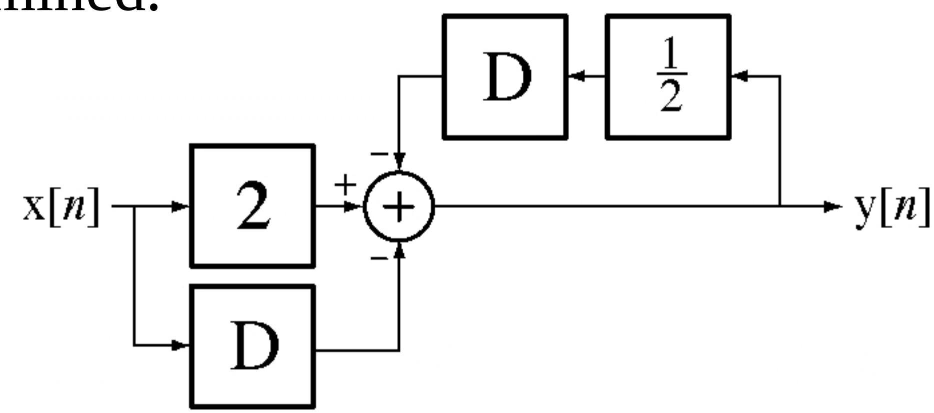
z Transform Signal and System Analysis

Block Diagrams and Transfer Functions

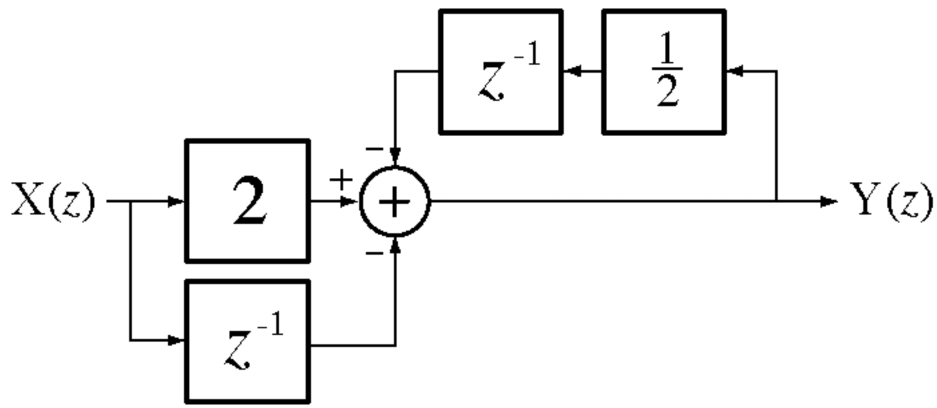
Just as with CT systems, DT systems are conveniently described by block diagrams and transfer functions can be determined from them. For example, from this DT system block diagram the difference equation can be determined.



$$y[n] = 2\,x[n] - x[n-1] - \frac{1}{2}y[n-1]$$

Block Diagrams and Transfer Functions

From a z-domain block diagram the transfer function can be determined.

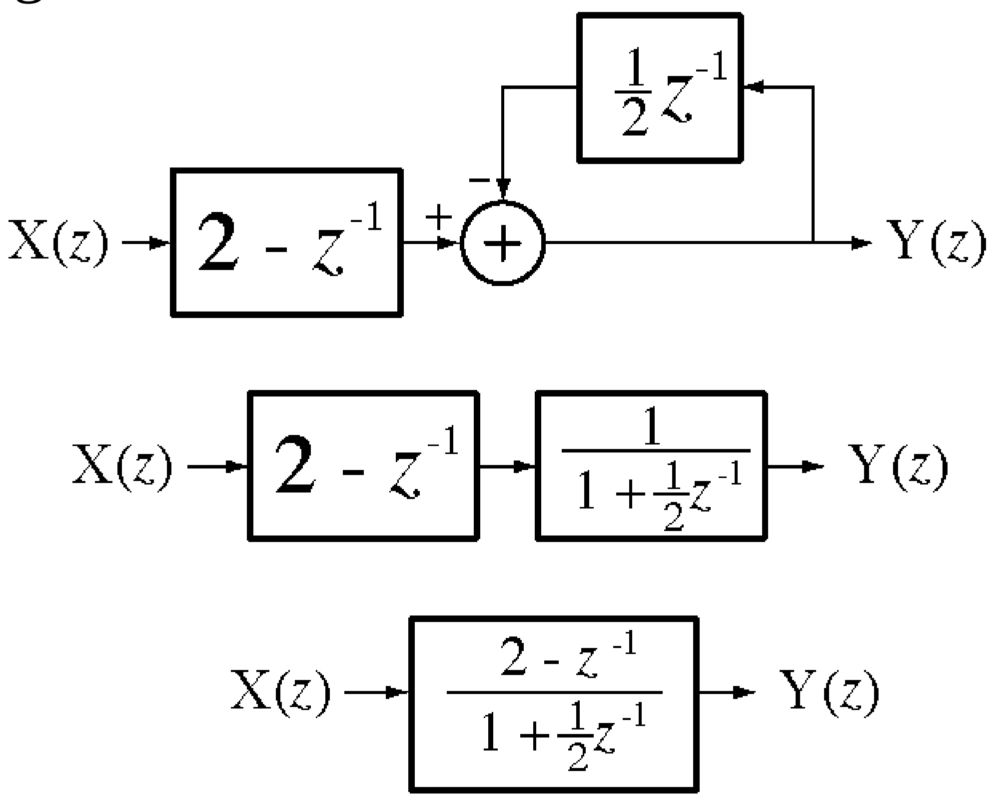


$$Y(z) = 2\,X(z) - z^{-1}\,X(z) - \frac{1}{2}z^{-1}\,Y(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 - z^{-1}}{1 + \frac{1}{2}z^{-1}} = \frac{2z - 1}{z + \frac{1}{2}}$$

Block Diagram Reduction

All the techniques for block diagram reduction introduced with the Laplace transform apply exactly to *z* transform block diagrams.

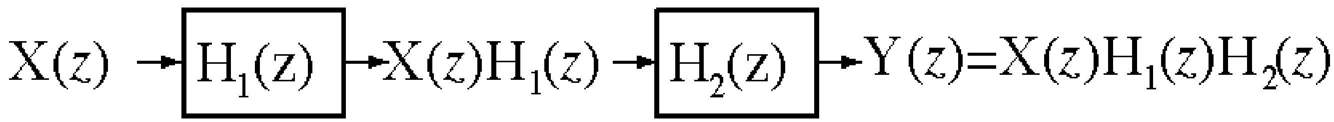


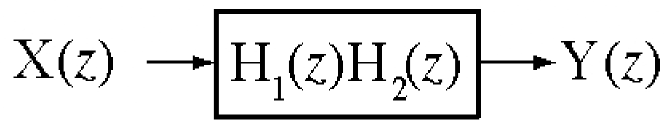
System Stability

A DT system is stable if its impulse response is absolutely summable. That requirement translates into the z-domain requirement that all the poles of the transfer function must lie in the <u>open</u> interior of the unit circle.

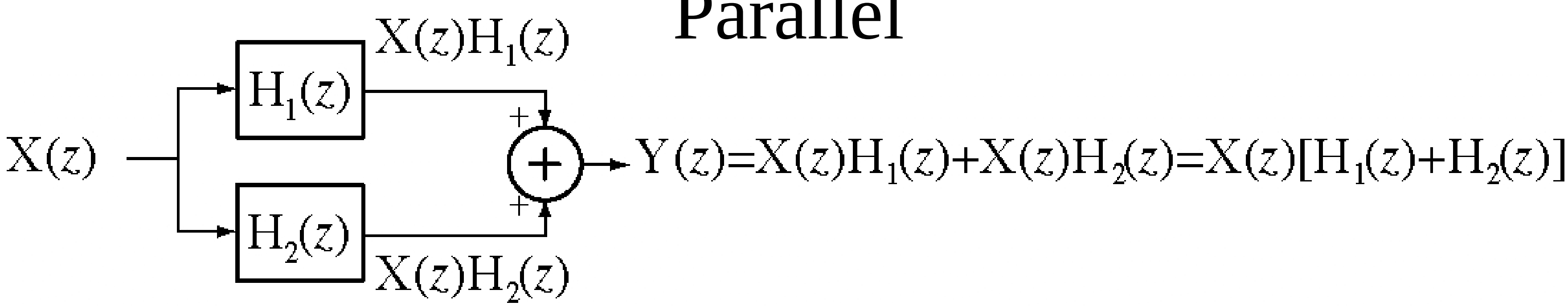
System Interconnections

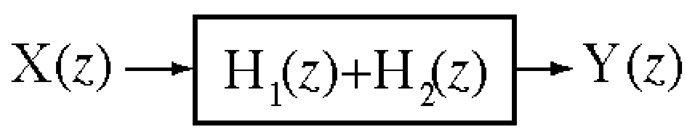
Cascade





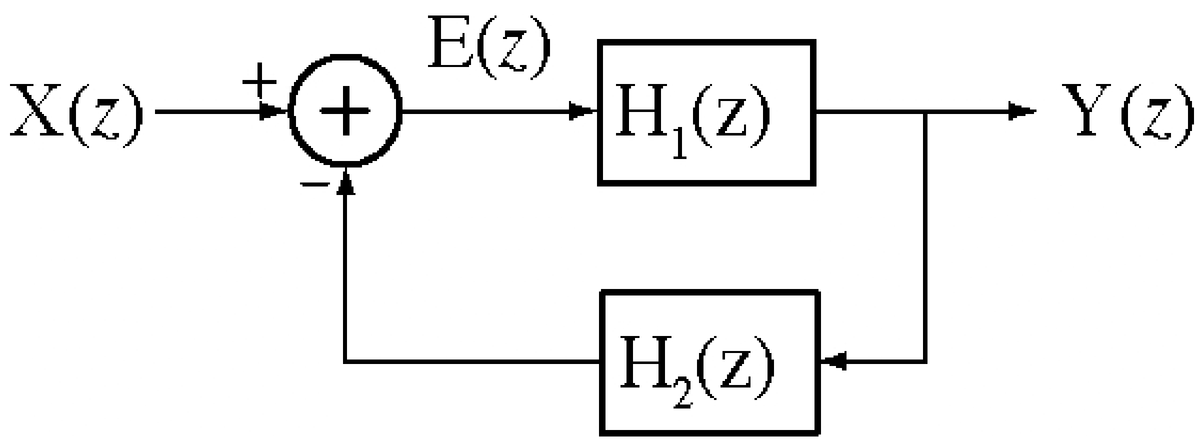
Parallel





System Interconnections

Feedback



$$H(z) = \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + H_1(z)H_2(z)} = \frac{H_1(z)}{1 + T(z)}$$

$$T(z) = H_1(z)H_2(z)$$

Responses to Standard Signals

If the system transfer function is $\mathrm{H}(z)=\dfrac{\mathrm{N}(z)}{\mathrm{D}(z)}$ the z transform of the unit-sequence response is $\mathrm{Y}(z)=\dfrac{z}{z-1}\dfrac{\mathrm{N}(z)}{\mathrm{D}(z)}$

which can be written in partial-fraction form as

$$\mathrm{Y}(z)=z\frac{\mathrm{N}_1(z)}{\mathrm{D}(z)}+\mathrm{H}(1)\frac{z}{z-1}$$

If the system is stable the transient term, $z\dfrac{\mathrm{N}_1(z)}{\mathrm{D}(z)}$, dies out and the steady-state response is $\mathrm{H}(1)\dfrac{z}{z-1}$.

Responses to Standard Signals

Let the system transfer function be $\mathrm{H}(z) = \dfrac{Kz}{z-p}$

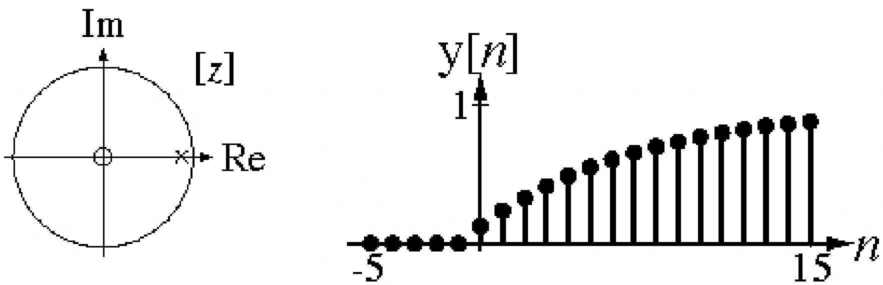
Then $\mathrm{Y}(z) = \dfrac{z}{z-1}\dfrac{Kz}{z-p} = \dfrac{K}{1-p}\left(\dfrac{z}{z-1} - \dfrac{pz}{z-p}\right)$

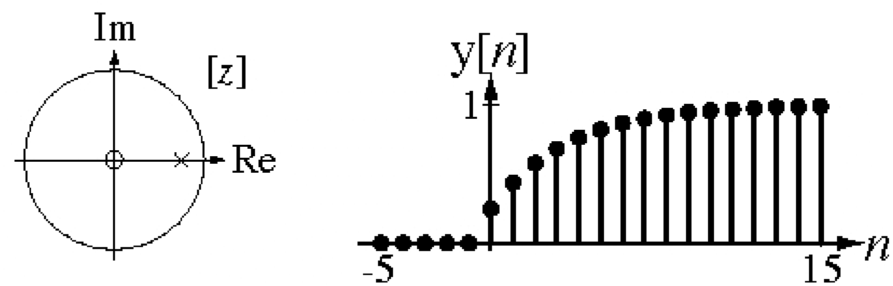
and $\mathrm{y}[n] = \dfrac{K}{1-p}\left(1-p^{n+1}\right)\mathrm{u}[n]$

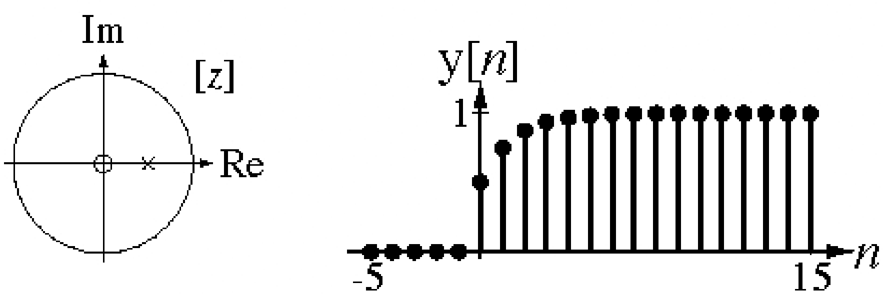
Let the constant, *K* be 1 - *p*. Then $\mathrm{y}[n] = \left(1-p^{n+1}\right)\mathrm{u}[n]$

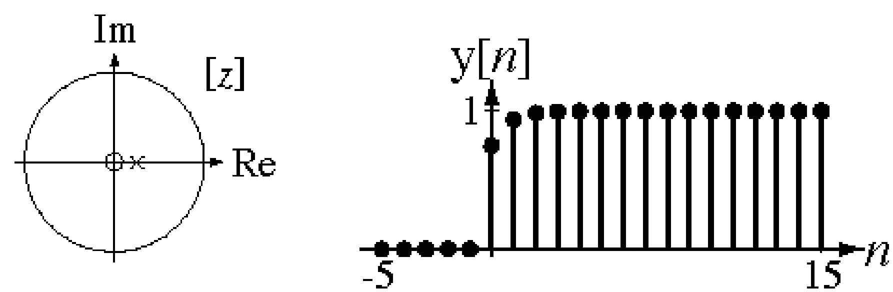
Responses to Standard Signals

Unit Sequence Response
One-Pole System



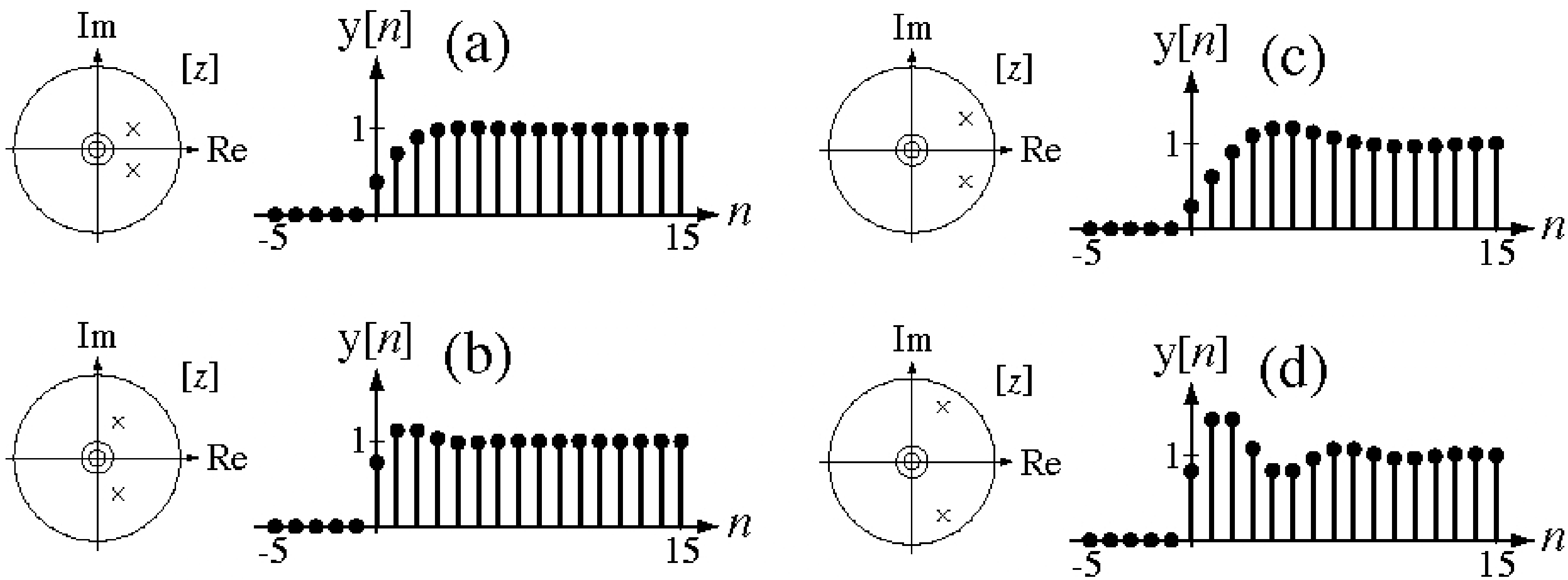






Responses to Standard Signals

Unit Sequence Response
Two-Pole System



Responses to Standard Signals

If the system transfer function is $\mathrm{H}(z) = \dfrac{\mathrm{N}(z)}{\mathrm{D}(z)}$ the z transform

of the response to a suddenly-applied sinusoid is

$$\mathrm{Y}(z) = \frac{\mathrm{N}(z)}{\mathrm{D}(z)} \frac{z[z - \cos(\Omega_0)]}{z^2 - 2z\cos(\Omega_0) + 1}$$

Let $p_1 = e^{j\Omega_0}$. Then the system response can be written as

$$\mathrm{y}[n] = \mathcal{Z}^{-1}\left(z\frac{\mathrm{N}_1(z)}{\mathrm{D}(z)} \right) + |\mathrm{H}(p_1)|\cos(\Omega_0 n + \angle \mathrm{H}(p_1))\mathrm{u}[n]$$

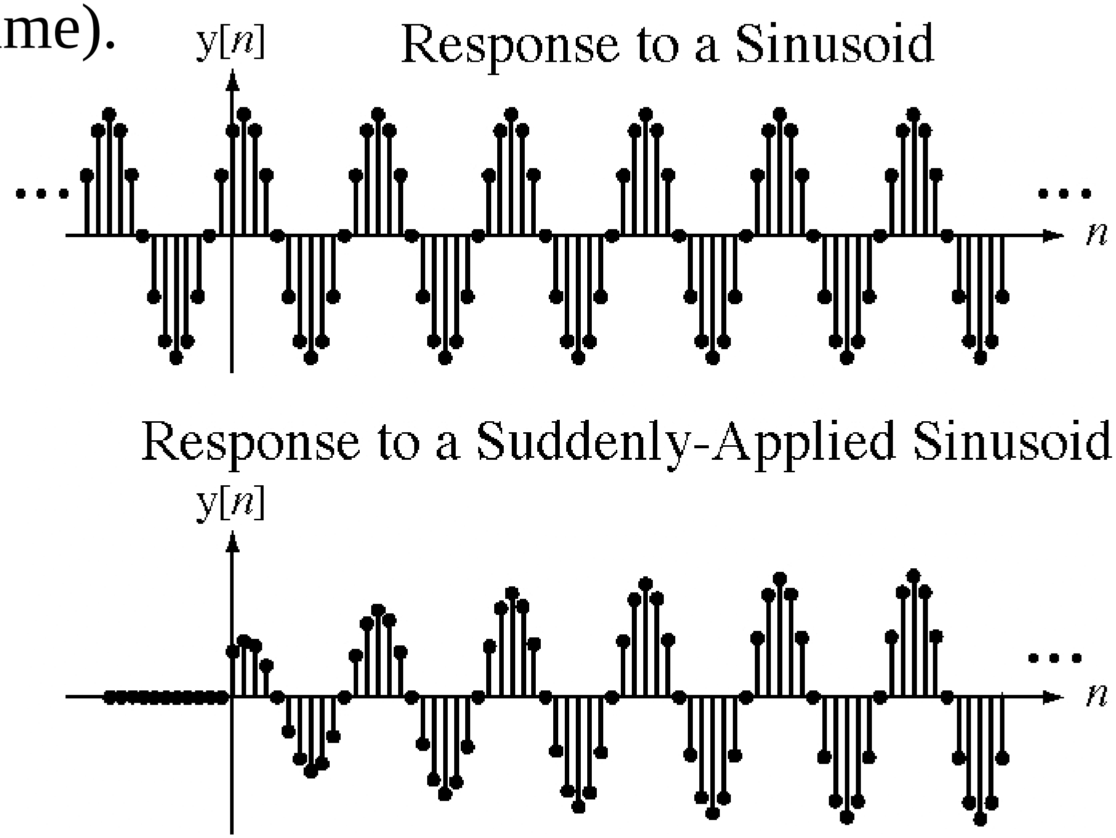
and, if the system is stable, the steady-state response is

$$|\mathrm{H}(p_1)|\cos(\Omega_0 n + \angle \mathrm{H}(p_1))\mathrm{u}[n]$$

a DT sinusoid with, generally, different magnitude and phase.

