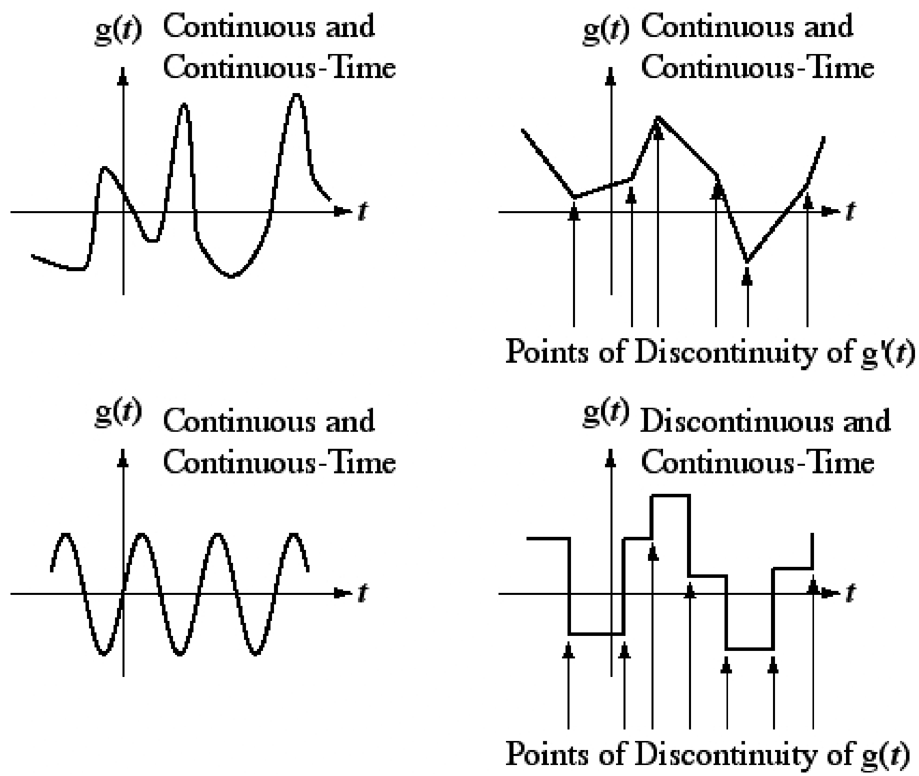
Mathematical Description of Signals

Continuous vs Continuous-Time Signals

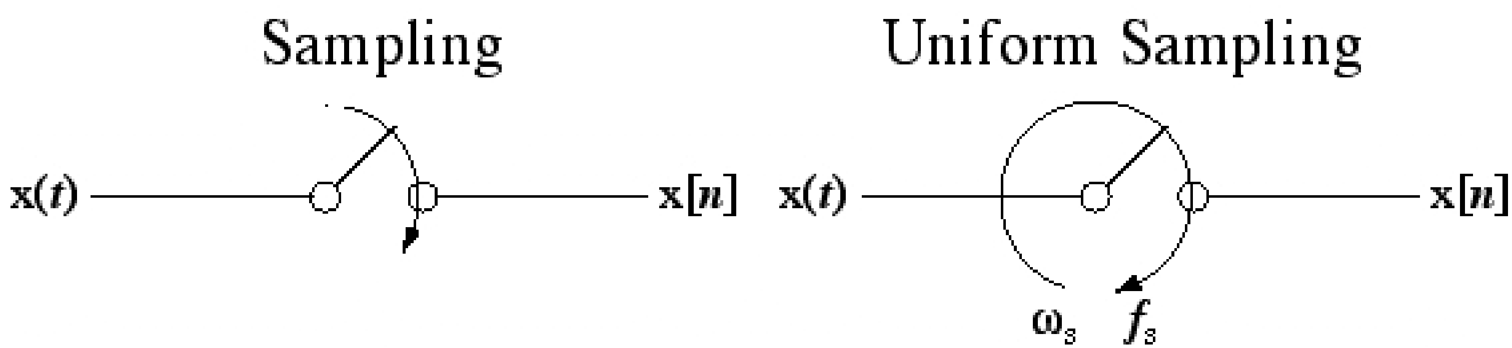
All continuous signals that are functions of time are *continuous-time (CT)* but not all CT signals are continuous



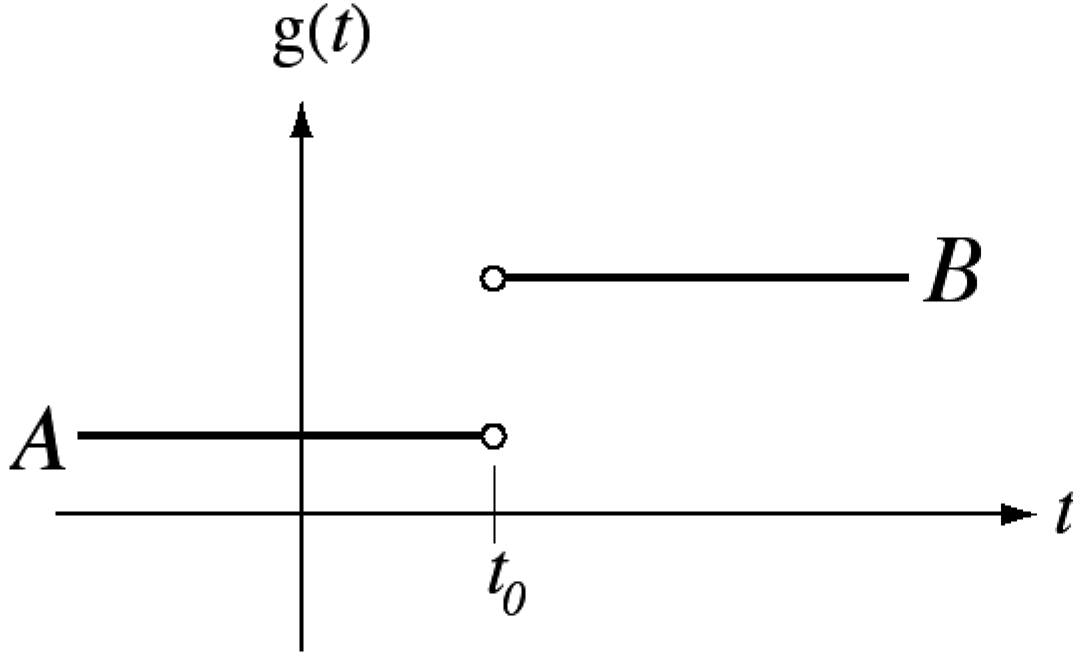
Sampling a CT Signal to Create a Discrete-Time (DT) Signal

- *Sampling* is the acquisition of the values of a CT signal at discrete points in time
- x(t) is a CT signal, x[n] is a DT signal

$\mathrm{x}[n] = \mathrm{x}(nT_s)$ where T_s is the time between samples



Introduction to the CT Unit Step



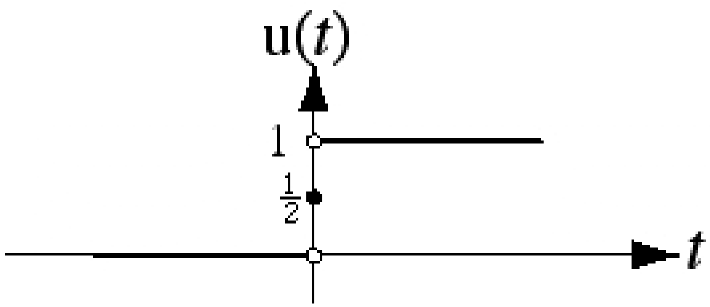
$$\lim_{\varepsilon \to 0} \int_{t_0-\varepsilon}^{t_0+\varepsilon} \mathrm{g}(t)\,dt = 0 \qquad \lim_{t \to t_0^-} \mathrm{g}(t) = A \neq B = \lim_{t \to t_0^+} \mathrm{g}(t)$$

There is no area under the single point, $\mathrm{g}(t_0)$, so the function value at that one point (if it is finite) does not affect the integral's value.

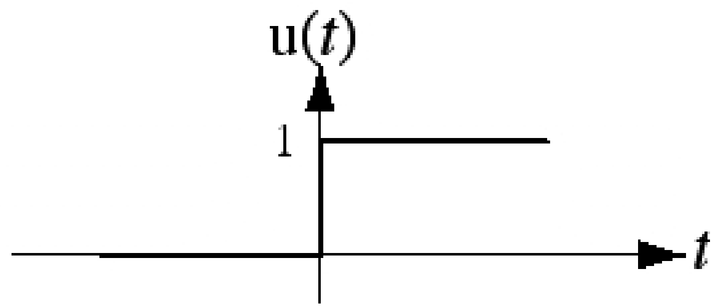
The CT Unit Step Function

$$\mathrm{u}(t) = \begin{cases} 1 \ , & t > 0 \\ \frac{1}{2} \ , & t = 0 \\ 0 \ , & t < 0 \end{cases}$$

Precise Graph



Commonly-Used Graph

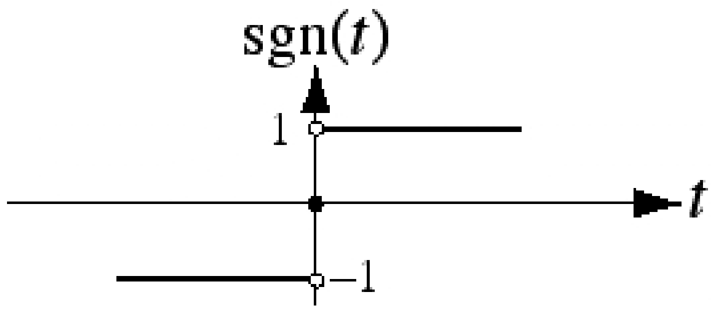


The product signal, g(t)u(t), can be thought of as the signal, g(t), "turned on" at time, $t = 0$.

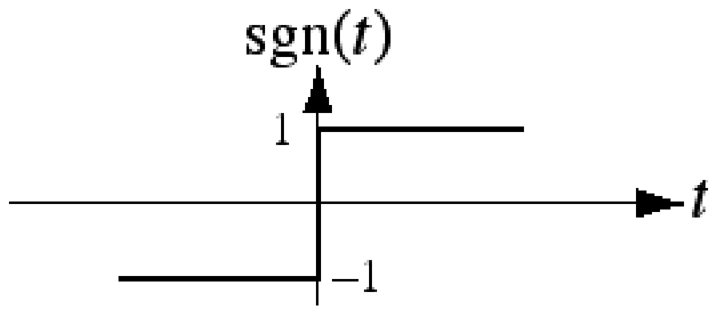
The CT Signum Function

$$\text{sgn}(t) = \begin{cases} 1 & , \ t > 0 \\ 0 & , \ t = 0 \\ -1 & , \ t < 0 \end{cases} = 2\,\text{u}(t) - 1$$

Precise Graph



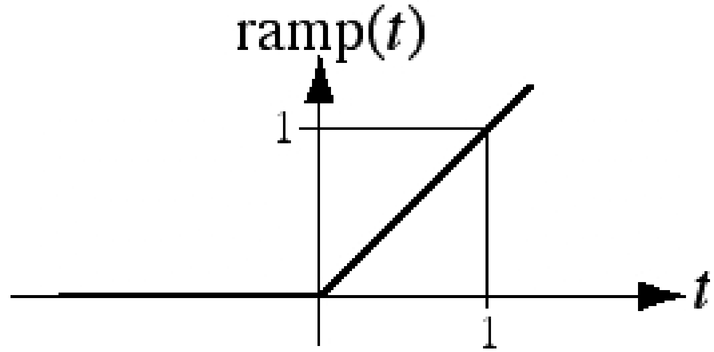
Commonly-Used Graph

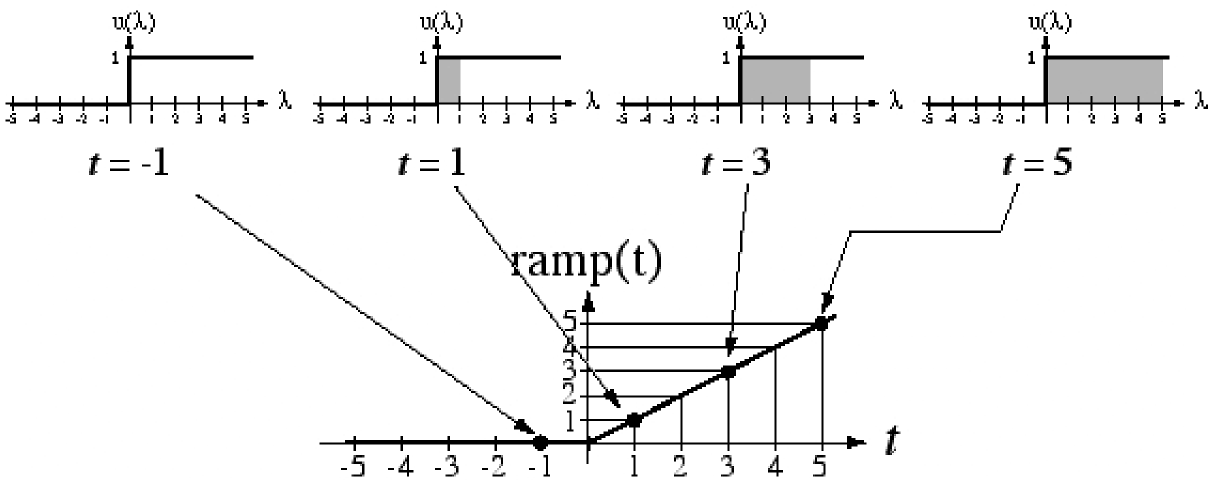


The signum function, in a sense, returns an indication of the sign of its argument.

The CT Unit Ramp Function

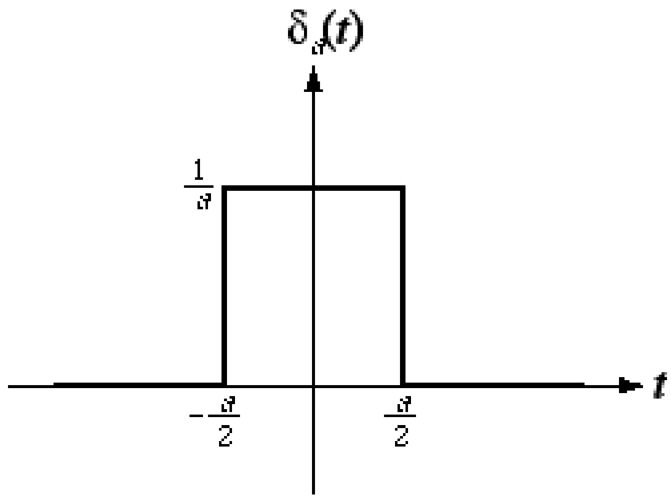
$$\text{ramp}(t) = \begin{Bmatrix} t & , & t > 0 \\ 0 & , & t \le 0 \end{Bmatrix} = \int_{-\infty}^{t} \text{u}(\lambda)\,d\lambda = t\,\text{u}(t)$$



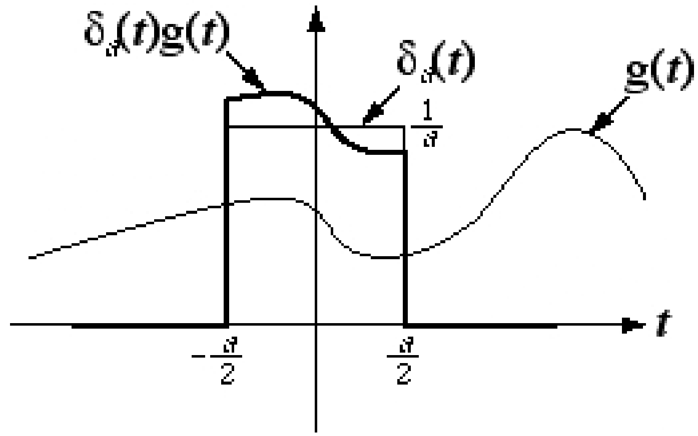


Introduction to the CT Impulse

Define a function, $\delta_a(t) = \begin{cases} \dfrac{1}{a} & , \ |t| < \dfrac{a}{2} \\ 0 & , \ |t| > \dfrac{a}{2} \end{cases}$



Let another function, g(t), be finite and continuous at t = 0.



Introduction to the CT Impulse

The area under the product of the two functions is

$$A = \frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} \mathrm{g}(t)\,dt$$

As the width of $\delta_a(t)$ approaches zero,

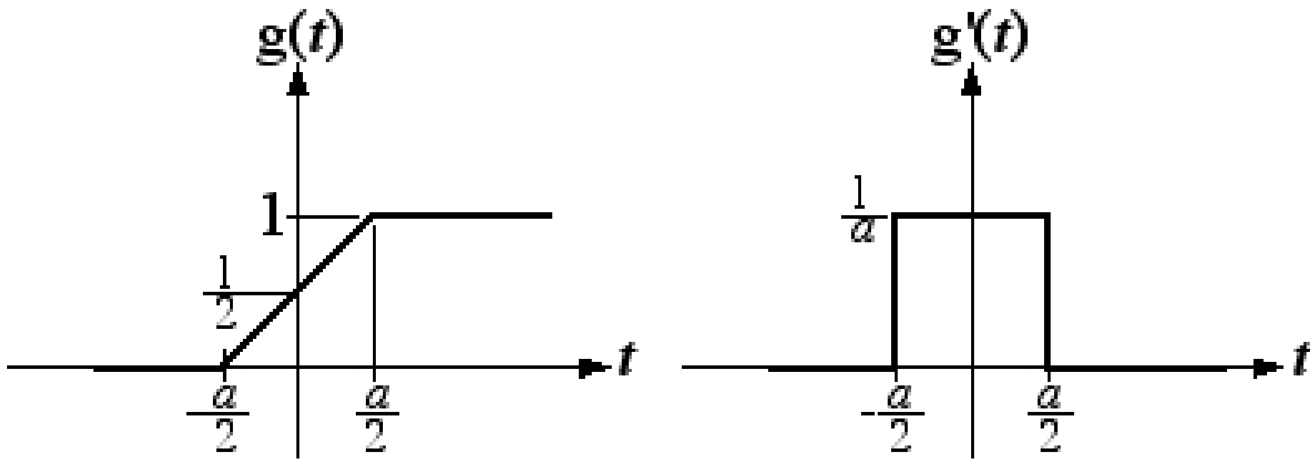
$$\lim_{a\to 0} A = \mathrm{g}(0)\lim_{a\to 0}\frac{1}{a}\int_{-\frac{a}{2}}^{\frac{a}{2}} dt = \mathrm{g}(0)\lim_{a\to 0}\frac{1}{a}(a) = \mathrm{g}(0)$$

The CT unit impulse is implicitly defined by

$$\mathrm{g}(0) = \int_{-\infty}^{\infty} \delta(t)\,\mathrm{g}(t)\,dt$$

The CT Unit Step and CT Unit Impulse

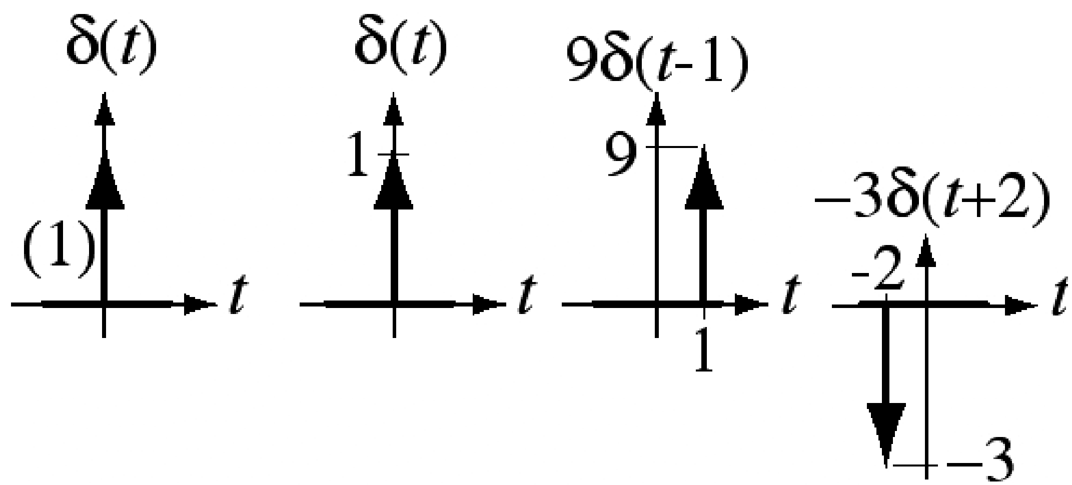
As a approaches zero, g(t) approaches a CT unit step and $g'(t)$ approaches a CT unit impulse



The CT unit step is the integral of the CT unit impulse and the CT unit impulse is the *generalized derivative* of the CT unit step

Graphical Representation of the CT Impulse

The CT impulse is not a function in the ordinary sense because its value at the time of its occurrence is not defined. It is represented graphically by a vertical arrow. Its strength is either written beside it or is represented by its length.



