp. 685-689).

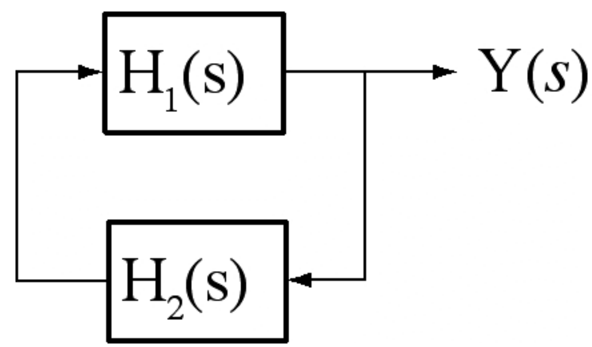
Analysis of Feedback Systems

Stable Oscillation Using Feedback

Prototype Feedback System

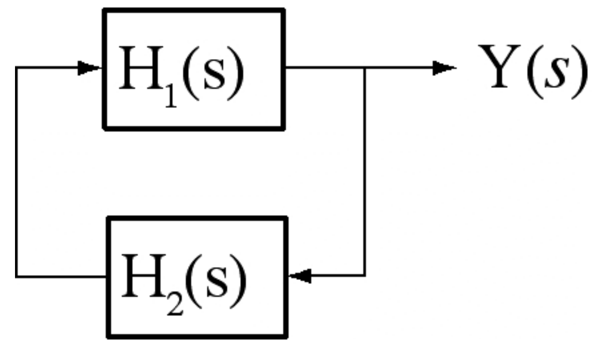


Feedback System Without Excitation



Analysis of Feedback Systems

Stable Oscillation Using Feedback

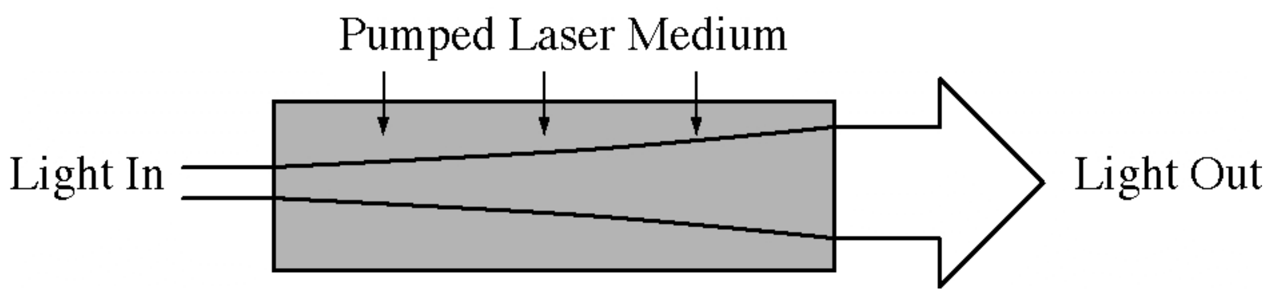


Can the response be non-zero when the excitation is zero? Yes, if the overall system gain is infinite. If the system transfer function has a pole pair on the ω axis, then the transfer function is infinite at the frequency of that pole pair and there can be a response without an excitation. In practical terms the trick is to be sure the poles stay on the ω axis. If the poles move into the left half-plane the response attenuates with time. If the poles move into the right half-plane the response grows with time (until the system goes non-linear).

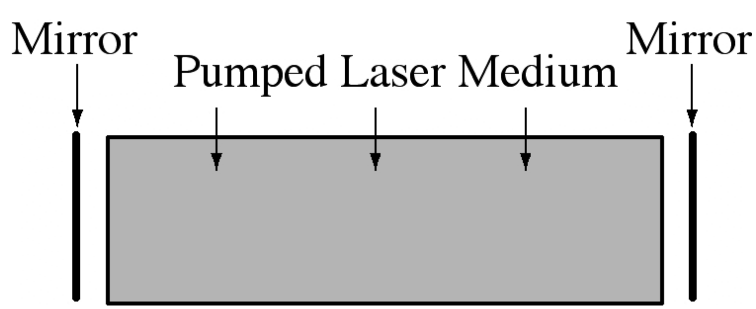
Analysis of Feedback Systems

Stable Oscillation Using Feedback

A real example of a system that oscillates stably is a laser. In a laser the forward path is an optical amplifier.



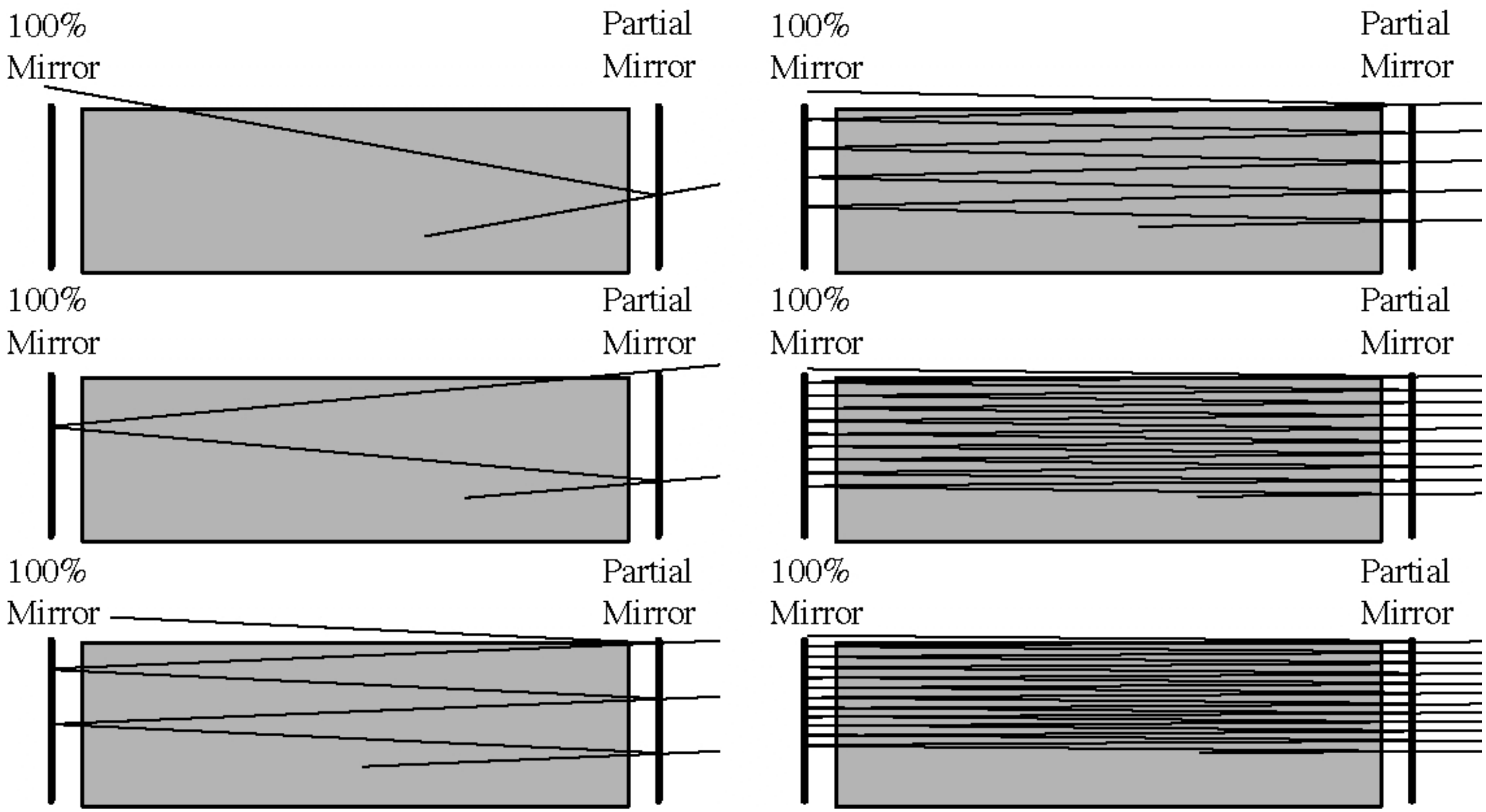
The feedback action is provided by putting mirrors at each end of the optical amplifier.



Analysis of Feedback Systems

Stable Oscillation Using Feedback

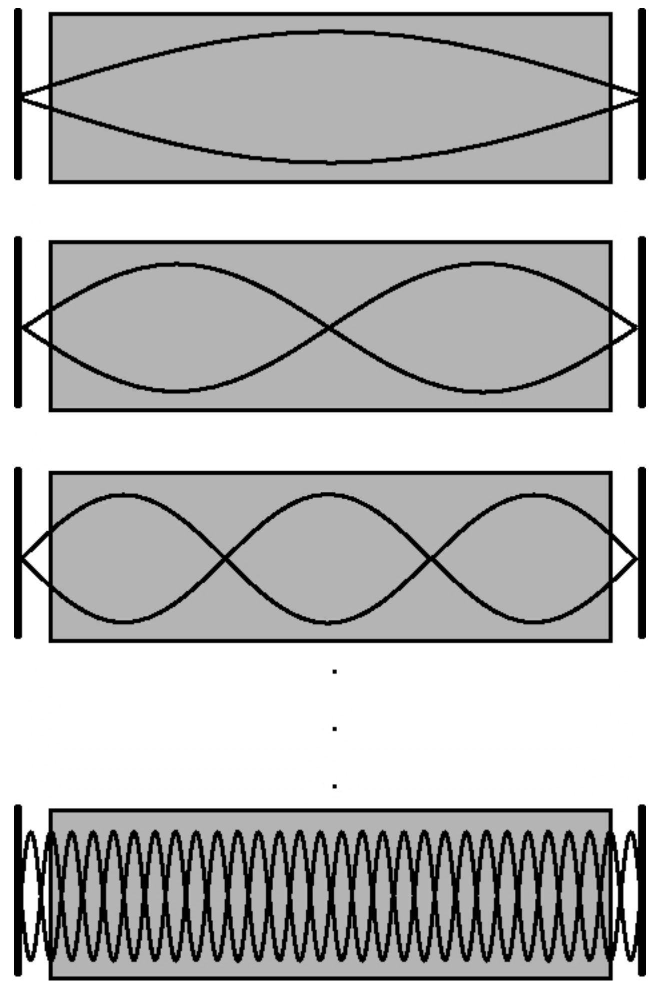
Laser action begins when a photon is spontaneously emitted from the pumped medium in a direction normal to the mirrors.



Analysis of Feedback Systems

Stable Oscillation Using Feedback

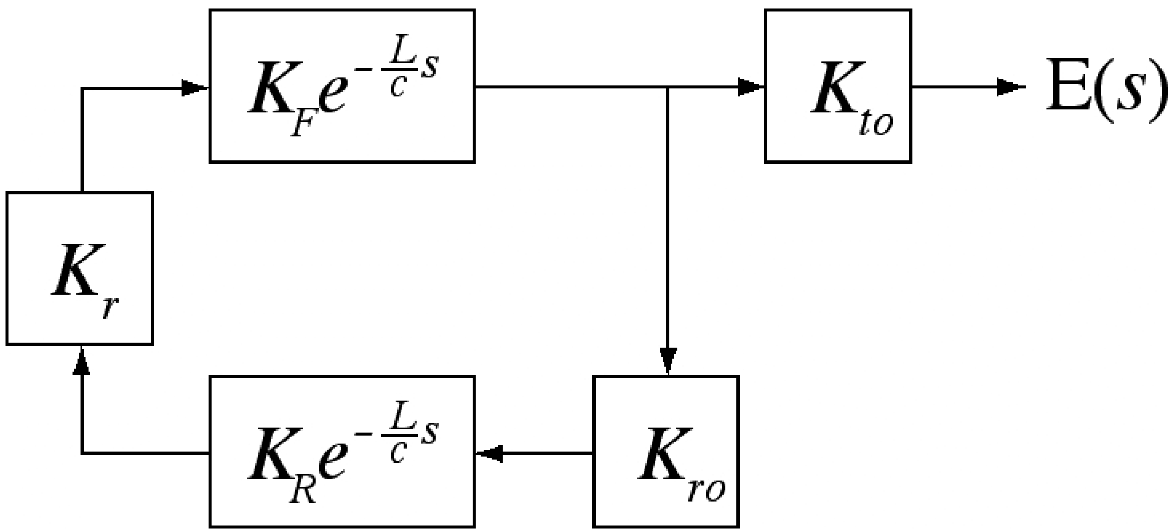
If the "round-trip" gain of the combination of pumped laser medium and mirrors is unity, sustained oscillation of light will occur. For that to occur the wavelength of the light must fit into the distance between mirrors an integer number of times.



Analysis of Feedback Systems

Stable Oscillation Using Feedback

A laser can be modeled by a block diagram in which the *K*'s represent the gain of the pumped medium or the reflection or transmission coefficient at a mirror, *L* is the distance between mirrors and *c* is the speed of light.



Analysis of Feedback Systems

The Routh-Hurwitz Stability Test

The *Routh-Hurwitz Stability Test* is a method for determining the stability of a system if its transfer function is expressed as a ratio of polynomials in s. Let the numerator be $\mathrm{N}(s)$ and let the denominator be

$$\mathrm{D}(s) = a_D s^D + a_{D-1} s^{D-1} + \cdots + a_1 s + a_0$$

Analysis of Feedback Systems

The Routh-Hurwitz Stability Test

The first step is to construct the "Routh array".

$$\begin{array}{cccccc|cccccc}
D & a_D & a_{D-2} & a_{D-4} & \cdots & a_0 & D & a_D & a_{D-2} & a_{D-4} & \cdots & a_1 \\
D-1 & a_{D-1} & a_{D-3} & a_{D-5} & \cdots & 0 & D-1 & a_{D-1} & a_{D-3} & a_{D-5} & \cdots & a_0 \\
D-2 & b_{D-2} & b_{D-4} & b_{D-6} & \cdots & 0 & D-2 & b_{D-2} & b_{D-4} & b_{D-6} & \cdots & 0 \\
D-3 & c_{D-3} & c_{D-5} & c_{D-7} & \cdots & 0 & D-3 & c_{D-3} & c_{D-5} & c_{D-7} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
2 & d_2 & d_0 & 0 & 0 & 0 & 2 & d_2 & d_0 & 0 & 0 & 0 \\
1 & e_1 & 0 & 0 & 0 & 0 & 1 & e_1 & 0 & 0 & 0 & 0 \\
0 & f_0 & 0 & 0 & 0 & 0 & 0 & f_0 & 0 & 0 & 0 & 0
\end{array}$$

D even   *D* odd

Analysis of Feedback Systems

The Routh-Hurwitz Stability Test

The first two rows contain the coefficients of the denominator polynomial. The entries in the following row are found by the formulas,

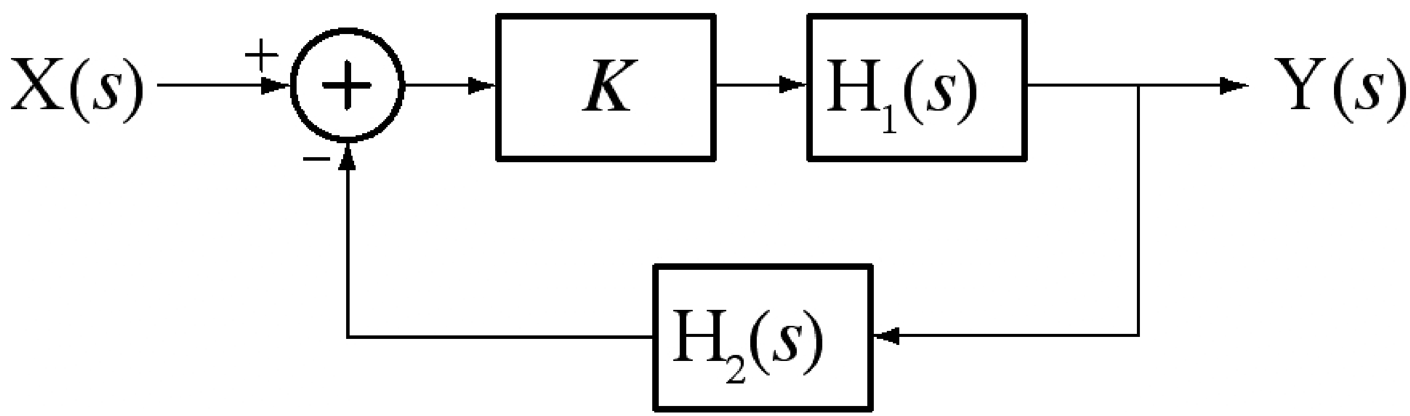
$$b_{D-2} = -\frac{\begin{vmatrix} a_D & a_{D-2} \\ a_{D-1} & a_{D-3} \end{vmatrix}}{a_{D-1}} \qquad b_{D-4} = -\frac{\begin{vmatrix} a_D & a_{D-4} \\ a_{D-1} & a_{D-5} \end{vmatrix}}{a_{D-1}} \quad \cdots$$

The entries on succeeding rows are computed by the same process based on previous row entries. If there are any zeros or sign changes in the a_D column, the system is unstable. The number of sign changes in the column is the number of poles in the right half-plane (pp. 693-694).