Pole-Zero Diagrams and Frequency Response

For a stable system, the response to a suddenly-applied sinusoid approaches the response to a true sinusoid (applied for all time).



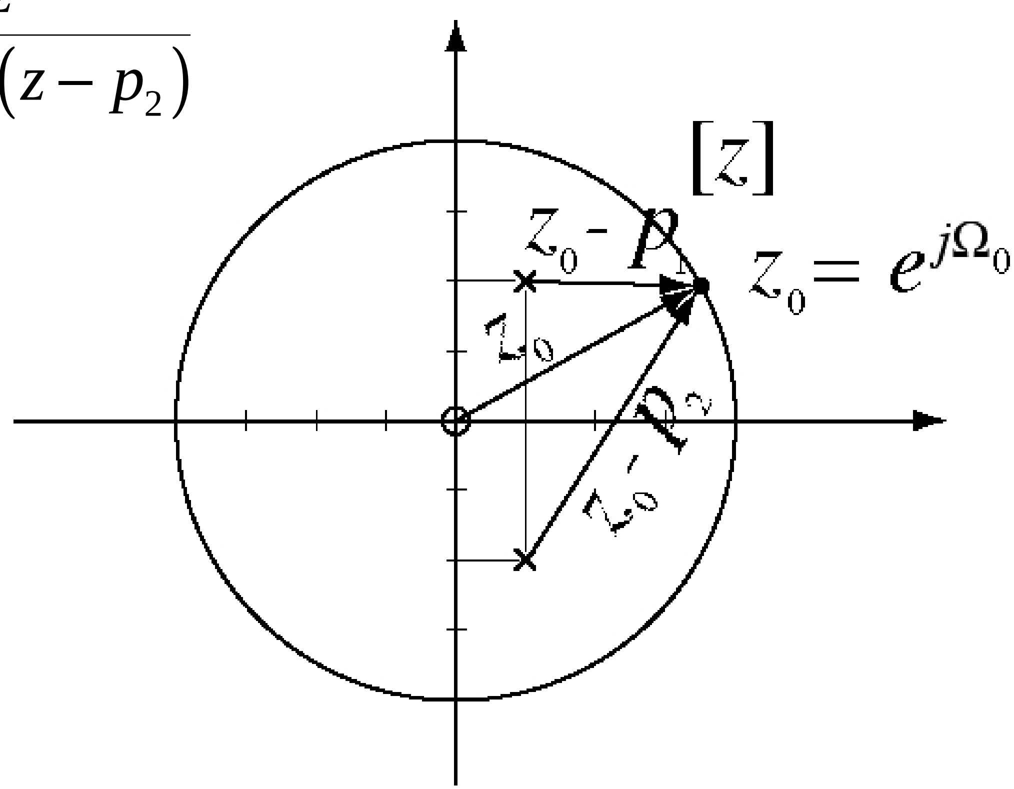
Pole-Zero Diagrams and Frequency Response

Let the transfer function of a DT system be

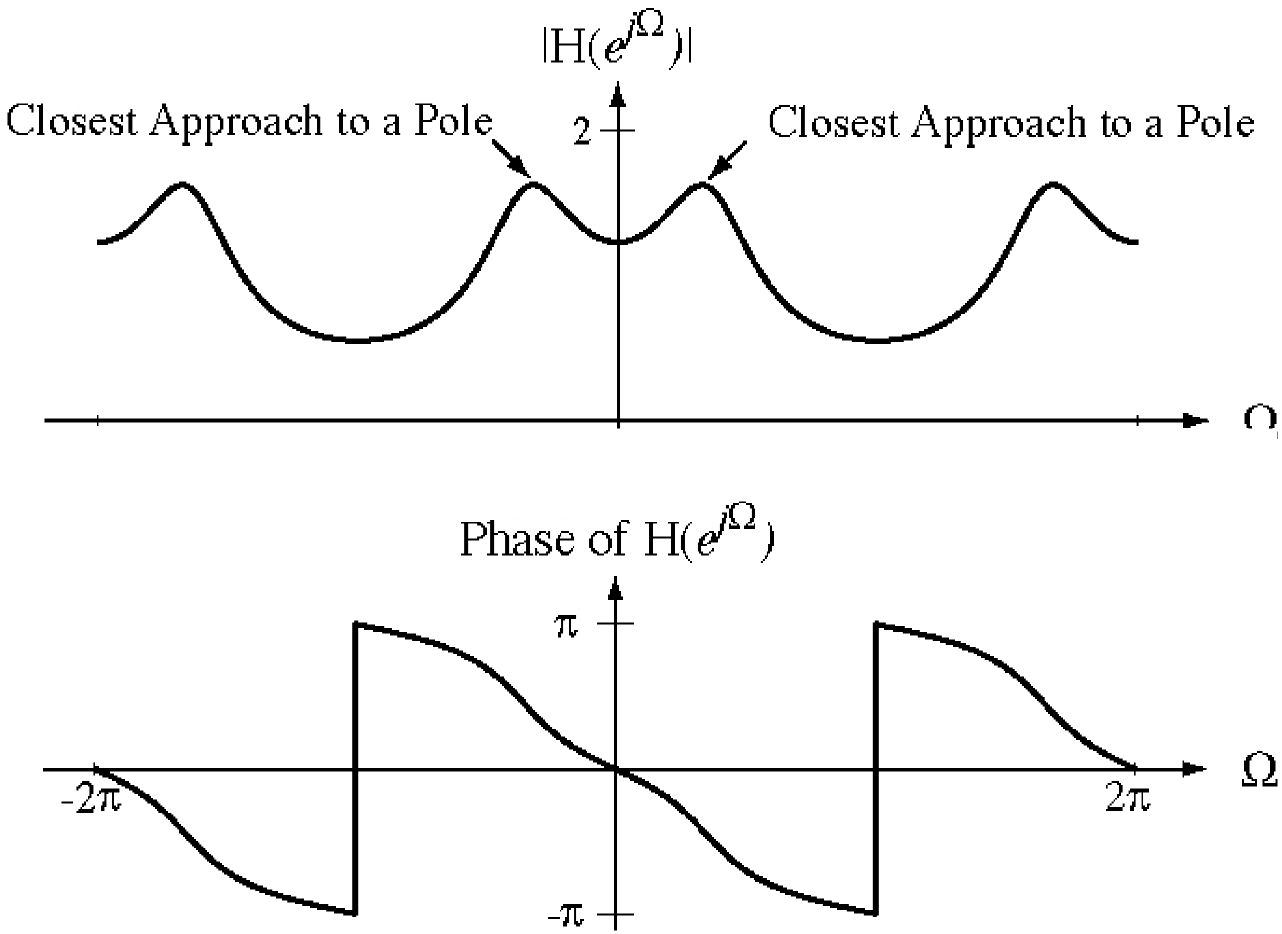
$$\mathrm{H}(z) = \frac{z}{z^2 - \frac{z}{2} + \frac{5}{16}} = \frac{z}{(z - p_1)(z - p_2)}$$

$$p_1 = \frac{1 + j2}{4} \qquad p_2 = \frac{1 - j2}{4}$$

$$\left|\mathrm{H}\left(e^{j\Omega}\right)\right| = \frac{\left|e^{j\Omega}\right|}{\left|e^{j\Omega} - p_1\right| \left|e^{j\Omega} - p_2\right|}$$



Pole-Zero Diagrams and Frequency Response



The Jury Stability Test

Let a transfer function be in the form, $\mathrm{H}(z) = \dfrac{\mathrm{N}(z)}{\mathrm{D}(z)}$

where $\mathrm{D}(z) = a_D z^D + a_{D-1} z^{D-1} + \cdots + a_1 z + a_0$

Form the "Jury" array

$$
\begin{array}{cccccccc}
1 & a_0 & a_1 & a_2 & \cdots & a_{D-2} & a_{D-1} & a_D \\
2 & a_D & a_{D-1} & a_{D-2} & \cdots & a_2 & a_1 & a_0 \\
3 & b_0 & b_1 & b_2 & \cdots & b_{D-2} & b_{D-1} & \\
4 & b_{D-1} & b_{D-2} & b_{D-3} & \cdots & b_1 & b_0 & \\
5 & c_0 & c_1 & c_2 & \cdots & c_{D-2} & & \\
6 & c_{D-2} & c_{D-3} & c_{D-4} & \cdots & c_0 & & \\
\vdots & \vdots & \vdots & \vdots & \iddots & & & \\
2D-3 & s_0 & s_1 & s_2 & & & &
\end{array}
$$

The Jury Stability Test

The third row is computed from the first two by

$$b_0 = \begin{vmatrix} a_0 & a_D \\ a_D & a_0 \end{vmatrix} ,\ b_1 = \begin{vmatrix} a_0 & a_{D-1} \\ a_D & a_1 \end{vmatrix} ,\ b_2 = \begin{vmatrix} a_0 & a_{D-2} \\ a_D & a_2 \end{vmatrix} , \cdots ,\ b_{D-1} = \begin{vmatrix} a_0 & a_1 \\ a_D & a_{D-1} \end{vmatrix}$$

The fourth row is the same set as the third row except in reverse order. Then the *c*'s are computed from the *b*'s in the same way the *b*'s are computed from the *a*'s. This continues until only three entries appear. Then the system is stable if

$$\mathrm{D}(1) > 0 \qquad\qquad (-1)^D\,\mathrm{D}(-1) > 0$$

$$a_D > |a_0| ,\ |b_0| > |b_{D-1}| ,\ |c_0| > |c_{D-2}| , \cdots , |s_0| > |s_2|$$

Root Locus

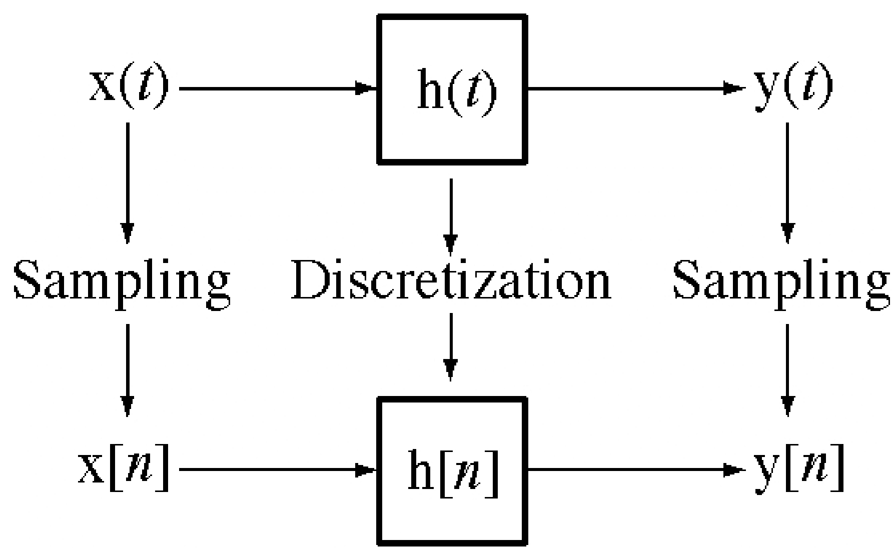
Root locus methods for DT systems are like root locus methods for CT systems except that the interpretation of the result is different.

CT systems: If the root locus crosses into the right half-plane the system goes unstable at that gain.

DT systems: If the root locus goes outside the unit circle the system goes unstable at that gain.

Simulating CT Systems with DT Systems

The ideal simulation of a CT system by a DT system would have the DT system's excitation and response be samples from the CT system's excitation and response. But that design goal is never achieved exactly in real systems at finite sampling rates.



Simulating CT Systems with DT Systems

One approach to simulation is to make the impulse response of the DT system be a sampled version of the impulse response of the CT system.

$$\mathrm{h}[n] = \mathrm{h}(nT_s)$$

With this choice, the response of the DT system to a DT unit impulse consists of samples of the response of the CT system to a CT unit impulse. This technique is called *impulse-invariant* design.

Simulating CT Systems with DT Systems

When $\mathrm{h}[n] = \mathrm{h}(nT_s)$ the impulse response of the DT system is a sampled version of the impulse response of the CT system *but the unit DT impulse is not a sampled version of the unit CT impulse.*

A CT impulse cannot be sampled. First, as a practical matter the probability of taking a sample at exactly the time of occurrence of the impulse is zero. Second, even if the impulse were sampled at its time of occurrence what would the sample value be? The functional value of the impulse is not defined at its time of occurrence *because the impulse is not an ordinary function.*

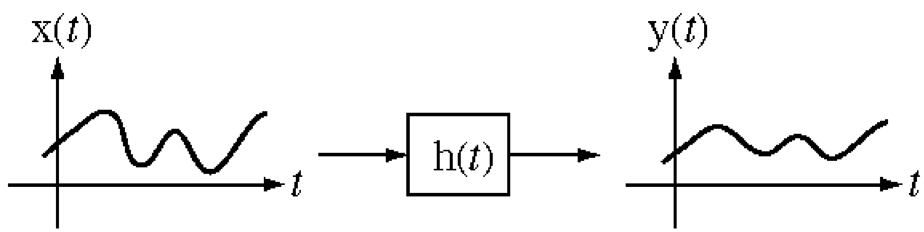
Simulating CT Systems with DT Systems

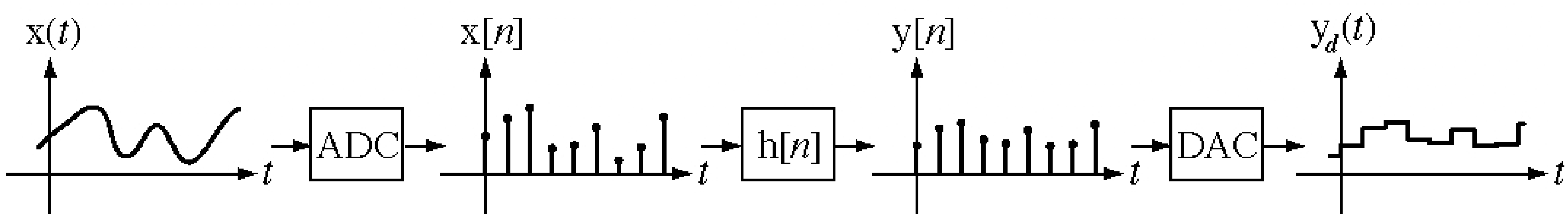
In impulse-invariant design, even though the impulse response is a sampled version of the CT system's impulse response *that does not mean that the response to samples from any arbitrary excitation will be a sampled version of the CT system's response to that excitation.*

All design methods for simulating CT systems with DT systems are approximations and whether or not the approximation is a good one depends on the design goals.

Sampled-Data Systems

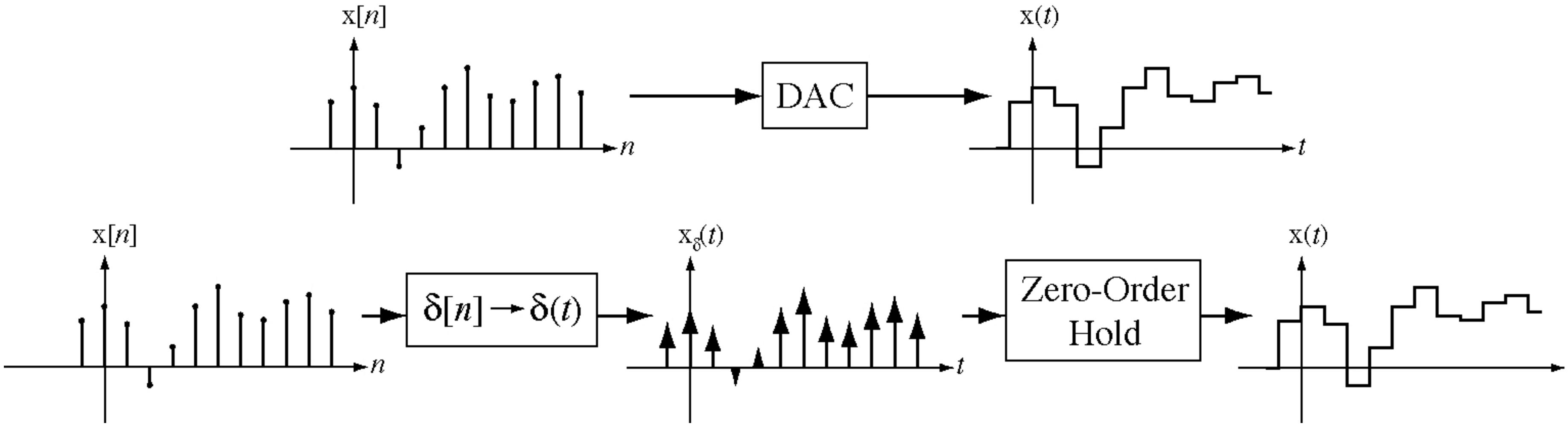
Real simulations of CT systems by DT systems usually sample the excitation with an ADC, process the samples and then produce a CT signal with a DAC.





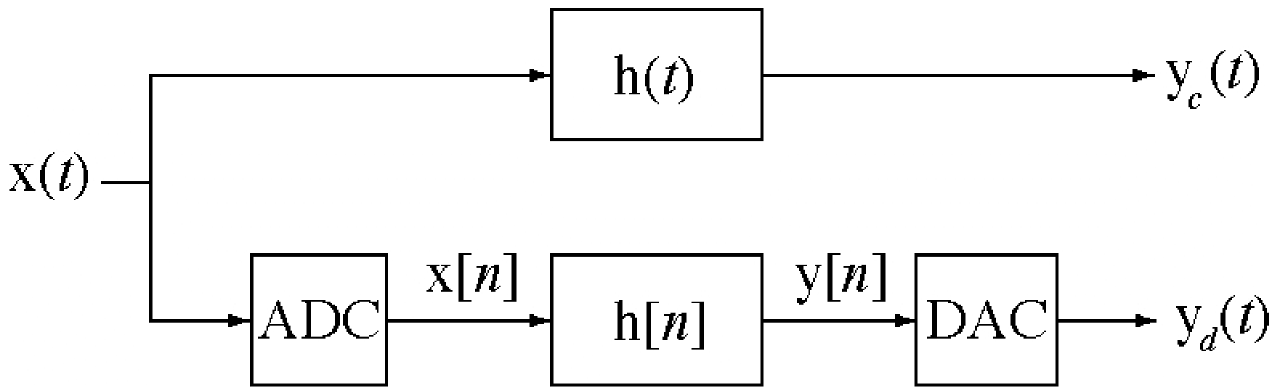
Sampled-Data Systems

An ADC simply samples a signal and produces numbers. A common way of modeling the action of a DAC is to imagine the DT impulses in the DT signal which drive the DAC are instead CT impulses of the same strength and that the DAC has the impulse response of a zero-order hold.



Sampled-Data Systems

The desired equivalence between a CT and a DT system is illustrated below.



The design goal is to make $y_d(t)$ look as much like $y_c(t)$ as possible by choosing $h[n]$ appropriately.