Properties of the CT Impulse

The sampling property

$$\int_{-\infty}^{\infty} \mathrm{g}(t)\delta(t-t_0)dt = \mathrm{g}(t_0)$$

The sampling property “extracts” the value of a function at a point.

The scaling property

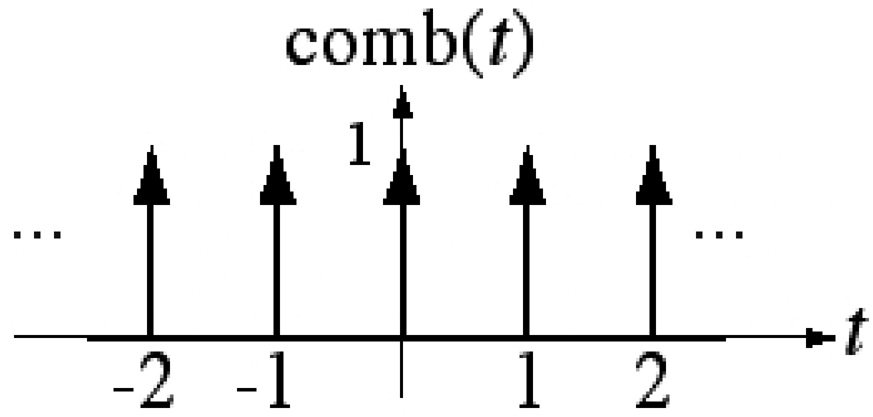
$$\delta(a(t-t_0)) = \frac{1}{|a|}\delta(t-t_0)$$

This property illustrates that the impulse is different from ordinary mathematical functions.

The CT Unit Comb

The CT unit comb is defined by

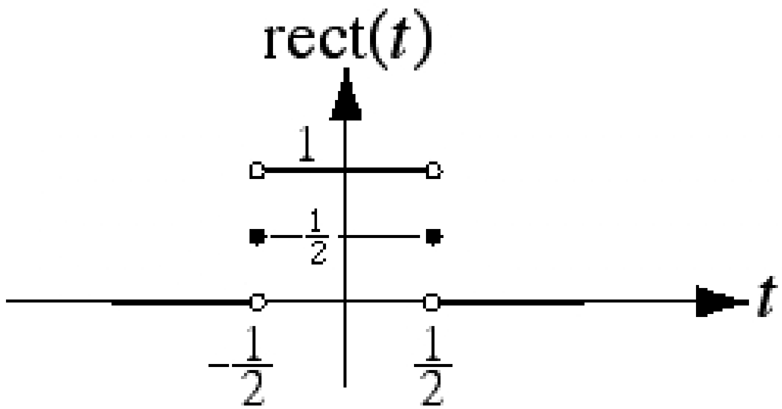
$$\text{comb}(t) = \sum_{n=-\infty}^{\infty} \delta(t-n) \quad , \quad n \text{ an integer}$$

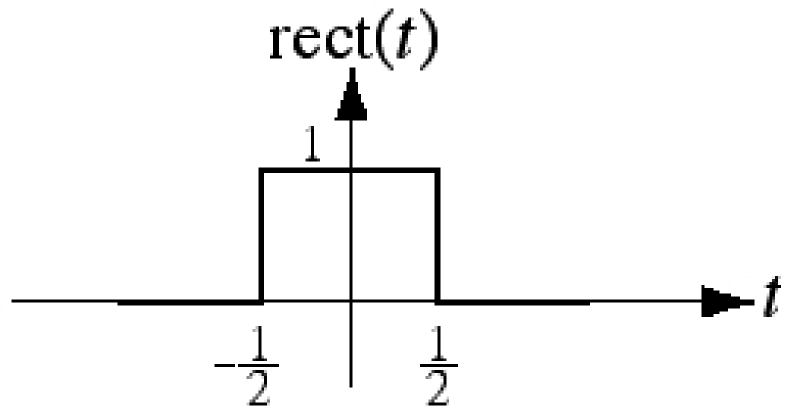


The comb is a sum of infinitely many uniformly-spaced impulses.

The CT Unit Rectangle Function

$$\operatorname{rect}(t) = \begin{cases} 1\,, & |t| < \frac{1}{2} \\ \frac{1}{2}\,, & |t| = \frac{1}{2} \\ 0\,, & |t| > \frac{1}{2} \end{cases} = \mathrm{u}\left(t + \frac{1}{2}\right) - \mathrm{u}\left(t - \frac{1}{2}\right)$$

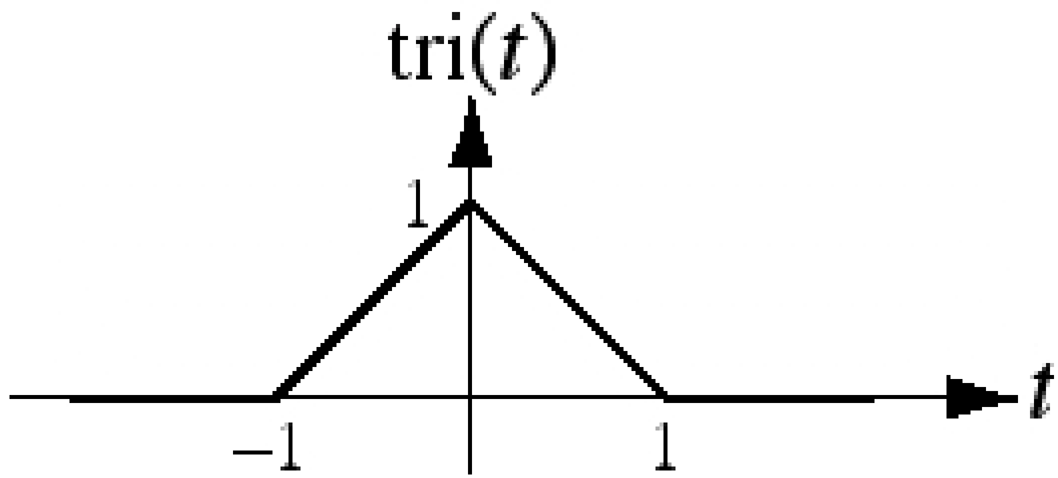




The product signal, g(t)rect(t), can be thought of as the signal, g(t), "turned on" at time, t = -1/2 and "turned back off" at time, t = +1/2.

The CT Unit Triangle Function

$$\text{tri}(t) = \begin{cases} 1 - |t| & , \ |t| < 1 \\ 0 & , \ |t| \geq 1 \end{cases}$$

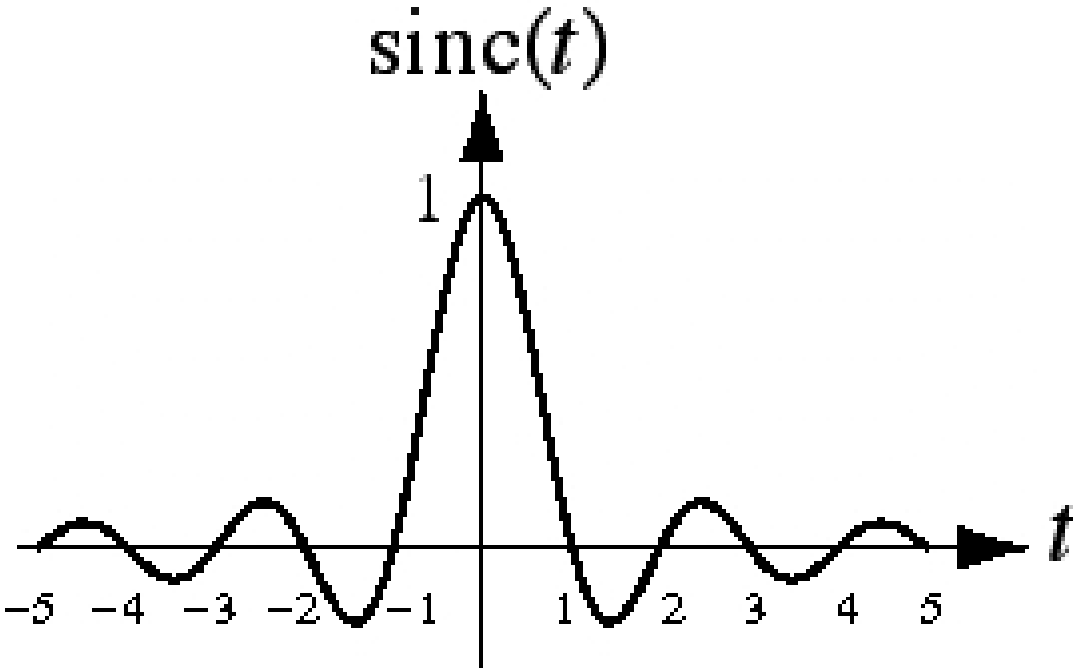


The unit triangle, defined this way, is related to the unit rectangle through an operation called *convolution* to be introduced in Chapter 3.

The CT Unit Sinc Function

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

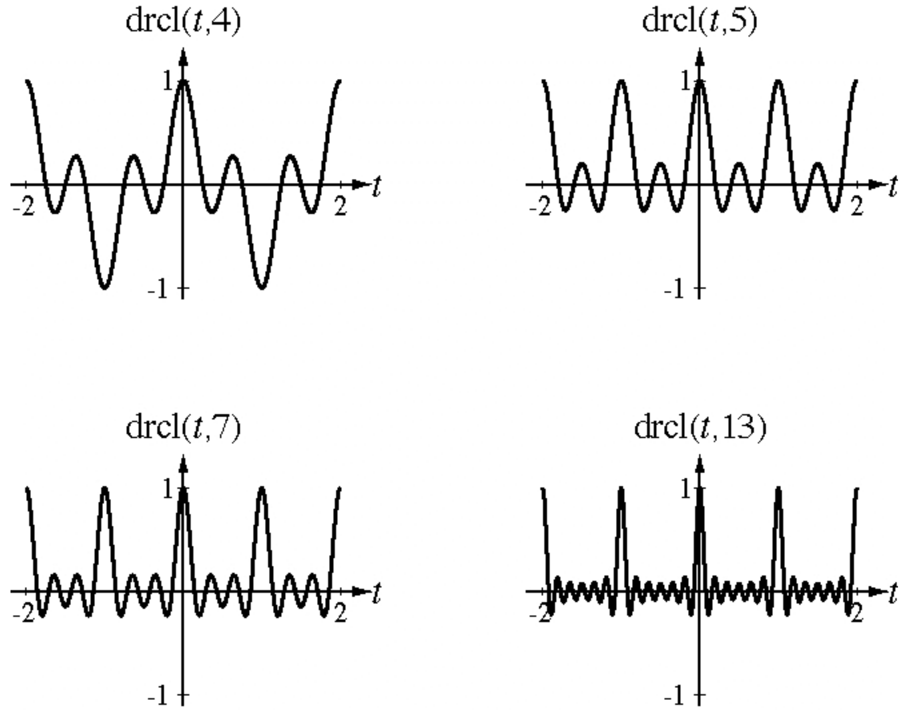
The unit sinc function is related to the unit rectangle function through the *Fourier transform*, to be introduced in Chapter 5.



$$\lim_{t\to 0}\mathrm{sinc}(t) = \lim_{t\to 0}\frac{\frac{d}{dt}(\sin(\pi t))}{\frac{d}{dt}(\pi t)} = \lim_{t\to 0}\frac{\pi\cos(\pi t)}{\pi} = 1$$

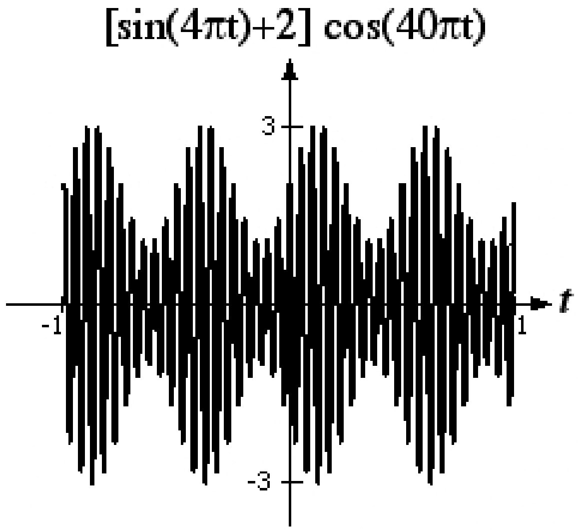
The CT Dirichlet Function

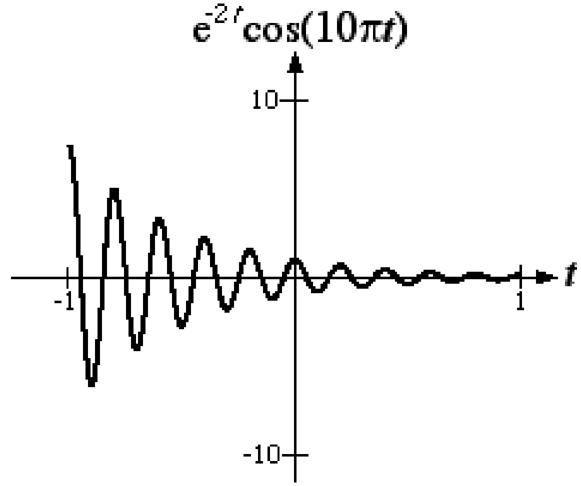
$$\mathrm{drcl}(t,N) = \frac{\sin(\pi N t)}{N \sin(\pi t)}$$

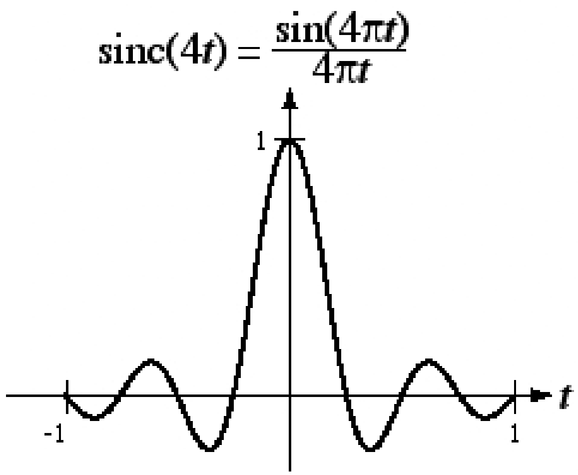


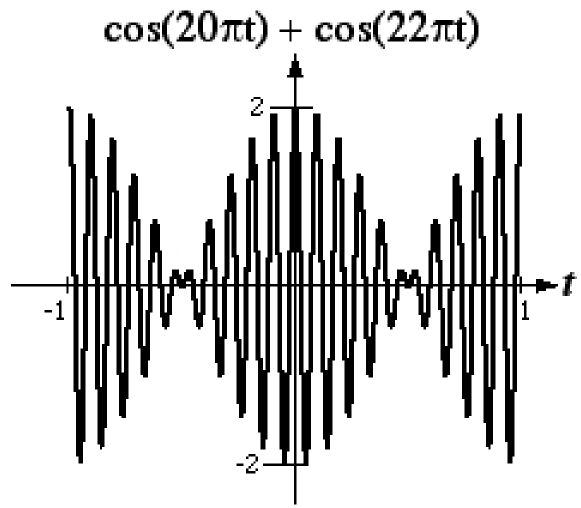
For odd *N*, the Dirichlet function is the sum of infinitely many uniformly-spaced sinc functions.

Combinations of CT Functions



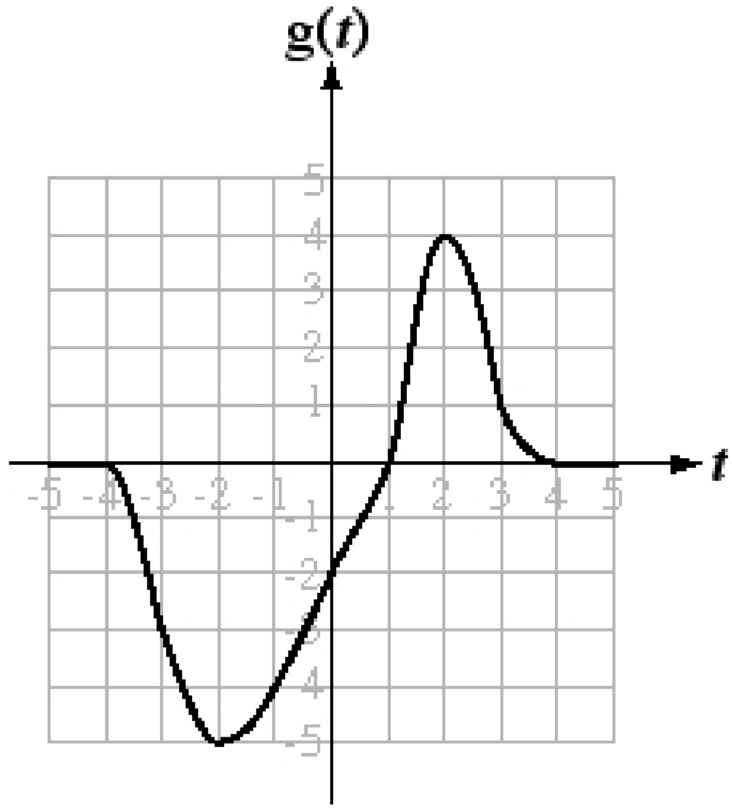






Transformations of CT Functions

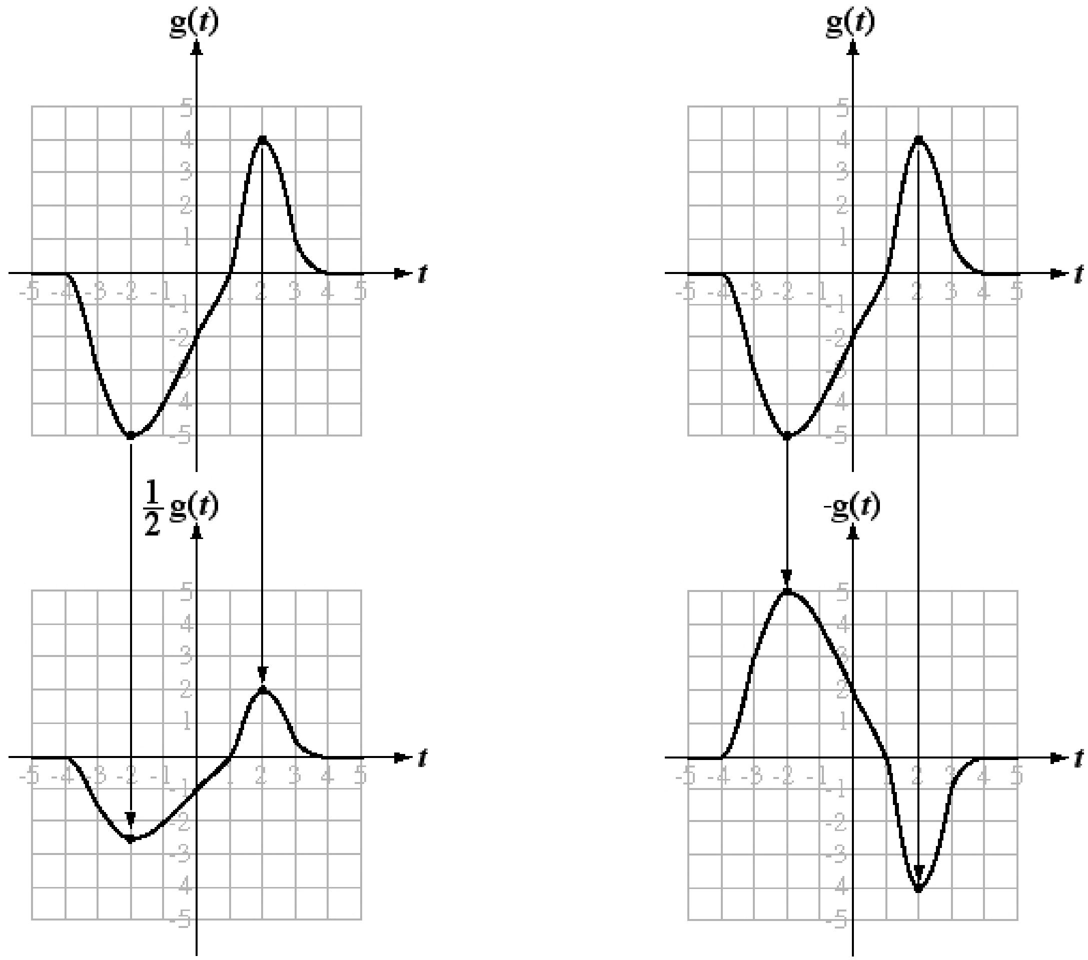
Let a CT function be defined graphically by



