The z Transform

Relation to the Laplace Transform

- The *z* transform is to DT signals and systems what the Laplace transform is to CT signals and systems

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Definition

The z transform can be viewed as a generalization of the DTFT or as natural result of exciting a discrete-time LTI system with its eigenfunction. The DTFT is defined by

$$\mathrm{x}[n] = \frac{1}{2\pi}\int_{2\pi} \mathrm{X}(j\Omega)e^{j\Omega n}\,d\Omega \xleftrightarrow{\mathcal{F}} \mathrm{X}(j\Omega) = \sum_{n=-\infty}^{\infty} \mathrm{x}[n]e^{-j\Omega n}$$

If a strict analogy with the Laplace transform were made Ω would replace ω, Σ would replace σ, S would replace s, a summation would replace the integral and the z transform would be defined by

$$\mathrm{X}(S) = \sum_{n=-\infty}^{\infty} \mathrm{x}[n]e^{-Sn} = \sum_{n=-\infty}^{\infty} \mathrm{x}[n]e^{-(\Sigma+j\Omega)n} = \sum_{n=-\infty}^{\infty} \left(\mathrm{x}[n]e^{-n\Sigma}\right)e^{-j\Omega n}$$

Definition

$$\mathrm{X}(S) = \sum_{n=-\infty}^{\infty} \mathrm{x}[n] e^{-Sn} = \sum_{n=-\infty}^{\infty} \mathrm{x}[n] e^{-(\Sigma + j\Omega)n} = \sum_{n=-\infty}^{\infty} \left(\mathrm{x}[n] e^{-n\Sigma} \right) e^{-j\Omega n}$$

Viewed this way the factor, $e^{-n\Sigma}$, would be a "convergence" factor in that same way that the factor, $e^{-\sigma t}$, was for the Laplace transform.

The other approach to defining the z transform is to excite a DT system with its eigenfunction, Az^n. The response would be

$$\mathrm{y}[n] = \mathrm{x}[n] * \mathrm{h}[n] = Az^n * \mathrm{h}[n] = \sum_{m=-\infty}^{\infty} \mathrm{h}[m] A z^{(n-m)} = \underbrace{Az^n}_{\mathrm{x}[n]} \underbrace{\sum_{m=-\infty}^{\infty} \mathrm{h}[m] z^{-m}}_{z \text{ transform of } \mathrm{h}[n]}$$

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Definition

The universally accepted definition of the *z* transform of a DT function, x, is

$$\mathrm{X}(z) = \sum_{n=-\infty}^{\infty} \mathrm{x}[n] z^{-n}$$

and x and X form a "z-transform pair",

$$\mathrm{x}[n] \xleftrightarrow{\mathcal{Z}} \mathrm{X}(z)$$

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Convergence

The DTFT's of some common functions do not, in the strict sense, converge. The DTFT of the unit sequence would be

$$\mathrm{X}(j\Omega) = \sum_{n=-\infty}^{\infty} \mathrm{u}[n] e^{-j\Omega n} = \sum_{n=0}^{\infty} e^{-j\Omega n}$$

which does not converge. But the *z* transform of the unit sequence does exist. It is

$$\mathrm{X}(z) = \sum_{n=-\infty}^{\infty} \mathrm{u}[n] z^{-n} = \sum_{n=0}^{\infty} z^{-n}$$

and the *z* transform exists for values of *z* whose magnitudes are greater than one. This defines a *region of convergence (ROC)* for the *z* transform of the unit sequence, the exterior of the unit circle in the *z* plane.

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Convergence

The series, $\sum_{n=0}^{\infty} z^{-n}$, is a geometric series. The general formula for the summation of a finite geometric series is

$$\sum_{n=0}^{N-1} r^n = \begin{cases} 1 & , \quad r = 1 \\ \dfrac{1-r^N}{1-r} & , \quad r \neq 1 \end{cases}$$

This formula also applies to the infinite series above *if* the magnitude of z is greater than one. In that case the z transform of the unit sequence is

$$\mathrm{X}(z) = \frac{z}{z-1} = \frac{1}{1-z^{-1}} \quad , \quad |z| > 1$$

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Transfer Functions

If x is the excitation, h is the impulse response and y is the system response of a discrete-time LTI system, then

$$\mathrm{Y}(z) = \mathrm{X}(z)\mathrm{H}(z)$$

and H is called the *transfer function* of the system. This is directly analogous to previous transfer functions,

$$\mathrm{Y}(j\omega) = \mathrm{X}(j\omega)\mathrm{H}(j\omega)$$

$$\mathrm{Y}(j\Omega) = \mathrm{X}(j\Omega)\mathrm{H}(j\Omega)$$

$$\mathrm{Y}(s) = \mathrm{X}(s)\mathrm{H}(s)$$

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Region of Convergence

Taking a path analogous to that used the development of the Laplace transform, the z transform of the causal DT signal