Sampled-Data Systems

Consider the response of the CT system *not to the actual signal*, $x(t)$, but rather to an impulse-sampled version of it,

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s) = x(t) f_s \,\mathrm{comb}(f_s t)$$

The response is

$$y(t) = h(t) * x_\delta(t) = h(t) * \sum_{m=-\infty}^{\infty} x[m]\delta(t - mT_s) = \sum_{m=-\infty}^{\infty} x[m]h(t - mT_s)$$

where $x[n] = x(nT_s)$ and the response at the nth multiple of T_s is

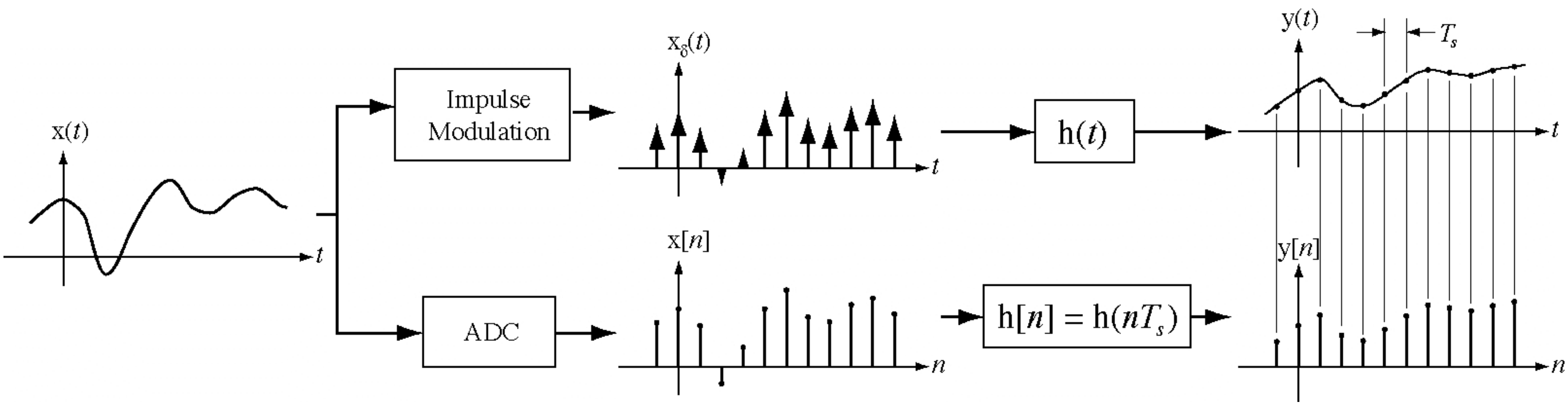
$$y(nT_s) = \sum_{m=-\infty}^{\infty} x[m]h((n-m)T_s)$$

The response of a DT system with $h[n] = h(nT_s)$ to the excitation, $x[n] = x(nT_s)$ is

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]$$

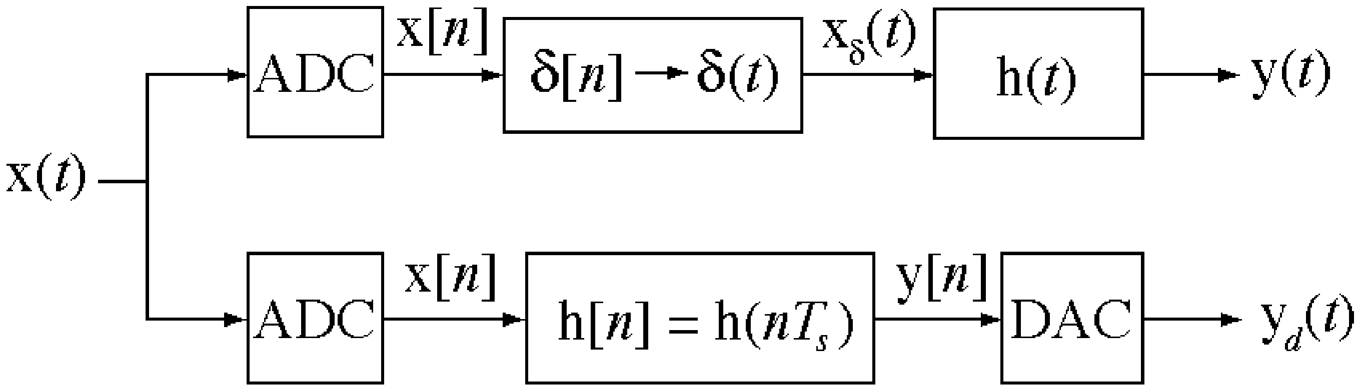
Sampled-Data Systems

The two responses are equivalent in the sense that the values at corresponding DT and CT times are the same.

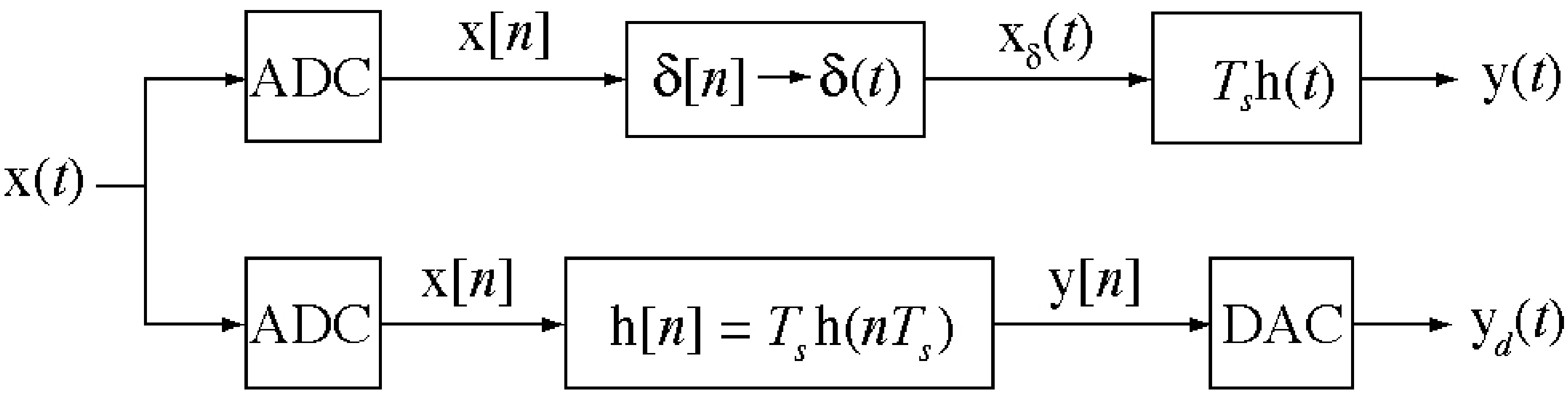


Sampled-Data Systems

Modify the CT system to reflect the last analysis.



Then multiply the impulse responses of both systems by T_s



Sampled-Data Systems

In the modified CT system,

$$\mathrm{y}(t)=\mathrm{x}_\delta(t)*T_s\,\mathrm{h}(t)=\left[\sum_{n=-\infty}^{\infty}\mathrm{x}(nT_s)\delta(t-nT_s)\right]*\mathrm{h}(t)T_s=\sum_{n=-\infty}^{\infty}\mathrm{x}(nT_s)\mathrm{h}(t-nT_s)T_s$$

In the modified DT system,

$$\mathrm{y}[n]=\sum_{m=-\infty}^{\infty}\mathrm{x}[m]\mathrm{h}[n-m]=\sum_{m=-\infty}^{\infty}\mathrm{x}[m]T_s\,\mathrm{h}((n-m)T_s)$$

where $\mathrm{h}[n]=T_s\,\mathrm{h}(nT_s)$ and h(t) still represents the impulse response of the original CT system. Now let T_s approach zero.

$$\lim_{T_s\to 0}\mathrm{y}(t)=\lim_{T_s\to 0}\sum_{n=-\infty}^{\infty}\mathrm{x}(nT_s)\mathrm{h}(t-nT_s)T_s=\int_{-\infty}^{\infty}\mathrm{x}(\tau)\mathrm{h}(t-\tau)d\tau$$

This is the response, $\mathrm{y}_c(t)$, of the original CT system.

Sampled-Data Systems

Summarizing, if the impulse response of the DT system is chosen to be $T_s \, \mathrm{h}(nT_s)$ then, in the limit as the sampling rate approaches infinity, the response of the DT system is exactly the same as the response of the CT system.

Of course the sampling rate can never be infinite in practice. Therefore this design is an approximation which gets better as the sampling rate is increased.

Digital Filters

- Digital filter design is simply DT system design applied to filtering signals
- A popular method of digital filter design is to simulate a proven CT filter design
- There many design approaches each of which yields a better approximation to the ideal as the sampling rate is increased

Digital Filters

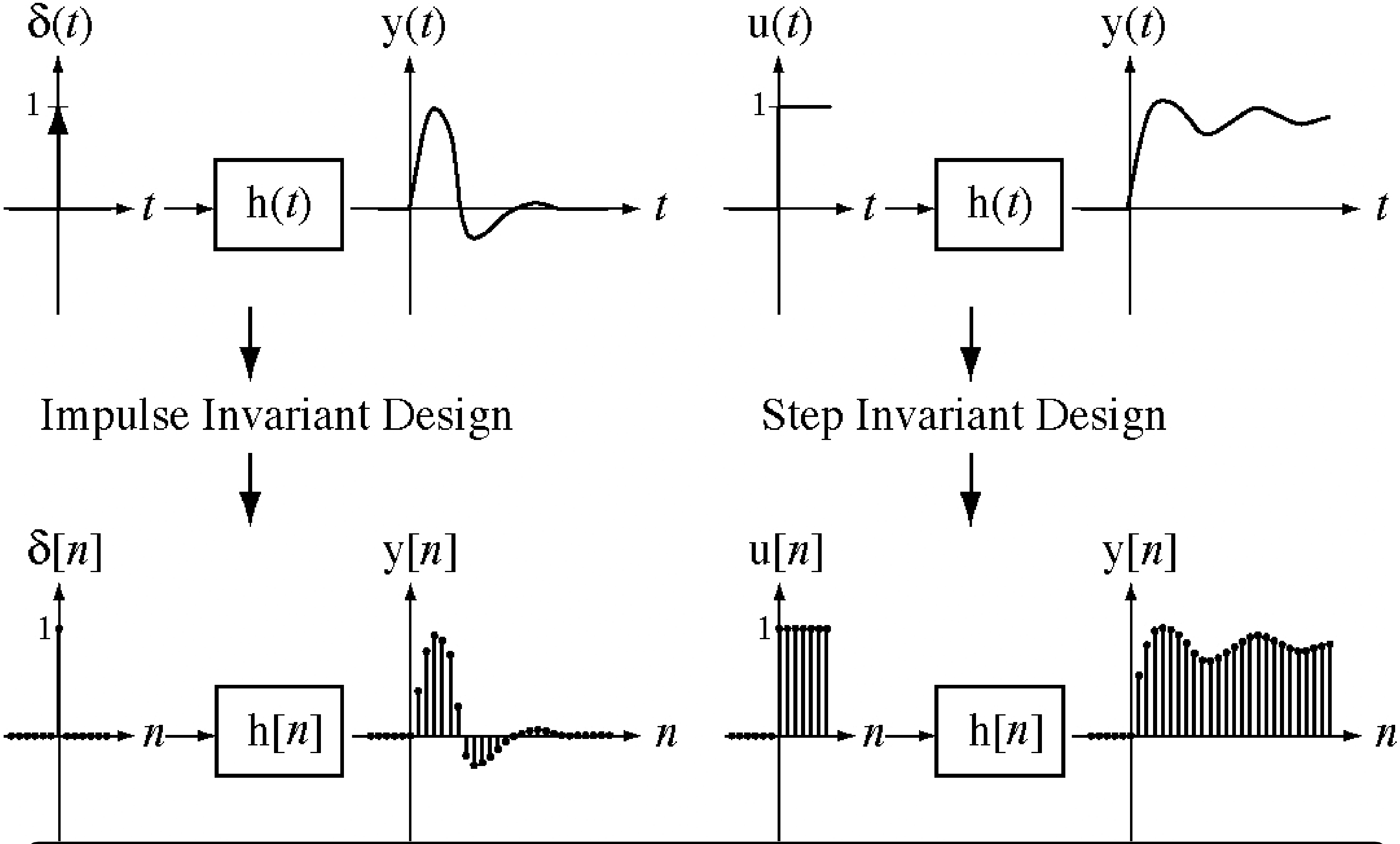
- Practical CT filters have *infinite-duration impulse responses*, impulse responses which never actually go to zero and stay there
- Some digital filter designs produce DT filters with infinite-duration impulse responses and these are called *IIR* filters
- Some digital filter designs produce DT filters with finite-duration impulse responses and these are called *FIR* filters

Digital Filters

- Some digital filter design methods use time-domain approximation techniques
- Some digital filter design methods use frequency-domain approximation techniques

Digital Filters

Impulse and Step Invariant Design



Digital Filters

Impulse and Step Invariant Design

Impulse invariant:

$$\mathrm{H}_s(s) \xrightarrow{\mathcal{L}^{-1}} \mathrm{h}(t) \xrightarrow{\text{Sample}} \mathrm{h}[n] \xrightarrow{\mathcal{Z}} \mathrm{H}_z(z)$$

Step invariant:

$$\mathrm{H}_s(s) \xrightarrow{\times\frac{1}{s}} \frac{\mathrm{H}_s(s)}{s} \xrightarrow{\mathcal{L}^{-1}} \mathrm{h}_{-1}(t) \xrightarrow{\text{Sample}} \mathrm{h}_{-1}[n]$$

$$\xrightarrow{\mathcal{Z}} \frac{z}{z-1}\mathrm{H}_z(z) \xrightarrow{\times\frac{z-1}{z}} \mathrm{H}_z(z)$$