and let $g(t) = 0 \quad , \quad |t| > 5$

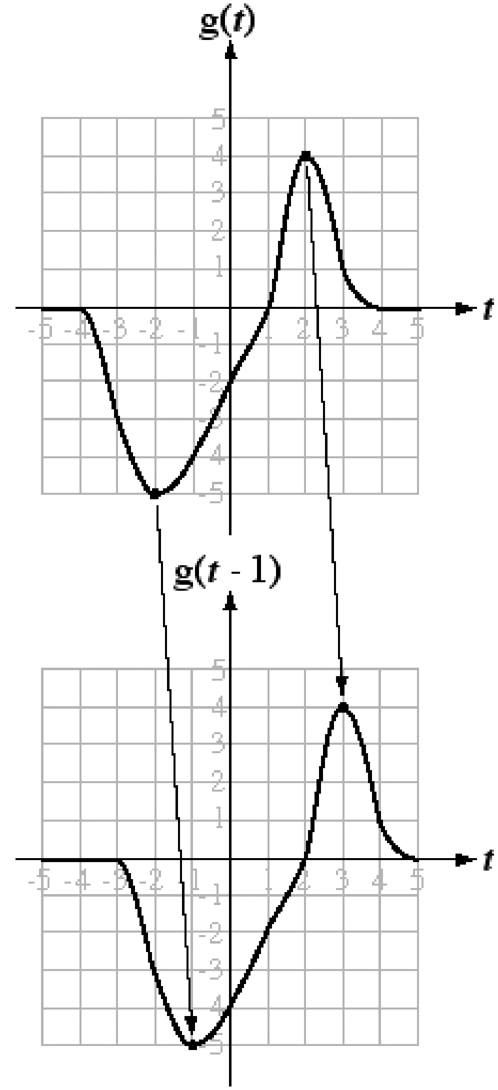
Transformations of CT Functions

Amplitude Scaling, $\mathrm{g}(t) \rightarrow A\,\mathrm{g}(t)$



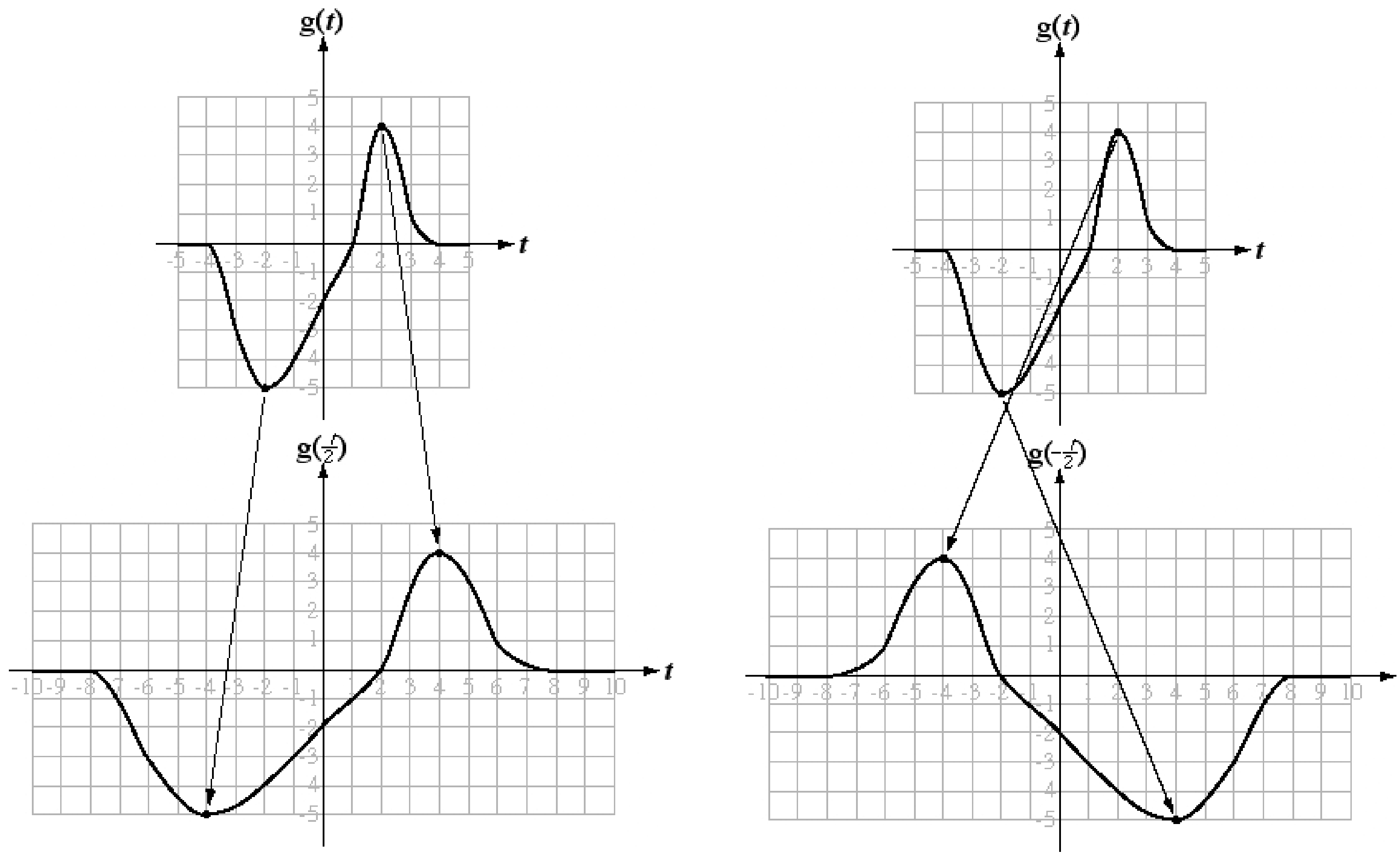
Transformations of CT Functions

Time shifting, $t \rightarrow t - t_0$



Transformations of CT Functions

Time scaling, $t \to \frac{t}{a}$



Transformations of CT Functions

Multiple transformations, $\mathrm{g}(t) \rightarrow A\,\mathrm{g}\left(\dfrac{t-t_0}{a}\right)$

A multiple transformation can be done in steps

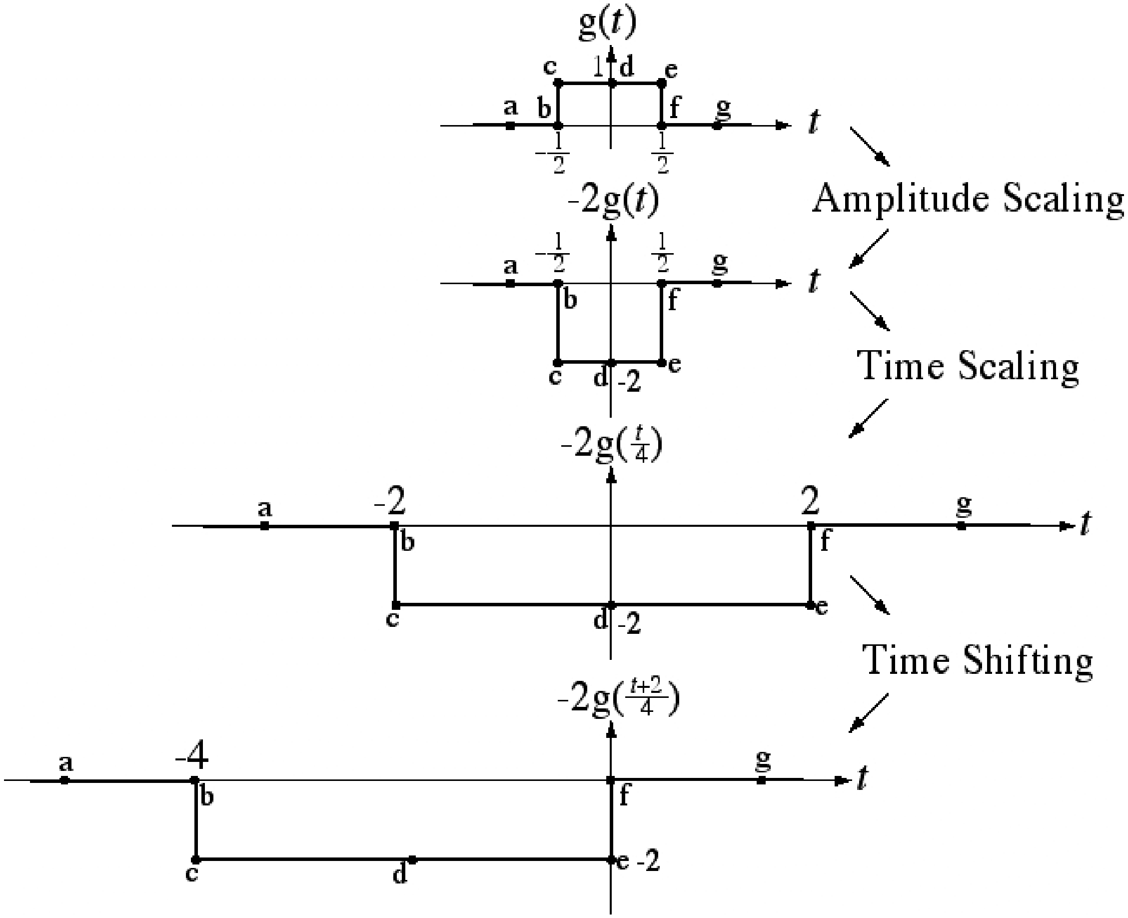
$$\mathrm{g}(t) \xrightarrow[\text{scaling, } A]{\text{amplitude}} A\,\mathrm{g}(t) \xrightarrow{t \rightarrow \frac{t}{a}} A\,\mathrm{g}\left(\frac{t}{a}\right) \xrightarrow{t \rightarrow t-t_0} A\,\mathrm{g}\left(\frac{t-t_0}{a}\right)$$

The sequence of the steps is significant

$$\mathrm{g}(t) \xrightarrow[\text{scaling, } A]{\text{amplitude}} A\,\mathrm{g}(t) \xrightarrow{t \rightarrow t-t_0} A\,\mathrm{g}(t-t_0) \xrightarrow{t \rightarrow \frac{t}{a}} A\,\mathrm{g}\left(\frac{t}{a}-t_0\right) \neq A\,\mathrm{g}\left(\frac{t-t_0}{a}\right)$$