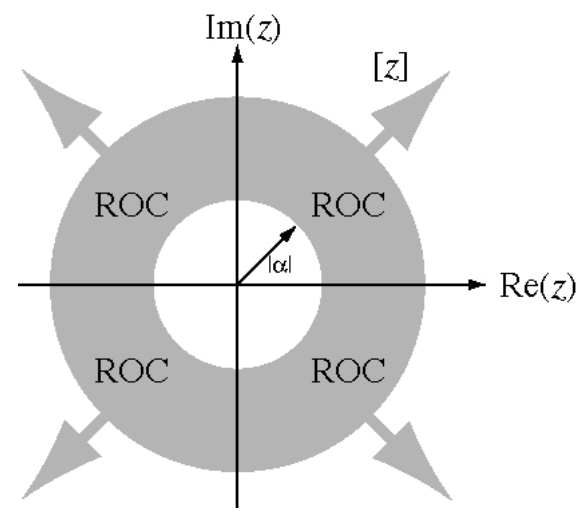
$$A\alpha^n \mathrm{u}[n] \, , \, |\alpha| > 0$$

is

$$\mathrm{X}(z) = A\sum_{n=-\infty}^{\infty} \alpha^n \mathrm{u}[n] z^{-n} = A\sum_{n=0}^{\infty} \alpha^n z^{-n} = A\sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n$$

and the series converges if $|z| > |\alpha|$. This defines the region of convergence as the exterior of a circle in the z plane centered at the origin, of radius, $|\alpha|$. The z transform is

$$\mathrm{X}(z) = A\frac{z}{z-\alpha} \, , \quad |z| > |\alpha|$$



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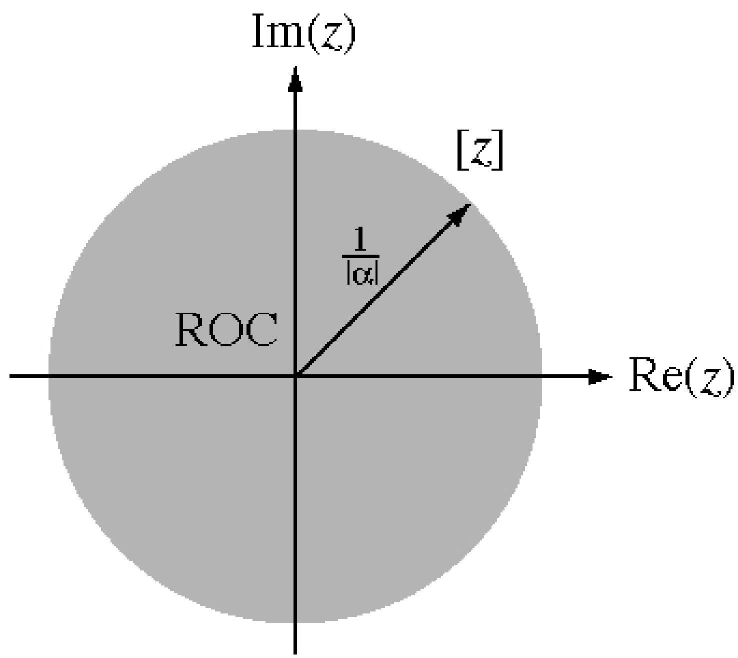
Region of Convergence

By similar reasoning, the *z* transform and region of convergence of the anti-causal signal, $A\alpha^{-n}\,\mathrm{u}[-n]$, $|\alpha| > 0$ are

$$X(z) = \frac{A}{1-\alpha z} = \frac{Az^{-1}}{z^{-1}-\alpha} \quad , \quad |z| < \frac{1}{|\alpha|}$$



The Unilateral *z* Transform

Just as it was convenient to define a unilateral Laplace transform it is convenient for analogous reasons to define a unilateral *z* transform

$$\mathrm{X}(z) = \sum_{n=0}^{\infty} \mathrm{x}[n] z^{-n}$$

which will simply be referred to as *the z* transform from this point on.