Digital Filters

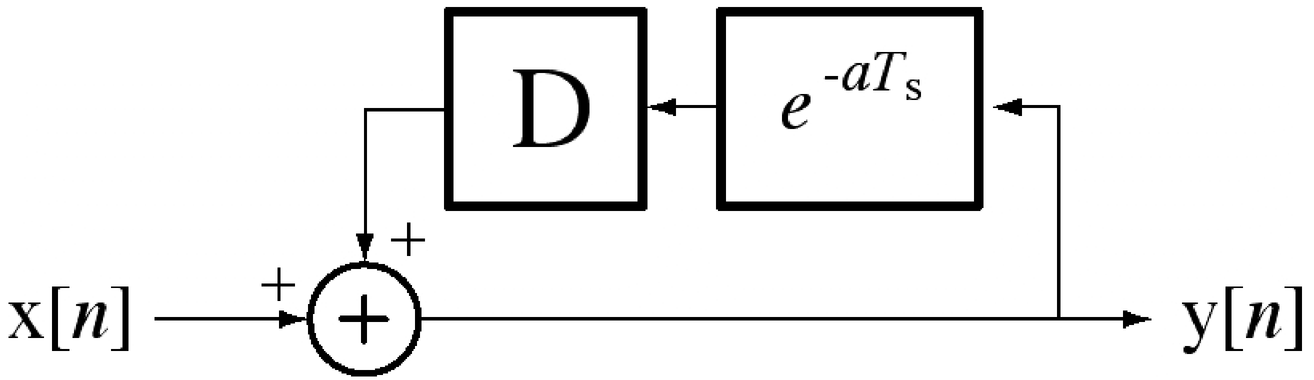
Impulse and Step Invariant Design

Impulse invariant approximation of the one-pole system,

$$\mathrm{H}_s(s) = \frac{1}{s+a}$$

yields

$$\mathrm{H}_z(z) = \frac{1}{1 - e^{-aT_s} z^{-1}}$$

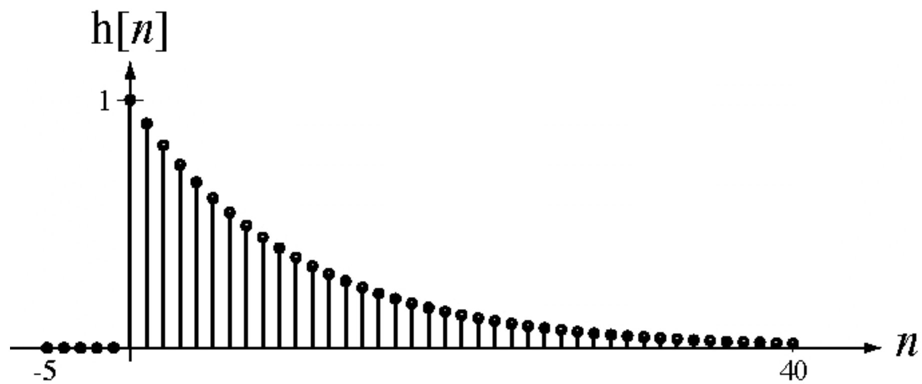


Digital Filters

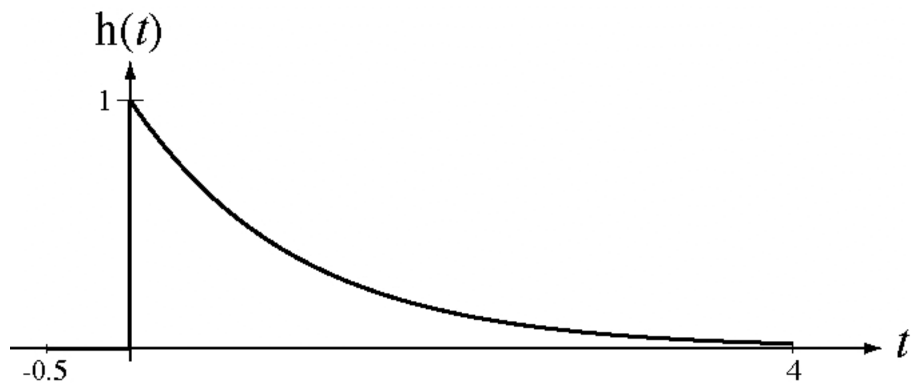
Impulse and Step Invariant Design

Let a be one and let $T_s = 0.1$ in $H_z(z) = \frac{1}{1 - e^{-aT_s} z^{-1}}$

Digital Filter
Impulse Response



CT Filter
Impulse Response



Digital Filters

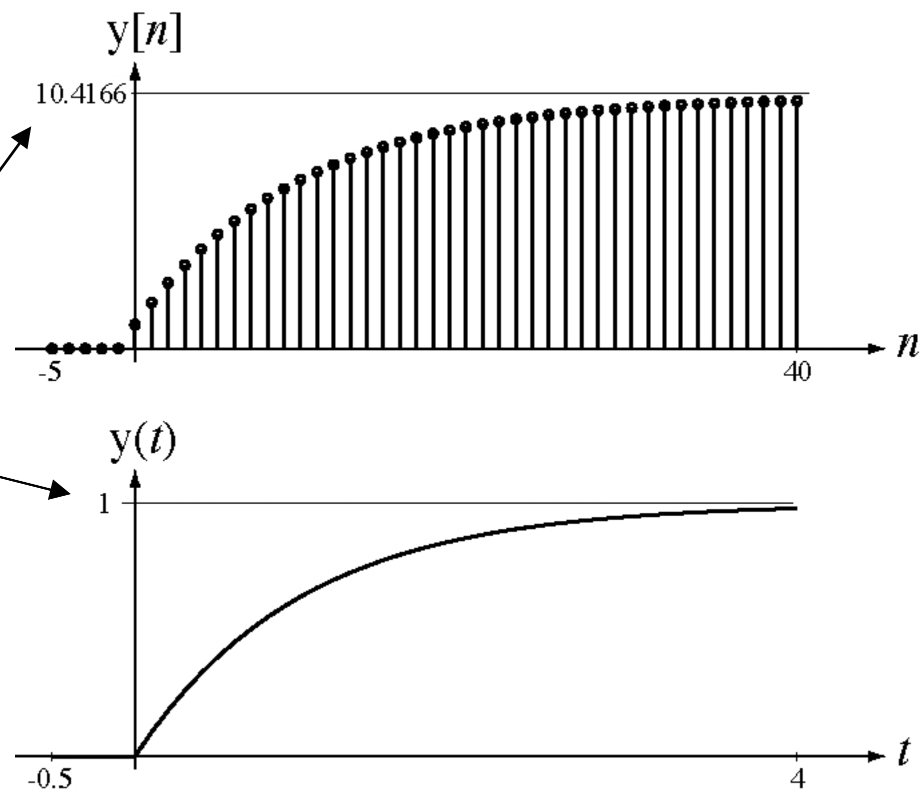
Impulse and Step Invariant Design

Step response of $\mathrm{H}_z(z) = \dfrac{1}{1 - e^{-aT_s} z^{-1}}$

Digital Filter
Step Response

Notice scale difference

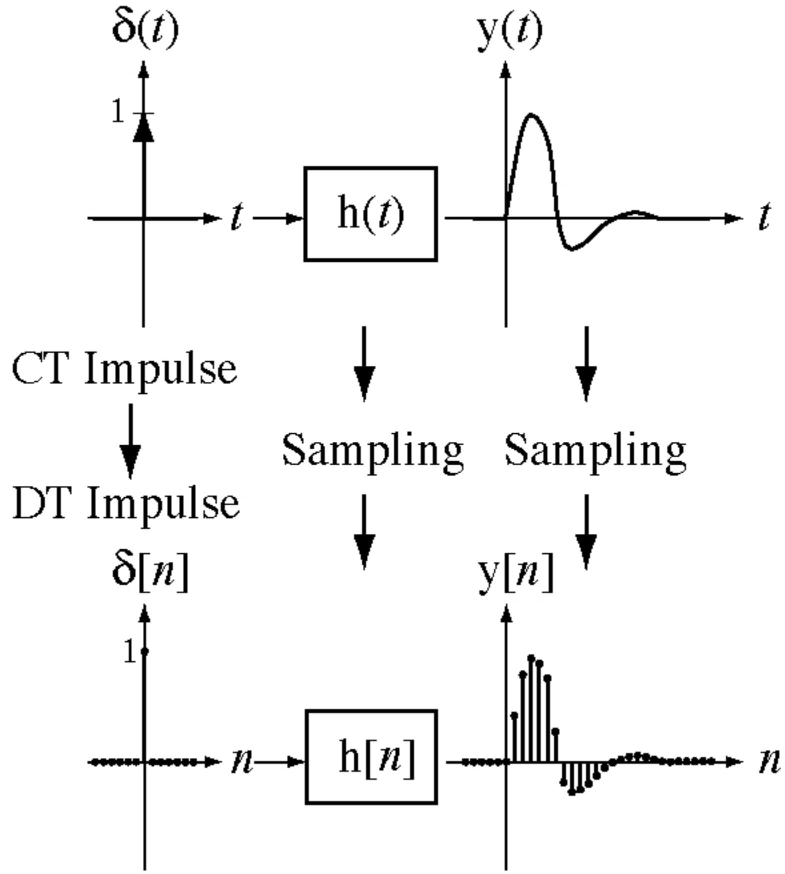
CT Filter
Step Response



Digital Filters

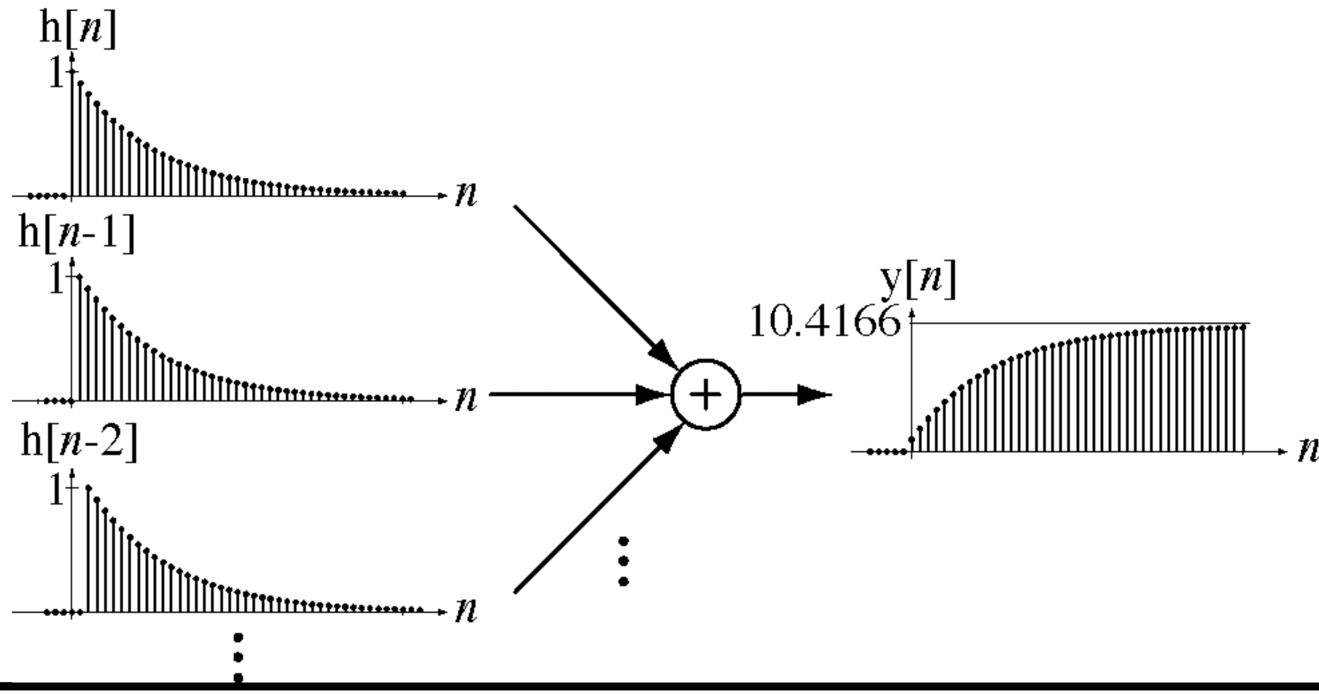
Why is the impulse response exactly right while the step response is wrong?

This design method forces an equality between the impulse strength of a CT excitation, a unit CT impulse at zero, and the impulse strength of the corresponding DT signal, a unit DT impulse at zero. It also makes the impulse response of the DT system, h[*n*], be samples from the impulse response of the CT system, h(*t*).



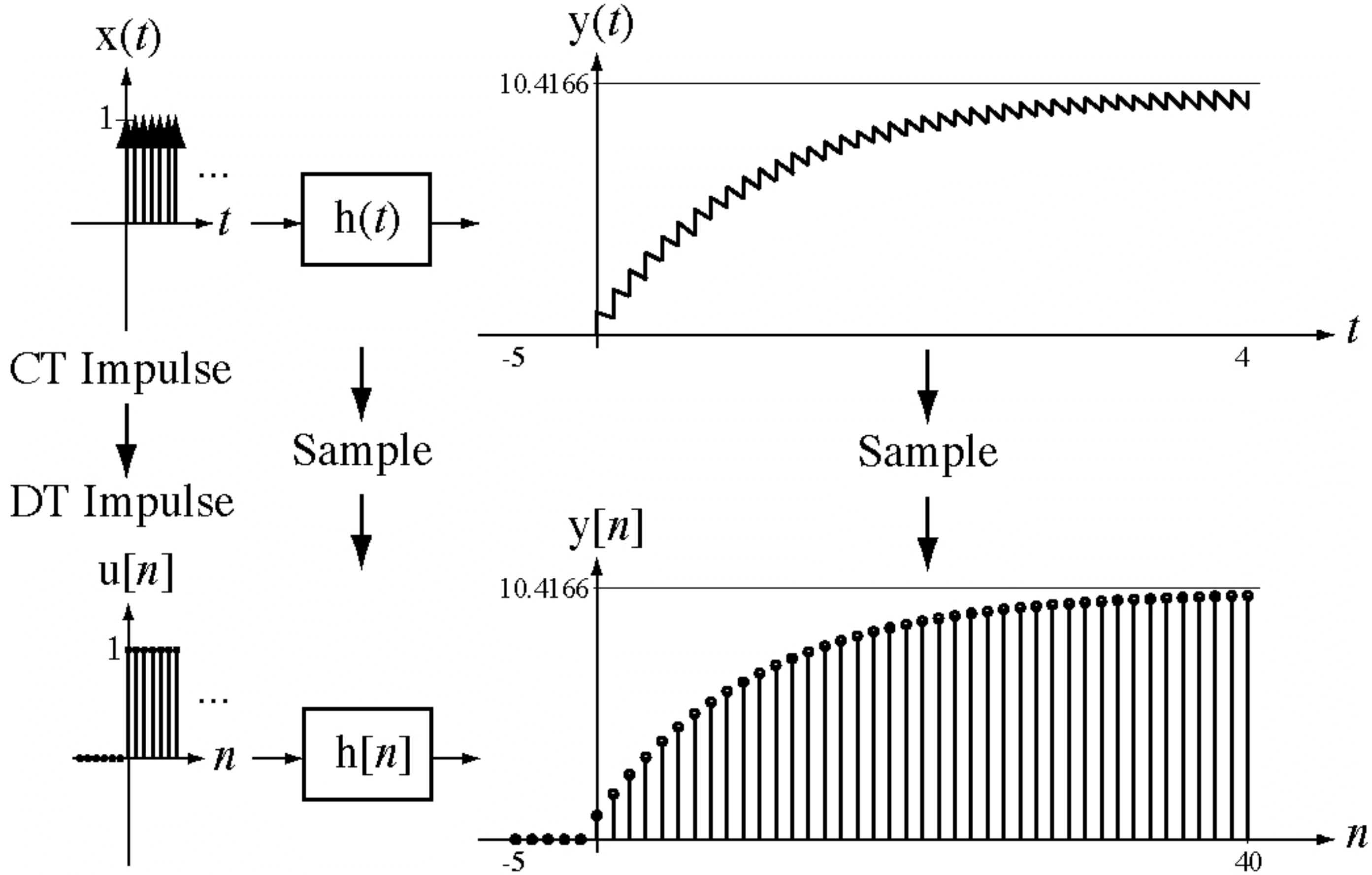
Digital Filters

A CT step excitation is not an impulse. So what should the correspondence between the CT and DT excitations be now? If the step excitation is sampled at the same rate as the impulse response was sampled, the resulting DT signal is the excitation of the DT system and the response of the DT system is the sum of the responses to all those DT impulses.



Digital Filters

If the excitation of the CT system were a sequence of CT unit impulses, occurring at the same sampling rate used to form h[n], then the response of the DT system would be samples of the response of the CT system.

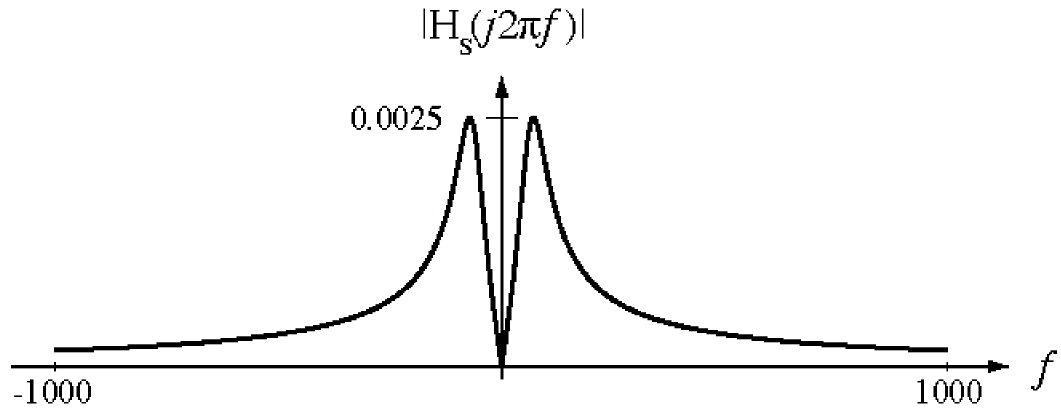


Digital Filters

Impulse and Step Invariant Design

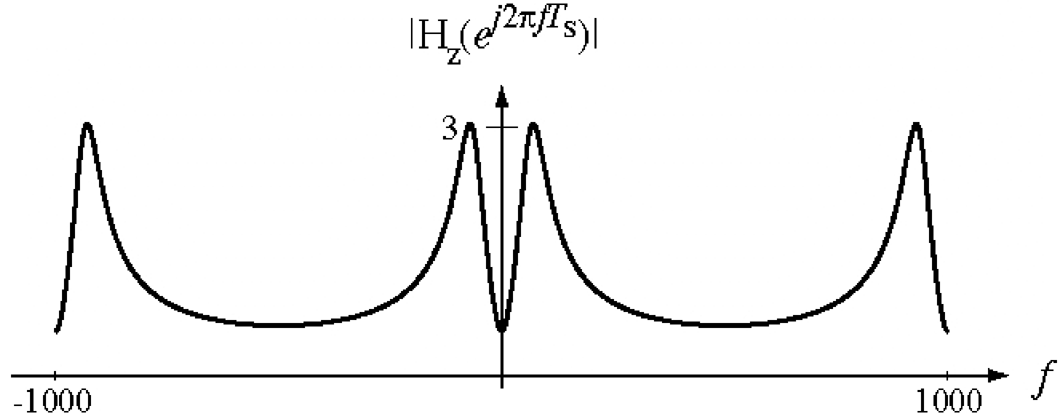
Impulse invariant approximation of

$$\mathrm{H}_s(s) = \frac{s}{s^2 + 400s + 2\times 10^5}$$



with a 1 kHz sampling rate yields

$$\mathrm{H}(z) = \frac{z(z-0.9135)}{z^2 - 1.508z + 0.6703}$$



Digital Filters

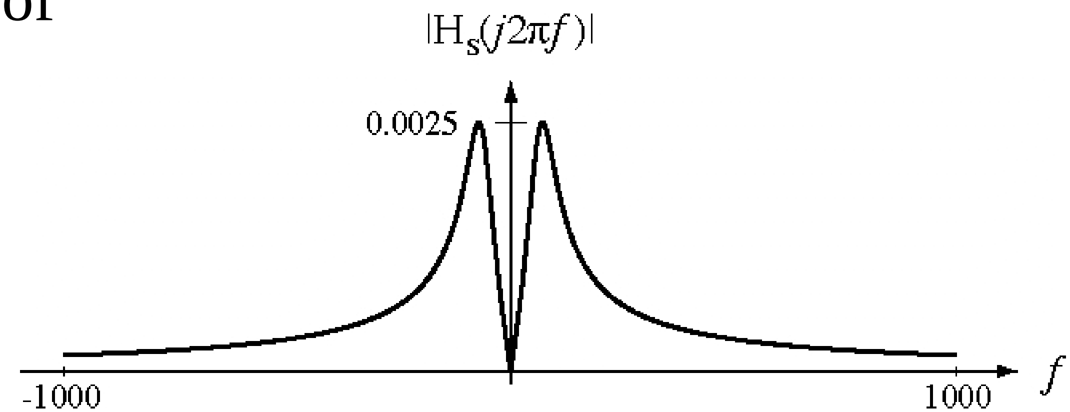
Impulse and Step Invariant Design

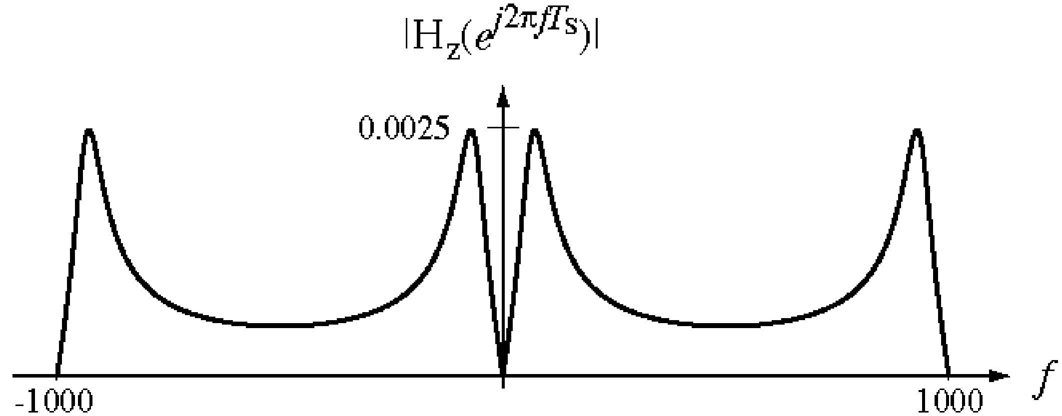
Step invariant approximation of

$$H_s(s) = \frac{s}{s^2 + 400s + 2 \times 10^5}$$

with a 1 kHz sampling rate yields

$$H_z(z) = \frac{7.97 \times 10^{-4}(z-1)}{z^2 - 1.509z + 0.6708}$$





Digital Filters

Finite Difference Design

Every CT transfer function implies a corresponding differential equation. For example,

$$\mathrm{H}_s(s) = \frac{1}{s+a} \Rightarrow \frac{d}{dt}(\mathrm{y}(t)) + a\,\mathrm{y}(t) = \mathrm{x}(t)$$

Derivatives can be approximated by finite differences.

Forward

$$\frac{d}{dt}(\mathrm{y}(t)) \cong \frac{\mathrm{y}[n+1] - \mathrm{y}[n]}{T_s}$$

Backward

$$\frac{d}{dt}(\mathrm{y}(t)) \cong \frac{\mathrm{y}[n] - \mathrm{y}[n-1]}{T_s}$$

Central

$$\frac{d}{dt}(\mathrm{y}(t)) \cong \frac{\mathrm{y}[n+1] - \mathrm{y}[n-1]}{2T_s}$$

Digital Filters
Finite Difference Design

Using a forward difference to approximate the derivative,

$$\mathrm{H}_s(s) = \frac{1}{s+a} \Rightarrow \frac{\mathrm{y}[n+1] - \mathrm{y}[n]}{T_s} + a\,\mathrm{y}[n] = \mathrm{x}[n]$$

A more systematic method is to realize that every *s* in a CT transfer function corresponds to a differentiation in the time domain which can be approximated by a finite difference.

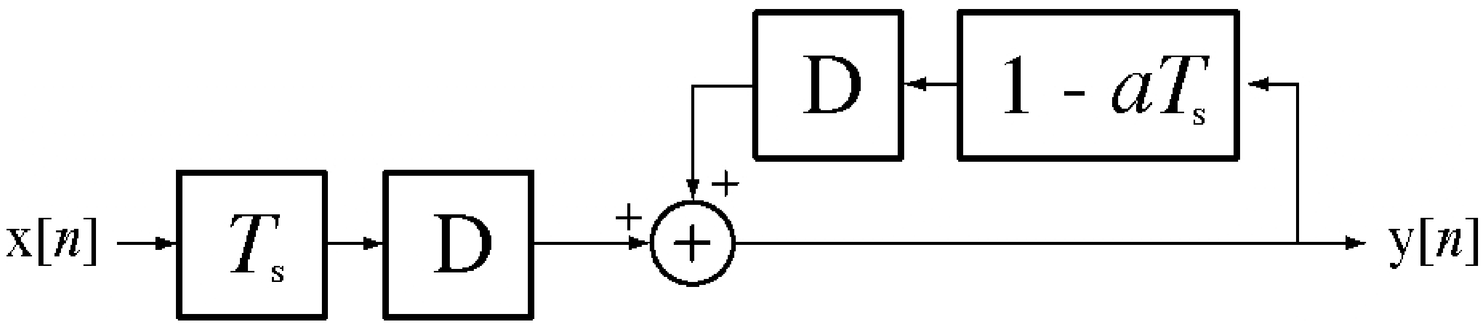
Forward	Backward	Central
$s \to \frac{z-1}{T_s}$	$s \to \frac{1-z^{-1}}{T_s}$	$s \to \frac{z-z^{-1}}{2T_s}$

Digital Filters

Finite Difference Design

Then

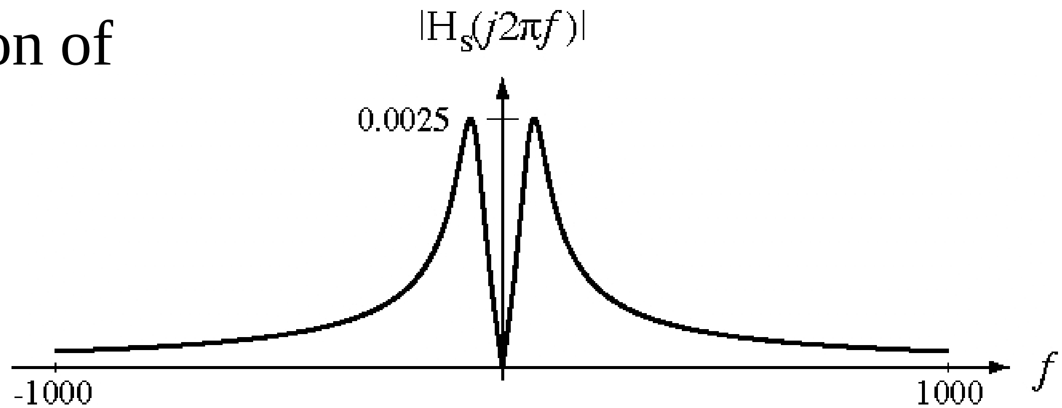
$$H_s(s) = \frac{1}{s+a} \Rightarrow H_z(z) = \left[\frac{1}{s+a}\right]_{s \to \frac{z-1}{T_s}} = \frac{T_s}{z-(1-aT_s)}$$