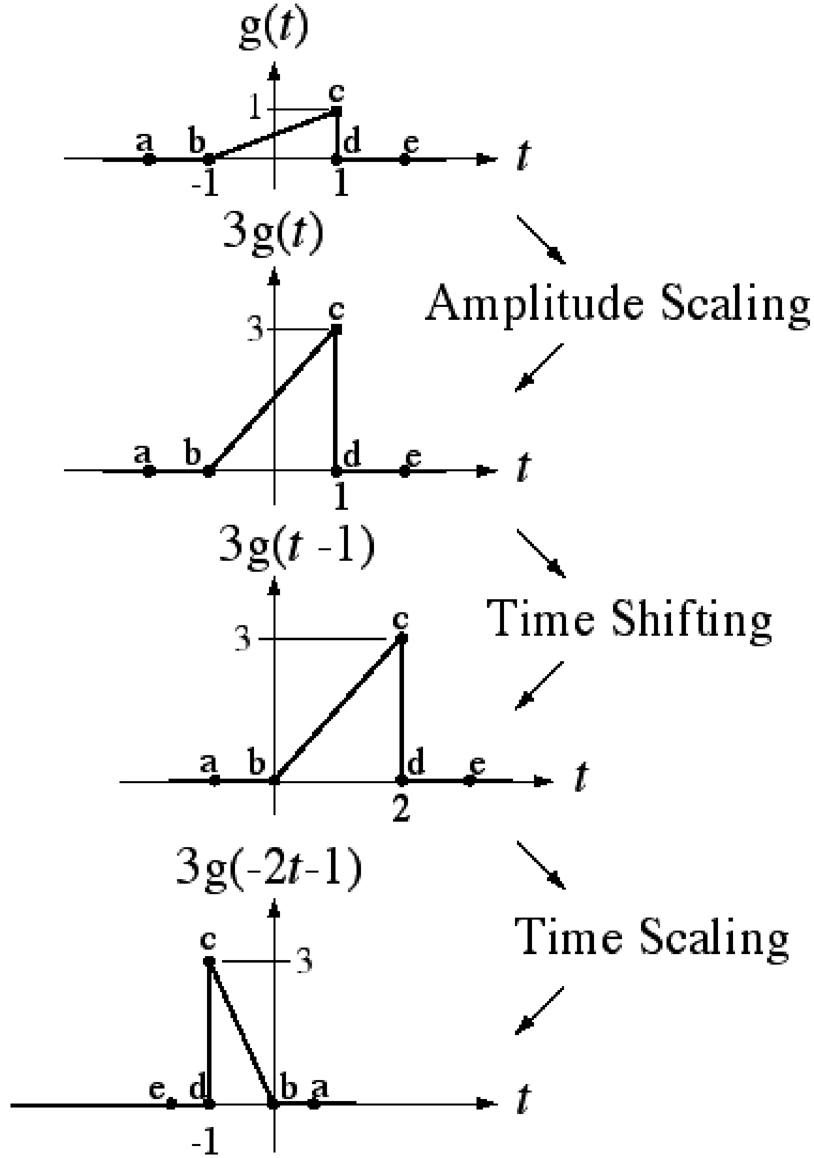
Transformations of CT Functions

Multiple transformations, $g(t) \to A\,g\left(\dfrac{t-t_0}{a}\right)$

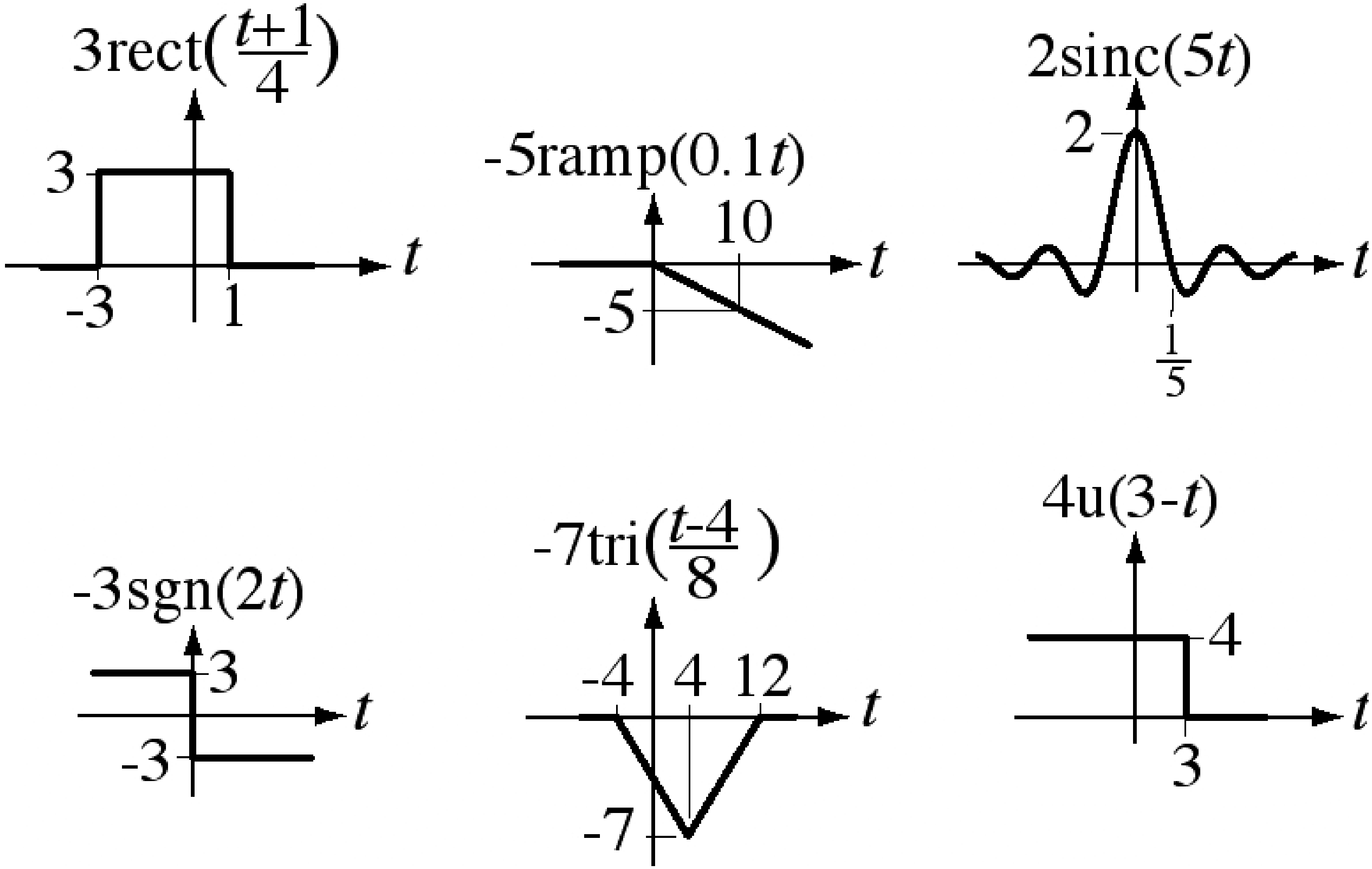


Transformations of CT Functions

Multiple transformations, $A\,g(bt - t_0)$

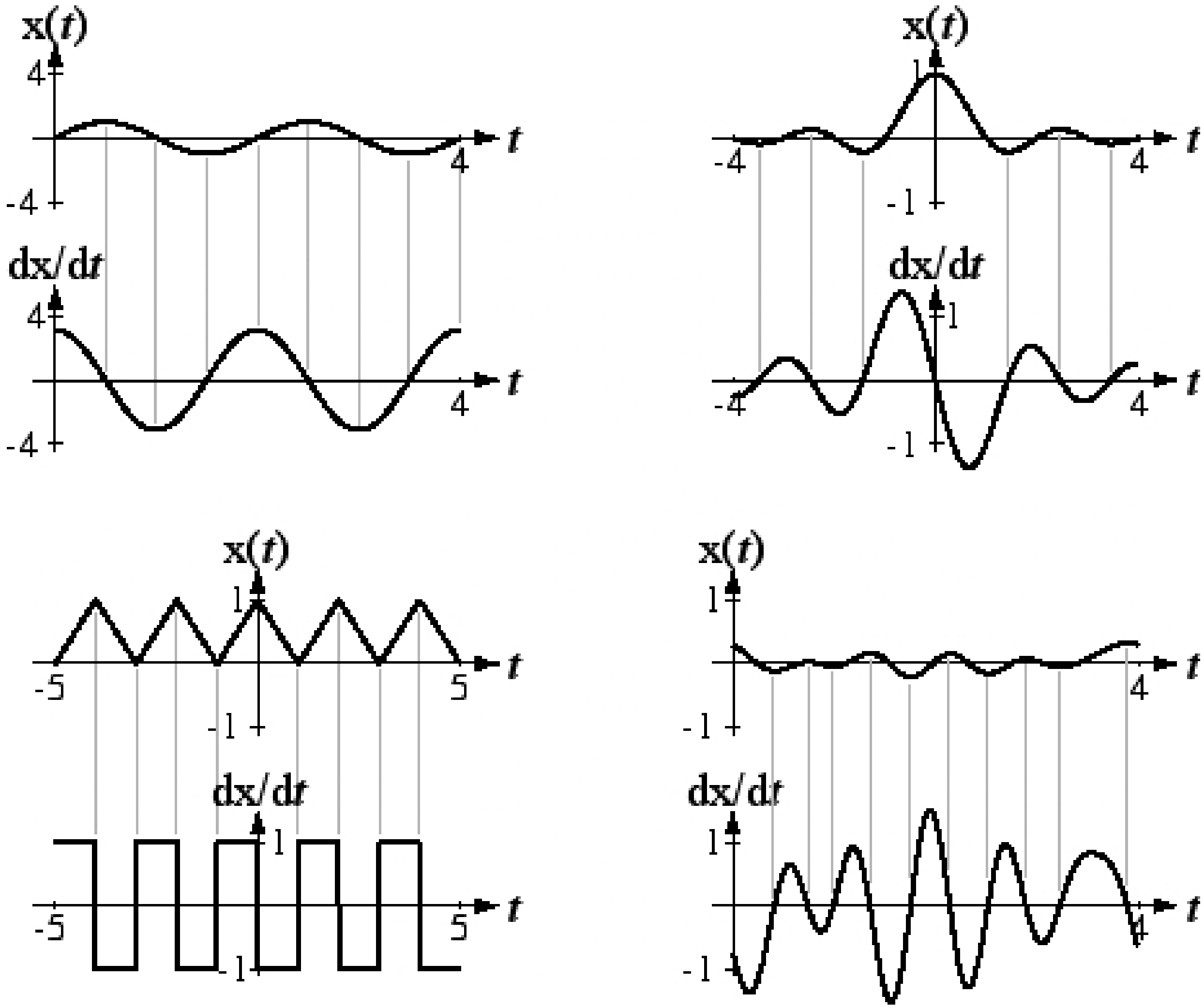


Transformations of CT Functions



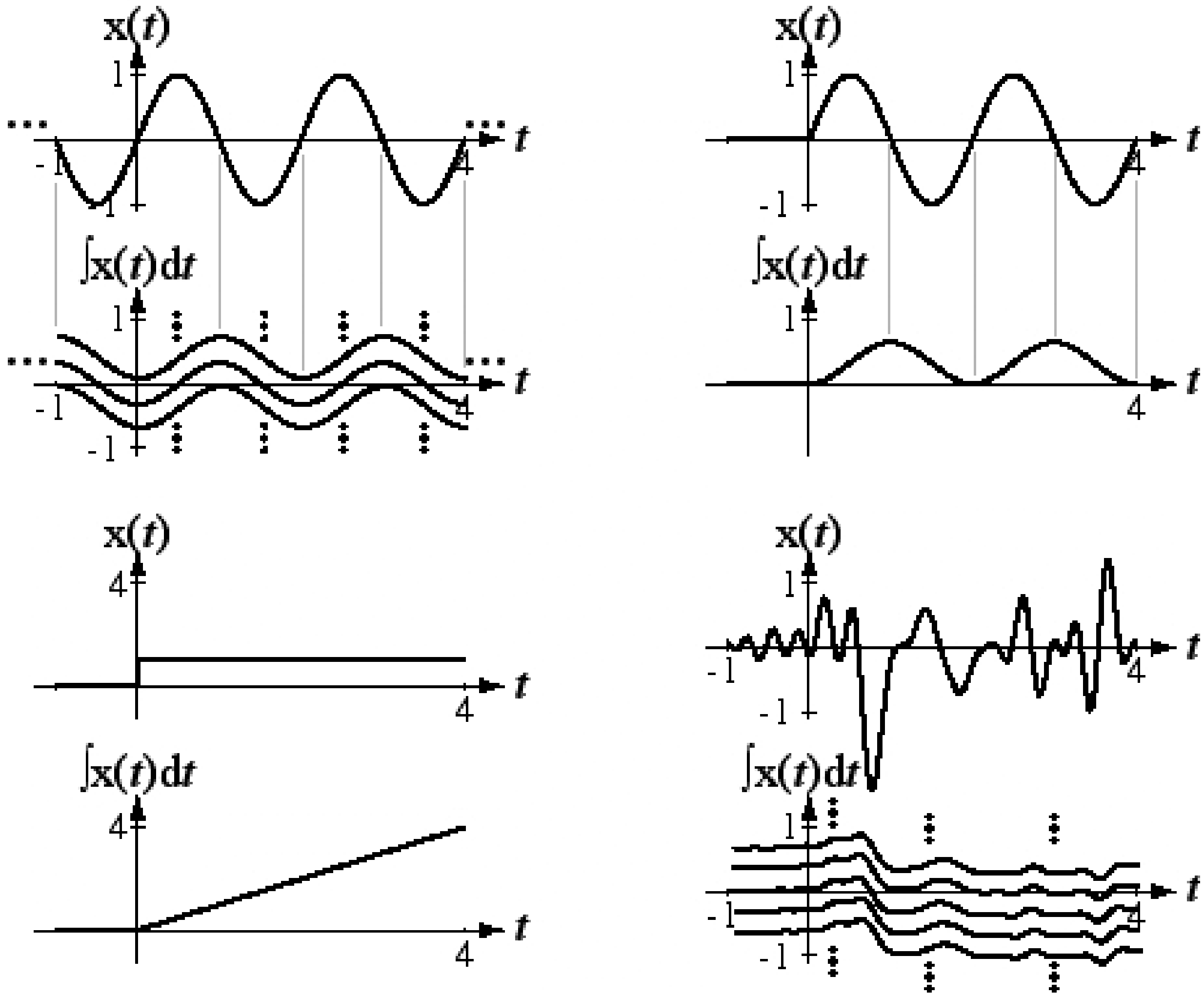
Transformations of CT Functions

Differentiation



Transformations of CT Functions

Integration



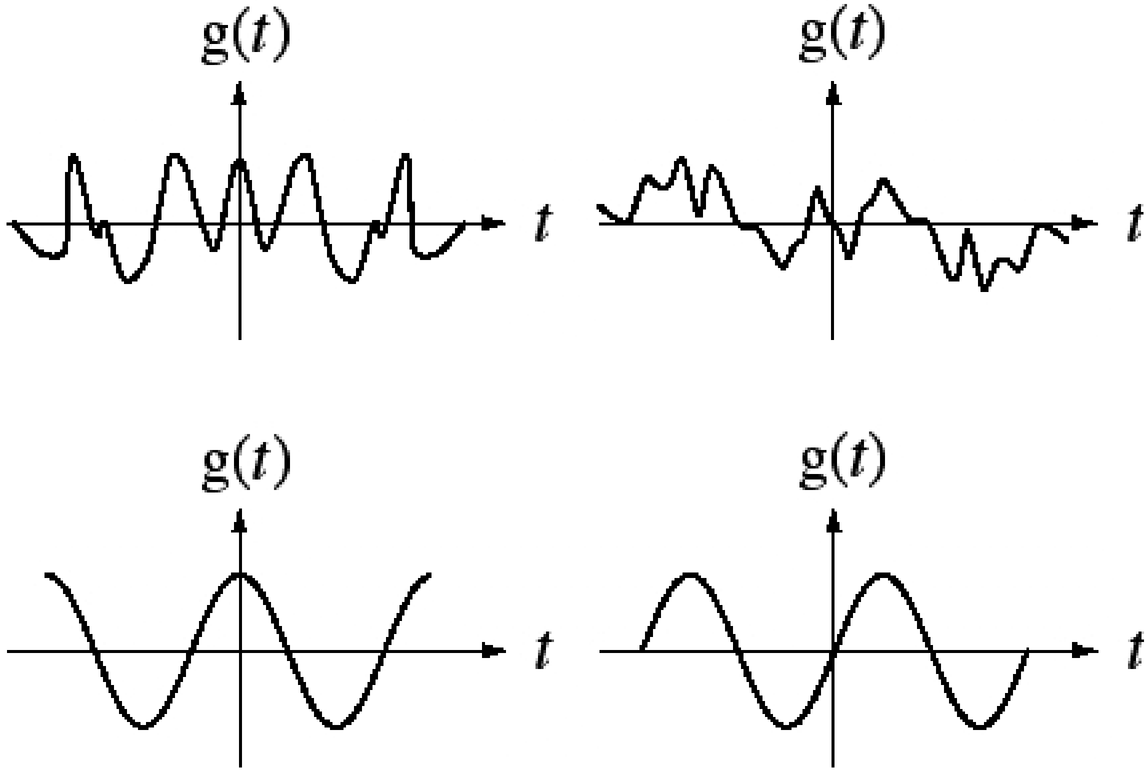
Even and Odd CT Functions

Even Functions

$$g(t) = g(-t)$$

Odd Functions

$$g(t) = -g(-t)$$



Even and Odd Parts of CT Functions

The even part of a CT function is $\mathrm{g}_e(t) = \dfrac{\mathrm{g}(t) + \mathrm{g}(-t)}{2}$

The odd part of a CT function is $\mathrm{g}_o(t) = \dfrac{\mathrm{g}(t) - \mathrm{g}(-t)}{2}$

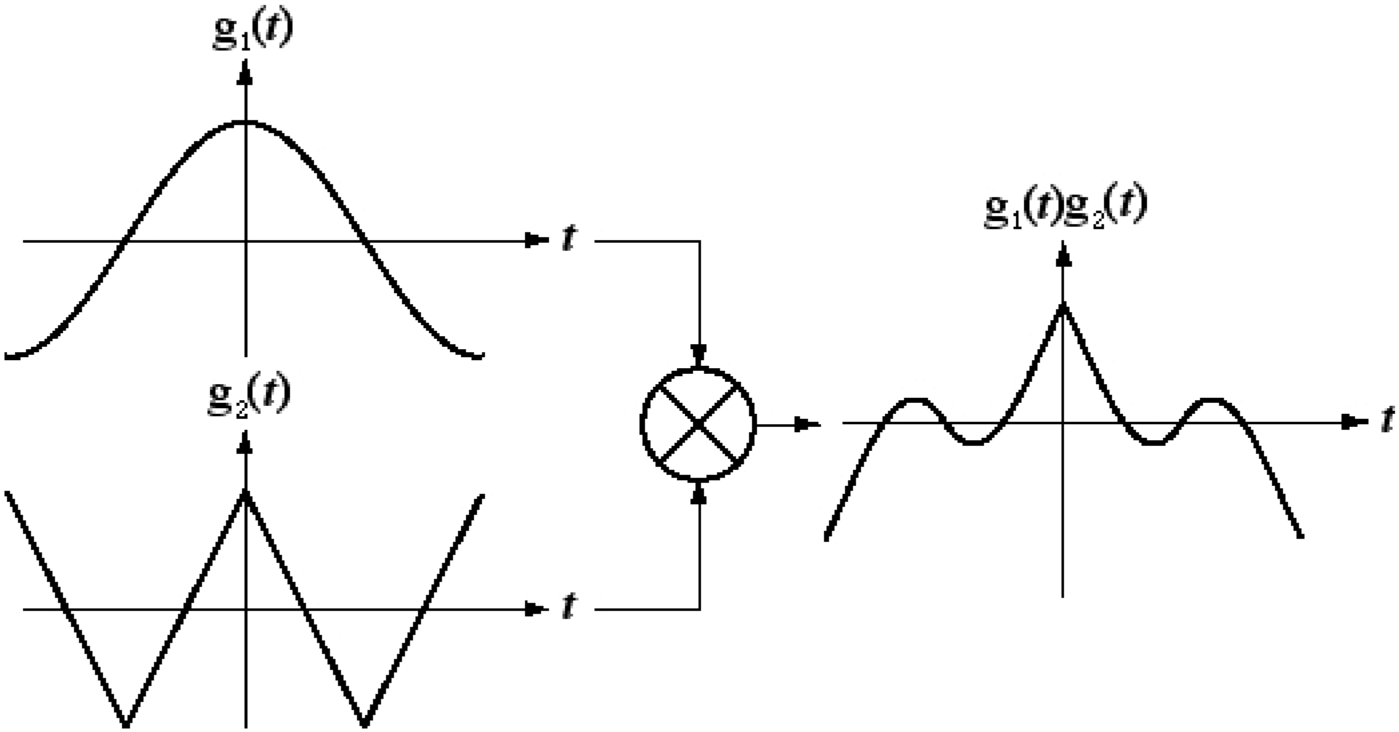
A function whose even part is zero is odd and a function whose odd part is zero is even.

The derivative of an even CT function is odd and the derivative of an odd CT function is even.

The integral of an even CT function is an odd CT function, *plus a constant*, and the integral of an odd CT function is even.