Analysis of Feedback Systems

Root Locus

Common Type of Feedback System



System Transfer Function $$\mathrm{H}(s)=\frac{K\,\mathrm{H}_1(s)}{1+K\,\mathrm{H}_1(s)\mathrm{H}_2(s)}$$

Loop Transfer Function $$\mathrm{T}(s)=K\,\mathrm{H}_1(s)\mathrm{H}_2(s)$$

Analysis of Feedback Systems

Root Locus

Poles of H(s) ⟶ Zeros of 1 + T(s)

T is of the form ⟶ $\mathrm{T}(s) = K \dfrac{\mathrm{P}(s)}{\mathrm{Q}(s)}$

Poles of H(s) ⟶ Zeros of 1 + $K \dfrac{\mathrm{P}(s)}{\mathrm{Q}(s)}$

Poles of H(s) ⟶

$$\mathrm{Q}(s) + K\,\mathrm{P}(s) = 0$$

or

$$\frac{\mathrm{Q}(s)}{K} + \mathrm{P}(s) = 0$$

Analysis of Feedback Systems

Root Locus

K can range from zero to infinity. For *K* approaching zero, using

$$\mathrm{Q}(s) + K\,\mathrm{P}(s) = 0$$

the poles of H are the same as the zeros of $\mathrm{Q}(s) = 0$ which are the poles of T. For *K* approaching infinity, using

$$\frac{\mathrm{Q}(s)}{K} + \mathrm{P}(s) = 0$$

the poles of H are the same as the zeros of $\mathrm{P}(s) = 0$ which are the zeros of T. So the poles of H start on the poles of T and terminate on the zeros of T, some of which may be at infinity. The curves traced by these pole locations as *K* is varied are called the *root locus*.

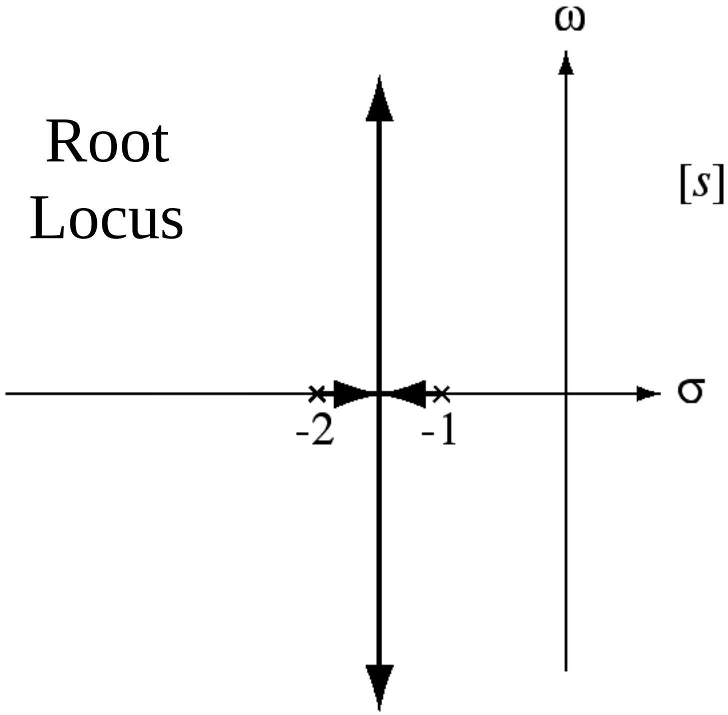
Analysis of Feedback Systems

Root Locus

Let $\mathrm{H}_1(s) = \dfrac{K}{(s+1)(s+2)}$ and let $\mathrm{H}_2(s) = 1$.

Then $\mathrm{T}(s) = \dfrac{K}{(s+1)(s+2)}$

Root Locus



No matter how large *K* gets this system is stable because the poles always lie in the left half-plane (although for large *K* the system may be very underdamped).

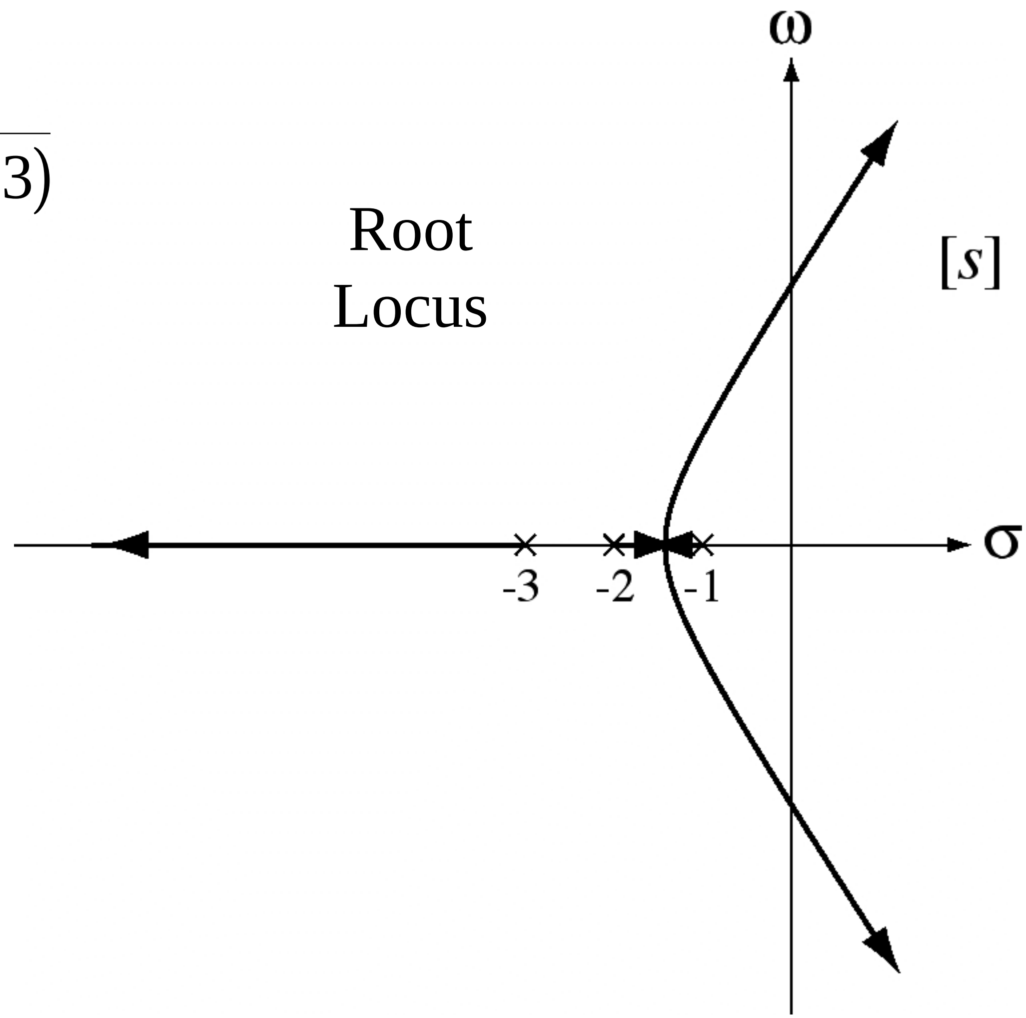
Analysis of Feedback Systems

Root Locus

Let $H_1(s) = \dfrac{K}{(s+1)(s+2)(s+3)}$

and let $H_2(s) = 1$.

At some finite value of K the system becomes unstable because two poles move into the right half-plane.



Analysis of Feedback Systems

Root Locus

Four Rules for Drawing a Root Locus

1. Each root-locus branch begins on a pole of T and terminates on a zero of T.

2. Any portion of the real axis for which the sum of the number of real poles and/or real zeros lying to its right on the real axis is odd, is a part of the root locus.

3. The root locus is symmetrical about the real axis.

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Analysis of Feedback Systems

Root Locus

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4. If the number of finite poles of T exceeds the number of finite zeros of T by an integer, *m*, then *m* branches of the root locus terminate on zeros of T which lie at infinity. Each of these branches approaches a straight-line asymptote and the angles of these asymptotes are at the angles,

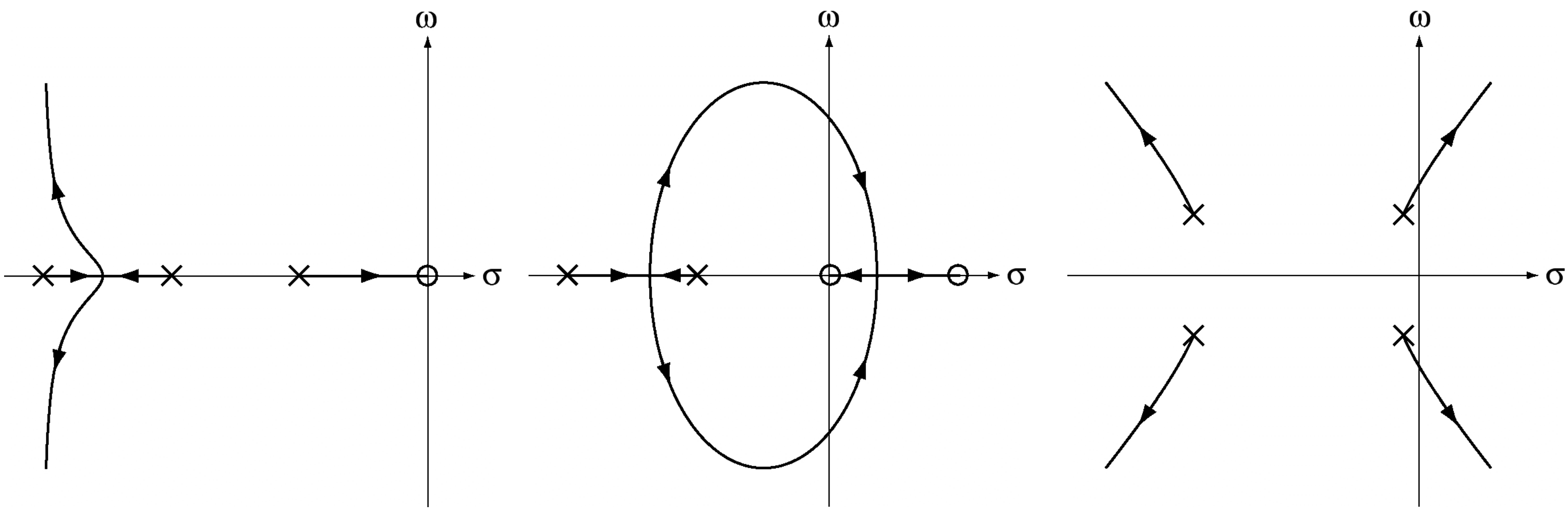
$$\frac{k\pi}{m} \quad , \quad k = 1, 3, 5, \ldots$$

with respect to the positive real axis. These asymptotes intersect on the real axis at the location,

$$\sigma = \frac{1}{m}\left(\sum \text{finite poles} - \sum \text{finite zeros}\right)$$

Analysis of Feedback Systems

Root Locus Examples



Analysis of Feedback Systems

Gain and Phase Margin

Real systems are usually designed with a margin of error to allow for small parameter variations and still be stable.

That "margin" can be viewed as a *gain margin* or a *phase margin*.

System instability occurs if, for any real ω,

$$\mathrm{T}(j\omega) = -1$$

a number with a magnitude of one and a phase of $-\pi$ radians.

Analysis of Feedback Systems

Gain and Phase Margin

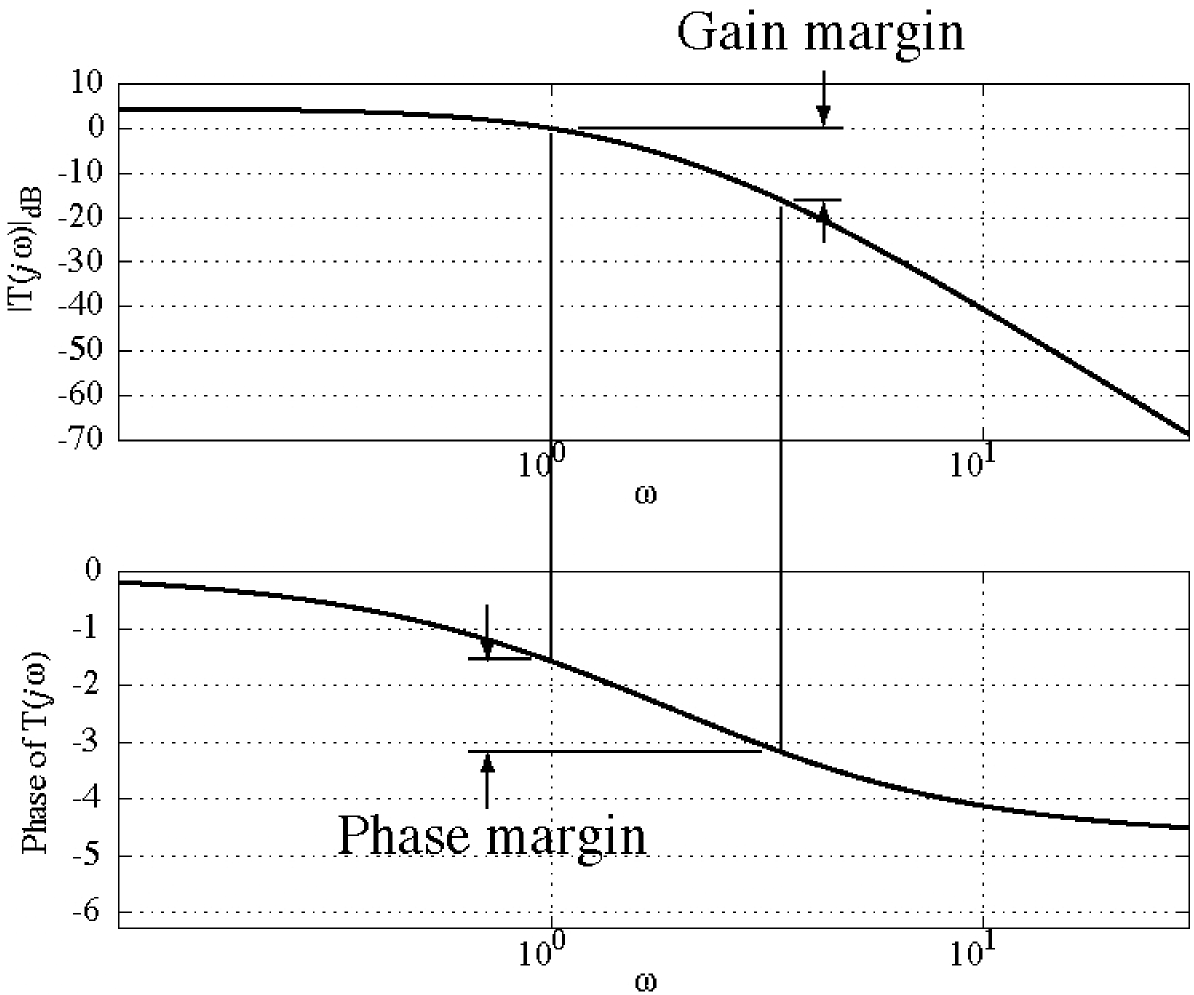
So to be guaranteed stable, a system must have a T whose magnitude, as a function of frequency, is less than one when the phase hits $-\pi$ or, seen another way, T must have a phase, as a function of frequency, more positive than $-\pi$ for all $|T|$ greater than one.