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Properties

If two causal DT signals form these transform pairs,

$$\mathrm{g}[n] \xleftrightarrow{\mathcal{Z}} \mathrm{G}(z) \quad \text{and} \quad \mathrm{h}[n] \xleftrightarrow{\mathcal{Z}} \mathrm{H}(z)$$

then the following properties hold for the z transform.

Linearity

$$\alpha\,\mathrm{g}[n] + \beta\,\mathrm{h}[n] \xleftrightarrow{\mathcal{Z}} \alpha\,\mathrm{G}(z) + \beta\,\mathrm{H}(z)$$

Time Shifting

Delay: $\mathrm{g}[n - n_0] \xleftrightarrow{\mathcal{Z}} z^{-n_0}\,\mathrm{G}(z)\ ,\ n_0 \geq 0$

Advance: $\mathrm{g}[n + n_0] \xleftrightarrow{\mathcal{Z}} z^{n_0}\left(\mathrm{G}(z) - \sum_{m=0}^{n_0-1} \mathrm{g}[m]z^{-m}\right)\ ,\ \ n_0 > 0$

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Properties

Change of Scale

$$\alpha^n \mathrm{g}[n] \xleftrightarrow{z} \mathrm{G}\left(\frac{z}{\alpha}\right)$$

Initial Value Theorem

$$\mathrm{g}[0] = \lim_{z \to \infty} \mathrm{G}(z)$$

z-Domain Differentiation

$$-n\,\mathrm{g}[n] \xleftrightarrow{z} z\frac{d}{dz}\mathrm{G}(z)$$

Convolution in Discrete Time

$$\mathrm{g}[n] * \mathrm{h}[n] \xleftrightarrow{z} \mathrm{H}(z)\mathrm{G}(z)$$

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Properties

Differencing

$$\mathrm{g}[n]-\mathrm{g}[n-1] \xleftrightarrow{\mathcal{Z}} \left(1-z^{-1}\right)\mathrm{G}(z)$$

Accumulation

$$\sum_{m=0}^{n}\mathrm{g}[m] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-1}\mathrm{G}(z)=\frac{1}{1-z^{-1}}\mathrm{G}(z)$$

Final Value Theorem

$$\lim_{n\to\infty}\mathrm{g}[n]=\lim_{z\to 1}(z-1)\mathrm{G}(z)$$

(if the limit exists)

The Inverse *z* Transform

There is an inversion integral for the *z* transform,

$$\mathrm{x}[n] = \frac{1}{j2\pi} \oint_{\mathrm{C}} \mathrm{X}(z) z^{n-1} dz$$

but doing it requires integration in the complex plane and it is rarely used in engineering practice.

There are two other common methods,

Synthetic Division
Partial-Fraction Expansion

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Synthetic Division

Suppose it is desired to find the inverse z transform of

$$\mathrm{H}(z) = \frac{z^3 - \frac{z^2}{2}}{z^3 - \frac{15}{12}z^2 + \frac{17}{36}z - \frac{1}{18}}$$

$$\begin{array}{r} 1 + \frac{3}{4}z^{-1} + \frac{67}{144}z^{-2} + \cdots \\ z^3 - \frac{15}{12}z^2 + \frac{17}{36}z - \frac{1}{18} \overline{\Big) z^3 - \frac{z^2}{2}} \\ \underline{z^3 - \frac{15}{12}z^2 + \frac{17}{36}z - \frac{1}{18}} \\ \frac{3}{4}z^2 - \frac{17}{36}z + \frac{1}{18} \\ \underline{\frac{3}{4}z^2 - \frac{45}{48}z + \frac{51}{144} - \frac{3}{72}z^{-1}} \\ \frac{67}{144}z \cdots \\ \vdots \quad \vdots \end{array}$$

Synthetically dividing the numerator by the denominator yields the infinite series

$$+\frac{3}{4}z^{-1} + \frac{67}{144}z^{-2} + \cdots$$

This will always work but the answer is not in closed form.

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Partial-Fraction Expansion

Algebraically, partial fraction expansion for finding inverse *z* transforms is identical to the same method applied to inverse Laplace transforms. For example,

$$\mathrm{H}(z)=\frac{z^2\left(z-\frac{1}{2}\right)}{\left(z-\frac{2}{3}\right)\left(z-\frac{1}{3}\right)\left(z-\frac{1}{4}\right)}$$

This fraction is improper in *z*. We could synthetically divide the numerator by the denominator once, yielding a remainder that is proper in *z* as with the Laplace transform but there is an alternate method that may be preferred in some situations.