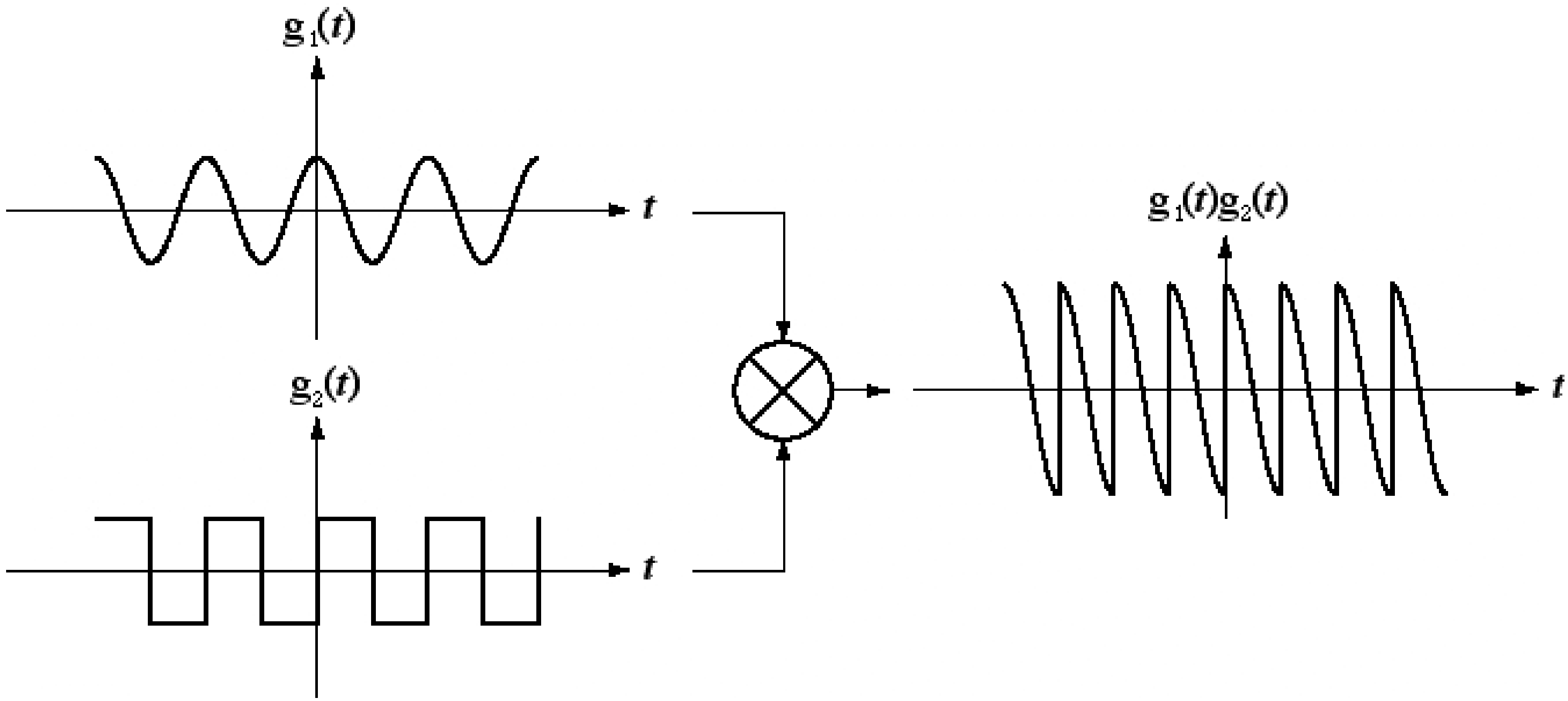
Products of Even and Odd CT Functions

Two Even Functions



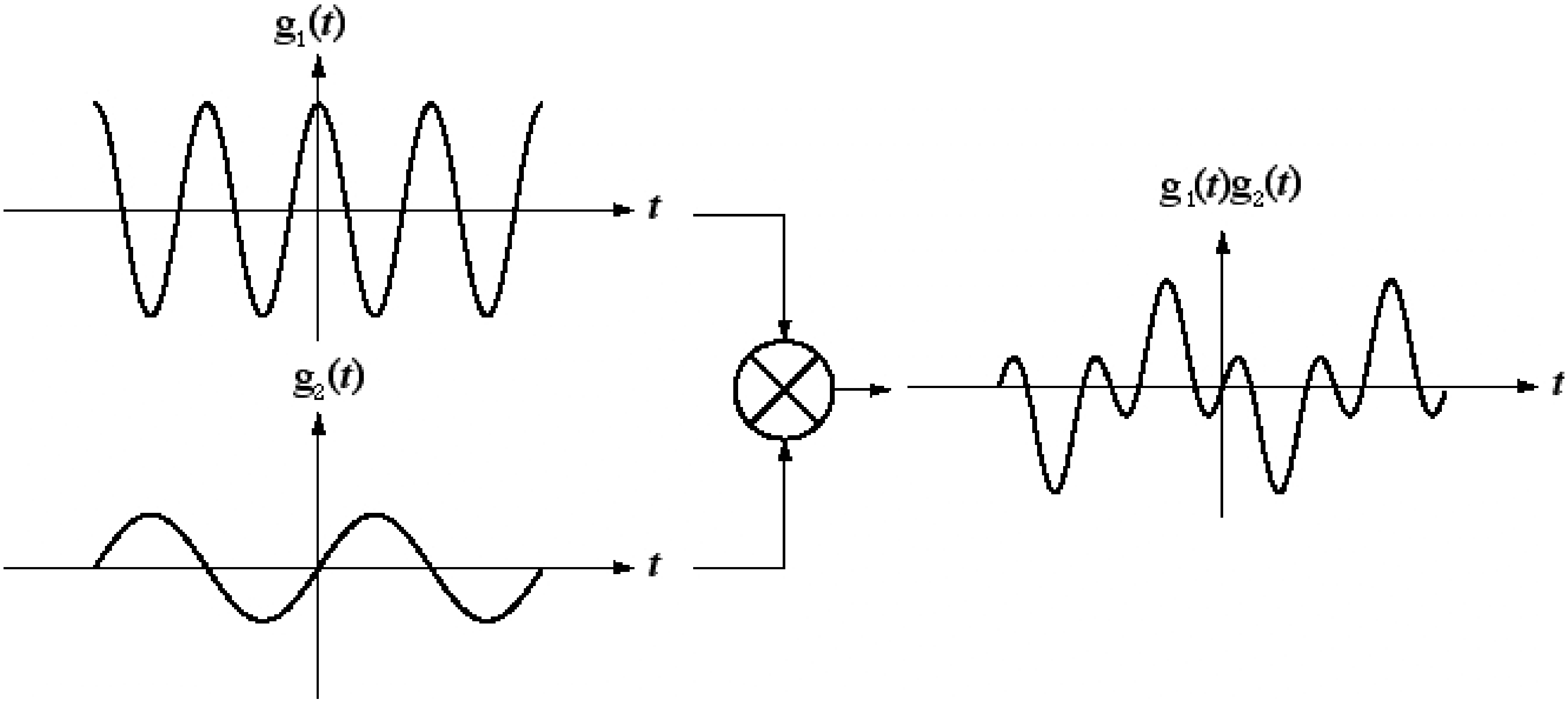
Products of Even and Odd CT Functions

An Even Function and an Odd Function



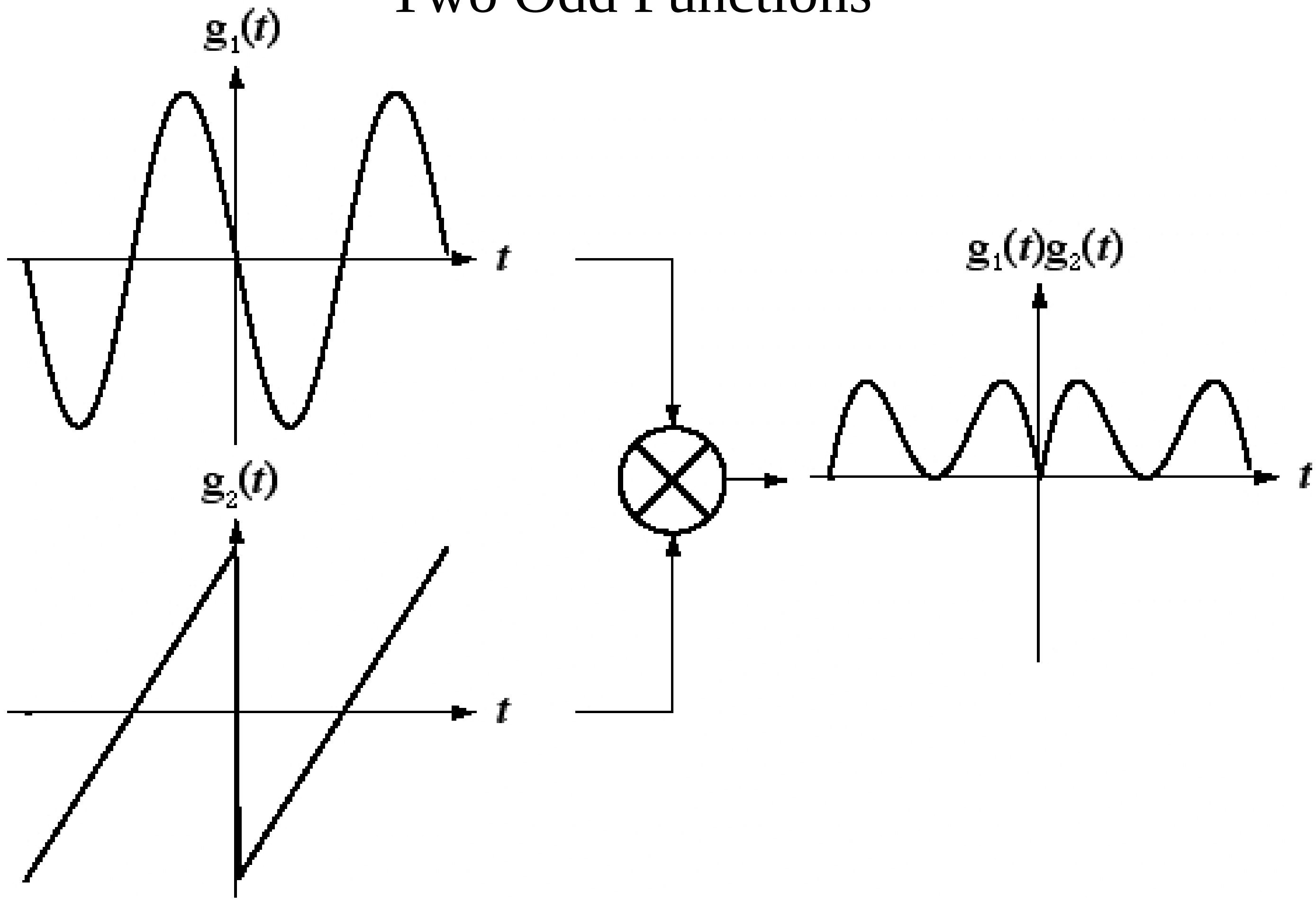
Products of Even and Odd CT Functions

An Even Function and an Odd Function

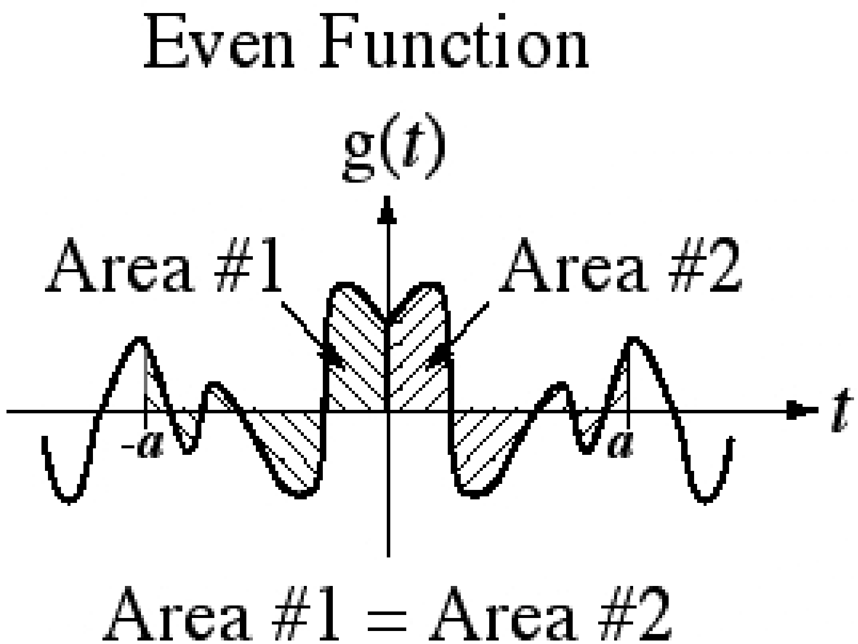


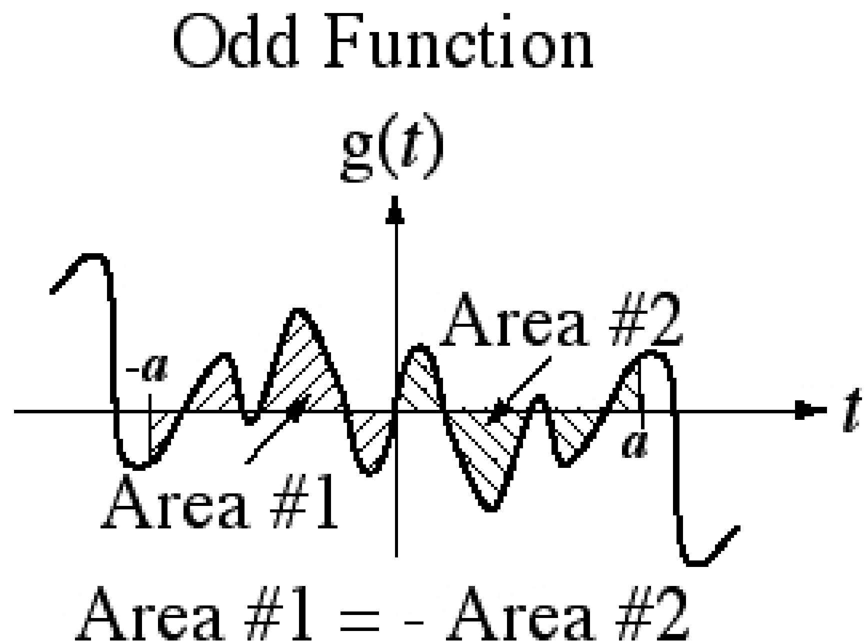
Products of Even and Odd CT Functions

Two Odd Functions



Integrals of Even and Odd CT Functions



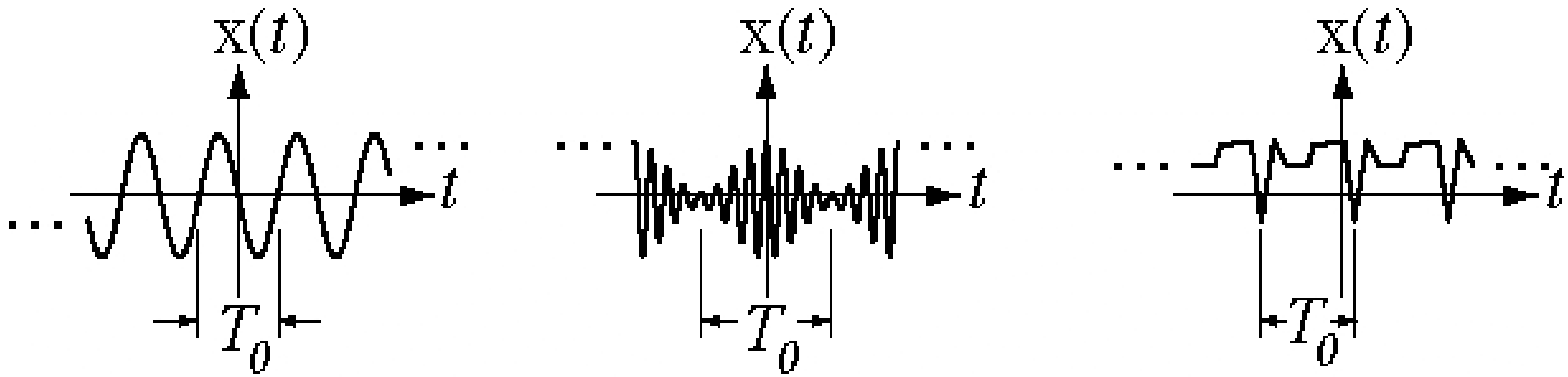


$$\int_{-a}^{a} \mathrm{g}(t)\,dt = 2\int_{0}^{a} \mathrm{g}(t)\,dt \qquad \int_{-a}^{a} \mathrm{g}(t)\,dt = 0$$

Periodic CT Functions

If a CT function, g(t), is periodic, $g(t) = g(t + nT)$, where n is any integer and T is a *period* of the function.

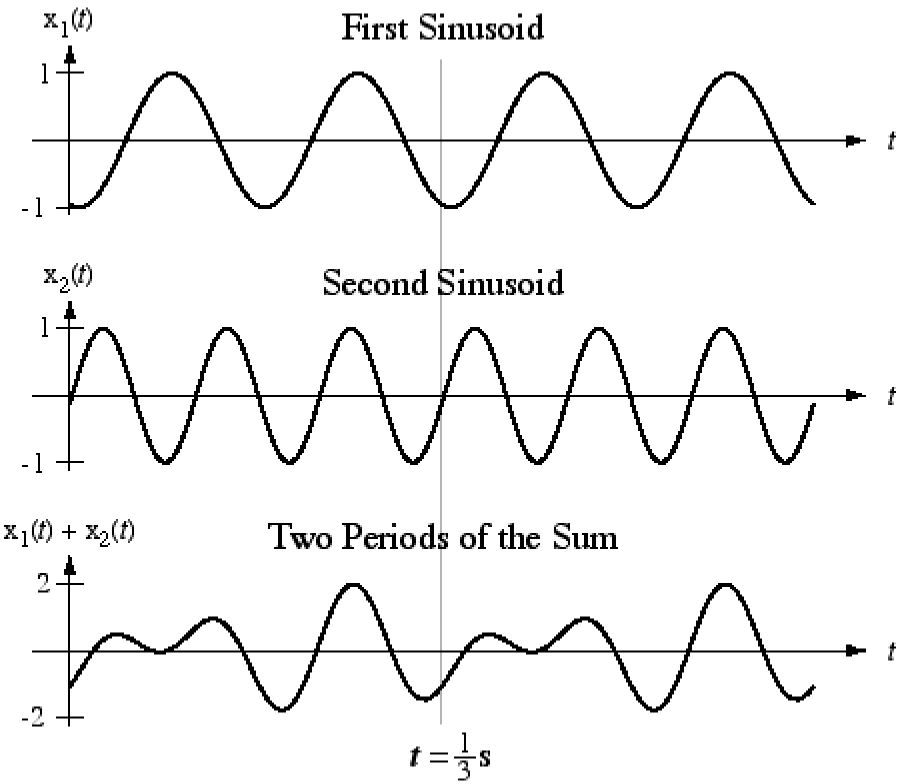
The minimum positive value of T for which $g(t) = g(t + T)$ is called the *fundamental period* of the function, T_0. The reciprocal of the fundamental period is the fundamental frequency, $f_0 = 1/T_0$.

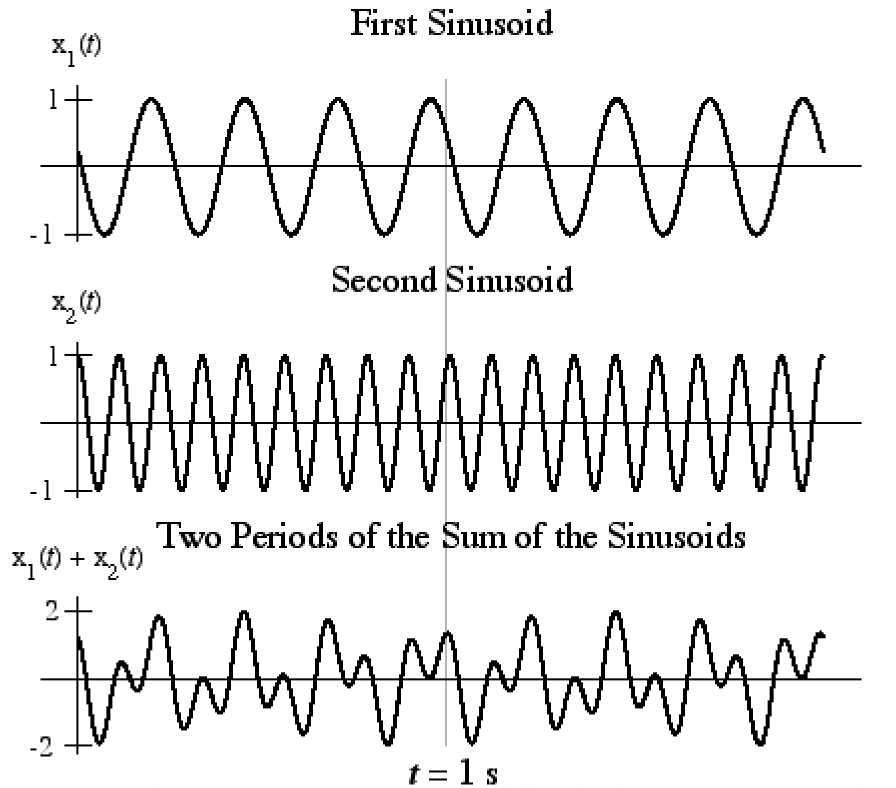


A function that is not periodic is *aperiodic*.

Sums of CT Periodic Functions

The period of the sum of CT periodic functions is the *least common multiple* of the periods of the individual functions summed. If the least common multiple is infinite, the sum function is aperiodic.





Discrete-Time Sinusoids

The general form of a periodic discrete-time (DT) sinusoid with fundamental period, N_0 , is

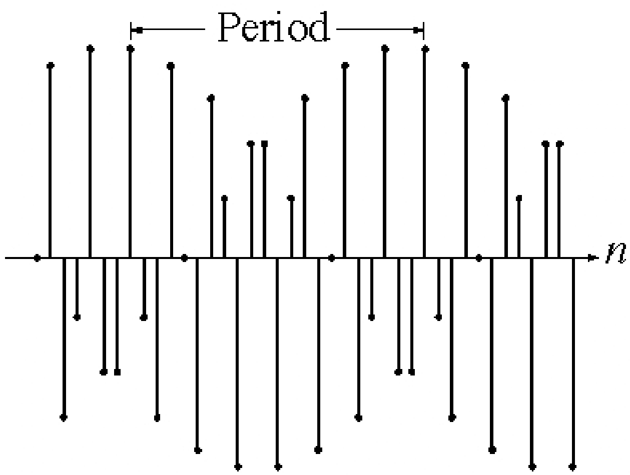
$$\mathrm{g}[n] = A\cos\left(\frac{2\pi mn}{N_0} + \theta\right) \text{ or } A\cos(2\pi mF_0 n + \theta) \text{ or } \mathrm{g}[n] = A\cos(m\Omega_0 n + \theta)$$

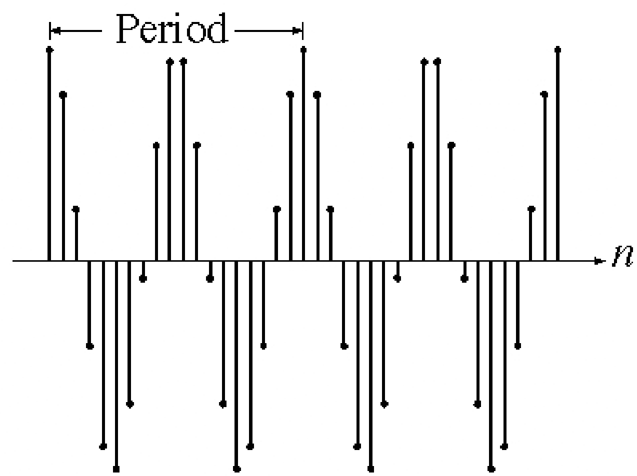
where m and N_0 are integers and F_0 is therefore the reciprocal of an integer. Unlike a CT sinusoid, a DT sinusoid is not necessarily periodic.

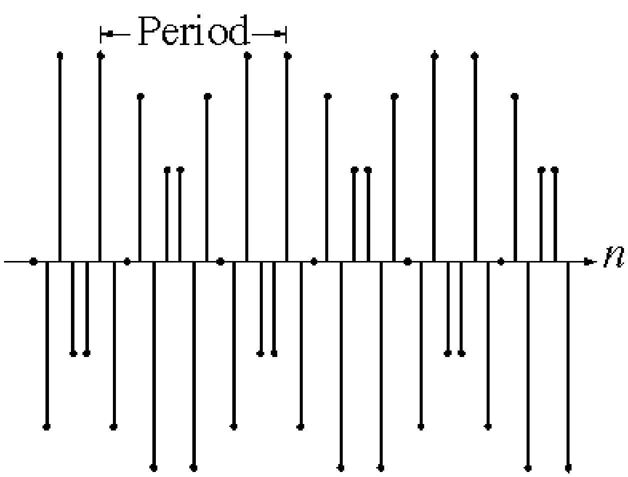
If a DT sinusoid has the form, $\mathrm{g}[n] = A\cos(2\pi Kn + \theta)$, then K must be a ratio of integers (a rational number) for $\mathrm{g}[n]$ to be periodic. If K is rational in the form, p/q, and all common factors in p and q have been cancelled, then the fundamental period of the sinusoid is q, not q/p (unless $p = 1$).