The difference between the a magnitude of T of 0 dB and the magnitude of T when the phase hits $-\pi$ is the gain margin.

The difference between the phase of T when the magnitude hits 0 dB and a phase of $-\pi$ is the phase margin.

Analysis of Feedback Systems

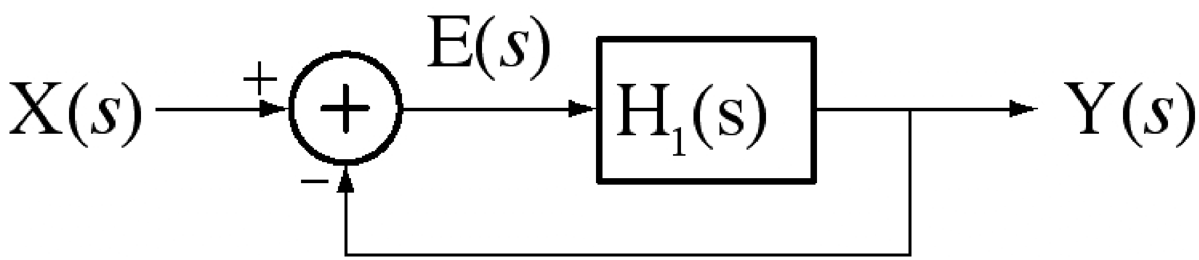
Gain and Phase Margin



Analysis of Feedback Systems

Steady-State Tracking Errors in Unity-Gain Feedback Systems

A very common type of feedback system is the *unity-gain* feedback connection.



The aim of this type of system is to make the response "track" the excitation. When the error signal is zero, the excitation and response are equal.

Analysis of Feedback Systems

Steady-State Tracking Errors in Unity-Gain Feedback Systems

The Laplace transform of the error signal is

$$E(s) = \frac{X(s)}{1 + H_1(s)}$$

The steady-state value of this signal is (using the final-value theorem)

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} s\,E(s) = \lim_{s\to 0} s\,\frac{X(s)}{1 + H_1(s)}$$

If the excitation is the unit step, $A\,u(t)$, then the steady-state error is

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{A}{1 + H_1(s)}$$

Analysis of Feedback Systems

Steady-State Tracking Errors in Unity-Gain Feedback Systems

If the forward transfer function is in the common form,

$$\mathrm{H}_1(s) = \frac{b_N s^N + b_{N-1} s^{N-1} + \cdots b_2 s^2 + b_1 s + b_0}{a_D s^D + a_{D-1} s^{D-1} + \cdots a_2 s^2 + a_1 s + a_0}$$

then

$$\lim_{t \to \infty} \mathrm{e}(t) = \lim_{s \to 0} \frac{1}{1 + \frac{b_N s^N + b_{N-1} s^{N-1} + \cdots b_2 s^2 + b_1 s + b_0}{a_D s^D + a_{D-1} s^{D-1} + \cdots a_2 s^2 + a_1 s + a_0}} = \frac{a_0}{a_0 + b_0}$$

If $a_0 = 0$ and $b_0 \neq 0$ the steady-state error is zero and the forward transfer function can be written as

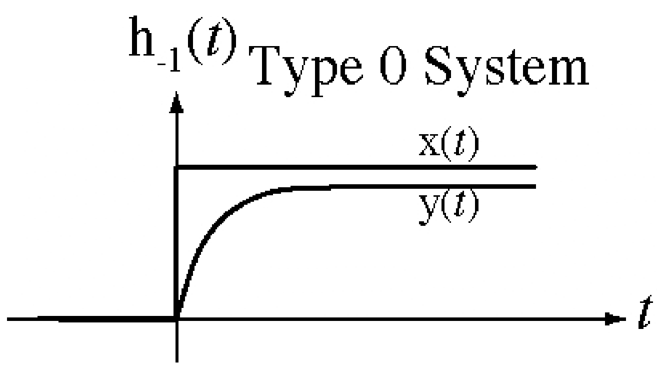
$$\mathrm{H}_1(s) = \frac{b_N s^N + b_{N-1} s^{N-1} + \cdots b_2 s^2 + b_1 s + b_0}{s\left(a_D s^{D-1} + a_{D-1} s^{D-2} + \cdots a_2 s + a_1\right)}$$

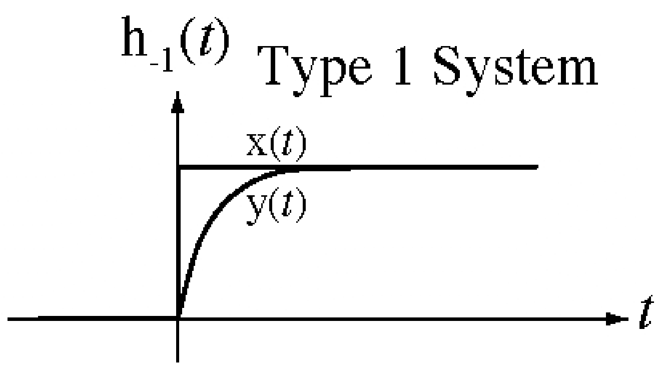
which has a pole at $s = 0$.

Analysis of Feedback Systems

Steady-State Tracking Errors in Unity-Gain Feedback Systems

If the forward transfer function of a unity-gain feedback system has a pole at zero and the system is stable, the steady-state error with step excitation is zero. This type of system is called a “type 1” system (one pole at $s = 0$ in the forward transfer function). If there are no poles at $s = 0$, it is called a “type 0” system and the steady-state error with step excitation is non-zero.





Analysis of Feedback Systems

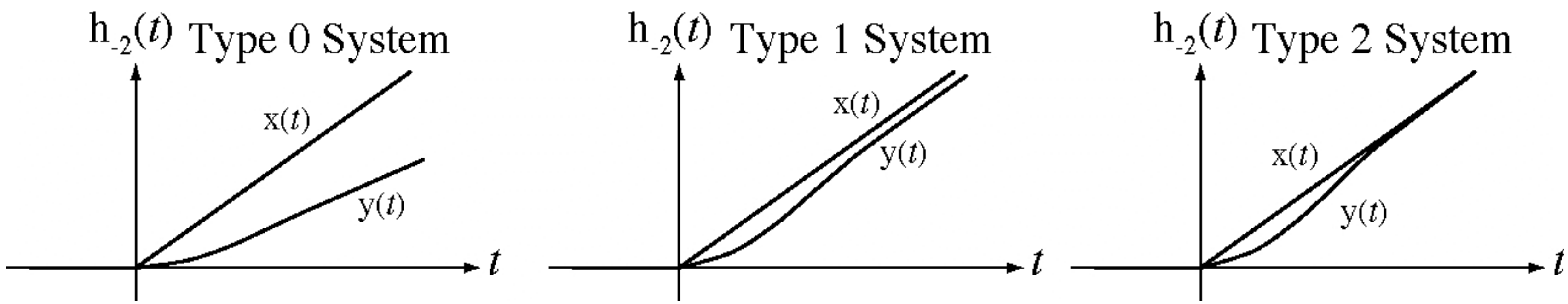
Steady-State Tracking Errors in Unity-Gain Feedback Systems

The steady-state error with ramp excitation is

Infinite for a stable type 0 system

Finite and non-zero for a stable type 1 system

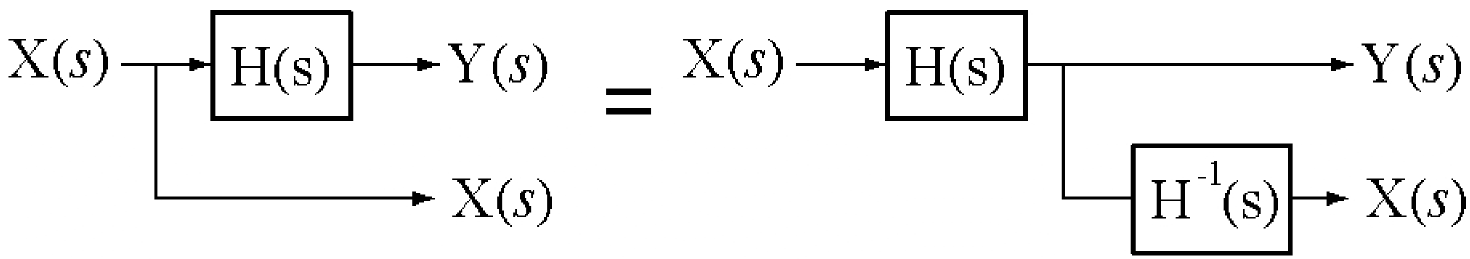
Zero for a stable type 2 system (2 poles at $s = 0$ in the forward transfer function)

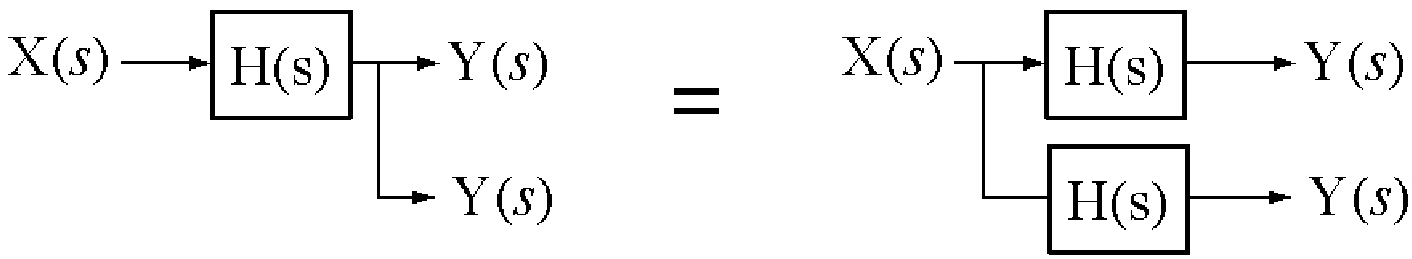


Block Diagram Reduction

It is possible, by a series of operations, to reduce a complicated block diagram down to a single block.