Digital Filters
Finite Difference Design

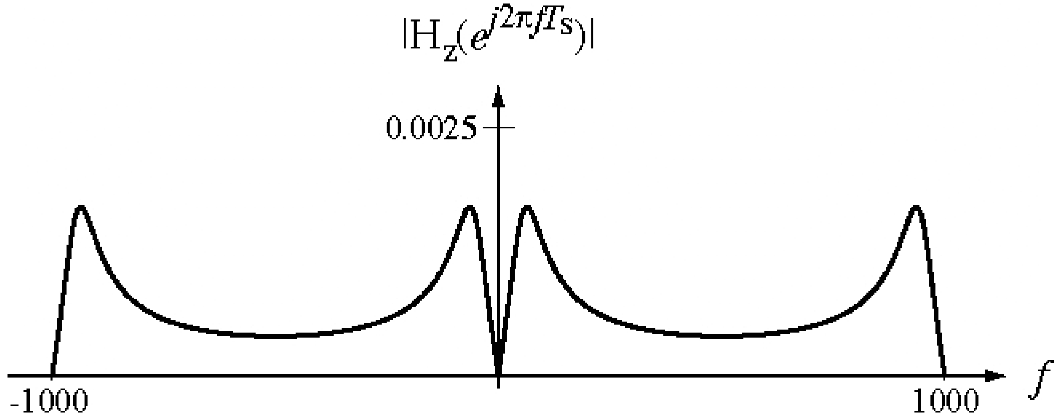
Finite difference approximation of

$$H_s(s) = \frac{s}{s^2 + 400s + 2\times 10^5}$$



with a 1 kHz sampling rate yields

$$H(z) = \frac{6.25\times 10^{-4} z(z-1)}{z^2 - 1.5z + 0.625}$$



Digital Filters

Direct Substitution and Matched *z*-Transform Design

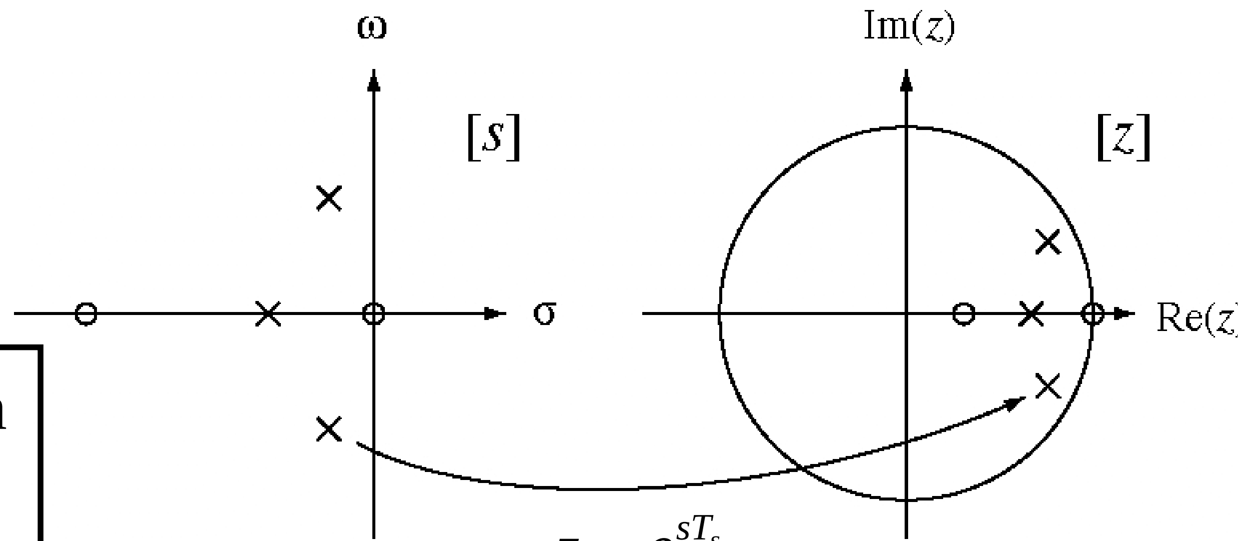
Direct substitution and matched filter design use the relationship, $z = e^{sT_s}$ to map the poles and zeros of an *s*-domain transfer function into corresponding poles and zeros of a *z*-domain transfer function. If there is an *s*-domain pole or zero at *a*, the *z*-domain pole or zero will be at e^{aT_s} .

Direct Substitution

$$s - a \to z - e^{aT_s}$$

Matched *z*-Transform

$$s - a \to 1 - e^{aT_s} z^{-1}$$



$$z = e^{sT_s}$$

Digital Filters

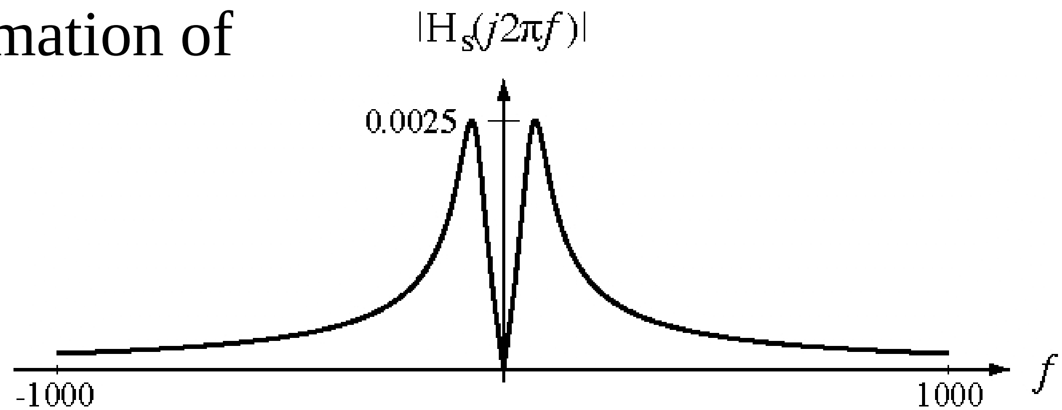
Direct Substitution and Matched *z*-Transform Design

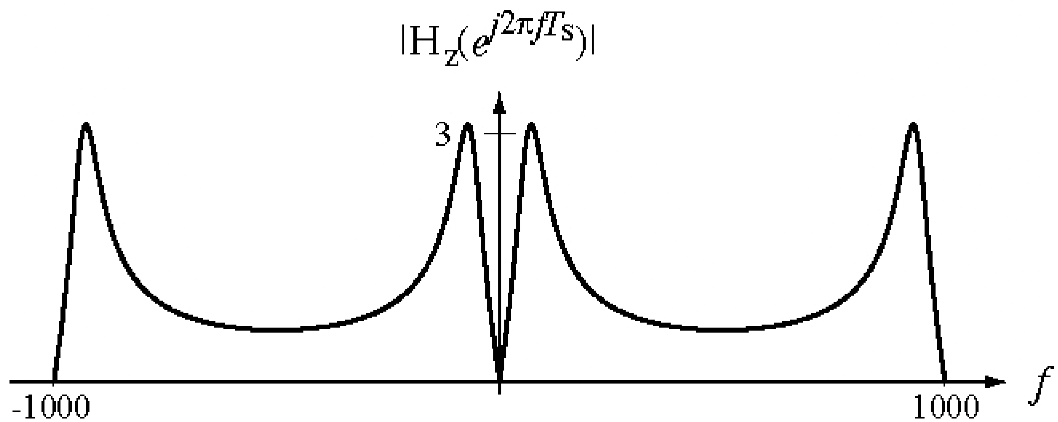
Matched *z*-transform approximation of

$$H_s(s) = \frac{s}{s^2 + 400s + 2\times10^5}$$

with a 1 kHz sampling rate yields

$$H_z(z) = \frac{z(z-1)}{z^2 - 1.509z + 0.6708}$$





Digital Filters
Bilinear Transformation

This method is based on trying to match the frequency response of a digital filter to that of the CT filter. As a practical matter it is impossible to match exactly because a digital filter has a periodic frequency response, but a good approximation can be made over a range of frequencies which can include all the expected signal power.

The basic idea is to use the transformation,

$$s \to \frac{1}{T_s}\ln(z) \qquad \text{or} \qquad e^{sT_s} \to z$$

to convert from the *s* to *z* domain.

Digital Filters

Bilinear Transformation

The straightforward application of the transformation, $s \to \frac{1}{T_s}\ln(z)$ would be the substitution,

$$\mathrm{H}_z(z) = \mathrm{H}_s(s)\big|_{s \to \frac{1}{T_s}\ln(z)}$$

But that yields a *z*-domain function that is a transcendental function of *z* with infinitely many poles. The exponential function can be expressed as the infinite series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and then approximated by truncating the series.

Digital Filters
Bilinear Transformation

Truncating the exponential series at two terms yields the transformation,

$$+ sT_s \to z$$

or

$$s \to \frac{z-1}{T_s}$$

This approximation is identical to the finite difference method using forward differences to approximate derivatives. This method has a problem. It is possible to transform a stable *s*-domain function into an unstable *z*-domain function.

Digital Filters

Bilinear Transformation

The stability problem can be solved by a very clever modification of the idea of truncating the series. Express the exponential as

$$e^{sT_s} = \frac{e^{s\frac{T_s}{2}}}{e^{-s\frac{T_s}{2}}} \rightarrow z$$

Then approximate both numerator and denominator with a truncated series.

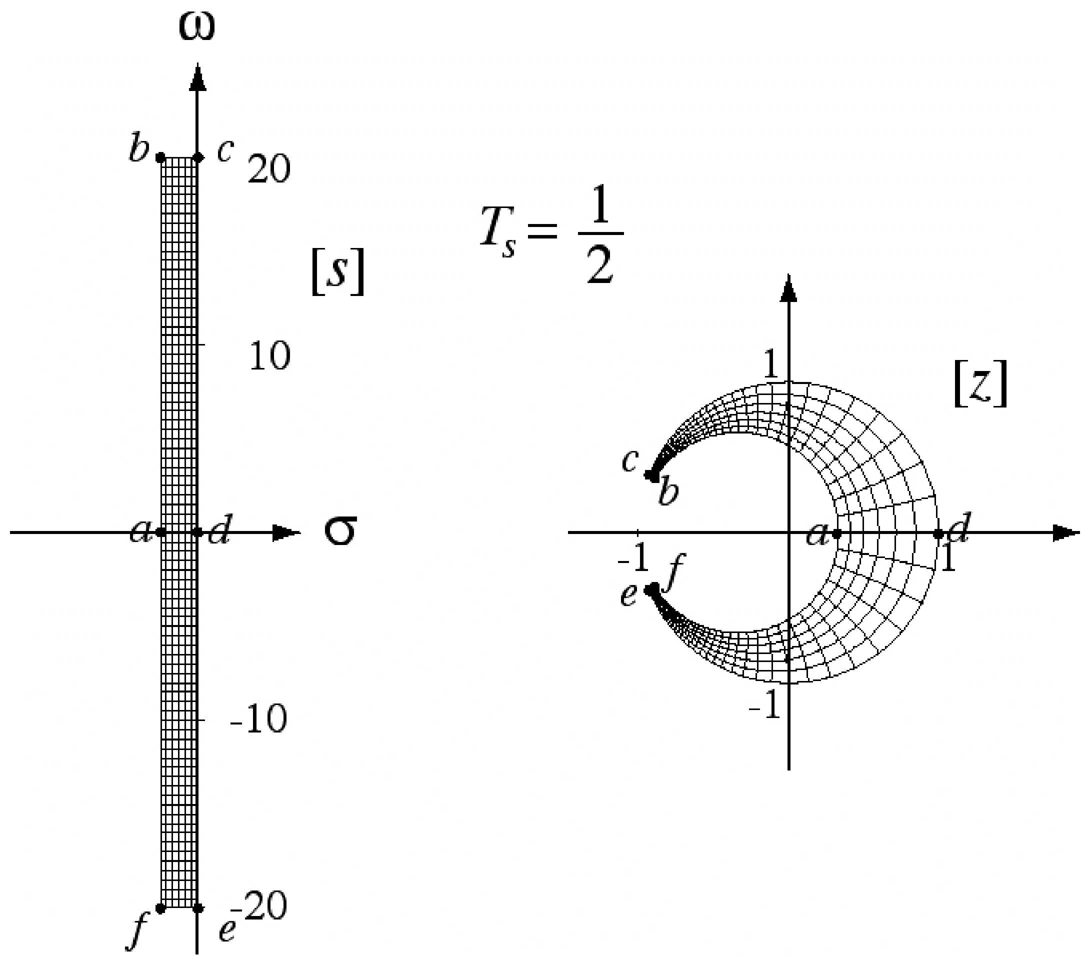
$$\frac{1+\frac{sT_s}{2}}{1-\frac{sT_s}{2}} \rightarrow z \quad \longrightarrow \quad s \rightarrow \frac{2}{T_s}\frac{z-1}{z+1}$$

This is called the *bilinear* transformation because both numerator and denominator are linear functions of *z*.

Digital Filters

Bilinear Transformation

The bilinear transformation has the quality that every point in the *s* plane maps into a unique point in the *z* plane, *and vice versa*. Also, the left half of the *s* plane maps into the interior of the unit circle in the *z* plane so a stable *s*-domain system is transformed into a stable *z*-domain system.



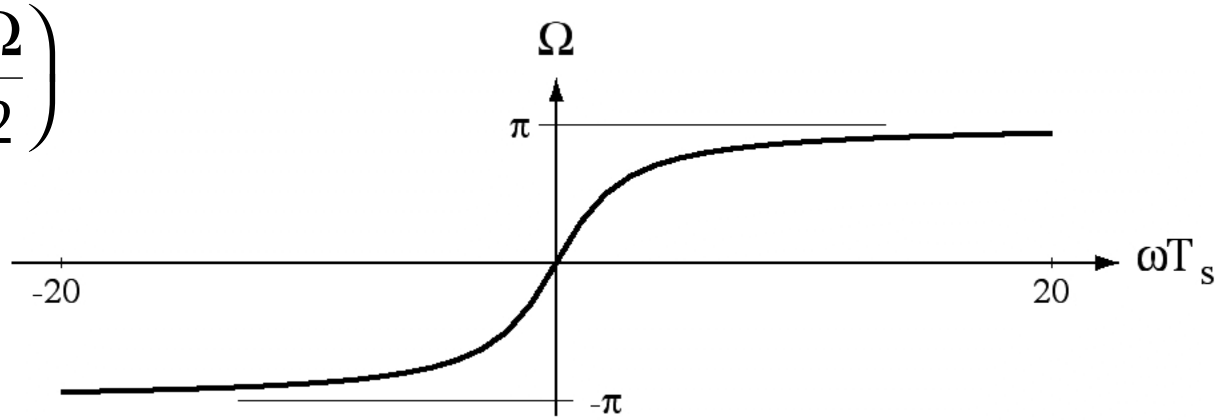
Digital Filters
Bilinear Transformation

The bilinear transformation is unique among the digital filter design methods because of the unique mapping of points between the two complex planes. There is however a "warping" effect. It can be seen by mapping real frequencies in the *z* plane (the unit circle) into corresponding points in the *s* plane (the ω axis). Letting $z = e^{j\Omega}$ with Ω real, the corresponding contour in the *s* plane is

$$s = \frac{2}{T_s}\frac{e^{j\Omega}-1}{e^{j\Omega}+1} = j\frac{2}{T_s}\tan\left(\frac{\Omega}{2}\right)$$

or

$$\Omega = 2\tan^{-1}\left(\frac{\omega T_s}{2}\right)$$



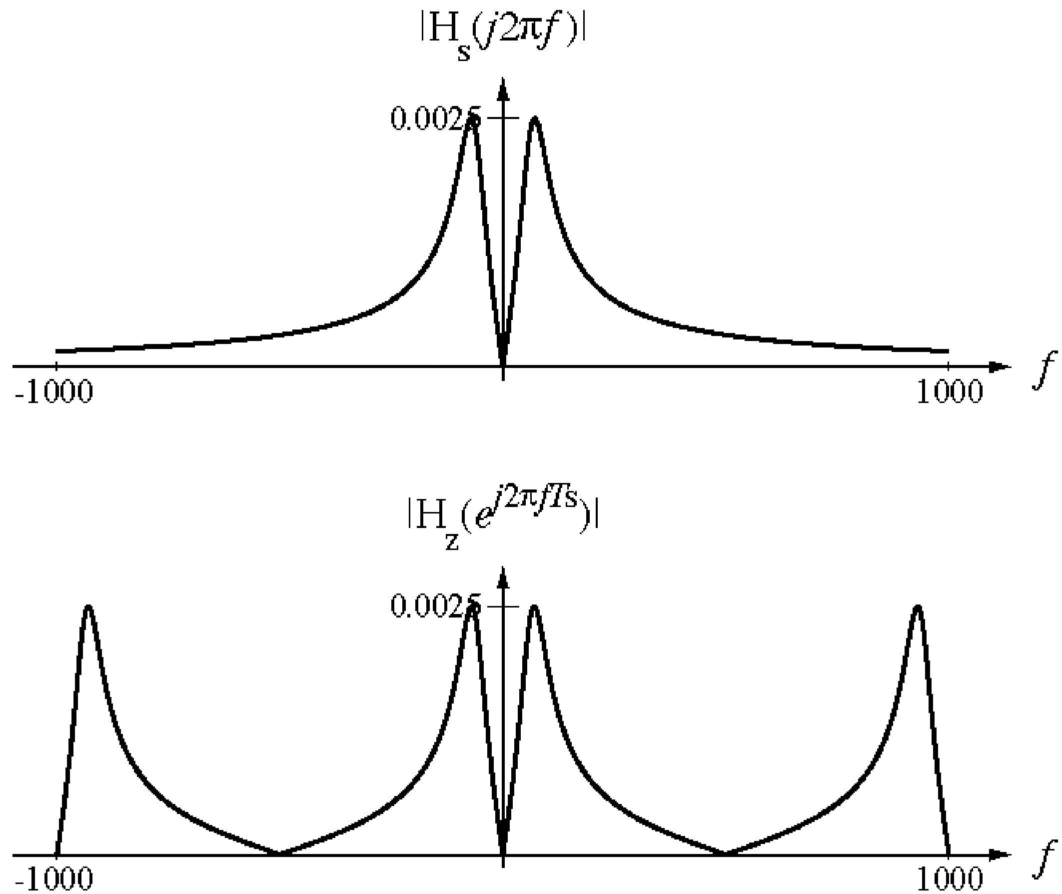
Digital Filters
Bilinear Transformation

Bilinear approximation of

$$\mathrm{H}_s(s) = \frac{s}{s^2 + 400s + 2\times10^5}$$

with a 1 kHz sampling rate yields

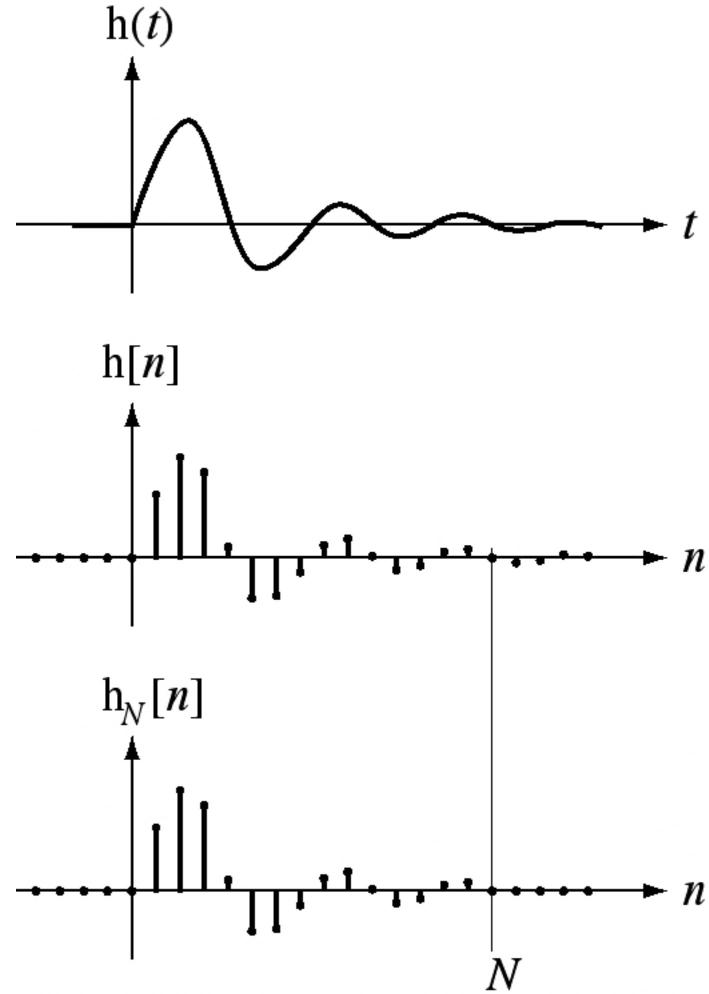
$$\mathrm{H}(z) = \frac{z^2 - 1}{z^2 - 1.52z + 0.68}$$



Digital Filters

FIR Filters

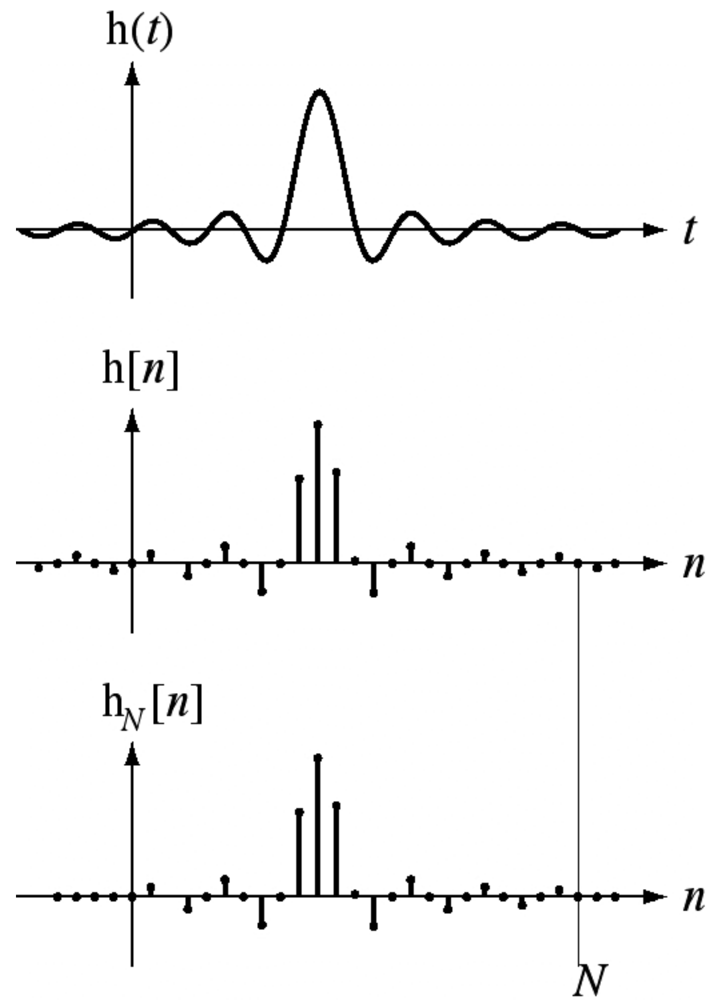
FIR digital filters are based on the idea of approximating an ideal impulse response. Practical CT filters have infinite-duration impulse responses. The FIR filter approximates this impulse by sampling it and then *truncating* it to a finite time (N impulses in the illustration).