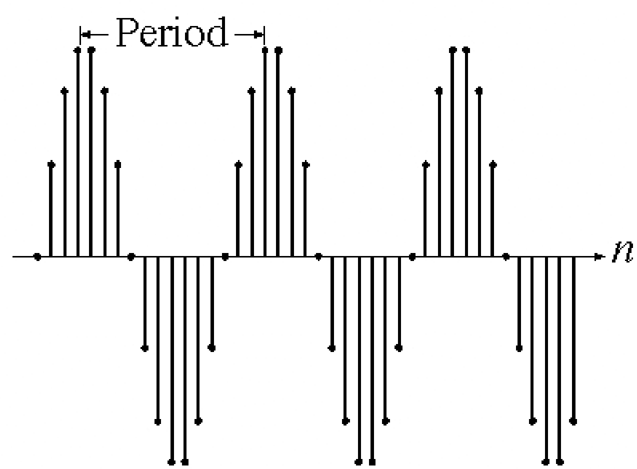
Discrete-Time Sinusoids

Periodic Sinusoids



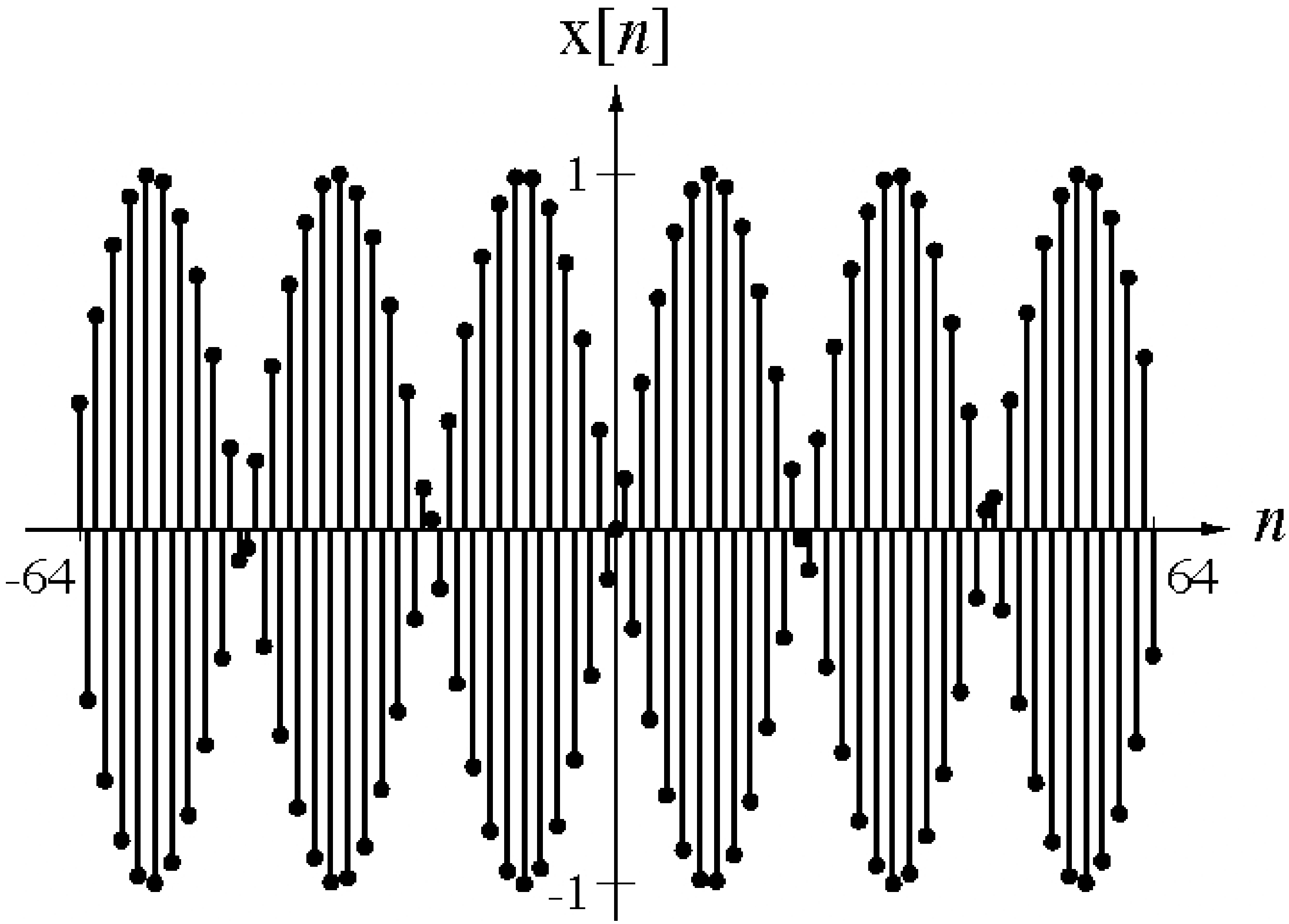






Discrete-Time Sinusoids

An Aperiodic Sinusoid



Discrete-Time Sinusoids

Two DT sinusoids whose analytical expressions look different,

$$\mathrm{g}_1[n] = A\cos(2\pi K_1 n + \theta) \quad \text{and} \quad \mathrm{g}_2[n] = A\cos(2\pi K_2 n + \theta)$$

may actually be the same. If

$$K_2 = K_1 + 2m\pi, \quad \text{where } m \text{ is an integer}$$

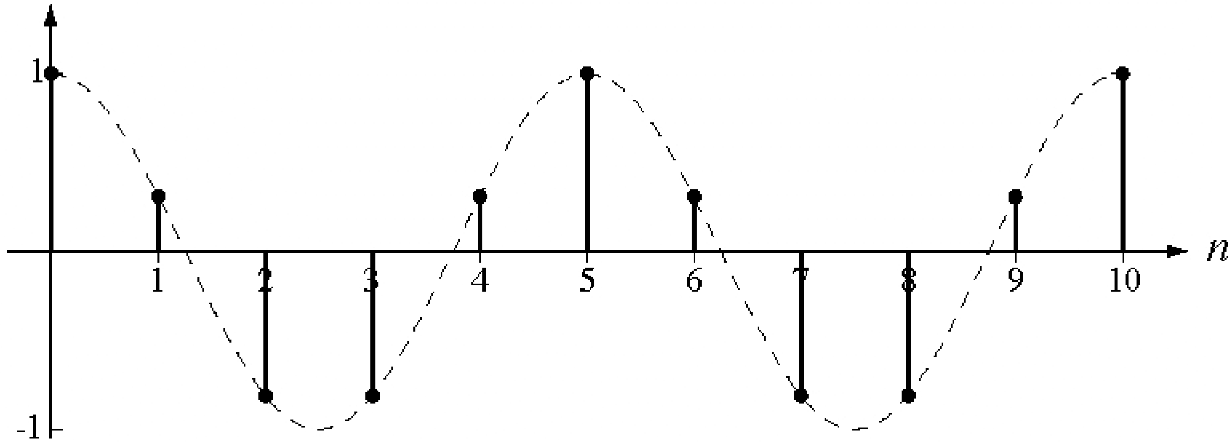
then (because *n* is discrete time and therefore an integer),

$$A\cos(2\pi K_1 n + \theta) = A\cos(2\pi K_2 n + \theta)$$

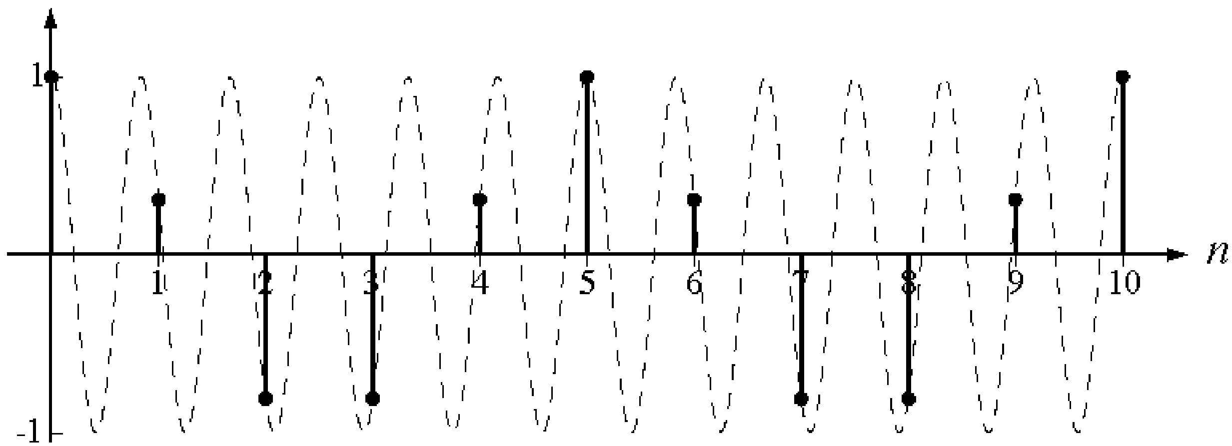
(Example on next slide)

Discrete-Time Sinusoids

$$g_1[n] = \cos\left(\frac{2\pi n}{5}\right)$$



$$g_2[n] = \cos\left(\frac{12\pi n}{5}\right)$$



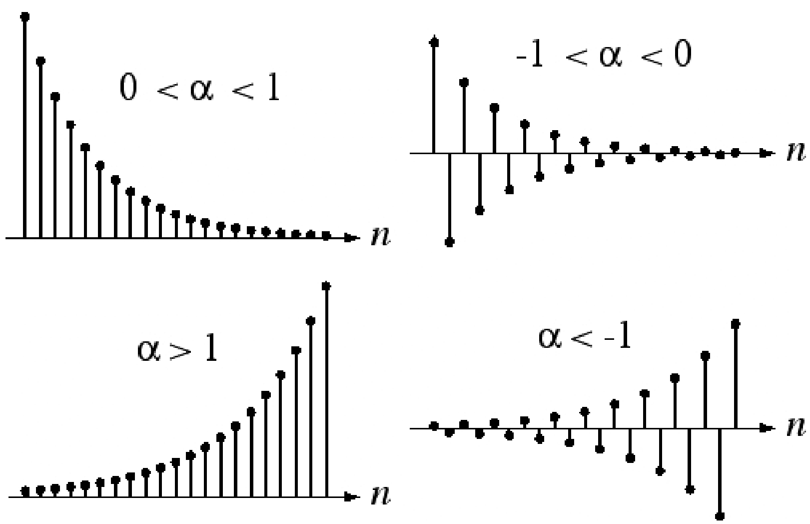
Discrete-Time Exponentials

The form of the discrete-time exponential is

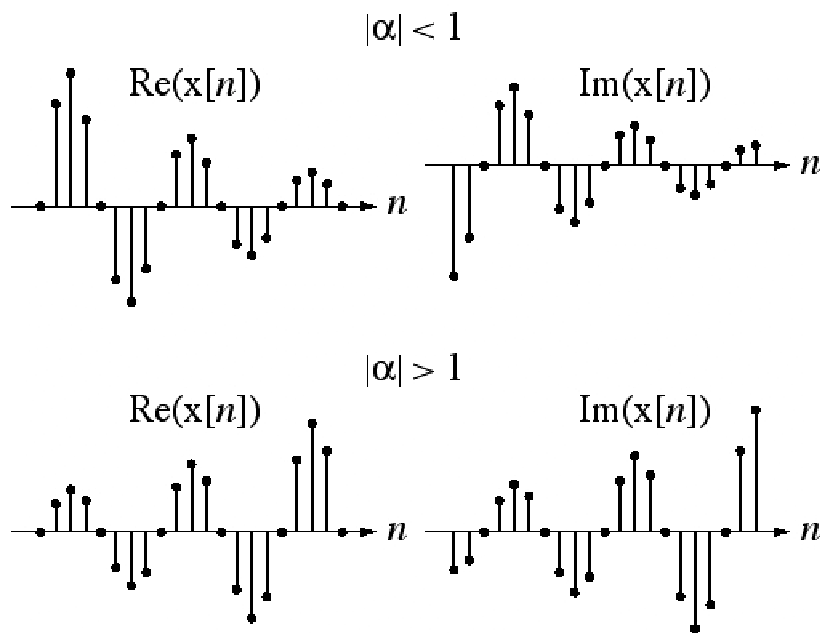
$$\mathrm{g}[n] = A\alpha^n \quad \text{or} \quad \mathrm{g}[n] = Ae^{\beta n}$$

where $\alpha = e^{\beta}$

Real α

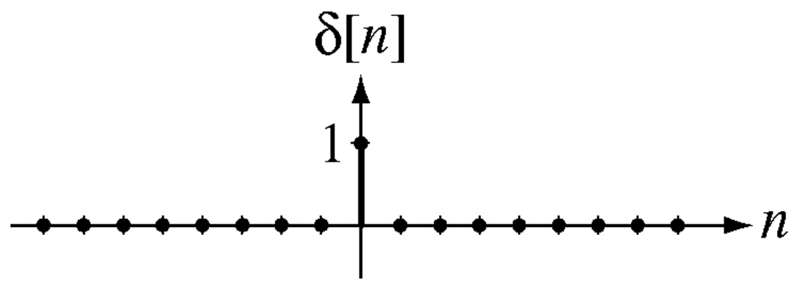


Complex α



The Discrete-Time Impulse Function

$$\delta[n] = \begin{cases} 1 \ , & n = 0 \\ 0 \ , & n \neq 0 \end{cases}$$



The DT unit impulse is a function in the ordinary sense (in contrast with the CT unit impulse). It has a sampling property,

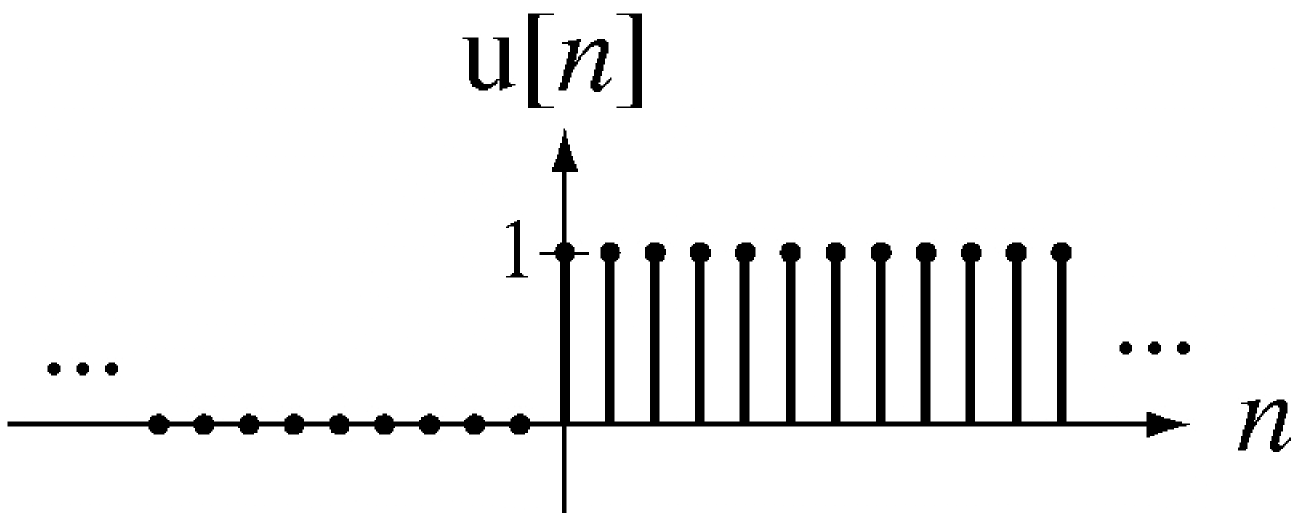
$$\sum_{n=-\infty}^{\infty} A\delta[n - n_0]\mathrm{x}[n] = A\,\mathrm{x}[n_0]$$

but no scaling property. That is,

$\delta[n] = \delta[an]$, for any non-zero, finite integer *a*.

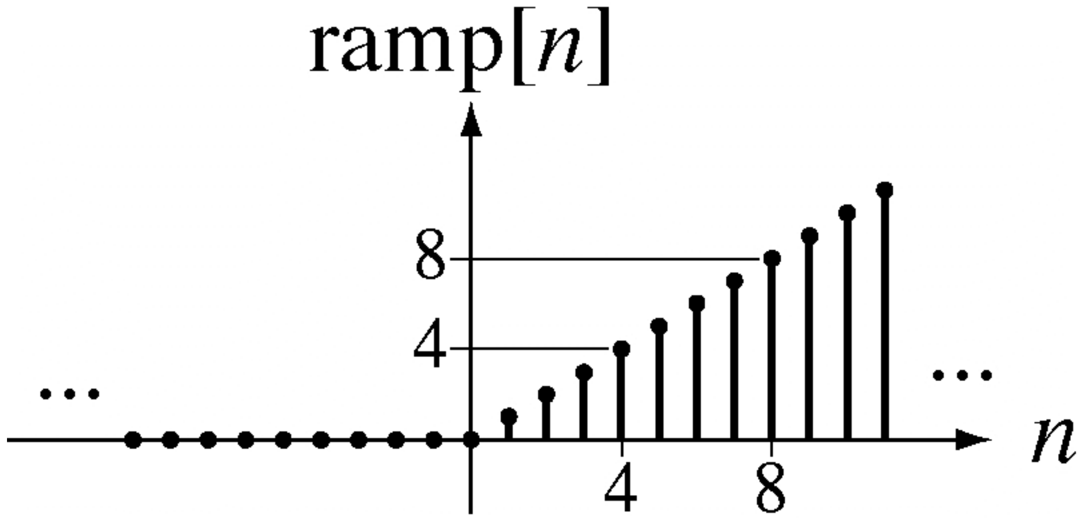
The DT Unit Sequence Function

$$\mathrm{u}[n] = \begin{cases} 1 \ , & n \geq 0 \\ 0 \ , & n < 0 \end{cases}$$



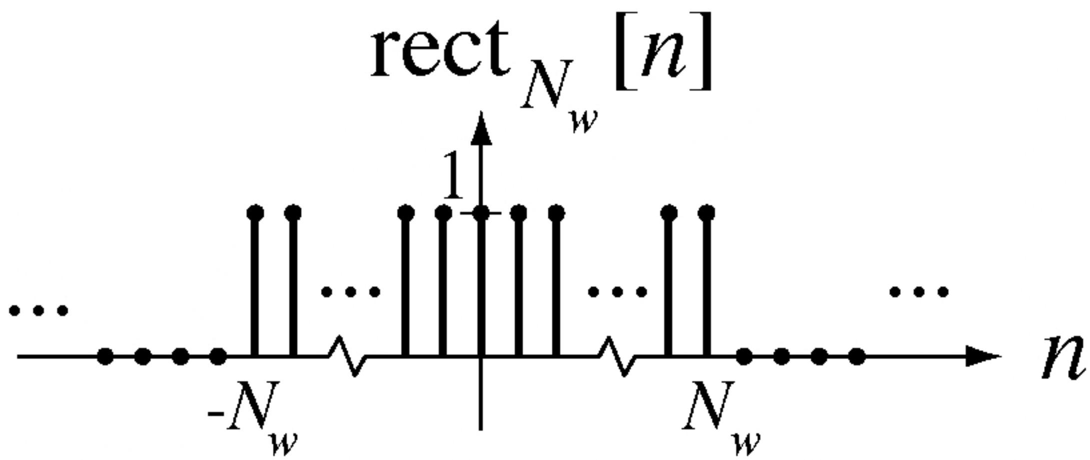
The DT Unit Ramp Function

$$\mathrm{ramp}[n] = \begin{Bmatrix} n & , & n \geq 0 \\ 0 & , & n < 0 \end{Bmatrix} = \sum_{m=-\infty}^{n} \mathrm{u}[m-1]$$



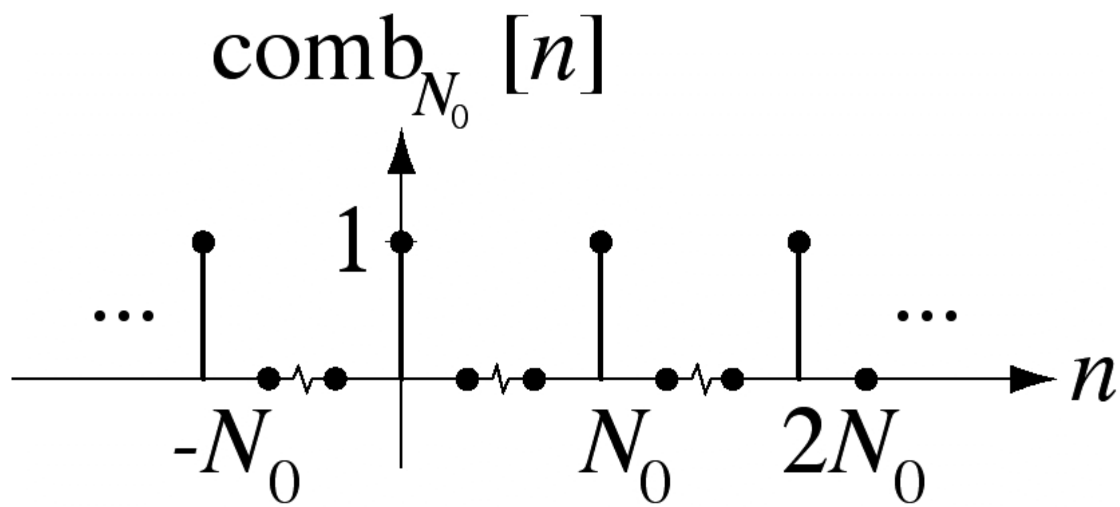
The DT Rectangle Function

$$\text{rect}_{N_w}[n] = \begin{cases} 1 \ , & |n| \le N_w \\ 0 \ , & |n| > N_w \end{cases} , \ N_w \ge 0 \ , \ N_w \text{ an integer}$$