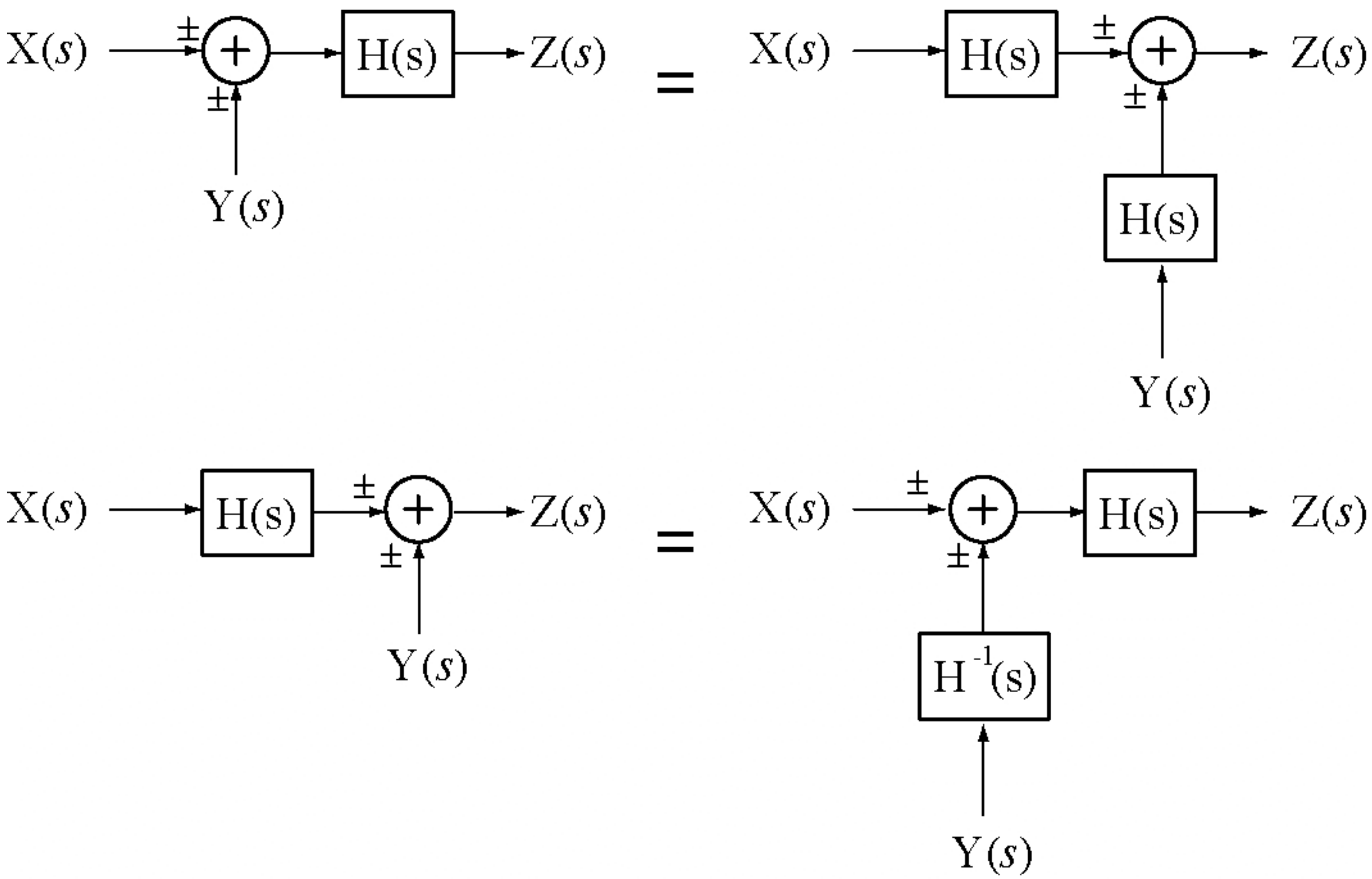
Moving a Pick-Off Point





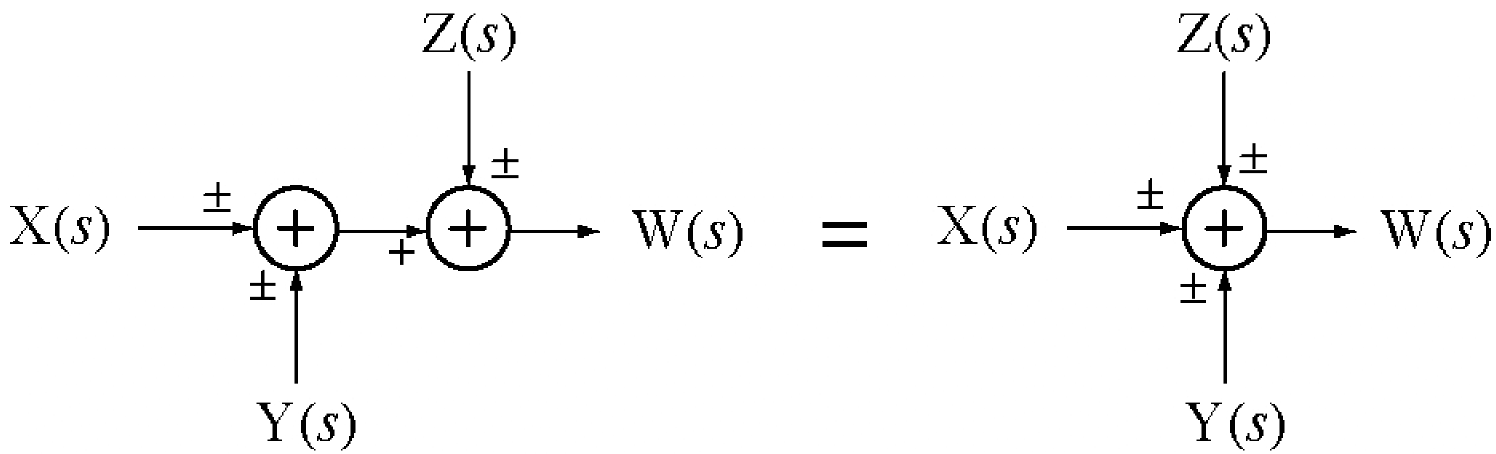
Block Diagram Reduction

Moving a Summer

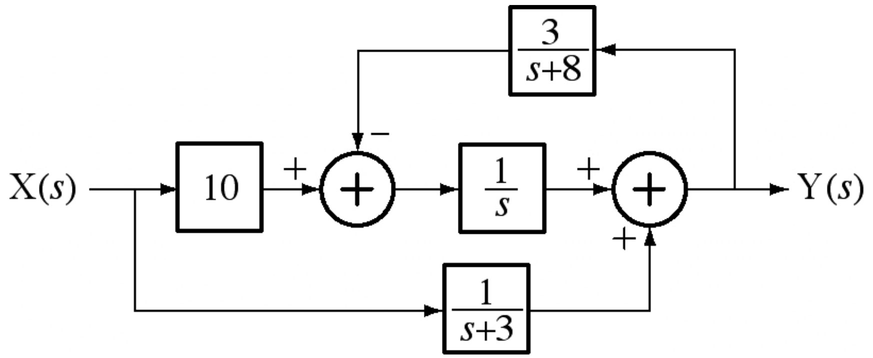


Block Diagram Reduction

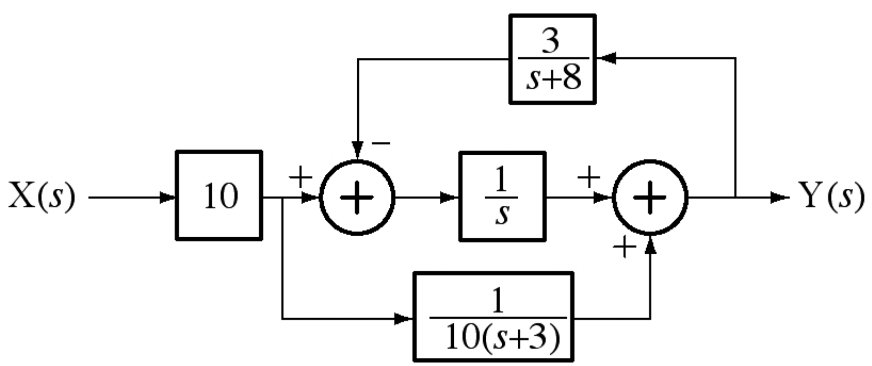
Combining Two Summers



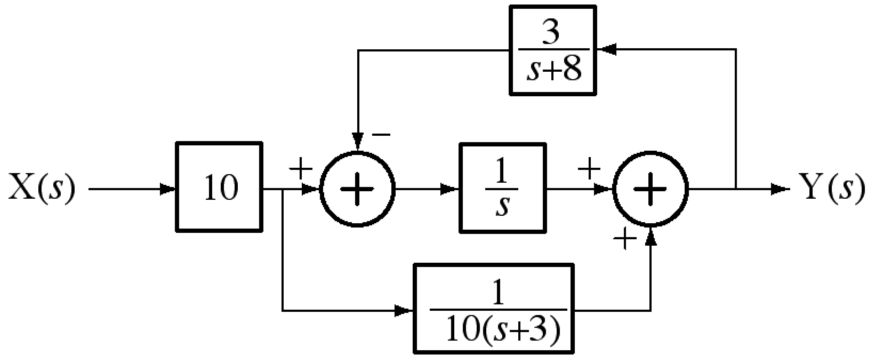
Block Diagram Reduction



Move Pick-Off Point



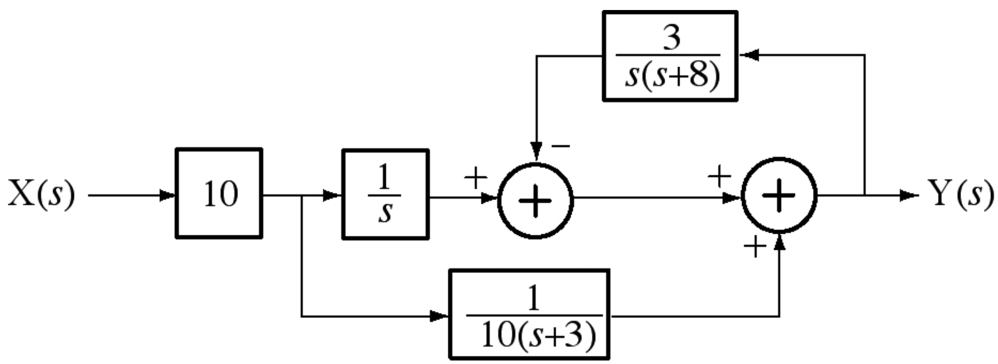
Block Diagram Reduction



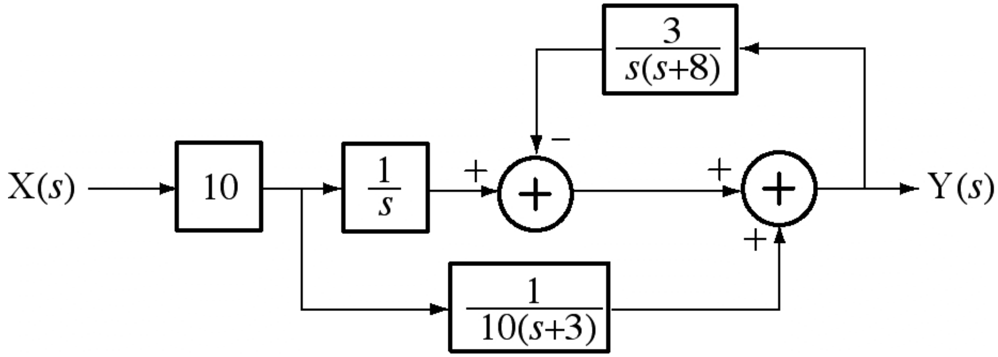
⇩

Move Summer

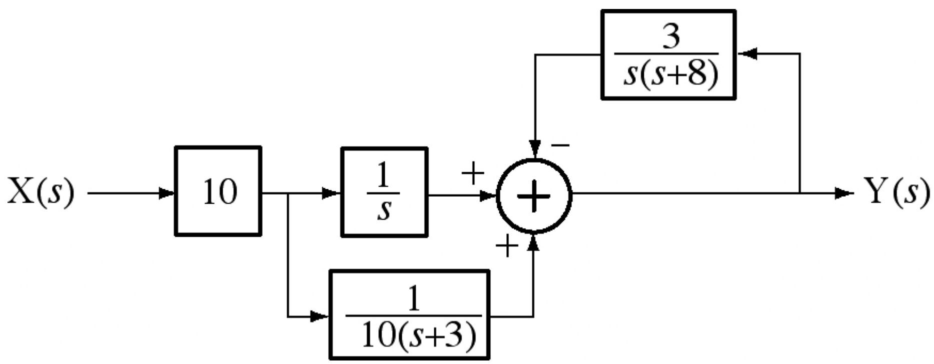
⇩



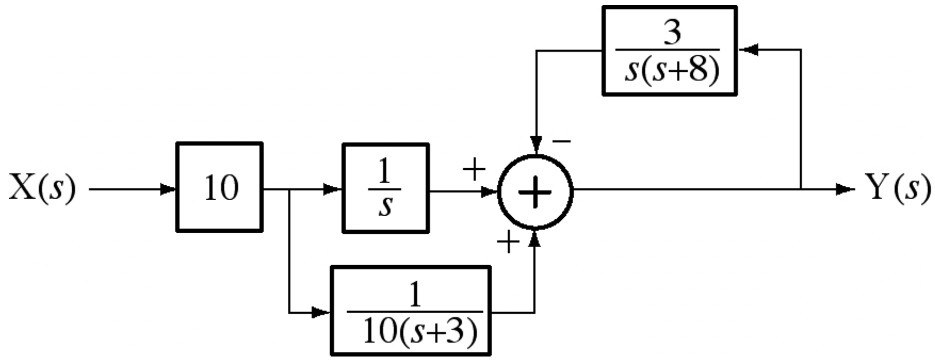
Block Diagram Reduction



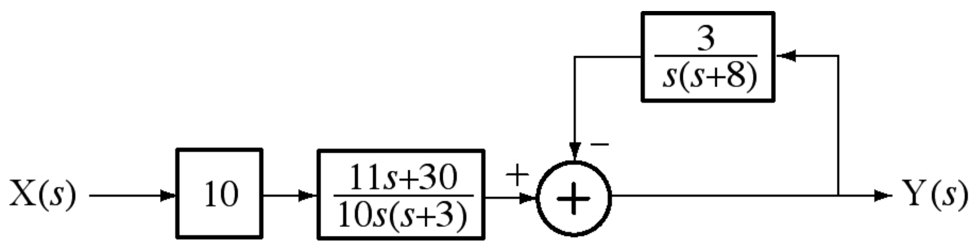
Combine Summers



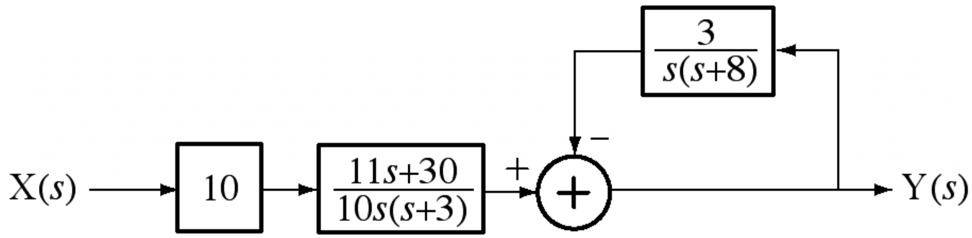
Block Diagram Reduction



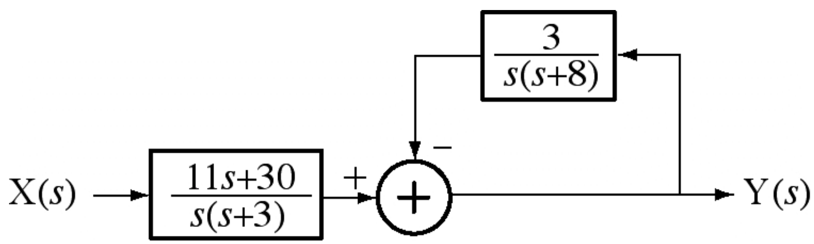
Combine Parallel Blocks



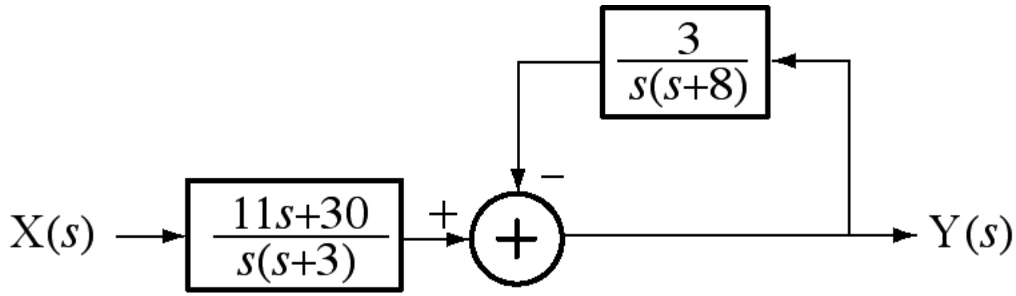
Block Diagram Reduction



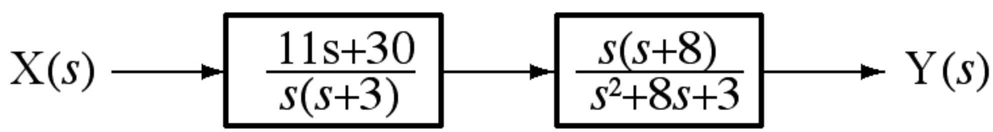
Combine Cascaded Blocks



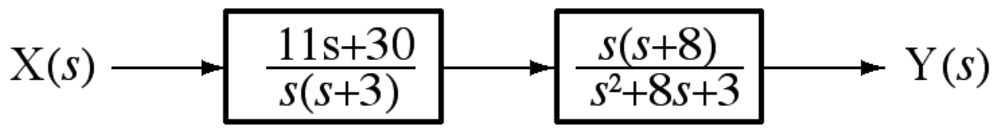
Block Diagram Reduction



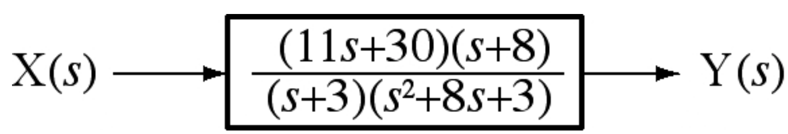
Reduce Feedback Loop



Block Diagram Reduction



Combine Cascaded Blocks



Mason's Theorem

Definitions:

Number of Paths from Input to Output - N_p

Number of Feedback Loops - N_L

Transfer Function of *i*th Path from Input to Output - $\mathrm{P}_i(s)$

Loop transfer Function of *i*th Feedback Loop - $\mathrm{T}_i(s)$

$$\Delta(s) = 1 + \sum_{i=1}^{N_L} \mathrm{T}_i(s) + \sum_{\substack{i\text{th loop and} \\ j\text{th loop not} \\ \text{sharing a signal}}} \mathrm{T}_i(s)\mathrm{T}_j(s) + \sum_{\substack{i\text{th}, j\text{th}, k\text{th} \\ \text{loops not} \\ \text{sharing a signal}}} \mathrm{T}_i(s)\mathrm{T}_j(s)\mathrm{T}_k(s) + \cdots$$

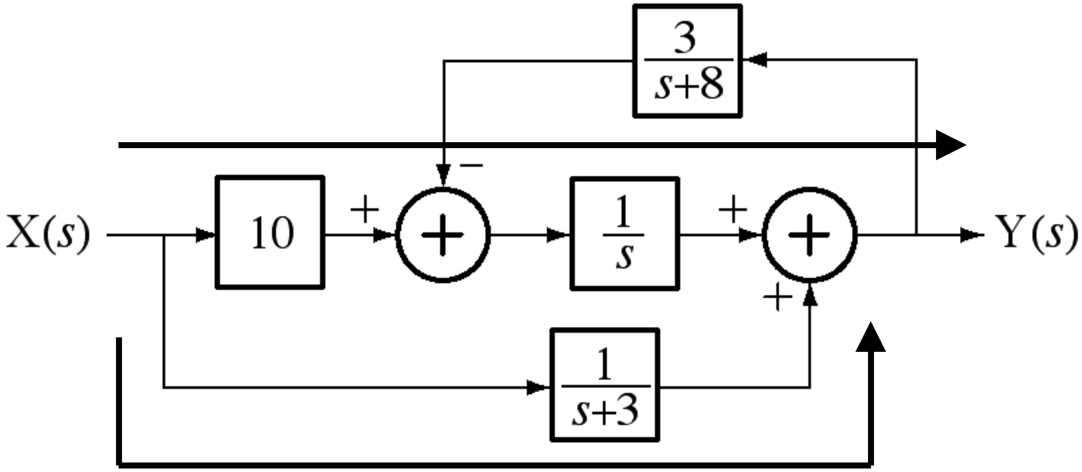
Mason's Theorem

The overall system transfer function is

$$\mathrm{H}(s) = \frac{\sum_{i=1}^{N_p} \mathrm{P}_i(s)\Delta_i(s)}{\Delta(s)}$$

where $\Delta_i(s)$ is the same as $\Delta(s)$ except that all feedback loops which share a signal with the *i*th path, $\mathrm{P}_i(s)$, are excluded.