Digital Filters

FIR Filters

FIR digital filters can also approximate *non-causal* filters by truncating the impulse response both before time $t = 0$ and after some later time which includes most of the signal energy of the ideal impulse response.

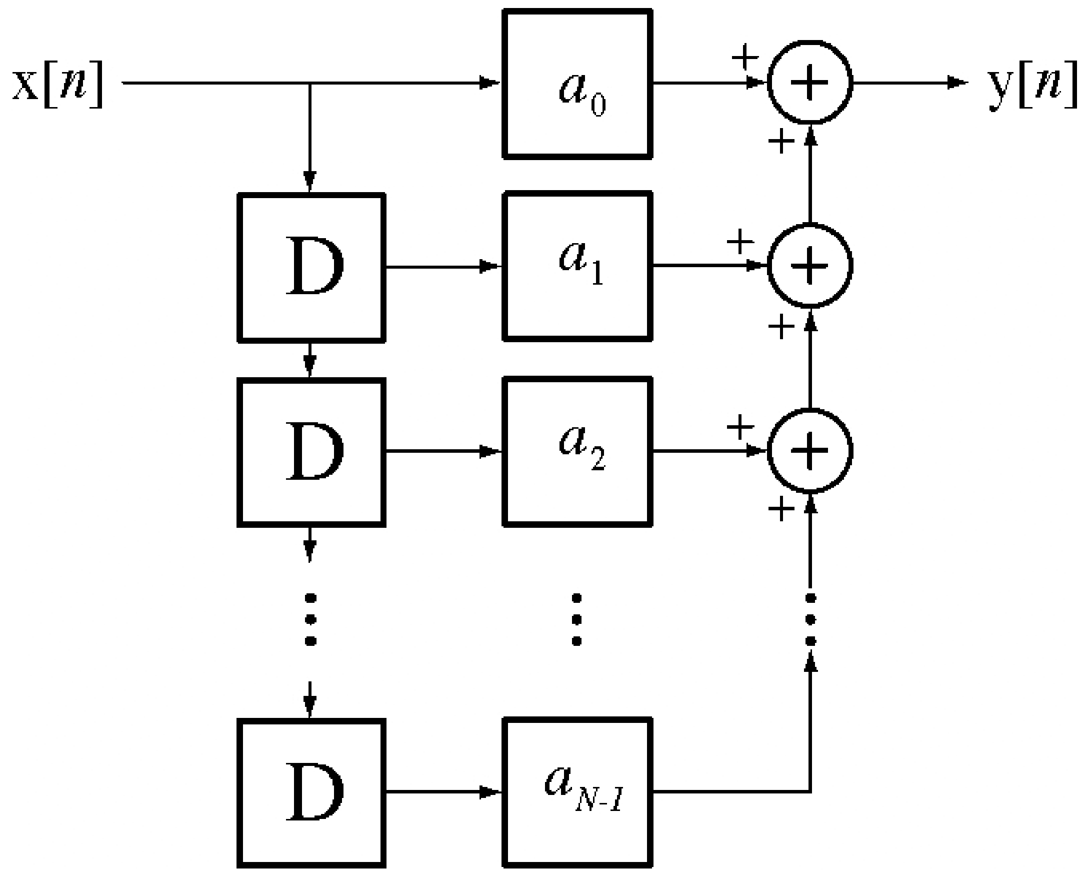


Digital Filters

FIR Filters

The design of an FIR filter is the essence of simplicity. It consists of multiple *feedforward* paths, each with a different delay and weighting factor and all of which are summed to form the response.

$$\mathrm{h}_N[n] = \sum_{m=0}^{N-1} a_m \delta[n-m]$$



Digital Filters

FIR Filters

Since this filter has no feedback paths its transfer function is of the form,

$$H_N(z) = \sum_{m=0}^{N-1} a_m z^{-m}$$

and it is guaranteed stable because it has N - 1 poles, all of which are located at $z = 0$.

Digital Filters
FIR Filters

The effect of truncating an impulse response can be modeled by multiplying the ideal impulse response by a “window” function. If a CT filter’s impulse response is truncated between $t = 0$ and $t = T$, the truncated impulse response is

$$\mathrm{h}_T(t) = \begin{Bmatrix} \mathrm{h}(t) & , & 0 < t < T \\ 0 & , & \text{otherwise} \end{Bmatrix} = \mathrm{h}(t)\mathrm{w}(t)$$

where, in this case,

$$\mathrm{w}(t) = \mathrm{rect}\left(\frac{t - \frac{T}{2}}{T}\right)$$

Digital Filters
FIR Filters

The frequency-domain effect of truncating an impulse response is to convolve the ideal frequency response with the transform of the window function.

$$\mathrm{H}_T(f) = \mathrm{H}(f) * W(f)$$

If the window is a rectangle,

$$\mathrm{W}(f) = T\,\mathrm{sinc}(Tf)e^{-j\pi fT}$$

Digital Filters

FIR Filters

Let the ideal transfer function be $\mathrm{H}(f) = \mathrm{rect}\left(\frac{f}{2B}\right)e^{-j\pi fT}$

The corresponding impulse response is

$$\mathrm{h}(t) = 2B\,\mathrm{sinc}\left(2B\left(t - \frac{T}{2}\right)\right)$$

The truncated impulse response is

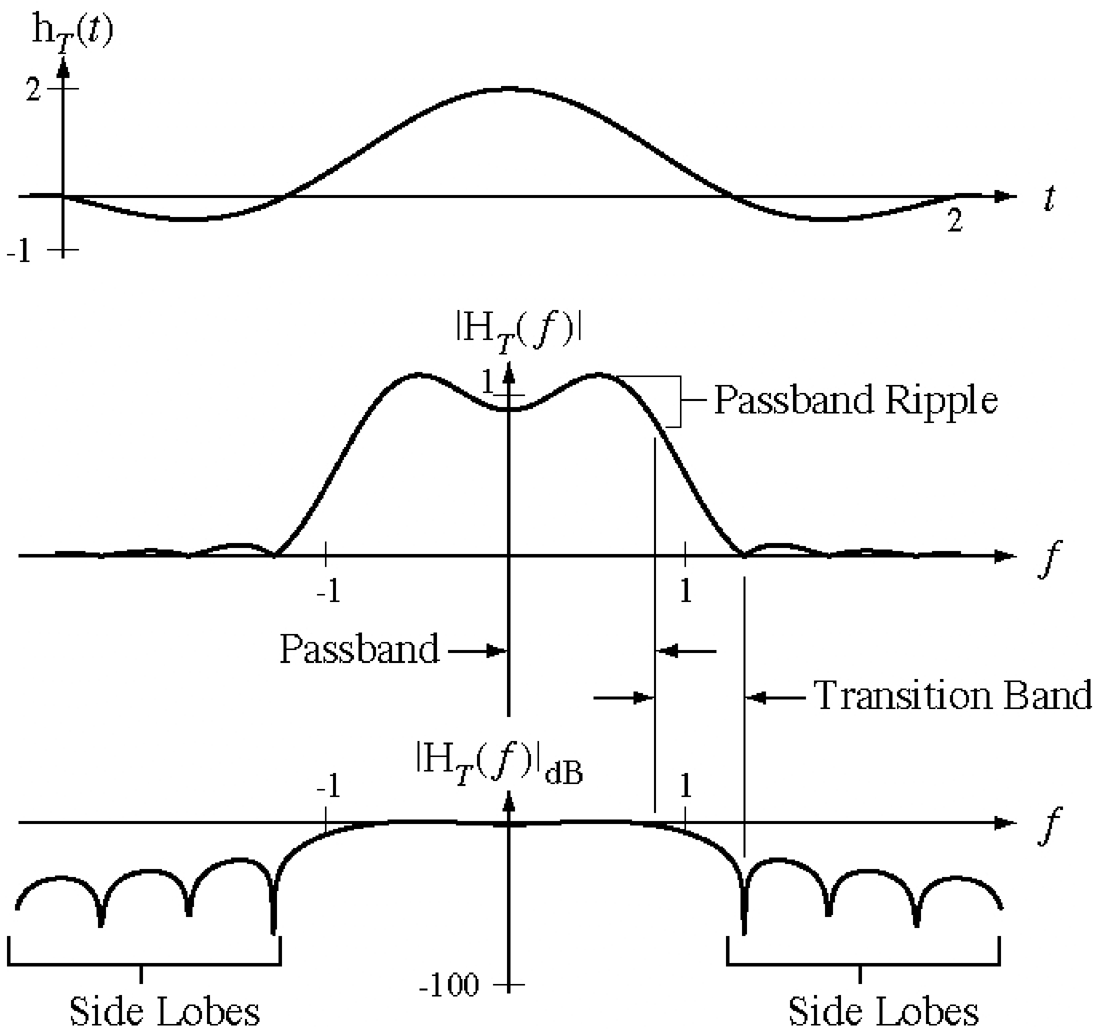
$$\mathrm{h}_T(t) = 2B\,\mathrm{sinc}\left(2B\left(t - \frac{T}{2}\right)\right)\mathrm{rect}\left(\frac{t - \frac{T}{2}}{T}\right)$$

The transfer function for the truncated impulse response is

$$\mathrm{H}_T(f) = \mathrm{rect}\left(\frac{f}{2B}\right)e^{-j\pi fT} * T\,\mathrm{sinc}(Tf)e^{-j\pi fT}$$

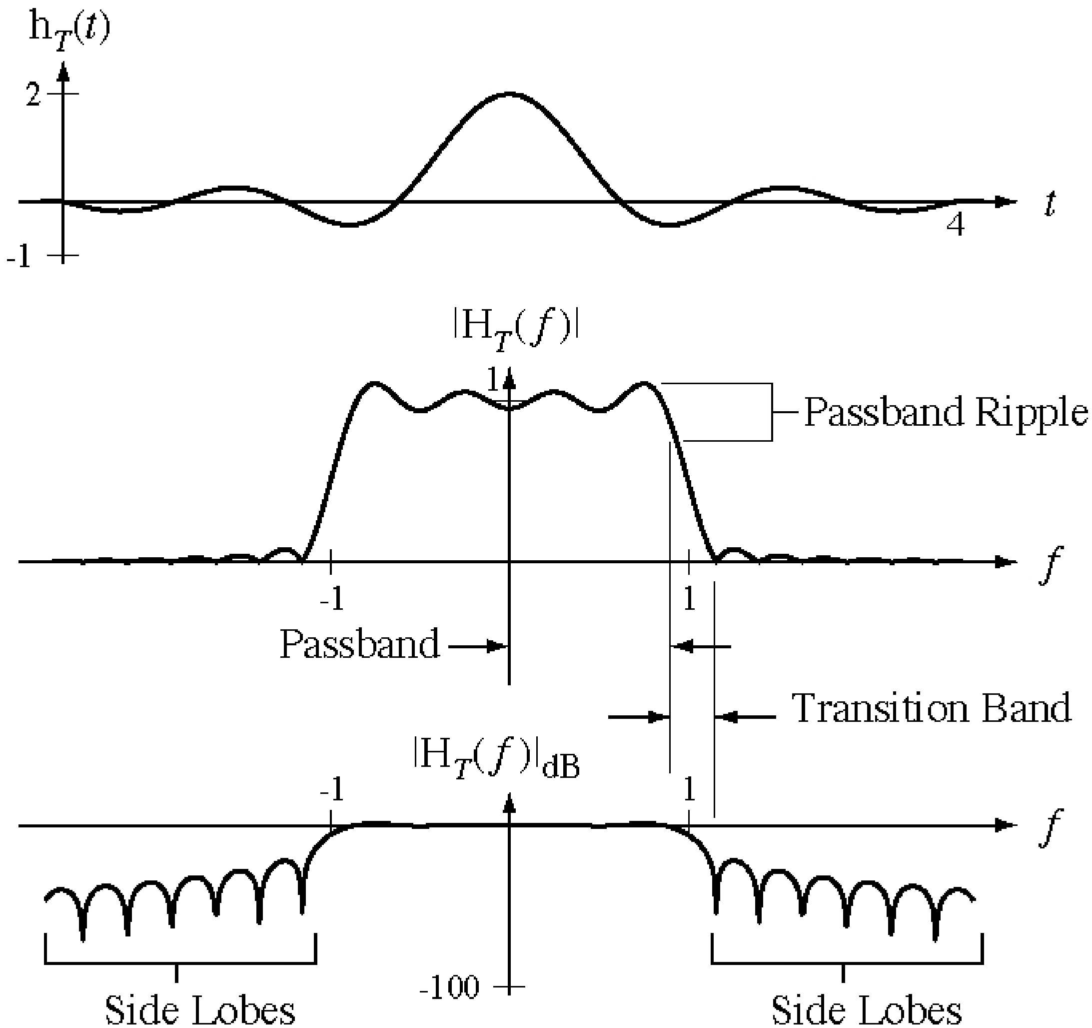
Digital Filters

FIR Filters



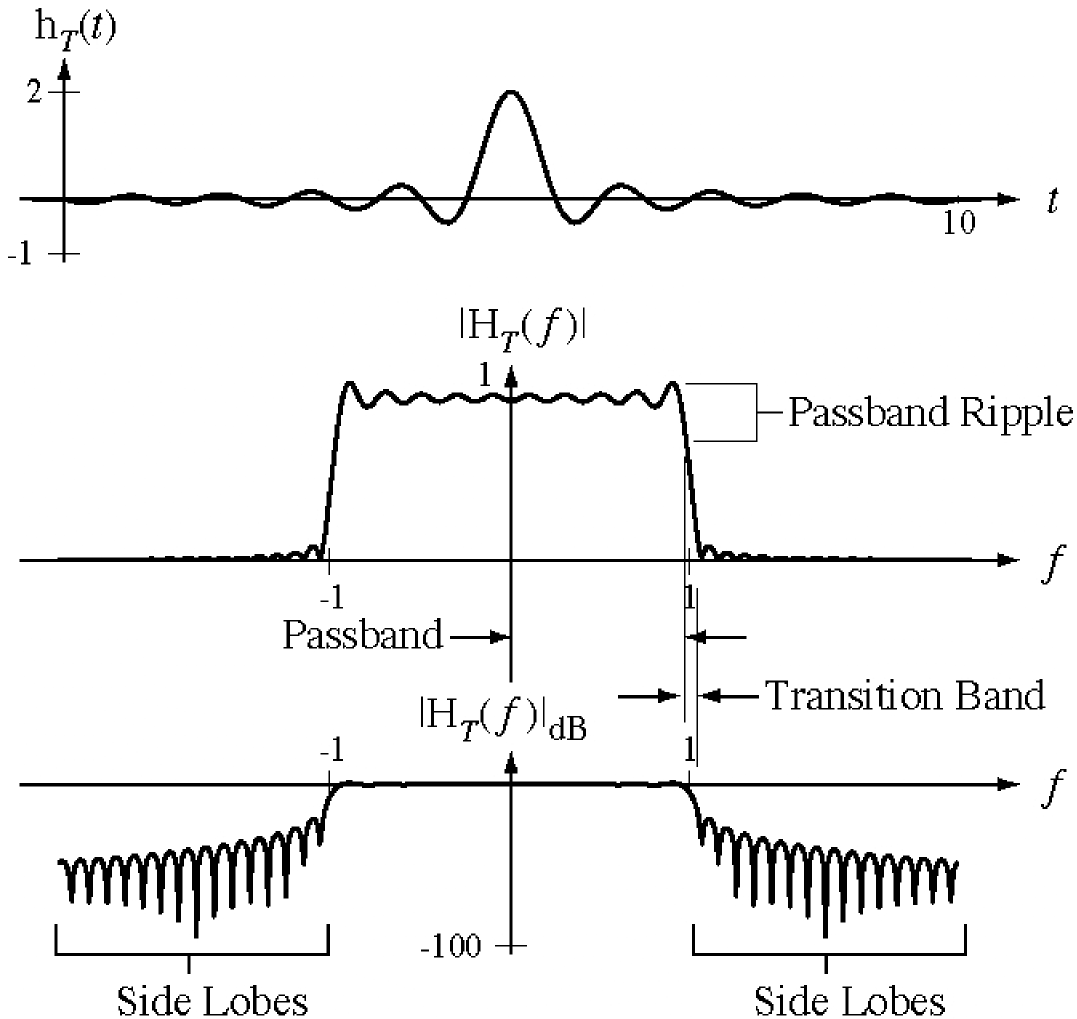
Digital Filters

FIR Filters



Digital Filters

FIR Filters



Digital Filters
FIR Filters

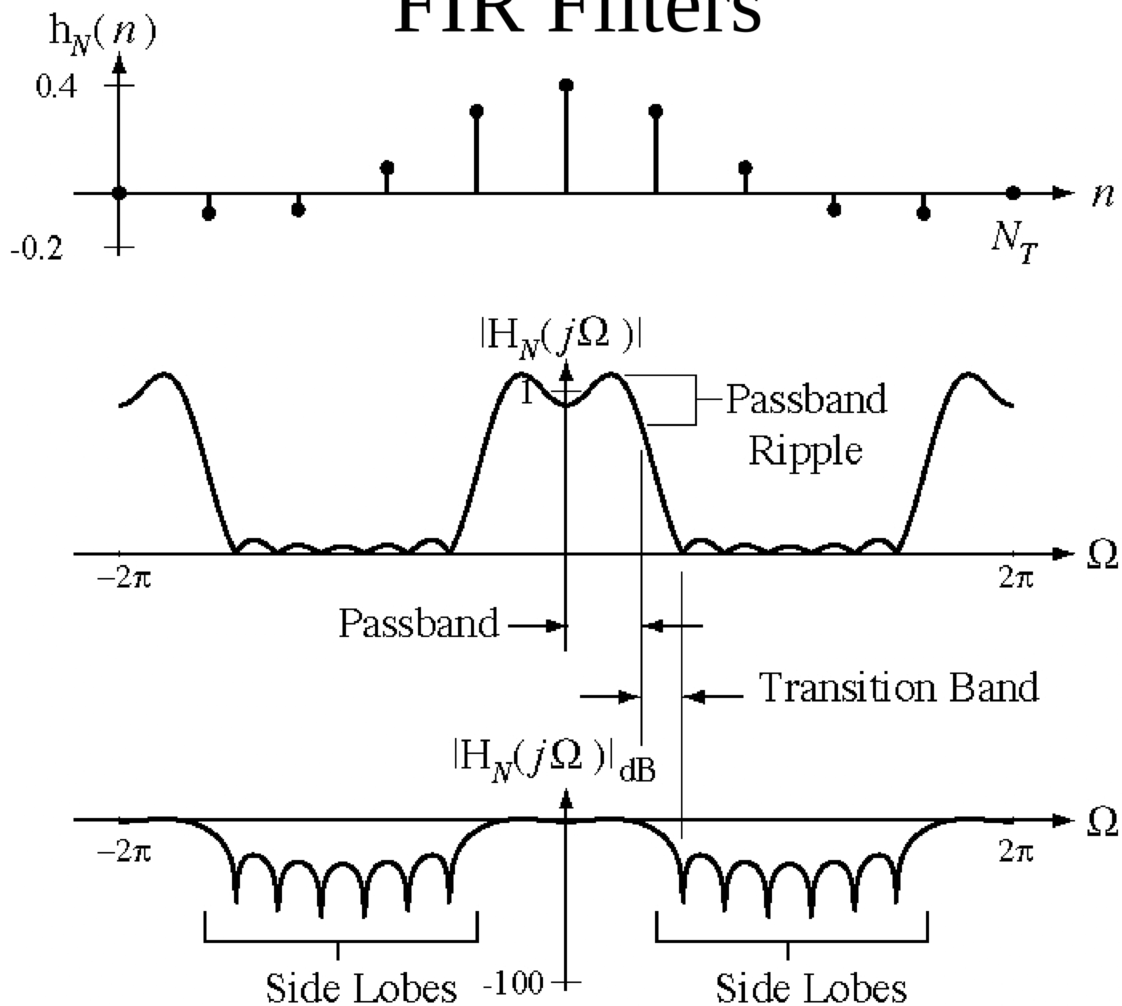
The effects of windowing a digital filter's impulse response are similar to the windowing effects on a CT filter.