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Partial-Fraction Expansion

$$\frac{\mathrm{H}(z)}{z} = \frac{z\left(z - \frac{1}{2}\right)}{\left(z - \frac{2}{3}\right)\left(z - \frac{1}{3}\right)\left(z - \frac{1}{4}\right)}$$

Dividing both sides by z makes the fraction proper in z and partial fraction expansion proceeds normally.

$$\frac{\mathrm{H}(z)}{z} = \frac{\frac{4}{5}}{z - \frac{2}{3}} + \frac{2}{z - \frac{1}{3}} - \frac{\frac{9}{5}}{z - \frac{1}{4}}$$

Then

$$\mathrm{H}(z) = \frac{\frac{4}{5}z}{z - \frac{2}{3}} + \frac{2z}{z - \frac{1}{3}} - \frac{\frac{9}{5}z}{z - \frac{1}{4}}$$

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Solving Difference Equations

The unilateral z transform is well suited to solving difference equations with initial conditions. For example,

$$\mathrm{y}[n+2]-\frac{3}{2}\mathrm{y}[n+1]+\frac{1}{2}\mathrm{y}[n]=\left(\frac{1}{4}\right)^n \quad , \quad \text{for } n \geq 0$$

$$\mathrm{y}[0]=10 \quad \text{and} \quad \mathrm{y}[1]=4$$

z transforming both sides,

$$z^2\left[\mathrm{Y}(z)-\mathrm{y}[0]-z^{-1}\mathrm{y}[1]\right]-\frac{3}{2}z\left[\mathrm{Y}(z)-\mathrm{y}[0]\right]+\frac{1}{2}\mathrm{Y}(z)=\frac{z}{z-\frac{1}{4}}$$

the initial conditions are called for systematically.

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Solving Difference Equations

Applying initial conditions and solving,

$$\mathrm{Y}(z) = z\left(\frac{\frac{16}{3}}{z - \frac{1}{4}} + \frac{4}{z - \frac{1}{2}} + \frac{\frac{2}{3}}{z - 1} \right)$$

and

$$\mathrm{y}[n] = \left[\frac{16}{3}\left(\frac{1}{4}\right)^n + 4\left(\frac{1}{2}\right)^n + \frac{2}{3} \right] \mathrm{u}[n]$$

This solution satisfies the difference equation and the initial conditions.

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z Transform - Laplace Transform Relationships

Let a signal, x(t), be sampled to form

$$\mathrm{x}[n] = \mathrm{x}(nT_s)$$

and impulse sampled to form

$$\mathrm{x}_\delta(t) = \mathrm{x}(t) f_s \,\mathrm{comb}(f_s t)$$

These two signals are equivalent in the sense that their impulse strengths are the same at corresponding times and the correspondence between times is $t = nT_s$.

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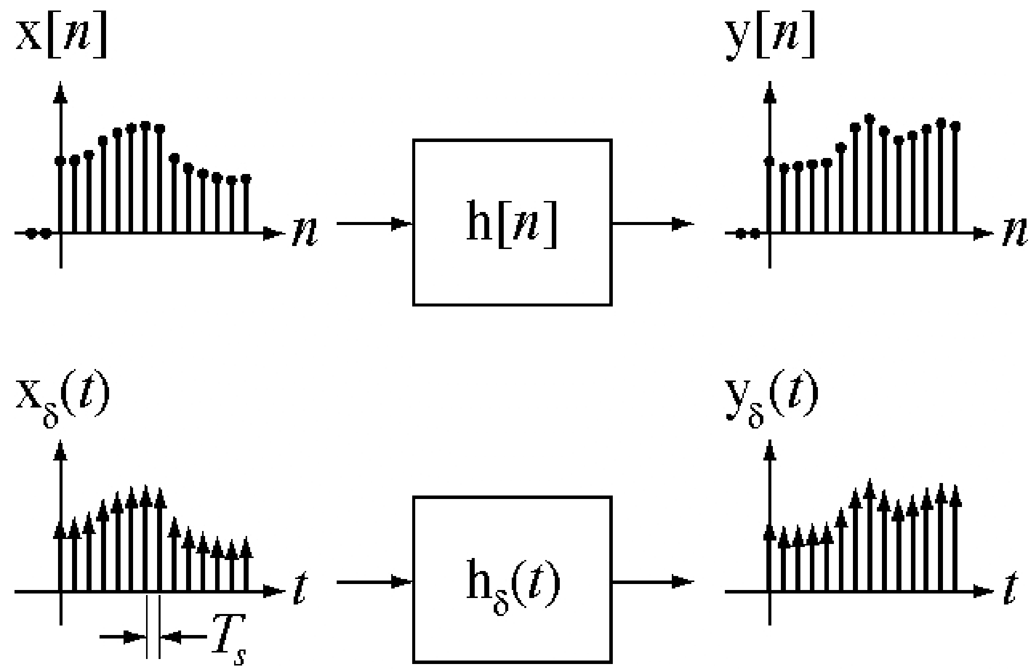
z Transform - Laplace Transform Relationships