Mason's Theorem



$$P_1(s) = \frac{10}{s}$$

$$P_2(s) = \frac{1}{s+3}$$

$$N_p = 2 \qquad N_L = 1$$

$$\Delta(s) = 1 + \frac{1}{s}\frac{3}{s+8} = 1 + \frac{3}{s(s+8)} \qquad \Delta_1(s) = \Delta_2(s) = 1$$

$$H(s) = \frac{\sum_{i=1}^{N_p} P_i(s)\Delta_i(s)}{\Delta(s)} = \frac{\frac{10}{s} + \frac{1}{s+3}}{1 + \frac{3}{s(s+8)}} = \frac{(s+8)(11s+30)}{(s+3)(s^2+8s+3)}$$

System Responses to Standard Signals

Unit Step Response

Let $\mathrm{H}(s) = \dfrac{\mathrm{N}(s)}{\mathrm{D}(s)}$ be proper in *s*. Then the Laplace transform of the unit step response is

$$\mathrm{Y}(s) = \mathrm{H}_{-1}(s) = \frac{\mathrm{N}(s)}{s\,\mathrm{D}(s)} = \frac{\mathrm{N}_1(s)}{\mathrm{D}(s)} + \frac{K}{s} \qquad K = \mathrm{H}(0)$$

If the system is stable, the inverse Laplace transform of $\dfrac{\mathrm{N}_1(s)}{\mathrm{D}(s)}$ is called the *transient response* and the *steady-state response* is $\dfrac{\mathrm{H}(0)}{s}$.

System Responses to Standard Signals

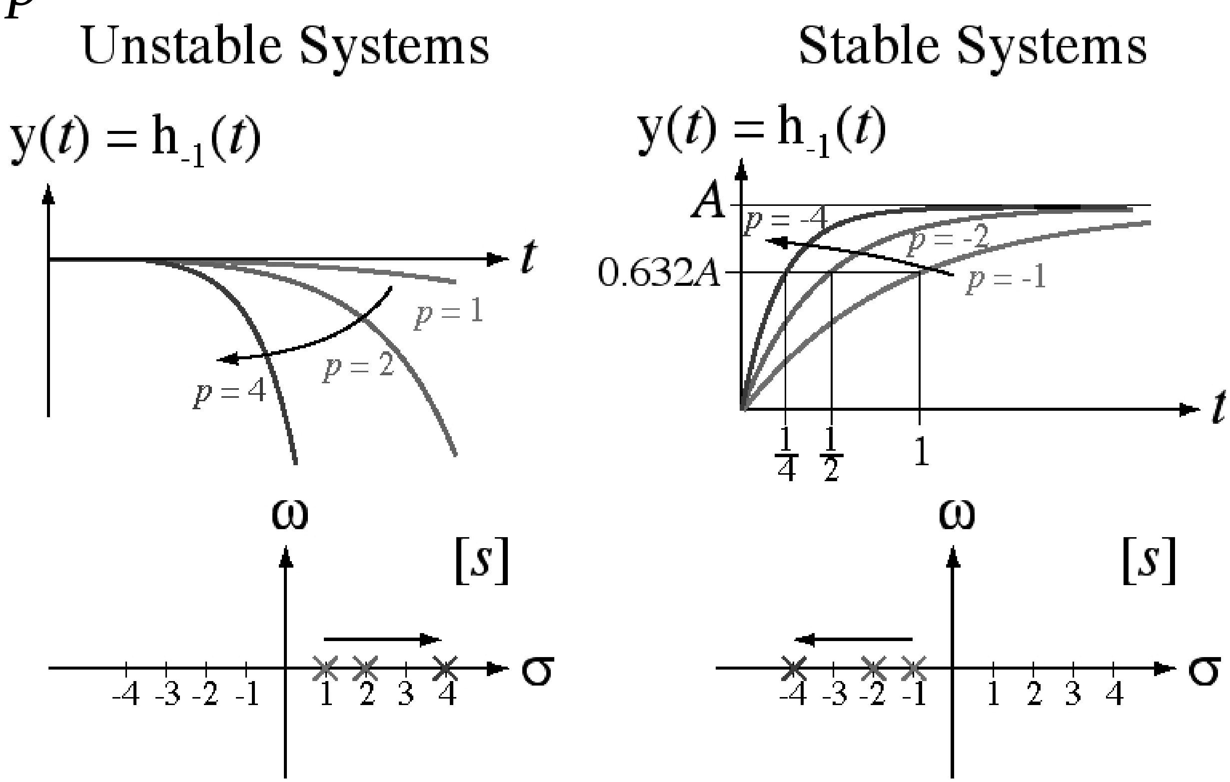
Let $\mathrm{H}(s) = \dfrac{\mathrm{N}(s)}{\mathrm{D}(s)}$ be proper in s. If the Laplace transform of the excitation is some general excitation, $\mathrm{X}(s)$, then the Laplace transform of the response is

$$\mathrm{Y}(s) = \frac{\mathrm{N}(s)}{\mathrm{D}(s)}\mathrm{X}(s) = \frac{\mathrm{N}(s)}{\mathrm{D}(s)}\frac{\mathrm{N}_x(s)}{\mathrm{D}_x(s)} = \underbrace{\frac{\mathrm{N}_1(s)}{\mathrm{D}(s)}}_{\substack{\text{same poles} \\ \text{as system}}} + \underbrace{\frac{\mathrm{N}_{x1}(s)}{\mathrm{D}_x(s)}}_{\substack{\text{same poles} \\ \text{as excitation}}}$$

System Responses to Standard Signals

Unit Step Response

Let $\mathrm{H}(s) = \dfrac{A}{1-\dfrac{s}{p}}$. Then the unit step response is $\mathrm{y}(t) = A\left(1-e^{pt}\right)\mathrm{u}(t)$



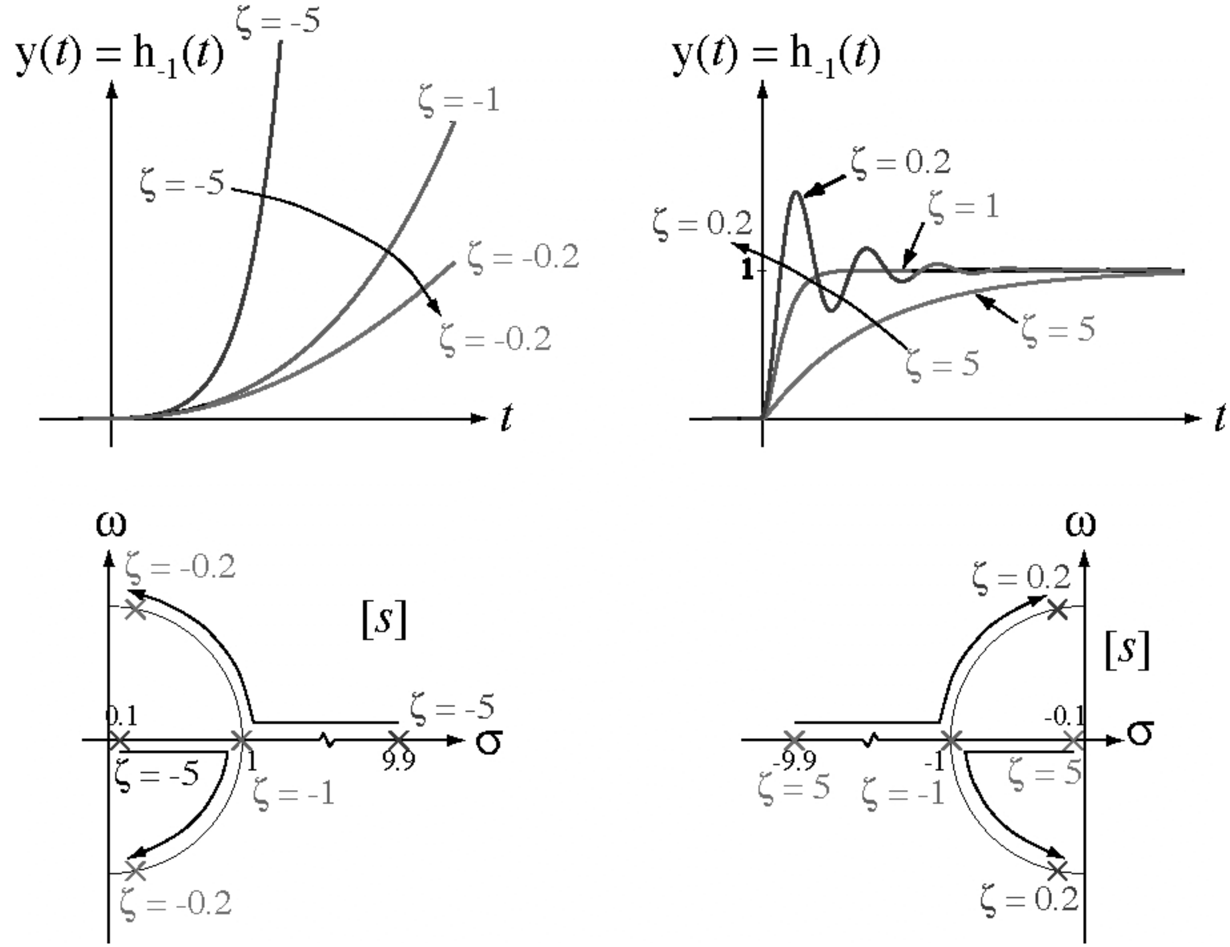
System Responses to Standard Signals

Unit Step Response

Let

$$\mathrm{H}(s) = \frac{A\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

(pp. 710-712)

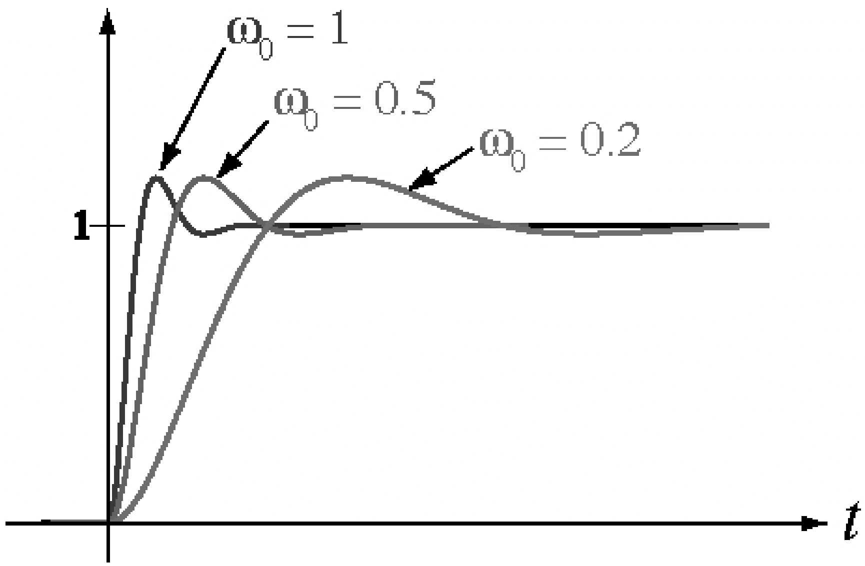


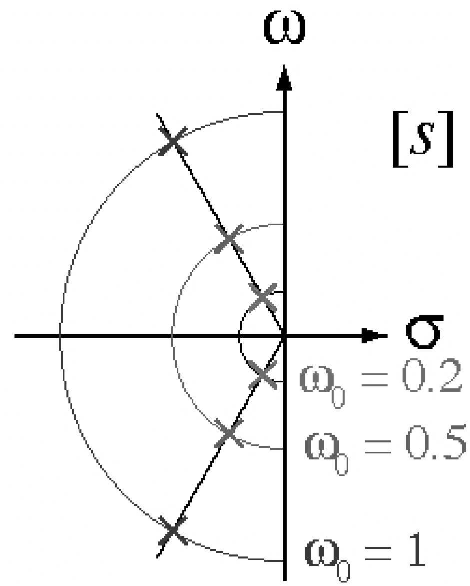
System Responses to Standard Signals

Unit Step Response

Let $\mathrm{H}(s) = \dfrac{A\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$

$\mathrm{y}(t) = \mathrm{h}_{-1}(t)$





System Responses to Standard Signals

Let $\mathrm{H}(s) = \dfrac{\mathrm{N}(s)}{\mathrm{D}(s)}$ be proper in *s*. If the excitation is a suddenly-applied, unit-amplitude cosine, the response is

$$\mathrm{Y}(s) = \frac{\mathrm{N}(s)}{\mathrm{D}(s)} \frac{s}{s^2 + \omega_0^2}$$

which can be reduced and inverse Laplace transformed into (pp. 713-714)

$$\mathrm{y}(t) = \mathcal{L}^{-1}\left(\frac{\mathrm{N}_1(s)}{\mathrm{D}(s)}\right) + \left|\mathrm{H}(j\omega_0)\right| \cos\left(\omega_0 t + \angle \mathrm{H}(j\omega_0)\right) \mathrm{u}(t)$$

If the system is stable, the steady-state response is a sinusoid of same frequency as the excitation but, generally, a different magnitude and phase.