$$\mathrm{h}_N[n] = \begin{cases} \mathrm{h}[n] & , \quad 0 \le n < N \\ 0 & , \quad \text{otherwise} \end{cases} = \mathrm{h}[n]\mathrm{w}[n]$$

$$\mathrm{H}_N(j\Omega) = \mathrm{H}(j\Omega) \circledast \mathrm{W}(j\Omega)$$

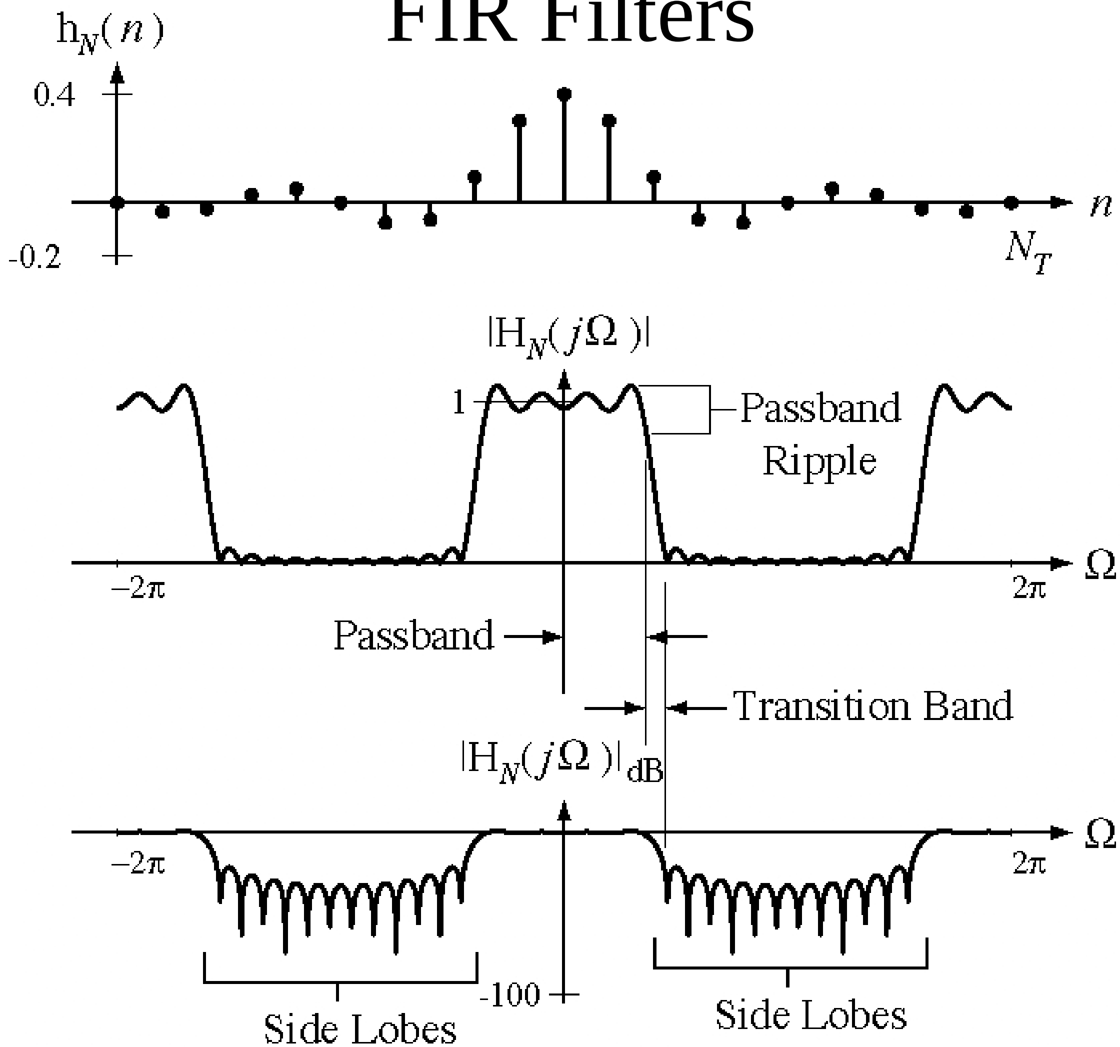
Digital Filters

FIR Filters



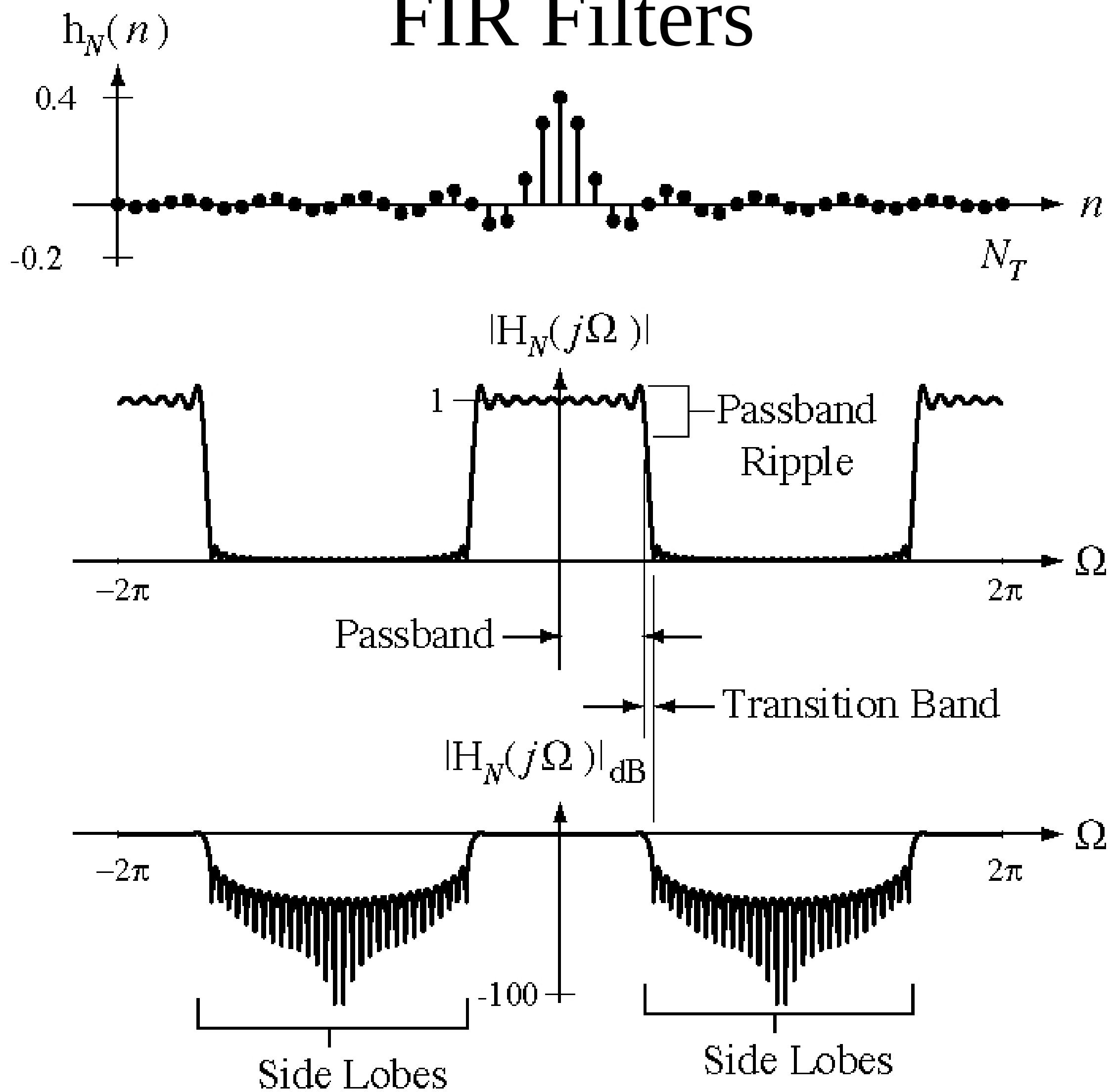
Digital Filters

FIR Filters



Digital Filters

FIR Filters



Digital Filters
FIR Filters

The “ripple” effect in the frequency domain can be reduced by using windows of different shapes. The shapes are chosen to have DTFT’s which are more confined to a narrow range of frequencies. Some commonly-used windows are

1. von Hann $\mathrm{w}[n] = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{N-1}\right)\right] \ , \ 0 \le n < N$

2. Bartlett

$$\mathrm{w}[n] = \begin{cases} \frac{2n}{N-1} & , \ 0 \le n \le \frac{N-1}{2} \\ 2 - \frac{2n}{N-1} & , \ \frac{N-1}{2} \le n < N \end{cases}$$

Digital Filters

FIR Filters

(windows continued)

3. Hamming $\mathrm{w}[n] = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)\ ,\ \ 0 \le n < N$

4. Blackman

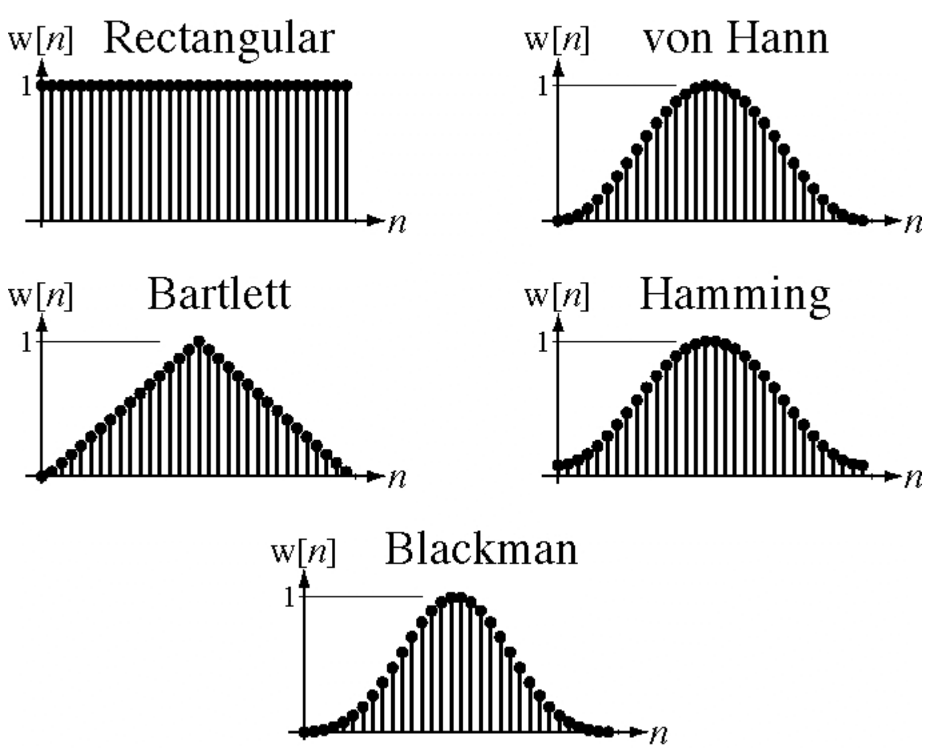
$$\mathrm{w}[n] = 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right)\ ,\ \ 0 \le n < N$$

5. Kaiser

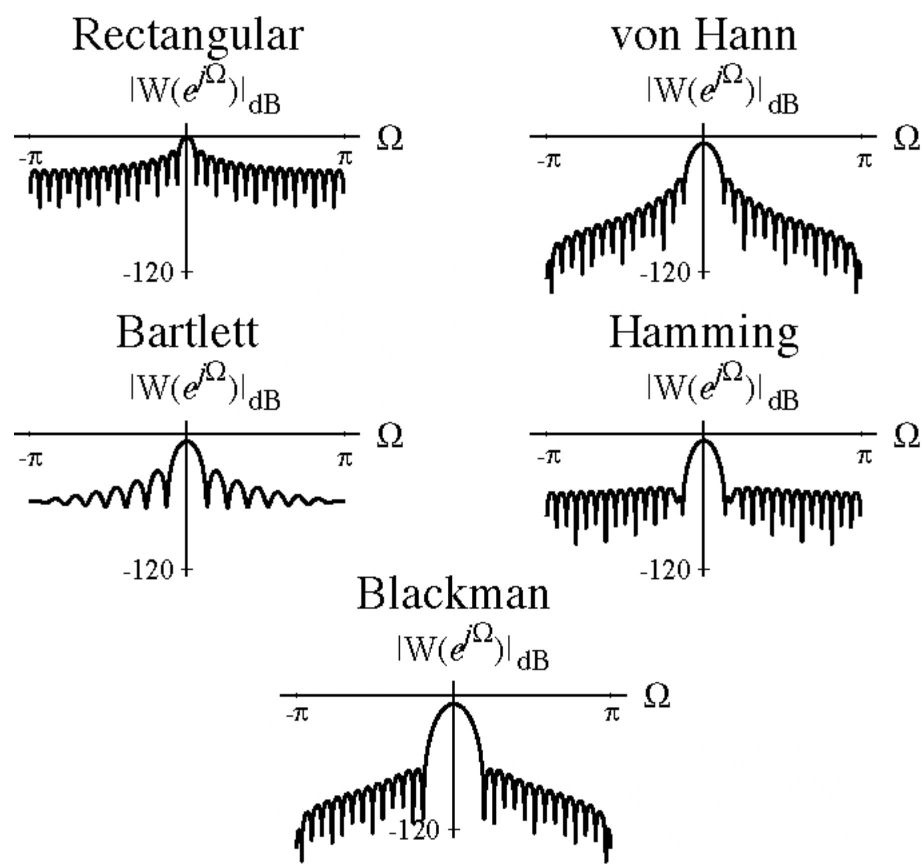
$$\mathrm{w}[n] = \frac{I_0\left(\omega_a\sqrt{\left(\frac{N-1}{2}\right)^2 - \left(n - \frac{N-1}{2}\right)^2}\right)}{I_0\left(\omega_a \frac{N-1}{2}\right)}$$

Digital Filters
FIR Filters

Windows



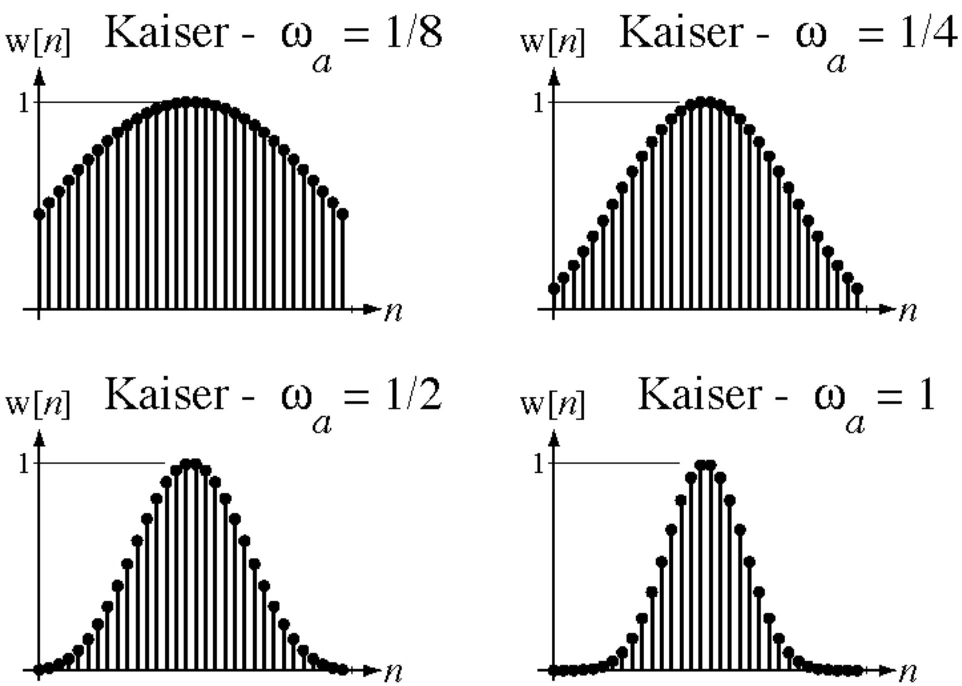
Window Transforms



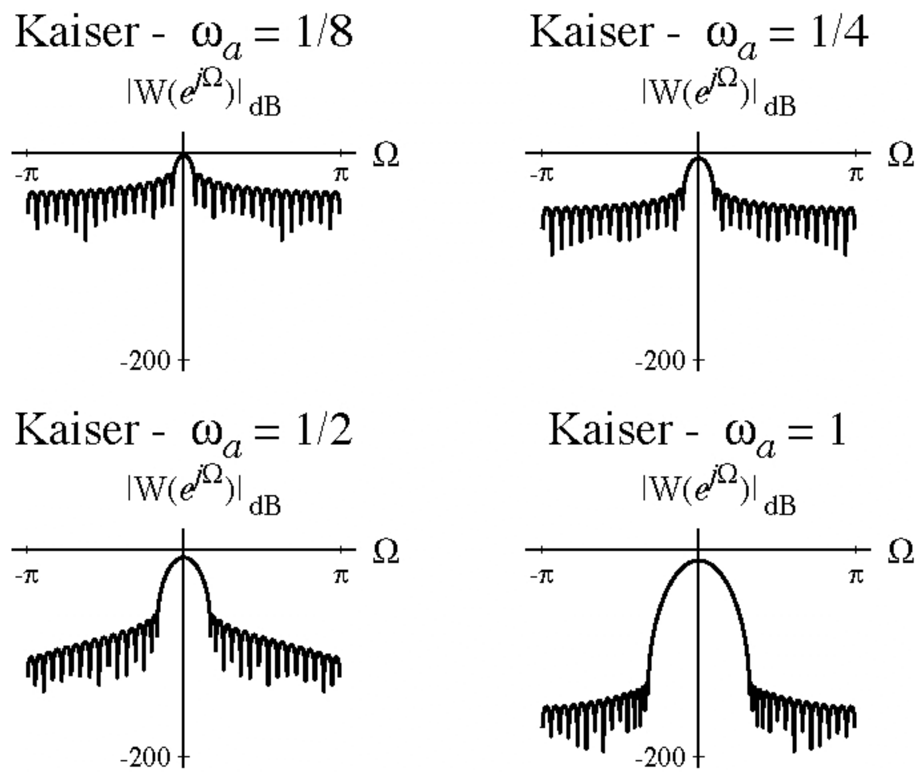
Digital Filters

FIR Filters

Windows



Window Transforms

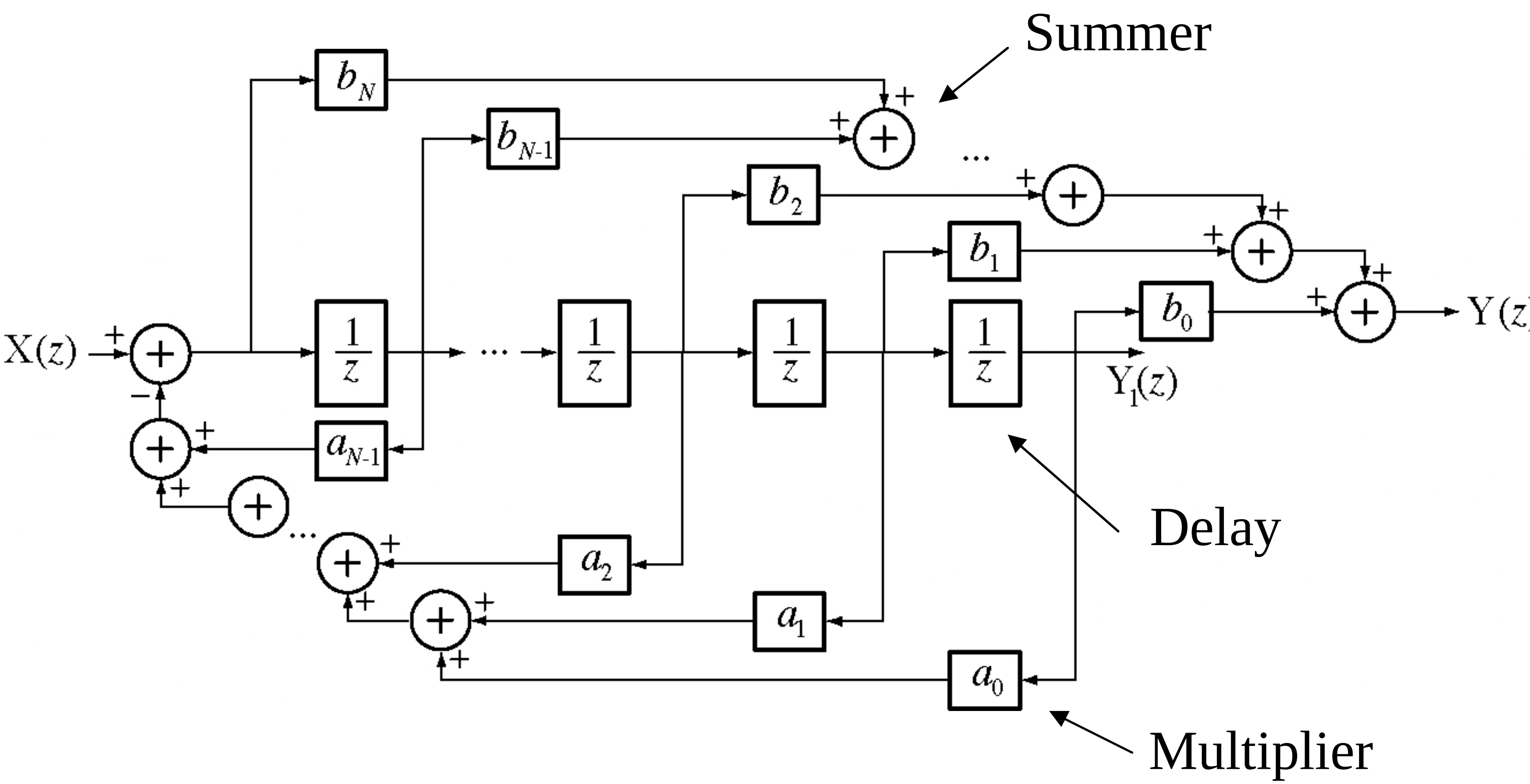


Standard Realizations

- Realization of a DT system closely parallels the realization of a CT system
- The basic forms, canonical, cascade and parallel have the same structure
- A CT system can be realized with integrators, summers and multipliers
- A DT system can be realized with delays, summers and multipliers