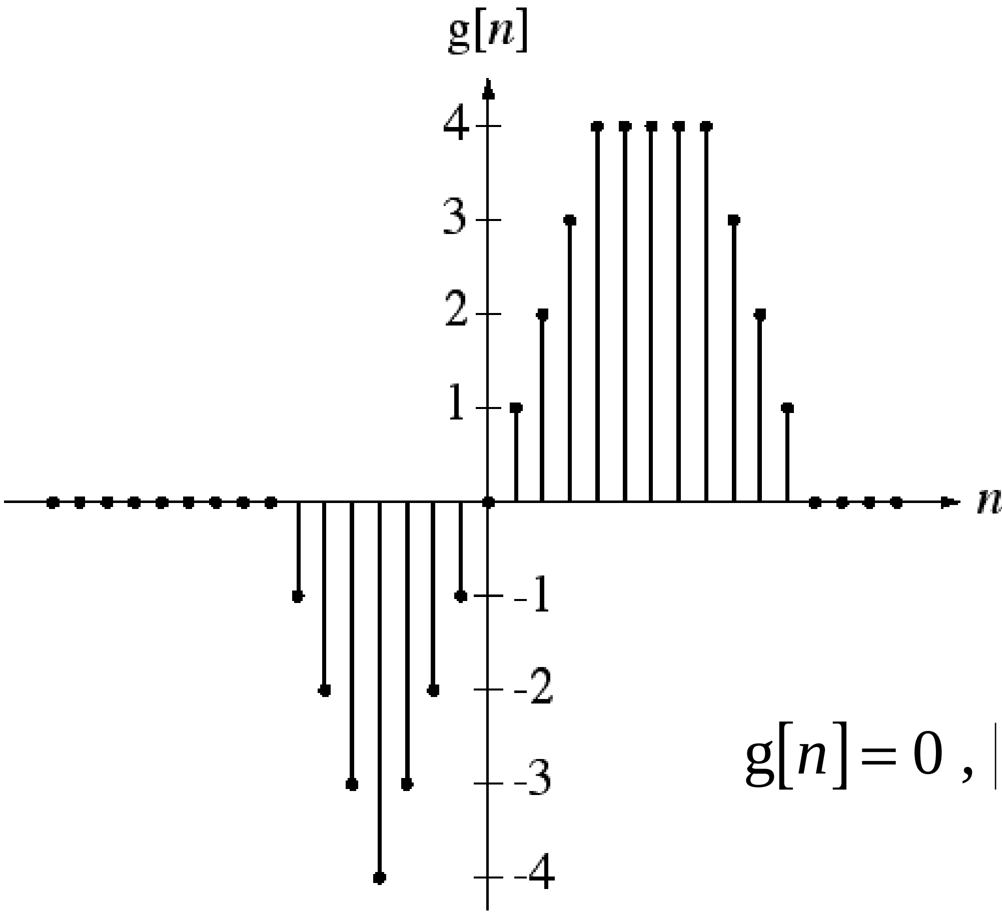
The DT Comb Function

$$\mathrm{comb}_{N_0}[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN_0]$$



Transformation of DT Functions

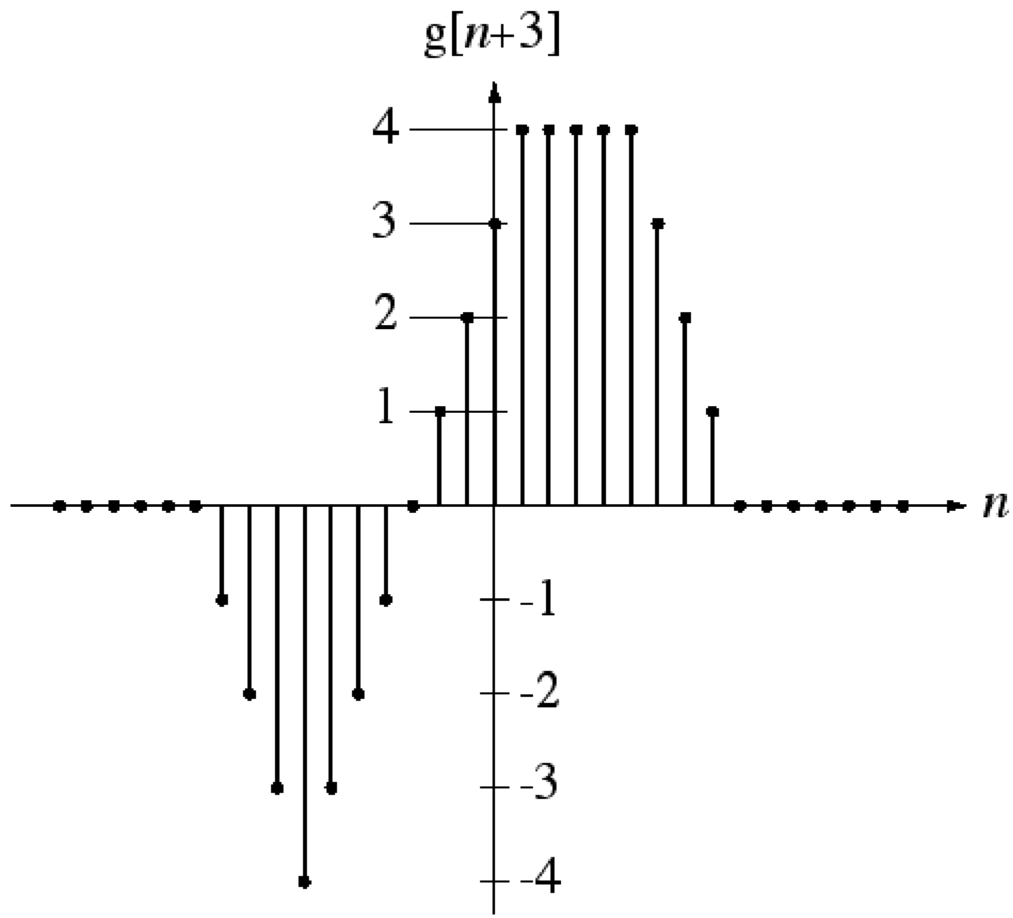
Let g[n] be graphically defined by



$$g[n] = 0 \;, \; |n| > 15$$

Transformation of DT Functions

Time Shifting $n \rightarrow n + n_0 \text{, } n_0$ an integer

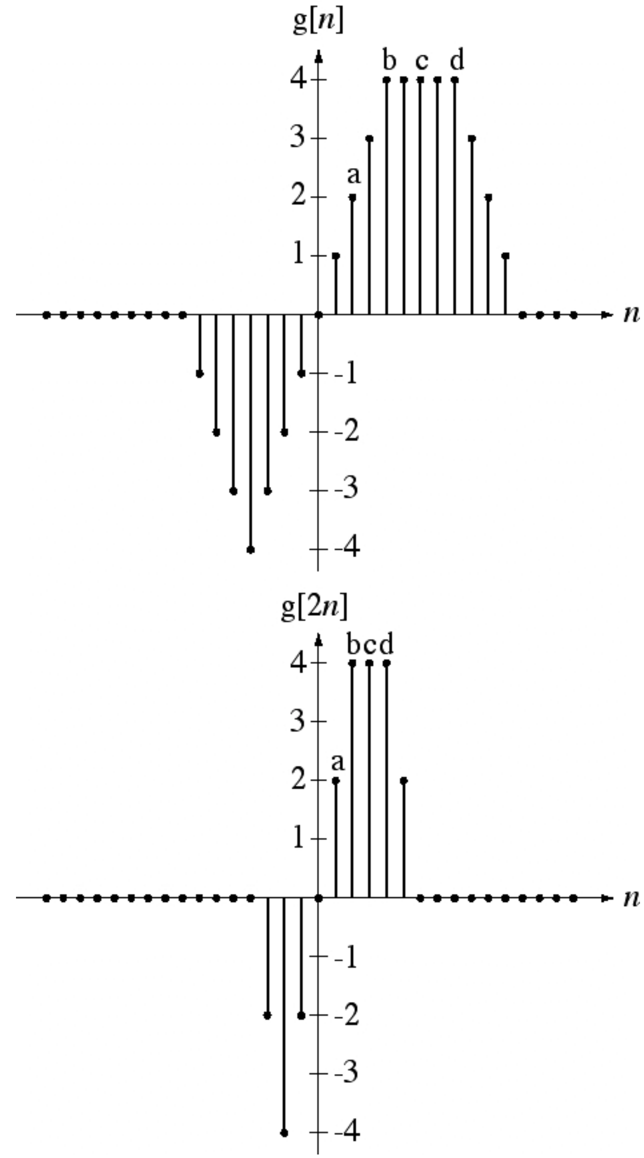


Transformation of DT Functions

Time compression

$$n \to Kn$$

K an integer > 1



Transformation of DT Functions

Time expansion $n \to \frac{n}{K}$, $K > 1$

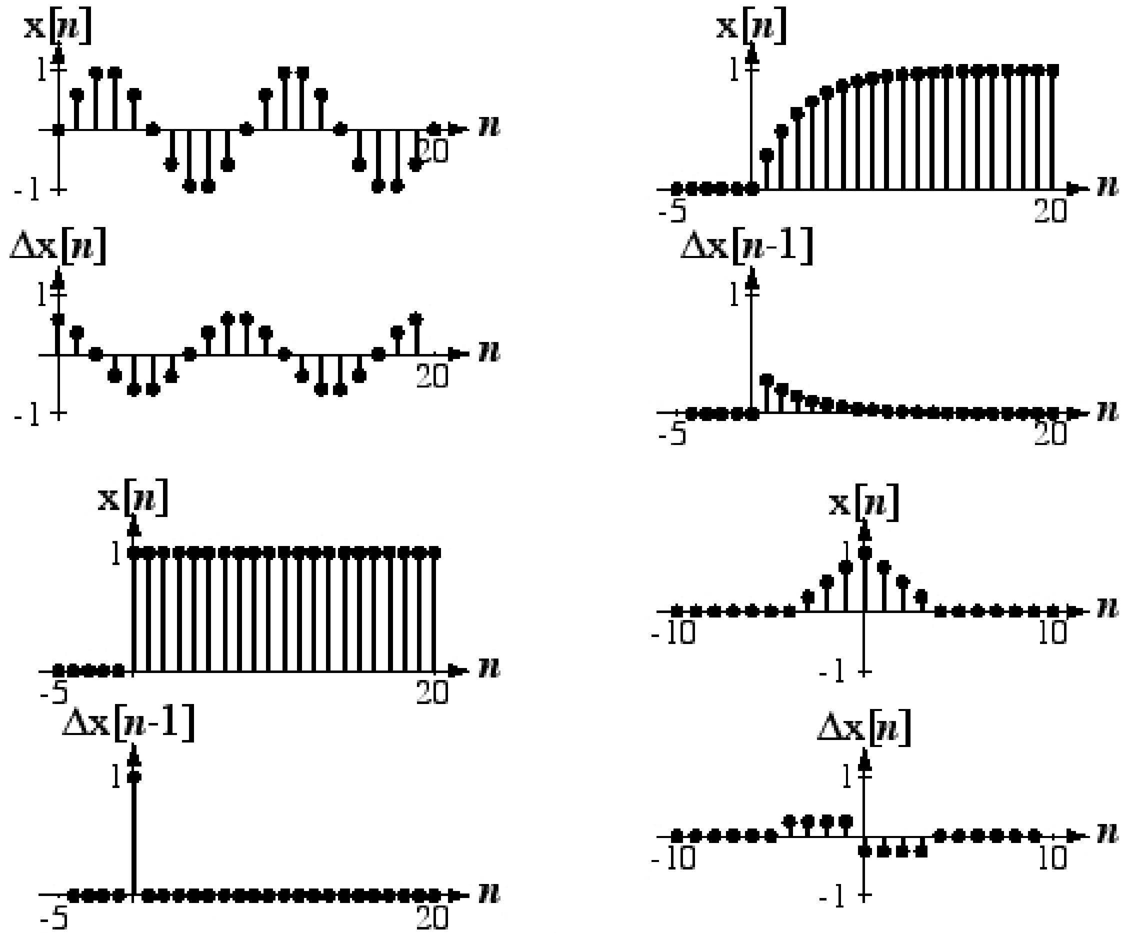
For all n such that n/K is an integer, $g\left[\frac{n}{K}\right]$ is defined.

For all n such that n/K is not an integer, $g\left[\frac{n}{K}\right]$ is not defined.