Pole-Zero Diagrams and Frequency Response

If the transfer function of a system is H(s), the frequency response is H($j\omega$). The most common type of transfer function is of the form,

$$\mathrm{H}(s) = A\frac{(s - z_1)(s - z_2)\cdots(s - z_N)}{(s - p_1)(s - p_2)\cdots(s - p_D)}$$

Therefore H($j\omega$) is

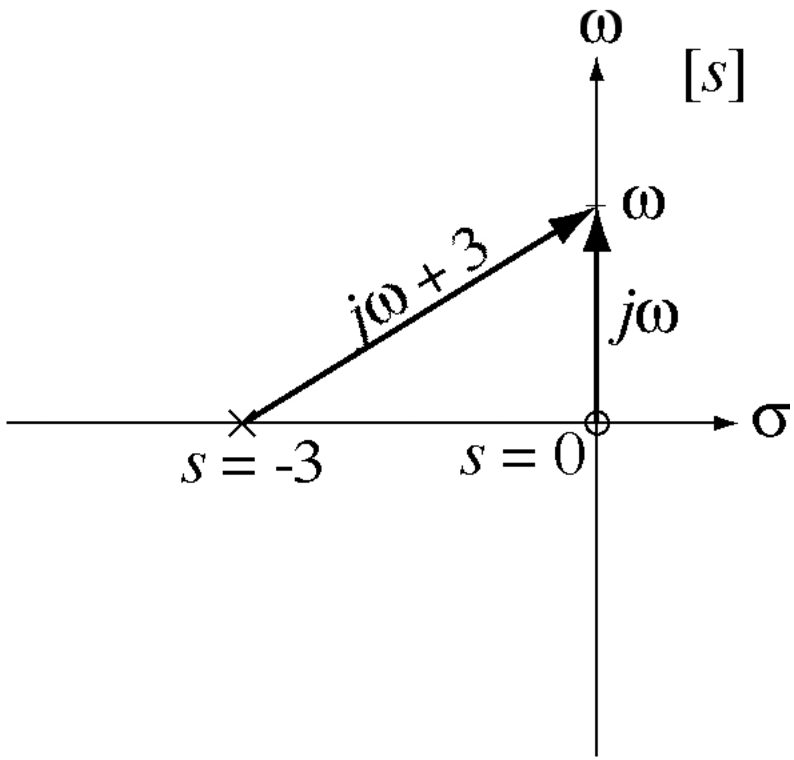
$$\mathrm{H}(j\omega) = A\frac{(j\omega - z_1)(j\omega - z_2)\cdots(j\omega - z_N)}{(j\omega - p_1)(j\omega - p_2)\cdots(j\omega - p_D)}$$

Pole-Zero Diagrams and Frequency Response

Let $\mathrm{H}(s) = \frac{3s}{s+3}$

$$\mathrm{H}(j\omega) = 3\frac{j\omega}{j\omega + 3}$$

The numerator, $j\omega$, and the denominator, $j\omega + 3$, can be conceived as vectors in the *s* plane.

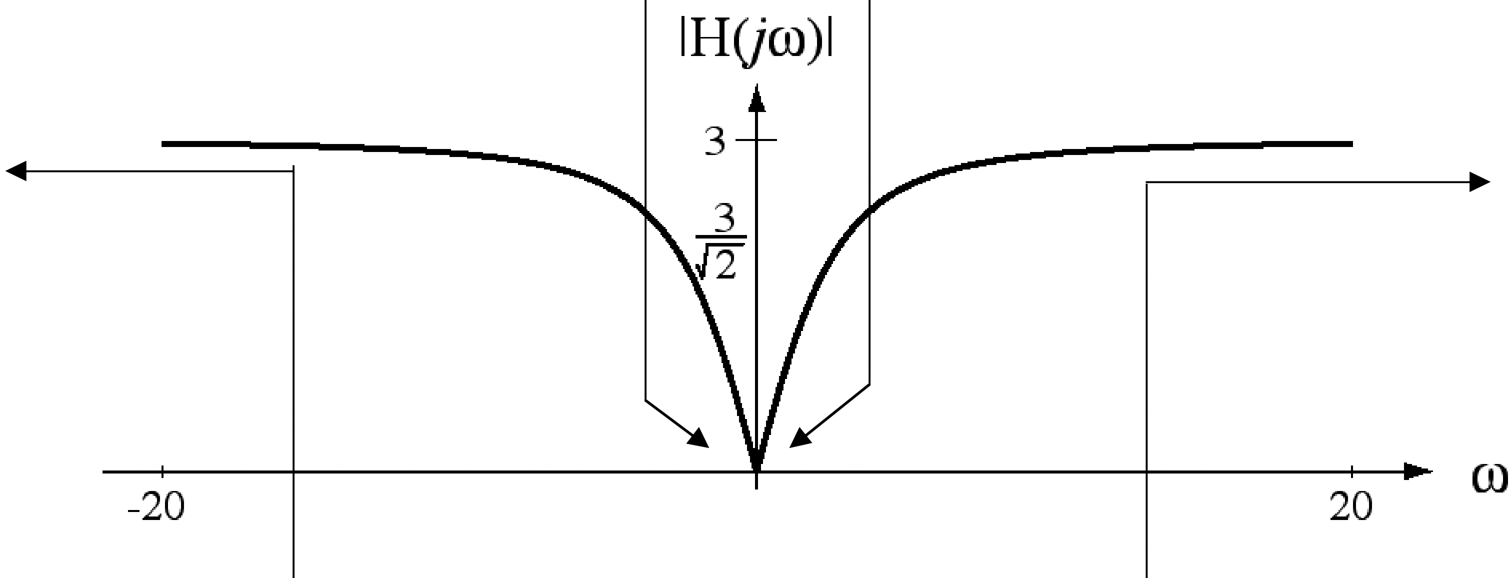


$$|\mathrm{H}(j\omega)| = 3\frac{|j\omega|}{|j\omega + 3|} \qquad \angle\mathrm{H}(j\omega) = \underbrace{\angle 3}_{=0} + \angle j\omega - \angle(j\omega + 3)$$

Pole-Zero Diagrams and Frequency Response

$$\lim_{\omega \to 0^-} |H(j\omega)| = \lim_{\omega \to 0^-} 3\frac{|j\omega|}{|j\omega + 3|} = 0$$

$$\lim_{\omega \to 0^+} |H(j\omega)| = \lim_{\omega \to 0^+} 3\frac{|j\omega|}{|j\omega + 3|} = 0$$



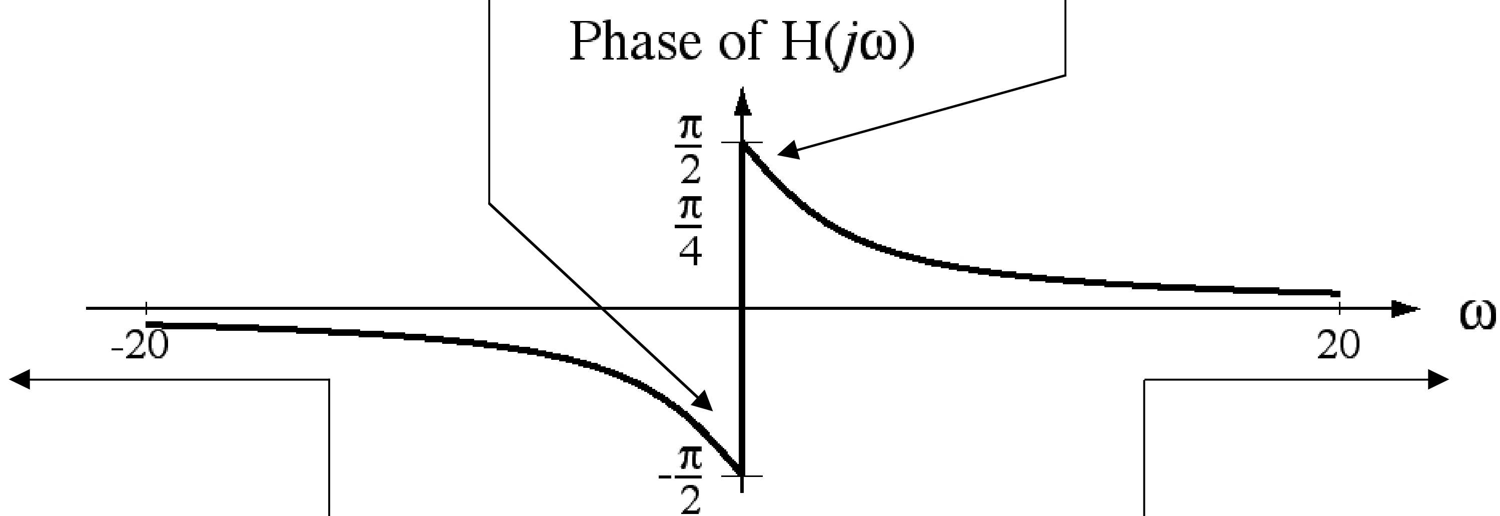
$$\lim_{\omega \to -\infty} |H(j\omega)| = \lim_{\omega \to -\infty} 3\frac{|j\omega|}{|j\omega + 3|} = 3$$

$$\lim_{\omega \to +\infty} |H(j\omega)| = \lim_{\omega \to +\infty} 3\frac{|j\omega|}{|j\omega + 3|} = 3$$

Pole-Zero Diagrams and Frequency Response

$$\lim_{\omega\to 0^-} \angle \mathrm{H}(j\omega) = -\frac{\pi}{2} - 0 = -\frac{\pi}{2}$$

$$\lim_{\omega\to 0^+} \angle \mathrm{H}(j\omega) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$



$$\lim_{\omega\to -\infty} \angle \mathrm{H}(j\omega) = -\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = 0$$

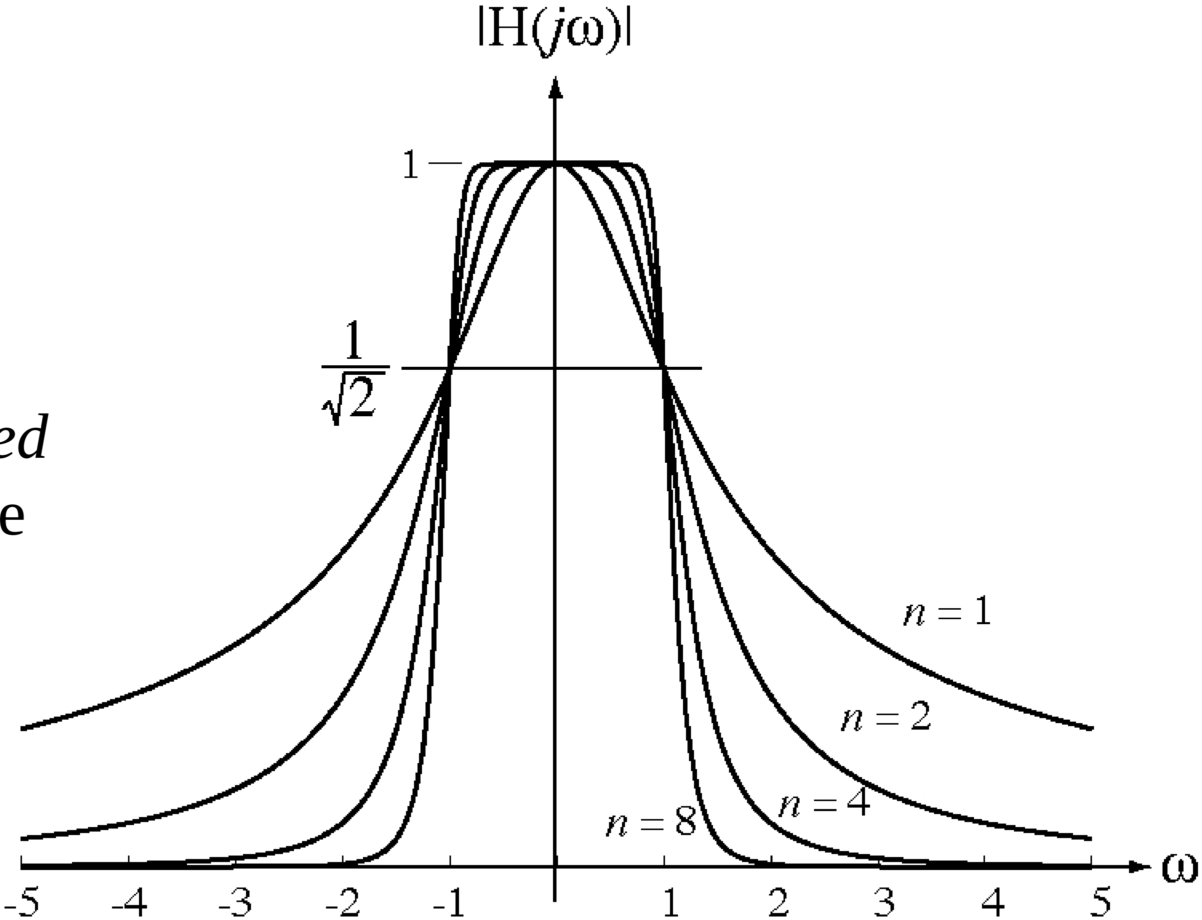
$$\lim_{\omega\to +\infty} \angle \mathrm{H}(j\omega) = \frac{\pi}{2} - \frac{\pi}{2} = 0$$

Butterworth Filters

The squared magnitude of the transfer function of an *n*th order, unity-gain, lowpass Butterworth filter with a corner frequency of 1 radian/s is

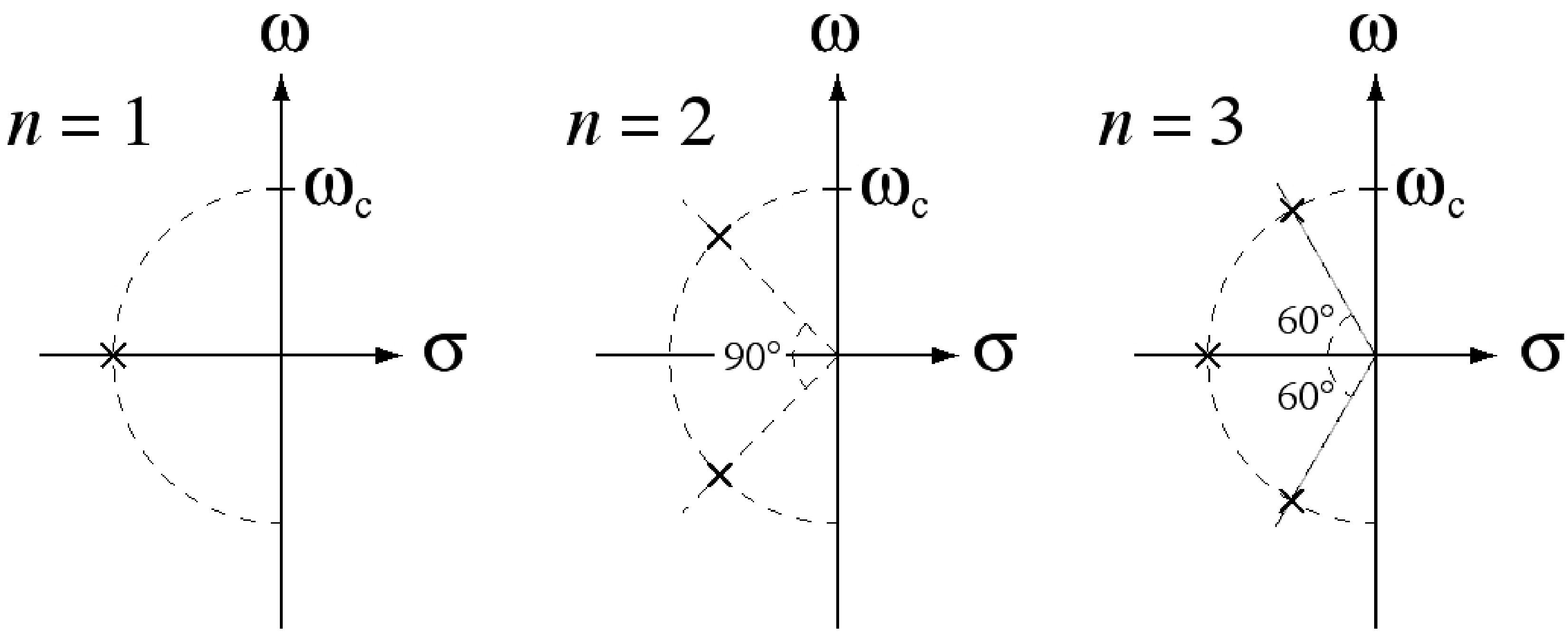
$$|H(j\omega)|^2 = \frac{1}{1+\omega^{2n}}$$

This is called a *normalized* Butterworth filter because its gain is normalized to one and its corner frequency
is normalized to
1 radian/s.