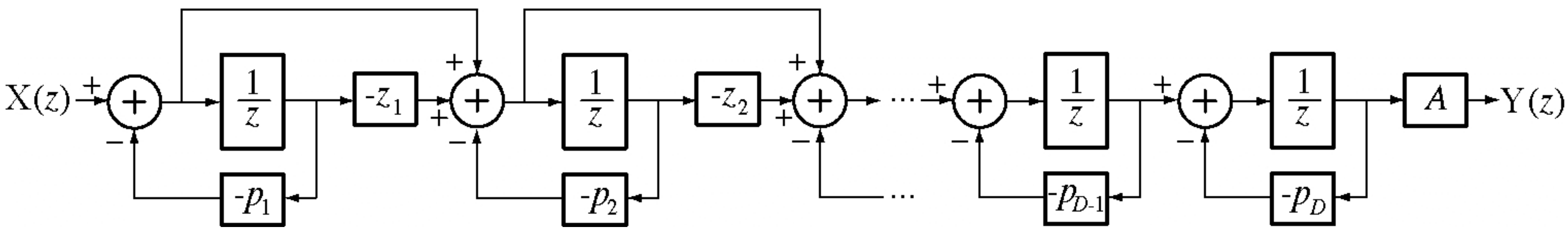
Standard Realizations

Canonical



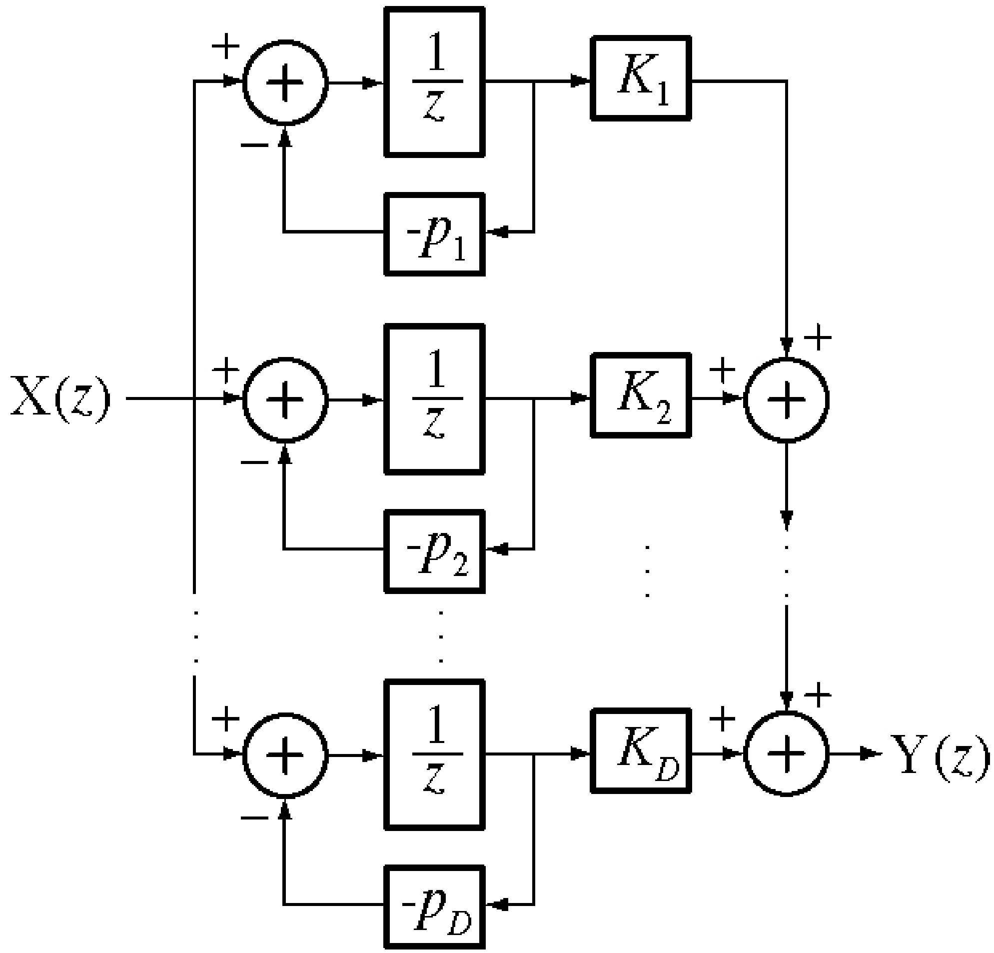
Standard Realizations

Cascade



Standard Realizations

Parallel



State-Space Analysis

In DT system state-space analysis the "next" state-variable values are set equal to a linear combination of the "present" state-variable values and the "present" excitations. The system and output equations are

$$\mathbf{q}[n+1] = \mathbf{A}\mathbf{q}[n] + \mathbf{B}\mathbf{x}[n] \quad , \quad \mathbf{y}[n] = \mathbf{C}\mathbf{q}[n] + \mathbf{D}\mathbf{x}[n]$$

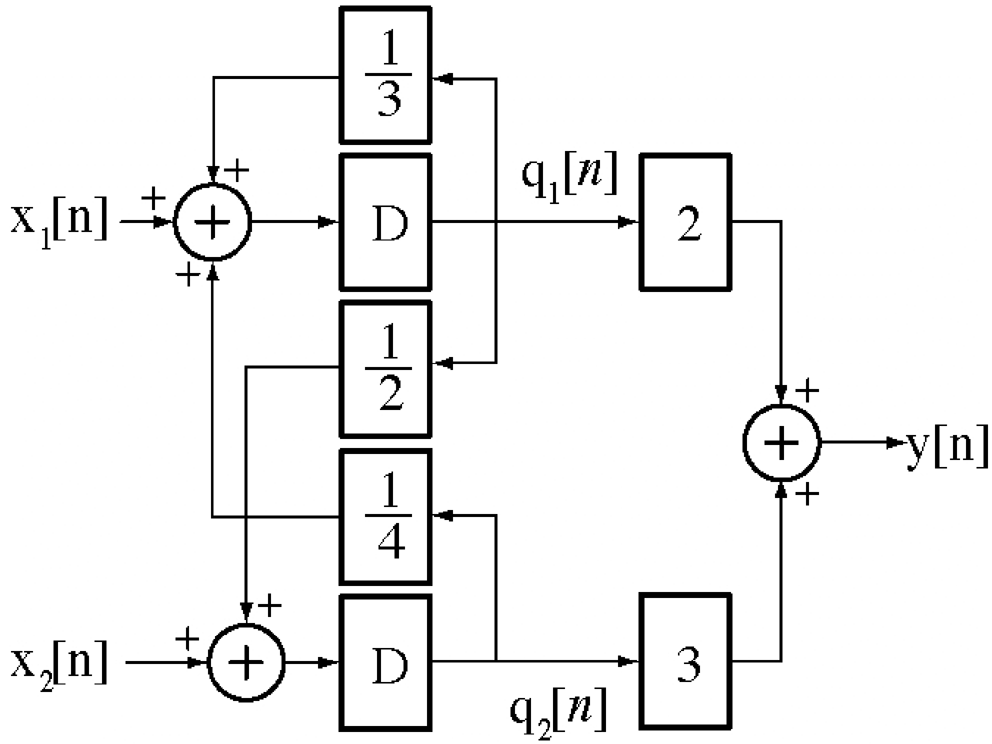
For this system,

$$\mathbf{q}[n] = \begin{bmatrix} q_1[n] \\ q_2[n] \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & 0 \end{bmatrix}$$

$$\mathbf{x}[n] = \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{y}[n] = [y[n]] \quad \mathbf{C} = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 0 \end{bmatrix}$$



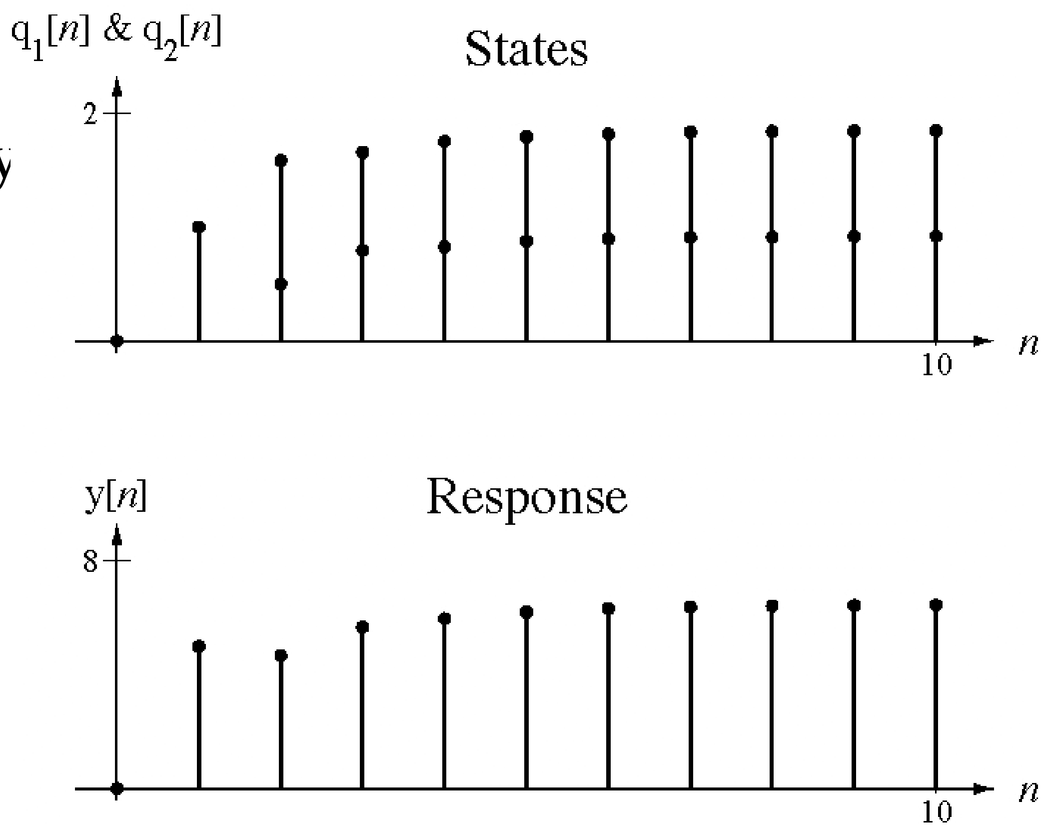
State-Space Analysis

For illustration purposed let the excitation vector be

$$\mathbf{x}[n] = \begin{bmatrix} \mathrm{u}[n] \\ \delta[n] \end{bmatrix}$$

and let the system be initially at rest. Then by direct recursion,

n	$q_1[n]$	$q_2[n]$	$y[n]$
0	0	0	0
1	1	1	5
2	1.5833	0.5	4.667
3	1.6528	0.7917	5.681
⋮	⋮	⋮	⋮



State-Space Analysis

The recursion process proceeds as follows

$$\mathbf{q}[1] = \mathbf{Aq}[0] + \mathbf{Bx}[0]$$

$$\mathbf{q}[2] = \mathbf{Aq}[1] + \mathbf{Bx}[1] = \mathbf{A}^2\mathbf{q}[0] + \mathbf{ABx}[0] + \mathbf{Bx}[1]$$

$$\mathbf{q}[3] = \mathbf{Aq}[2] + \mathbf{Bx}[2] = \mathbf{A}^3\mathbf{q}[0] + \mathbf{A}^2\mathbf{Bx}[0] + \mathbf{ABx}[1] + \mathbf{Bx}[2]$$

$$\vdots$$

$$\mathbf{q}[n] = \mathbf{A}^n\mathbf{q}[0] + \mathbf{A}^{n-1}\mathbf{Bx}[0] + \mathbf{A}^{n-2}\mathbf{Bx}[1] + \cdots + \mathbf{A}^1\mathbf{Bx}[n-2] + \mathbf{A}^0\mathbf{Bx}[n-1]$$

and

$$\mathbf{y}[1] = \mathbf{Cq}[1] + \mathbf{Dx}[1] = \mathbf{CAq}[0] + \mathbf{CBx}[0] + \mathbf{Dx}[1]$$

$$\mathbf{y}[2] = \mathbf{Cq}[2] + \mathbf{Dx}[2] = \mathbf{CA}^2\mathbf{q}[0] + \mathbf{CABx}[0] + \mathbf{CBx}[1] + \mathbf{Dx}[2]$$

$$\mathbf{y}[3] = \mathbf{Cq}[3] + \mathbf{Dx}[3] = \mathbf{CA}^3\mathbf{q}[0] + \mathbf{CA}^2\mathbf{Bx}[0] + \mathbf{CABx}[1] + \mathbf{CBx}[2] + \mathbf{Dx}[3]$$

$$\vdots$$

$$\mathbf{y}[n] = \mathbf{CA}^n\mathbf{q}[0] + \mathbf{CA}^{n-1}\mathbf{Bx}[0] + \mathbf{CA}^{n-2}\mathbf{Bx}[1] + \cdots + \mathbf{CA}^0\mathbf{Bx}[n-1] + \mathbf{Dx}[n]$$

State-Space Analysis

The recursions can be written in the more compact forms,

$$\mathbf{q}[n] = \underbrace{\mathbf{A}^n \mathbf{q}[0]}_{\text{Zero-Input Response}} + \underbrace{\sum_{m=0}^{n-1} \mathbf{A}^{n-m-1} \mathbf{B}\mathbf{x}[m]}_{\text{Zero-State Response}}$$

$$\mathbf{y}[n] = \mathbf{C}\mathbf{A}^n \mathbf{q}[0] + \mathbf{C} \sum_{m=0}^{n-1} \mathbf{A}^{n-m-1} \mathbf{B}\mathbf{x}[m] + \mathbf{D}\mathbf{x}[n]$$

These two equations can be written in the forms,

$$\mathbf{q}[n] = \underbrace{\phi[n]\mathbf{q}[0]}_{\text{zero-excitation response}} + \underbrace{\phi[n-1]\mathrm{u}[n-1] * \mathbf{B}\mathbf{x}[n]}_{\text{zero-state response}}$$

$$\mathbf{y}[n] = \mathbf{C}\phi[n]\mathbf{q}[0] + \mathbf{C}\phi[n-1]\mathrm{u}[n-1] * \mathbf{B}\mathbf{x}[n] + \mathbf{D}\mathbf{x}[n]$$

where $\mathbf{A}^n = \phi[n]$ (pp. 866-867).

State-Space Analysis

An alternate to the previous discrete-time-domain solution of the state and output equations is to solve them using the z transform. Transforming the system equation,

$$z\mathbf{Q}(z) - z\mathbf{q}[0] = \mathbf{A}\mathbf{Q}(z) + \mathbf{B}\mathbf{X}(z)$$

$$\mathbf{Q}(z) = [z\mathbf{I} - \mathbf{A}]^{-1}[\mathbf{B}\mathbf{X}(z) + z\mathbf{q}[0]] = \underbrace{[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}\mathbf{X}(z)}_{\substack{\text{zero-state} \\ \text{response}}} + \underbrace{z[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{q}[0]}_{\substack{\text{zero-excitation} \\ \text{response}}}$$

by comparing this equation with a previous one,

$$\mathbf{q}[n] = \underbrace{\phi[n]\mathbf{q}[0]}_{\substack{\text{zero-excitation} \\ \text{response}}} + \underbrace{\phi[n-1]\mathrm{u}[n-1] * \mathbf{B}\mathbf{x}[n]}_{\substack{\text{zero-state} \\ \text{response}}}$$

it is apparent that $\phi[n] \overset{\mathcal{Z}}{\longleftrightarrow} z[z\mathbf{I} - \mathbf{A}]^{-1}$ and therefore $\Phi(z) = z[z\mathbf{I} - \mathbf{A}]^{-1}$

State-Space Analysis

Let the excitation vector again be $\mathbf{x}[n] = \begin{bmatrix} \mathrm{u}[n] \\ \delta[n] \end{bmatrix}$ and let the system be initially at rest.

$$\mathbf{Q}(z) = \begin{bmatrix} z - \frac{1}{3} & -\frac{1}{4} \\ -\frac{1}{2} & z \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{z}{z-1} \\ 1 \end{bmatrix}$$

$$\mathbf{Q}(z) = \begin{bmatrix} \frac{1.846}{z-1} - \frac{0.578}{z-0.5575} - \frac{0.268}{z+0.2242} \\ \frac{0.923}{z-1} - \frac{0.519}{z-0.5575} + \frac{0.596}{z+0.2242} \end{bmatrix}$$

Inverse transforming (pg. 868),

$$\mathbf{q}[n] = \begin{bmatrix} 1.846 - 0.578(0.5575)^{(n-1)} - 0.268(-0.2242)^{(n-1)} \\ 0.923 - 0.519(0.5575)^{(n-1)} + 0.596(-0.2242)^{(n-1)} \end{bmatrix} \mathrm{u}[n-1]$$

State-Space Analysis

The response vector is easily found from the state-variable vector.

$$\mathbf{y}[n] = \left[6.461 - 2.713(0.5575)^{(n-1)} + 1.252(-0.2242)^{(n-1)}\right]\mathrm{u}[n-1]$$

The closed-form solution has the same initial values as the recursion solution indicating it is probably correct.

n	$\mathrm{q}_1[n]$	$\mathrm{q}_2[n]$	$\mathrm{y}[n]$
0	0	0	0
1	1	1	5
2	1.5833	0.5	4.667
3	1.6528	0.7917	5.681
⋮	⋮	⋮	⋮

State-Space Analysis

Some other results of state-space analysis that are similar to those from the CT-system case are

Transfer Function $\longrightarrow$ $\mathbf{H}(z) = \mathbf{C}[z\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B} + \mathbf{D}$

If $\mathbf{q}_2[n] = \mathbf{T}\mathbf{q}_1[n]$ and $\mathbf{q}_1[n+1] = \mathbf{A}_1\mathbf{q}_1[n] + \mathbf{B}_1\mathbf{x}[n]$ then

$$\mathbf{q}_2[n+1] = \mathbf{A}_2\mathbf{q}_2[n] + \mathbf{B}_2\mathbf{x}[n]$$

where $\mathbf{A}_2 = \mathbf{T}\mathbf{A}_1\mathbf{T}^{-1}$ and $\mathbf{B}_2 = \mathbf{T}\mathbf{B}_1$ and

$$\mathbf{y}[n] = \mathbf{C}_2\mathbf{q}_2[n] + \mathbf{D}_2\mathbf{x}[n]$$

where $\mathbf{C}_2 = \mathbf{C}_1\mathbf{T}^{-1}$ and $\mathbf{D}_2 = \mathbf{D}_1$.