Let a DT system have the impulse response, h[n], and let a CT system have the impulse response, $h_\delta(t) = \sum_{n=-\infty}^{\infty} h[n]\delta(t - nT_s)$.

If x[n] is applied to the DT system and $x_\delta(t)$ is applied to the CT system, their responses will be equivalent in the sense that the impulse strengths are the same.



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z Transform - Laplace Transform Relationships

The transfer function of the DT system is

$$\mathrm{H}(z) = \sum_{n=-\infty}^{\infty} \mathrm{h}[n] z^{-n}$$

and the transfer function of the CT system is

$$\mathrm{H}_{\delta}(s) = \sum_{n=-\infty}^{\infty} \mathrm{h}[n] e^{-nT_s s}$$

The equivalence between them is seen in the transformation,

$$\mathrm{H}_{\delta}(s) = \mathrm{H}(z)\big|_{z \to e^{sT_s}}$$

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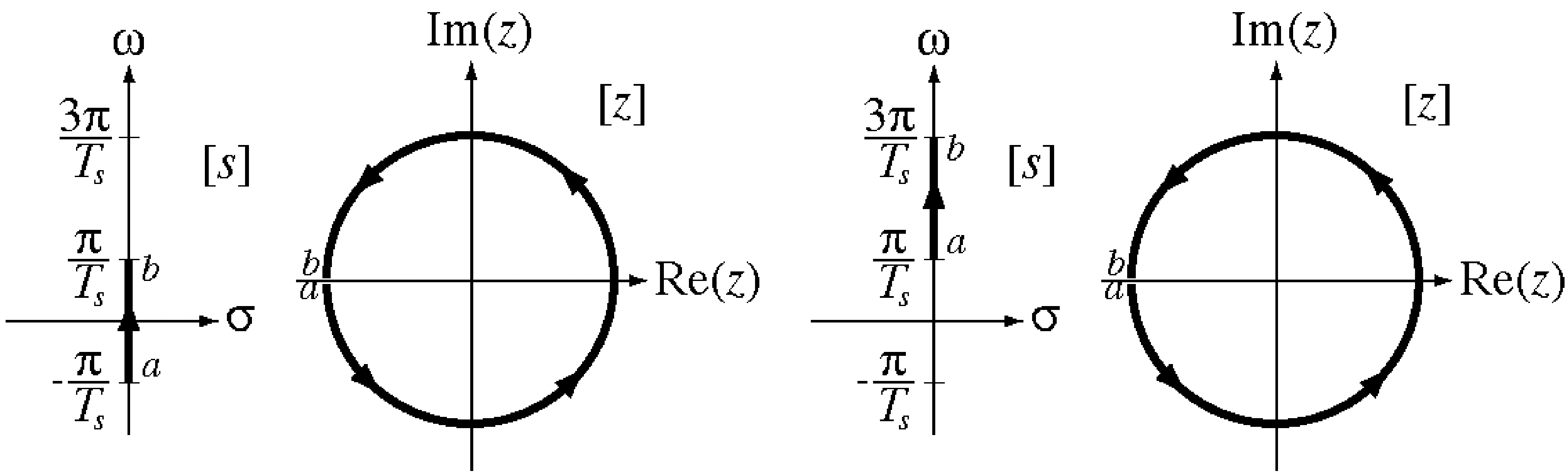
z Transform - Laplace Transform Relationships

The relationship, $z = e^{sT_s}$, *maps* points in the s plane into points in the z plane and vice versa.



Different contours in the s plane map into the same contour in the z plane.