Butterworth Filters

A Butterworth filter transfer function has no finite zeros and the poles all lie on a semicircle in the left-half plane whose radius is the corner frequency in radians/s and the angle between the pole locations is always π/n radians.



Butterworth Filters

Frequency Transformations

A normalized lowpass Butterworth filter can be transformed into an unnormalized highpass, bandpass or bandstop Butterworth filter through the following transformations (pp. 721-725).

Lowpass to Highpass $\quad s \rightarrow \frac{\omega_c}{s}$

Lowpass to Bandpass $\quad s \rightarrow \frac{s^2 + \omega_L \omega_H}{s(\omega_H - \omega_L)}$

Lowpass to Bandstop $\quad s \rightarrow \frac{s(\omega_H - \omega_L)}{s^2 + \omega_L \omega_H}$

Standard Realizations of Systems

There are multiple ways of drawing a system block diagram corresponding to a given transfer function of the form,

$$\mathrm{H}(s) = \frac{\mathrm{Y}(s)}{\mathrm{X}(s)} = \frac{\sum_{k=0}^{N} b_k s^k}{\sum_{k=0}^{N} a_k s^k} = \frac{b_N s^N + b_{N-1} s^{N-1} + \cdots + b_1 s + b_0}{s^N + a_{N-1} s^{N-1} + \cdots + a_1 s + a_0} \quad , \quad a_N = 1$$

Standard Realizations of Systems

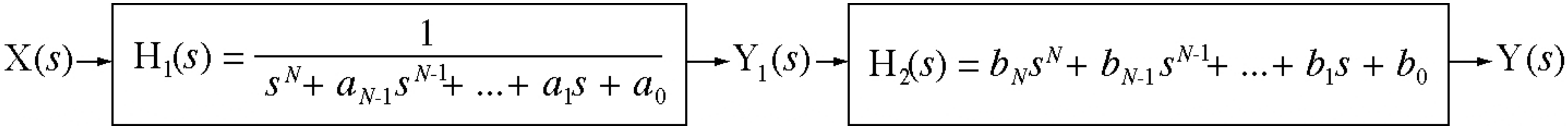
Canonical Form

The transfer function can be conceived as the product of two transfer functions,

$$\mathrm{H}_1(s) = \frac{\mathrm{Y}_1(s)}{\mathrm{X}(s)} = \frac{1}{s^N + a_{N-1}s^{N-1} + \cdots + a_1 s + a_0}$$

and

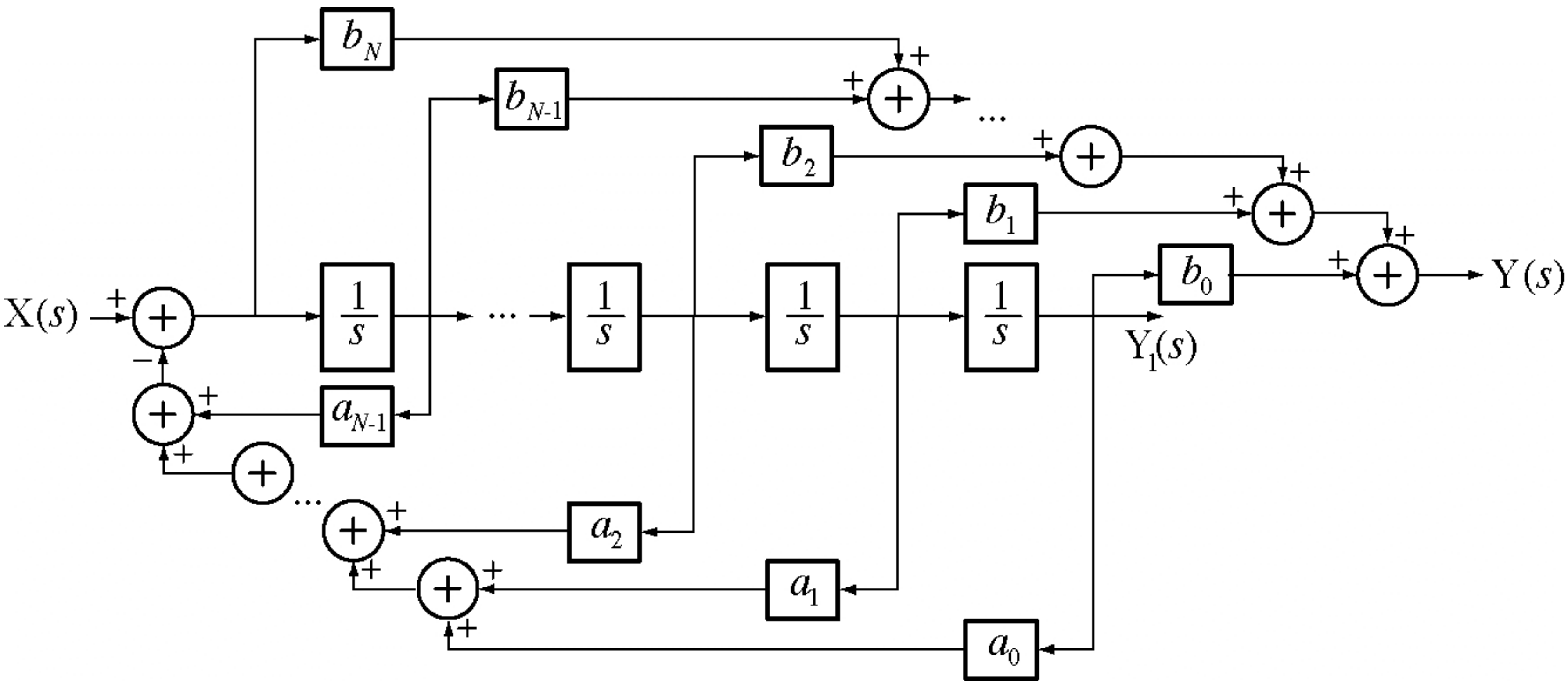
$$\mathrm{H}_2(s) = \frac{\mathrm{Y}(s)}{\mathrm{Y}_1(s)} = b_N s^N + b_{N-1}s^{N-1} + \cdots + b_1 s + b_0$$



Standard Realizations of Systems

Canonical Form

The system can then be realized in this form,



Standard Realizations of Systems

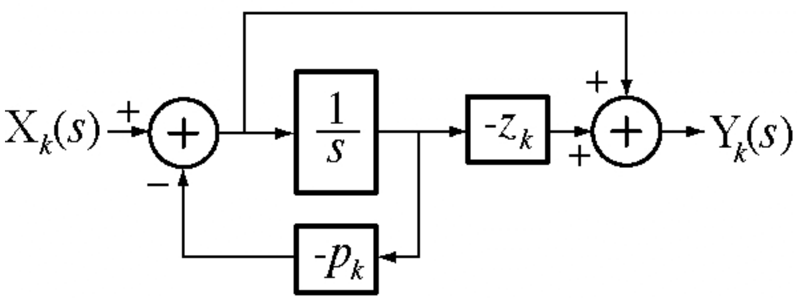
Cascade Form

The transfer function can be factored into the form,

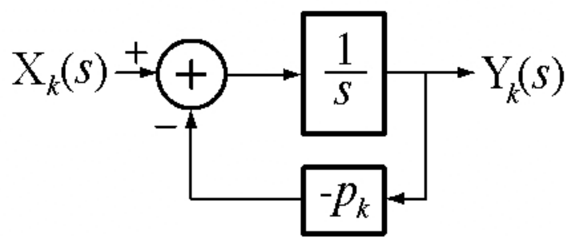
$$\mathrm{H}(s) = A\frac{s-z_1}{s-p_1}\frac{s-z_2}{s-p_2}\cdots\frac{s-z_N}{s-p_N}\frac{1}{s-p_{N+1}}\frac{1}{s-p_{N+2}}\cdots\frac{1}{s-p_D}$$

and each factor can be realized in a small canonical-form subsystem of either of the two forms,

$$\mathrm{H}_k(s) = \frac{s - z_k}{s - p_k}$$



$$\mathrm{H}_k(s) = \frac{1}{s - p_k}$$

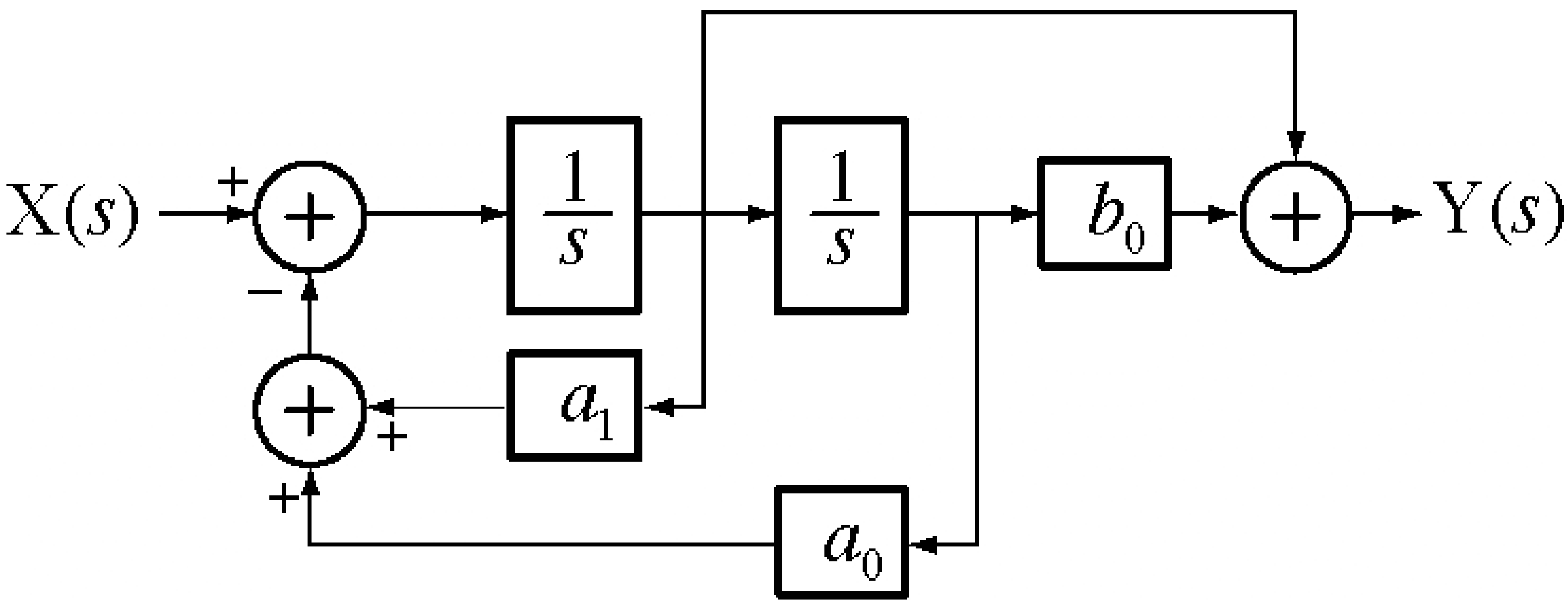


and these subsystems can then be cascade connected.

Standard Realizations of Systems

Cascade Form

A problem that arises in the cascade form is that some poles or zeros may be complex. In that case, a complex conjugate pair can be combined into one second-order subsystem of the form,



Standard Realizations of Systems

Parallel Form

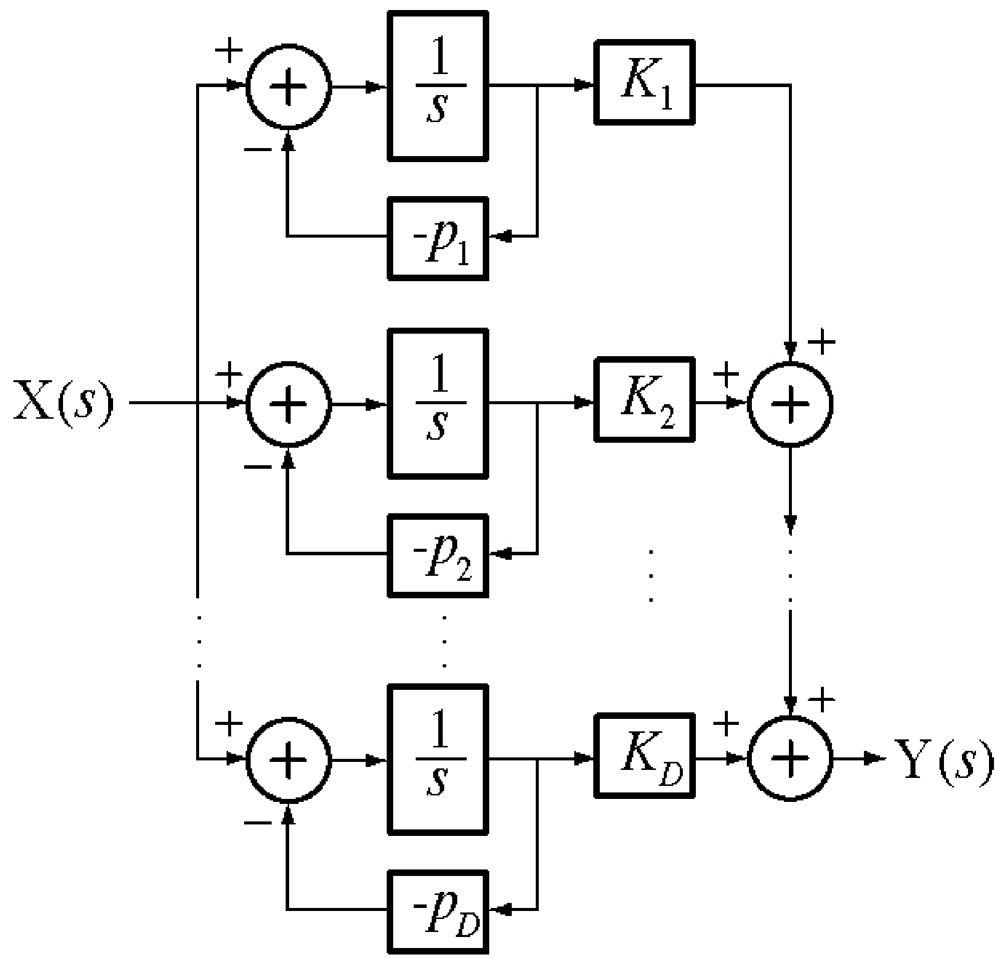
The transfer function can be expanded in partial fractions of the form,

$$\mathrm{H}(s) = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \cdots + \frac{K_D}{s - p_D}$$

Each of these terms describes a subsystem. When all the subsystems are connected in parallel the overall system is realized.