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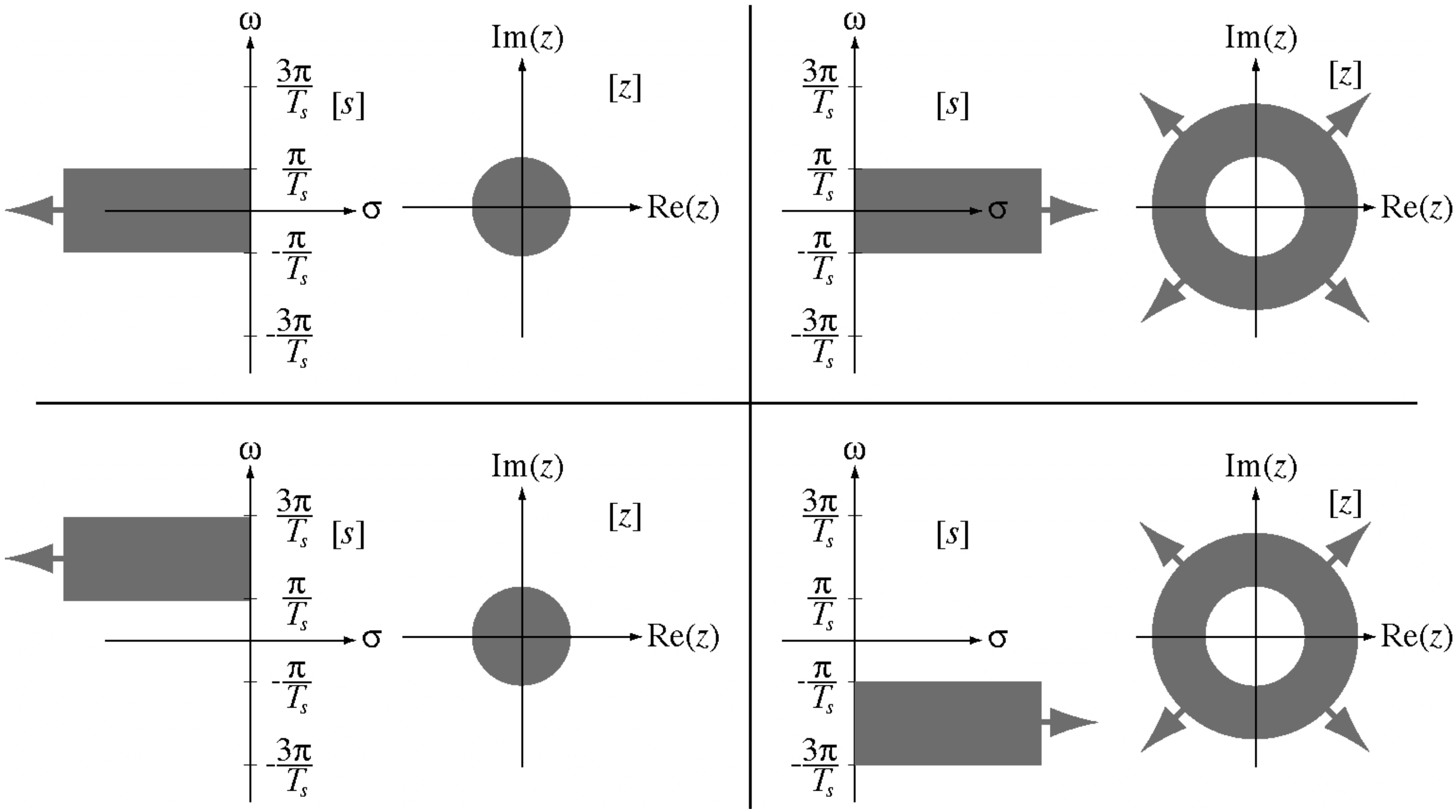
z Transform - Laplace Transform Relationships



The Bilateral z Transform

The bilateral z transform can be used to analyze non-causal signals and/or systems. It is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} x[n] z^{-n} + \sum_{n=-\infty}^{-1} x[n] z^{-n}$$

This can be manipulated into

$$X(z) = X_c(z) - x[0] + X_{ac}(z)$$

where

$$X_c(z) = \sum_{n=0}^{\infty} x[n] z^{-n} \quad \text{and} \quad X_{ac}(z) = \sum_{n=0}^{\infty} x[-n] z^{n}$$

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The Bilateral z Transform

The bilateral z transform can be found using the unilateral z transform by these four steps.

1. Find the unilateral z transform $X_c(z)$ and its ROC.

2. Find the unilateral z transform, $X_{ac}\left(\frac{1}{z}\right)$, of the discrete-time inverse of the anti-causal part of x[n].

3. Make the change of variable, $z \rightarrow \frac{1}{z}$, in the result of step 2 and in its ROC.

4. Add the results of steps 1 and 3 and subtract x[0] to form X(z).

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