Standard Realizations of Systems

Parallel Form



State-Space Analysis

- In larger systems it is important to keep the analysis methods systematic to avoid errors
- One popular method for doing this is through the use of *state variables*
- State variables are signals in a system which, together with the excitations, completely characterize the state of the system
- As the system changes dynamically the state variables change value and the system moves on a *trajectory* through *state space*

State-Space Analysis

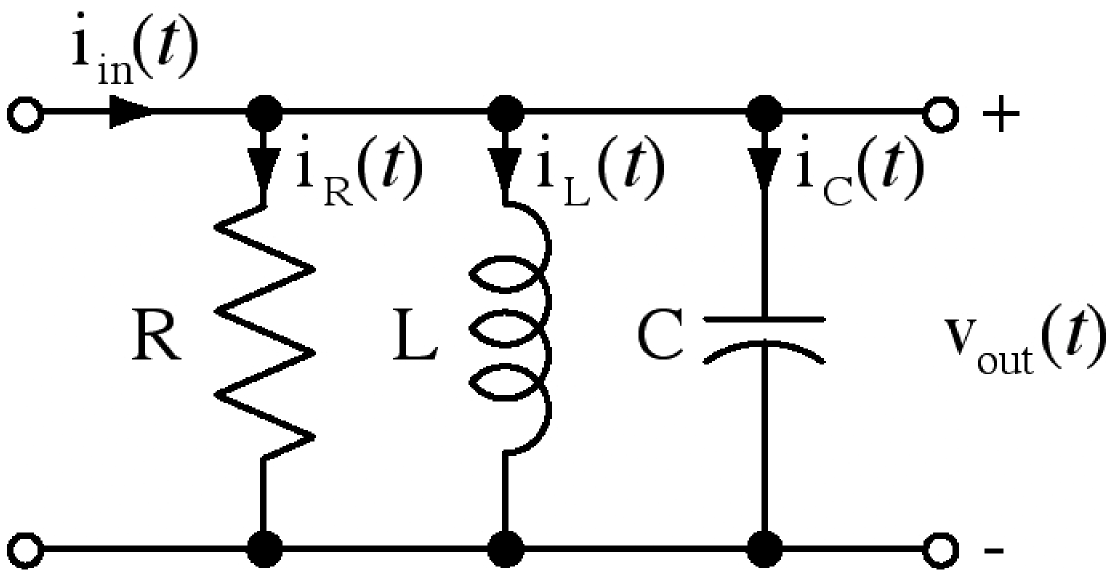
- State space is an *N*-dimensional space where *N* is the *order* of the system
- The order of a system is the number of state variables needed to characterize it
- State variables are not unique, there can be multiple correct sets

State-Space Analysis

- There are several advantages to state-space analysis
  - Reduction of the probability of analysis errors
  - Complete description of the system signals
  - Insight into system dynamics
  - Can be formulated using matrix methods and the system state can be expressed in two matrix equations
  - Combined with transform methods it is very powerful

State-Space Analysis

To illustrate state-space methods, let the system be this circuit



Let the state variables be the capacitor voltage and the inductor current and let the output signals be $v_{out}(t)$ and $i_R(t)$.

State-Space Analysis

Two differential equations in the two state variables, called the state equations, characterize the circuit,

$$\mathrm{i}'_L(t) = \frac{1}{L}\mathrm{v}_C(t) \qquad \mathrm{v}'_C(t) = -\frac{1}{C}\mathrm{i}_L(t) - \frac{G}{C}\mathrm{v}_C(t) + \frac{1}{C}\mathrm{i}_{in}(t)$$

Two more equations called the output equations define the responses in terms of the state variables,

$$\mathrm{v}_{out}(t) = \mathrm{v}_C(t) \qquad \mathrm{i}_R(t) = G\,\mathrm{v}_C(t)$$

State-Space Analysis

The state equations can be written in matrix form as

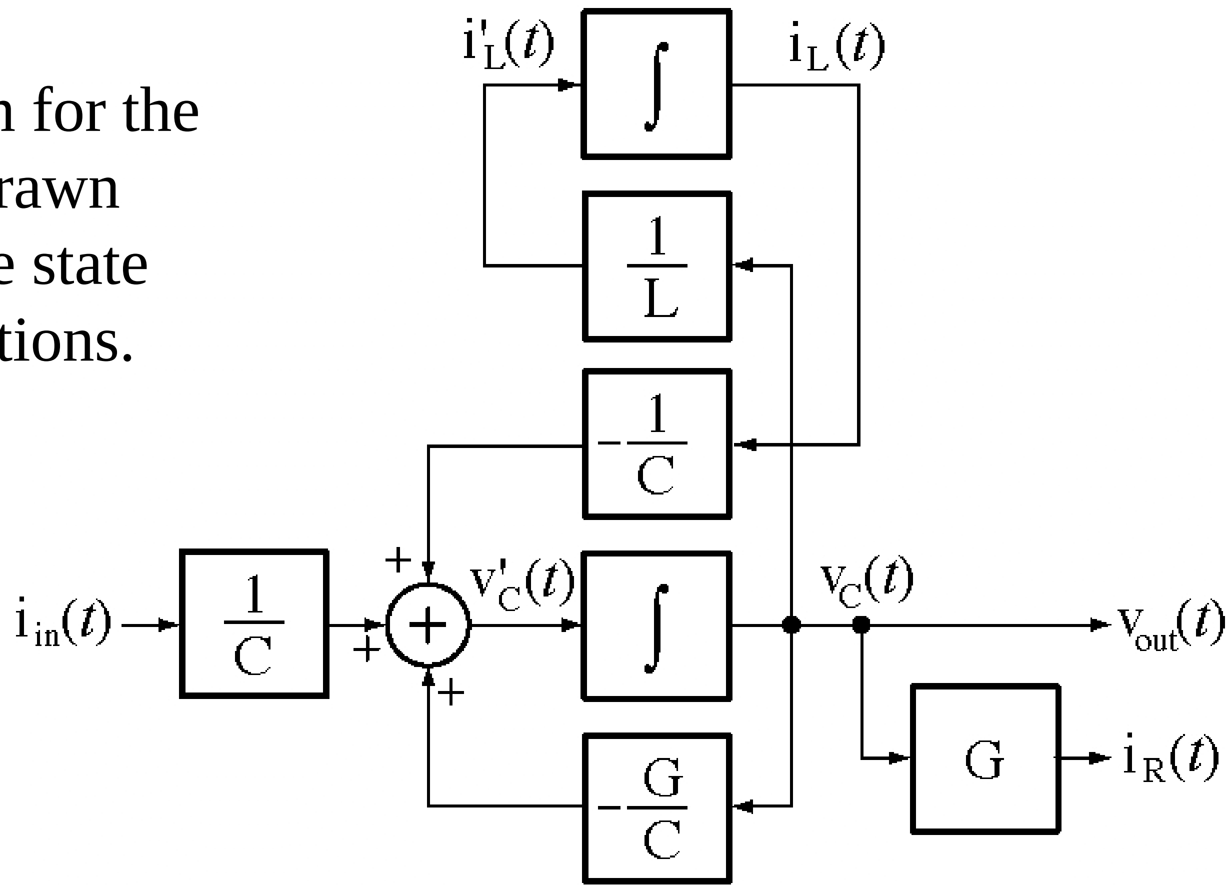
$$\begin{bmatrix} \mathrm{i}'_L(t) \\ \mathrm{v}'_C(t) \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{G}{C} \end{bmatrix} \begin{bmatrix} \mathrm{i}_L(t) \\ \mathrm{v}_C(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{C} \end{bmatrix} \left[\mathrm{i}_{in}(t) \right]$$

and the output equations can be written in matrix form as

$$\begin{bmatrix} \mathrm{v}_{out}(t) \\ \mathrm{i}_R(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & G \end{bmatrix} \begin{bmatrix} \mathrm{i}_L(t) \\ \mathrm{v}_C(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \left[\mathrm{i}_{in}(t) \right]$$

State-Space Analysis

A block diagram for the system can be drawn directly from the state and output equations.



State-Space Analysis

The state and output equations can be written compactly as

$$\mathbf{q}'(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}\mathbf{x}(t)$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{q}(t) + \mathbf{D}\mathbf{x}(t)$$

where

$$\mathbf{q}(t) = \begin{bmatrix} \mathrm{i}_L(t) \\ \mathrm{v}_C(t) \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{G}{C} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{C} \end{bmatrix} \quad \mathbf{x}(t) = \left[\mathrm{i}_{in}(t)\right]$$

$$\mathbf{y}(t) = \begin{bmatrix} \mathrm{v}_{out}(t) \\ \mathrm{i}_R(t) \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \\ 0 & G \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

State-Space Analysis

The solution of the state and output equations can be found using the Laplace transform.

$$s\mathbf{Q}(s) - \mathbf{q}(0^+) = \mathbf{AQ}(s) + \mathbf{BX}(s)$$

or

$$[s\mathbf{I} - \mathbf{A}]\mathbf{Q}(s) = \mathbf{BX}(s) + \mathbf{q}(0^+)$$

Multiplying both sides by $[s\mathbf{I} - \mathbf{A}]^{-1}$

$$\mathbf{Q}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\left[\mathbf{BX}(s) + \mathbf{q}(0^+)\right]$$

The matrix, $[s\mathbf{I} - \mathbf{A}]^{-1}$, is conventionally designated by the symbol, $\Phi(s)$. Then

$$\mathbf{Q}(s) = \Phi(s)\left[\mathbf{BX}(s) + \mathbf{q}(0^+)\right] = \underbrace{\Phi(s)\mathbf{BX}(s)}_{\substack{\text{zero-state} \\ \text{response}}} + \underbrace{\Phi(s)\mathbf{q}(0^+)}_{\substack{\text{zero-input} \\ \text{response}}}$$

State-Space Analysis

The time-domain solution is then

$$\mathbf{q}(t) = \underbrace{\phi(t) * \mathbf{B}\mathbf{x}(t)}_{\substack{\text{zero-state} \\ \text{response}}} + \underbrace{\phi(t)\mathbf{q}(0^+)}_{\substack{\text{zero-input} \\ \text{response}}}$$

where $\phi(t) \xleftrightarrow{\mathcal{L}} \Phi(s)$ and $\phi(t)$ is called the *state transition matrix*.

State-Space Analysis

To make the example concrete, let the excitation and initial conditions be

$$\mathrm{i}(t) = A\,\mathrm{u}(t) \qquad \mathbf{q}(0^+) = \begin{bmatrix} \mathrm{i}_L(0^+) \\ \mathrm{v}_C(0^+) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and let $R = \frac{1}{3}$, $C = 1$, $L = 1$

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & -\frac{1}{L} \\ \frac{1}{C} & s + \frac{G}{C} \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s + \frac{G}{C} & \frac{1}{L} \\ -\frac{1}{C} & s \end{bmatrix}}{s^2 + \frac{G}{C}s + \frac{1}{LC}}$$

State-Space Analysis

Solving for the states in the Laplace domain,

$$\mathbf{Q}(s)=\begin{bmatrix} \dfrac{1}{sLC\left(s^2+\frac{G}{C}s+\frac{1}{LC}\right)}+\dfrac{1}{L\left(s^2+\frac{G}{C}s+\frac{1}{LC}\right)} \\ \dfrac{1}{C\left(s^2+\frac{G}{C}s+\frac{1}{LC}\right)}+\dfrac{s}{s^2+\frac{G}{C}s+\frac{1}{LC}} \end{bmatrix}$$

Substituting in numerical component values,

$$\mathbf{Q}(s)=\begin{bmatrix} \dfrac{1}{s\left(s^2+3s+1\right)}+\dfrac{1}{s^2+3s+1} \\ \dfrac{1}{s^2+3s+1}+\dfrac{s}{s^2+3s+1} \end{bmatrix}$$

State-Space Analysis

Inverse Laplace transforming, the state variables are

$$\mathbf{q}(t) = \begin{bmatrix} 1 - 0.277e^{-2.62t} - 0.723e^{-0.382t} \\ 0.723e^{-0.382t} + 0.277e^{-2.62t} \end{bmatrix} \mathrm{u}(t)$$

and the response is

$$\mathbf{y}(t) = \begin{bmatrix} 0 & 1 \\ 0 & G \end{bmatrix} \mathbf{q} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 - 0.277e^{-2.62t} - 0.723e^{-0.382t} \\ 0.723e^{-0.382t} + 0.277e^{-2.62t} \end{bmatrix} \mathrm{u}(t)$$

or

$$\mathbf{y}(t) = \begin{bmatrix} 0.723e^{-0.382t} + 0.277e^{-2.62t} \\ 2.169e^{-0.382t} + 0.831e^{-2.62t} \end{bmatrix} \mathrm{u}(t)$$

State-Space Analysis

In a system in which the initial conditions are zero (the zero-input response is zero), the matrix transfer function can be found from the state and output equations.

$$s\mathbf{Q}(s) - \underbrace{\mathbf{q}(0^+)}_{=0} = \mathbf{A}\mathbf{Q}(s) + \mathbf{B}\mathbf{X}(s)$$

or

$$\mathbf{Q}(s) = [s\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{X}(s) = \Phi(s)\mathbf{B}\mathbf{X}(s)$$

The response is

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{Q}(s) + \mathbf{D}\mathbf{X}(s) = \mathbf{C}\Phi(s)\mathbf{B}\mathbf{X}(s) + \mathbf{D}\mathbf{X}(s) = [\mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D}]\mathbf{X}(s)$$

and the matrix transfer function is

$$\mathbf{H}(s) = \mathbf{C}\Phi(s)\mathbf{B} + \mathbf{D}$$

State-Space Analysis

Any set of state variables can be transformed into another valid set through a linear transformation. Let $\mathbf{q}_1(t)$ be the initial set and let $\mathbf{q}_2(t)$ be the new set, related to $\mathbf{q}_1(t)$ by

$$\mathbf{q}_2(t) = \mathbf{T}\mathbf{q}_1(t)$$

Then

$$\mathbf{q}_2'(t) = \mathbf{T}\mathbf{q}_1'(t) = \mathbf{T}(\mathbf{A}_1\mathbf{q}_1(t) + \mathbf{B}_1\mathbf{x}(t)) = \mathbf{T}\mathbf{A}_1\mathbf{q}_1(t) + \mathbf{T}\mathbf{B}_1\mathbf{x}(t)$$

and using $\mathbf{q}_1(t) = \mathbf{T}^{-1}\mathbf{q}_2(t)$,

$$\mathbf{q}_2'(t) = \mathbf{T}\mathbf{A}_1\mathbf{T}^{-1}\mathbf{q}_2(t) + \mathbf{T}\mathbf{B}_1\mathbf{x}(t) = \mathbf{A}_2\mathbf{q}_2(t) + \mathbf{B}_2\mathbf{x}(t)$$

where

$$\mathbf{A}_2 = \mathbf{T}\mathbf{A}_1\mathbf{T}^{-1} \qquad \mathbf{B}_2 = \mathbf{T}\mathbf{B}_1$$

State-Space Analysis

By a similar argument,

$$\mathbf{y}(t) = \mathbf{C}_2\mathbf{q}_2(t) + \mathbf{D}_2\mathbf{x}(t)$$

where

$$\mathbf{C}_2 = \mathbf{C}_1\mathbf{T}^{-1} \qquad \mathbf{D}_2 = \mathbf{D}_1$$

Transformation to a new set of state variables does not change the eigenvalues of the system.