Transformation of DT Functions

Differencing

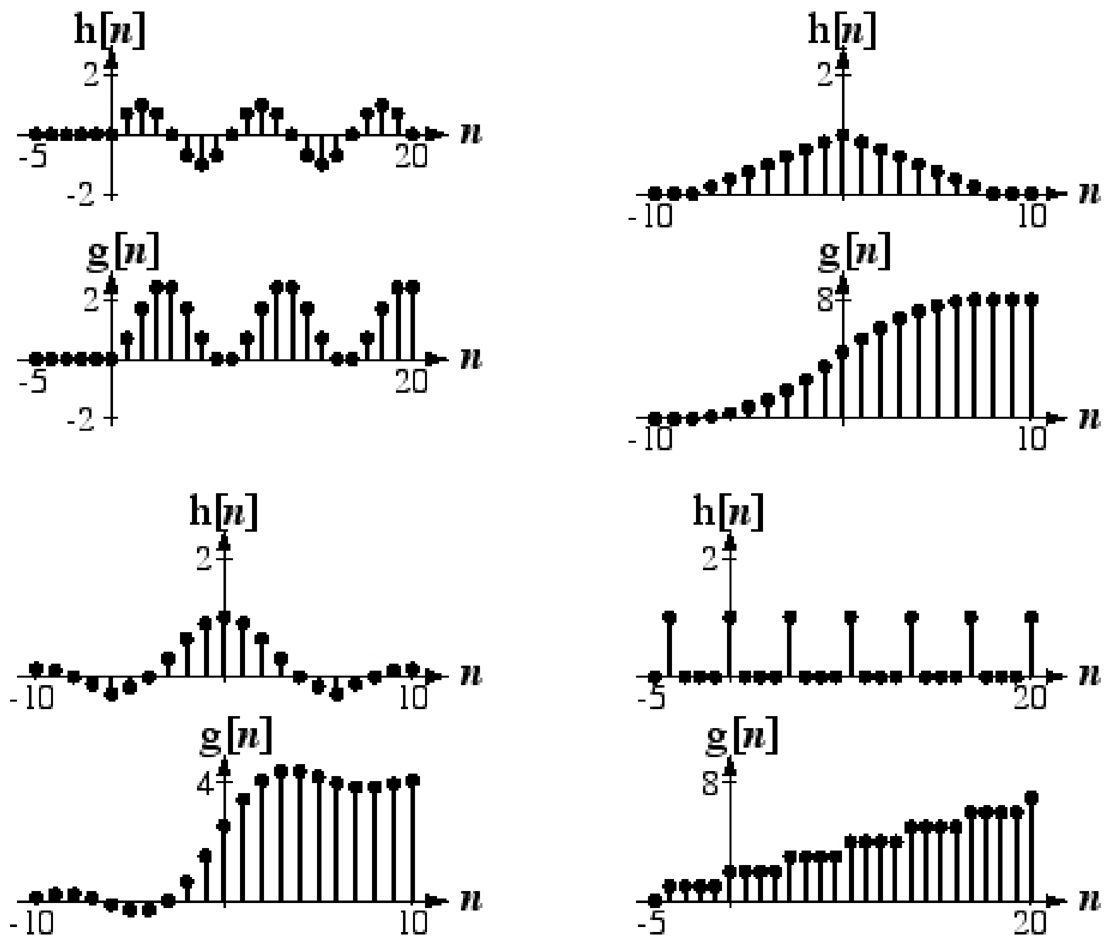
$$\Delta g[n] = g[n+1] - g[n]$$



Transformation of DT Functions

Accumulation

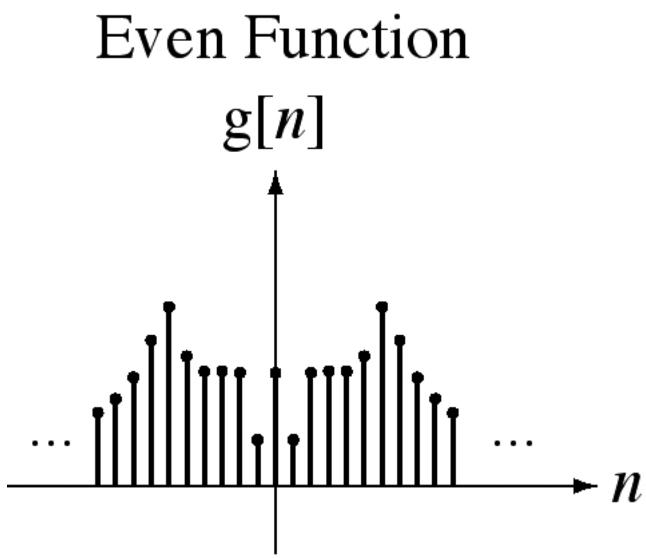
$$g[n] = \sum_{m=-\infty}^{n} h[m]$$

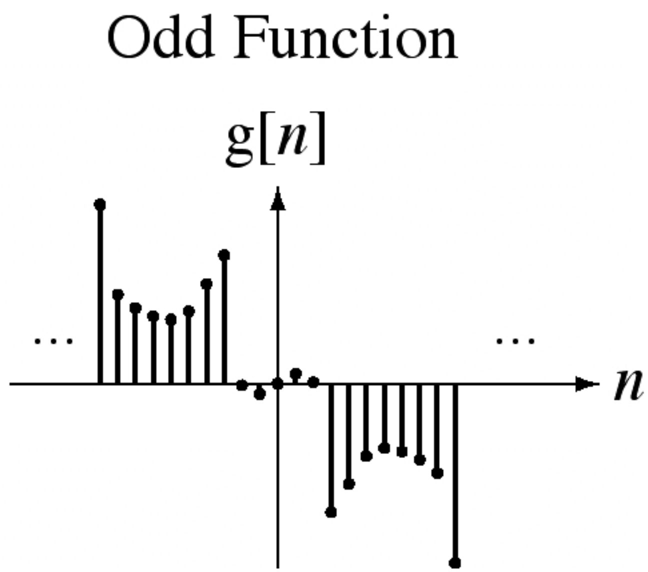


Even and Odd DT Functions

$$g[n] = g[-n]$$

$$g[n] = -g[-n]$$



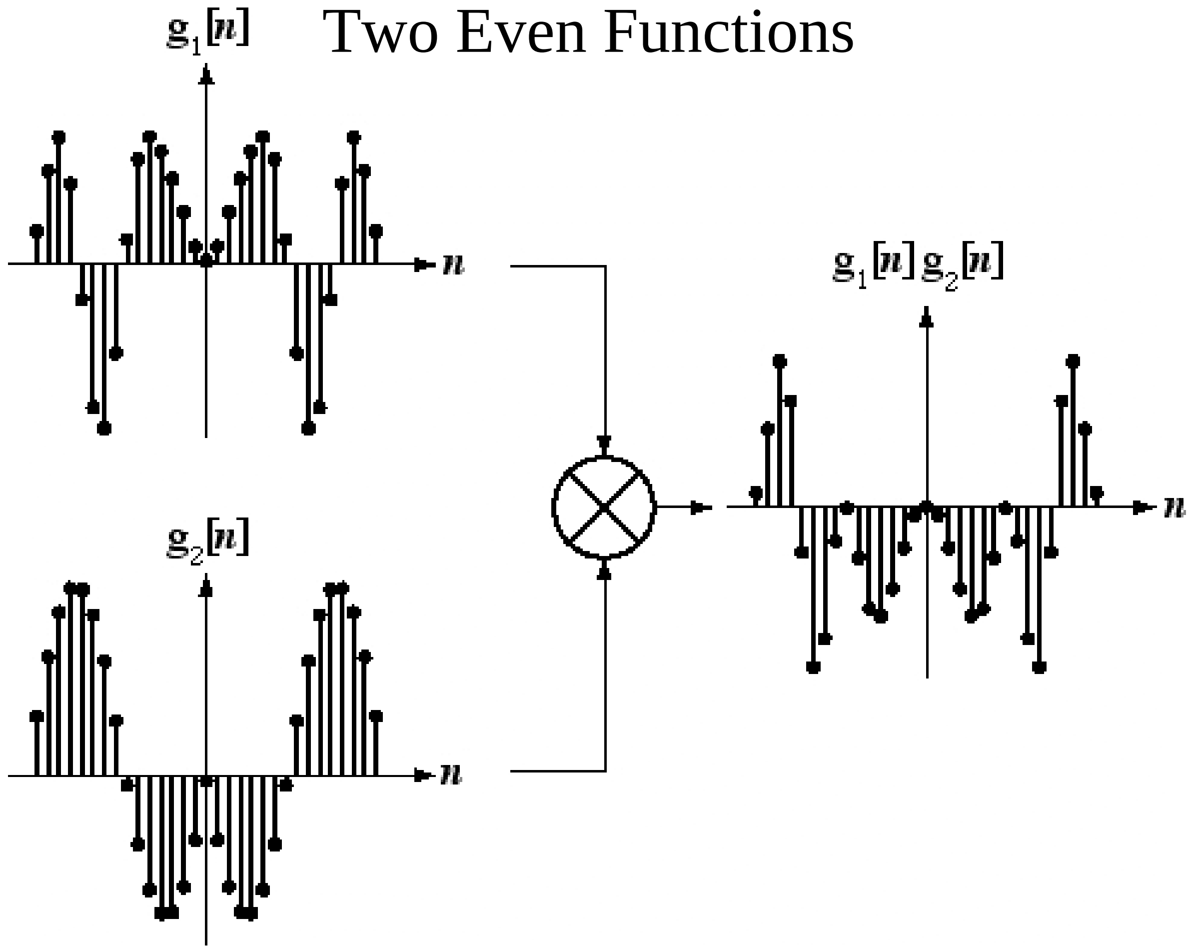


$$g_e[n] = \frac{g[n] + g[-n]}{2}$$

$$g_o[n] = \frac{g[n] - g[-n]}{2}$$

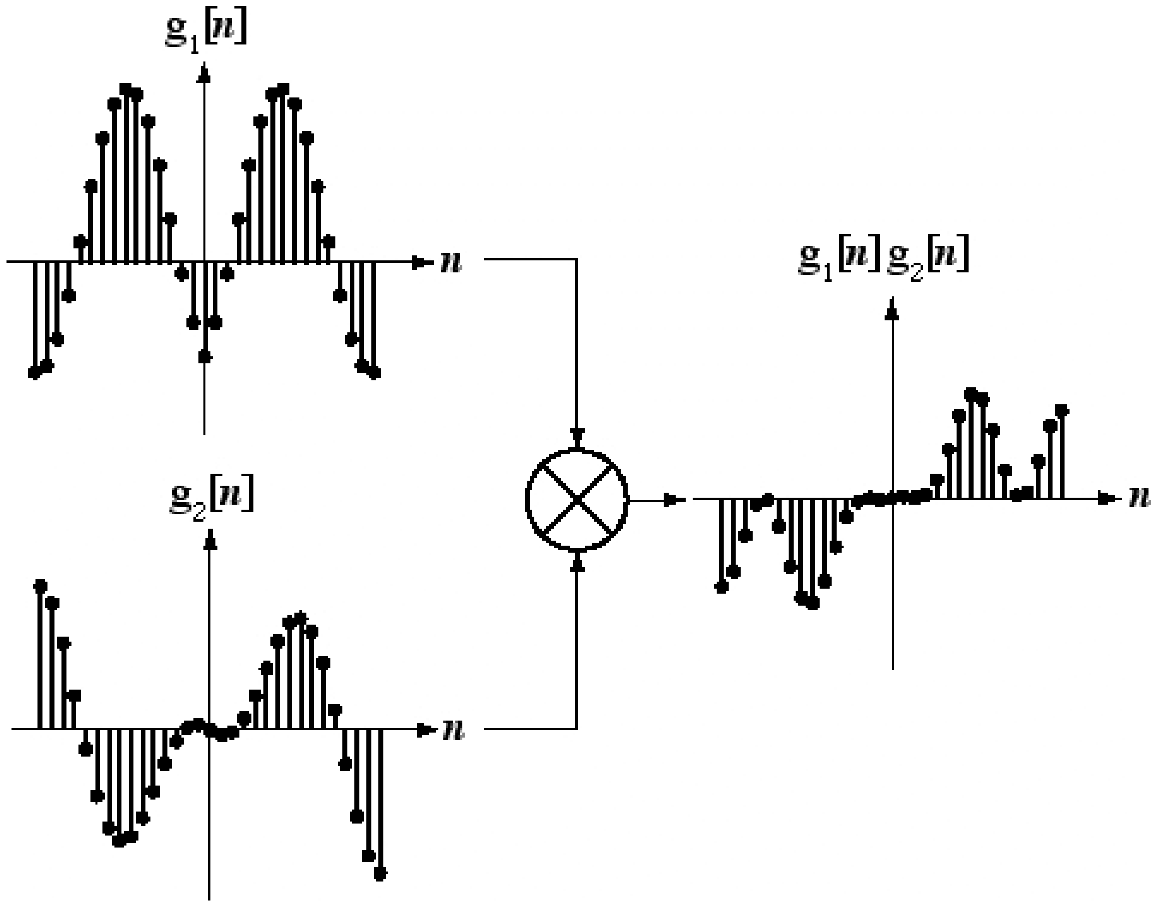
Products of Even and Odd DT Functions

Two Even Functions



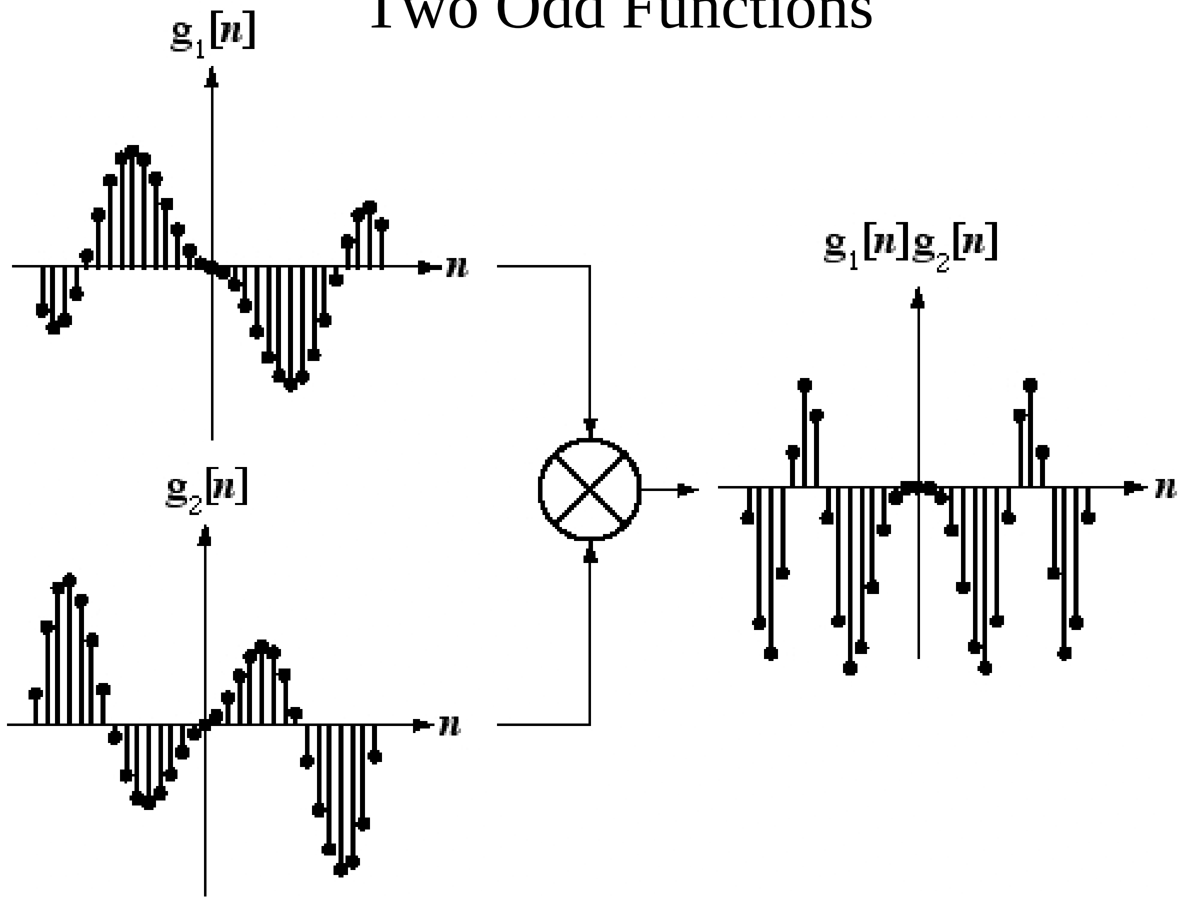
Products of Even and Odd DT Functions

An Even Function and an Odd Function

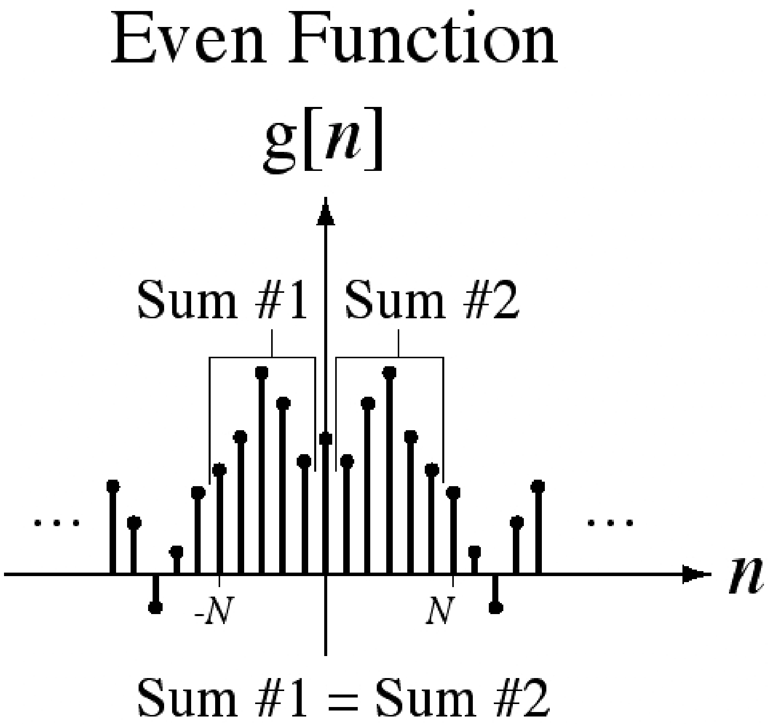


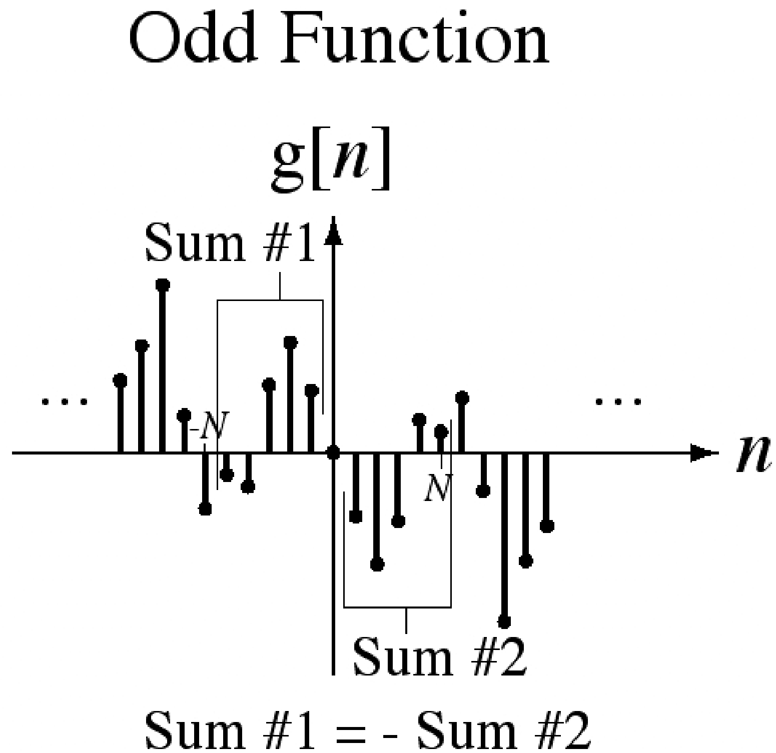
Products of Even and Odd DT Functions

Two Odd Functions



Accumulation of Even and Odd DT Functions





$$\sum_{n=-N}^{N} g[n] = g[0] + 2\sum_{n=1}^{N} g[n]$$

$$\sum_{n=-N}^{N} g[n] = 0$$

Periodic DT Functions

A periodic DT function is one which is invariant to the transformation, $n \to n + mN$, where N is a period of the function and m is any integer.

The minimum positive integer value of N for which $g[n] = g[n+N]$ is called the *fundamental period*, N_0

Signal Energy and Power

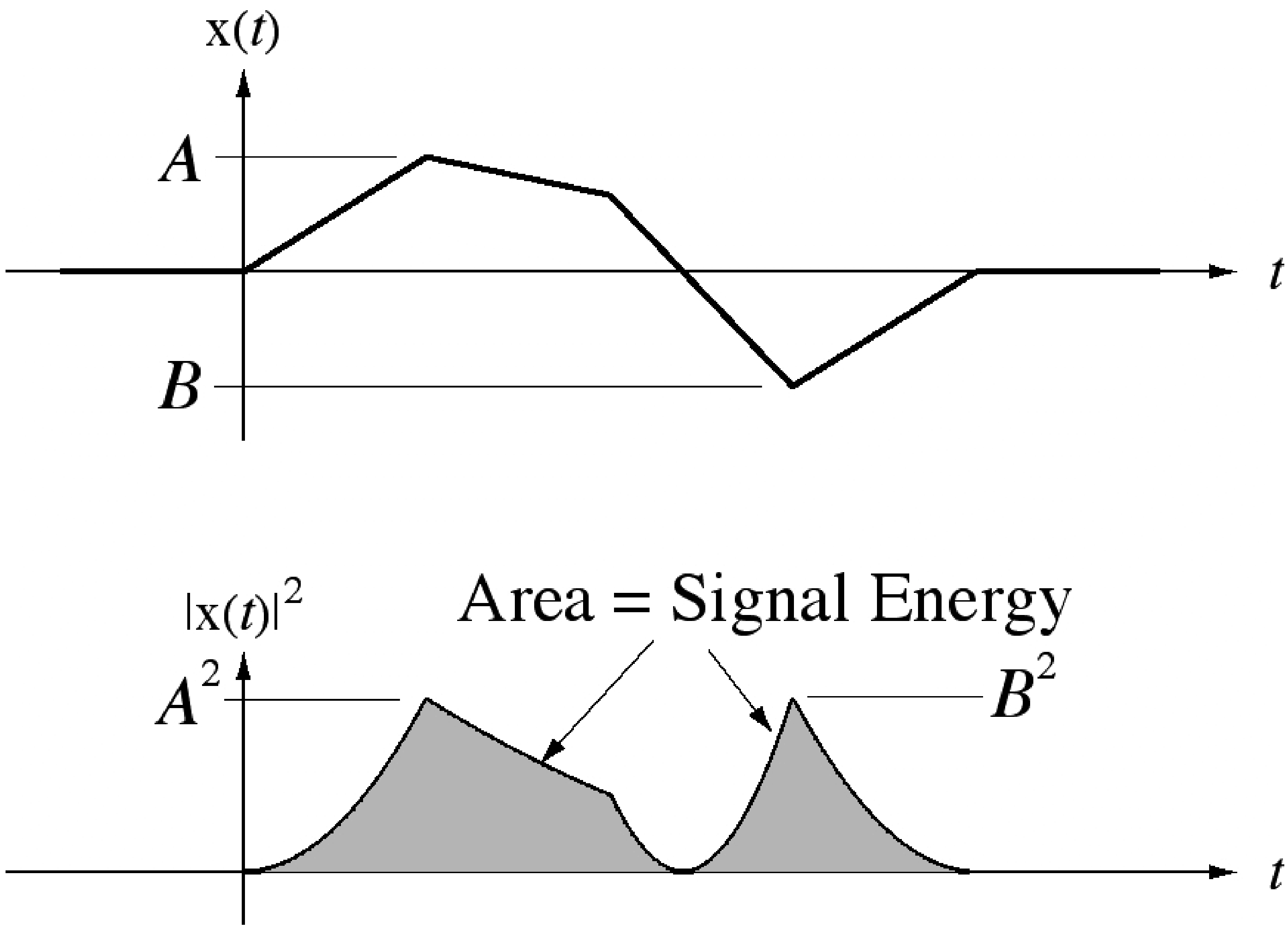
The *signal energy* of a CT signal, x(t), is

$$E_x = \int_{-\infty}^{\infty} |\mathrm{x}(t)|^2 \, dt$$

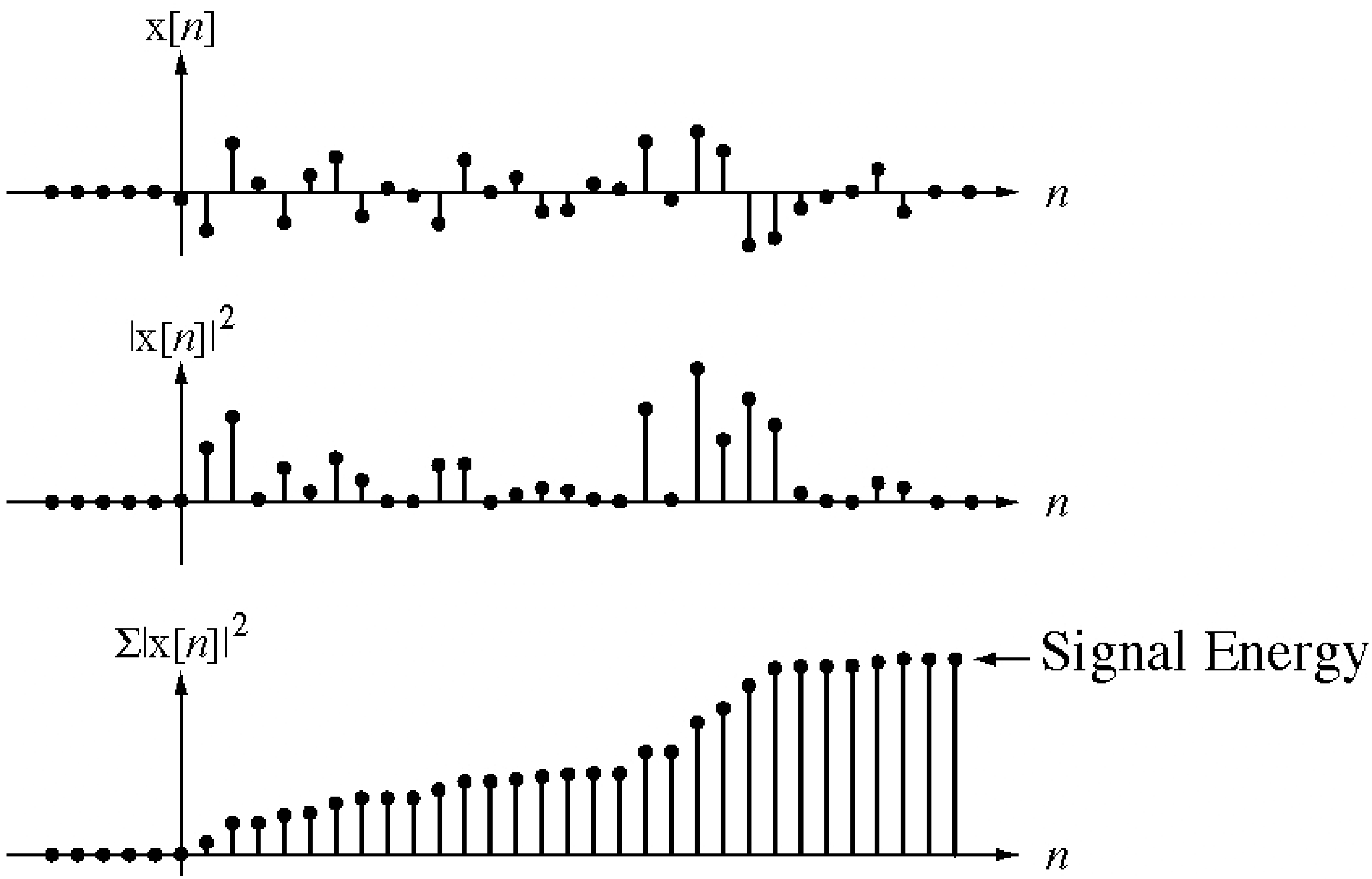
The signal energy of a DT signal, x[n], is

$$E_x = \sum_{n=-\infty}^{\infty} |\mathrm{x}[n]|^2$$

Signal Energy and Power



Signal Energy and Power



Signal Energy and Power

Some signals have infinite signal energy. In that case
It is more convenient to deal with average *signal power*.

The average signal power of a CT signal, x(t), is

$$P_x = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \mathrm{x}(t) \right|^2 dt$$

The average signal power of a DT signal, x[n], is

$$P_x = \lim_{N \to \infty} \frac{1}{2N} \sum_{n=-N}^{N-1} \left| \mathrm{x}[n] \right|^2$$

Signal Energy and Power

For a periodic CT signal, x(t), the average signal power is

$$P_x = \frac{1}{T}\int_T |\mathrm{x}(t)|^2 \, dt$$

where T is any period of the signal.

For a periodic DT signal, x[n], the average signal power is

$$P_x = \frac{1}{N}\sum_{n=\langle N \rangle} |\mathrm{x}[n]|^2$$

where N is any period of the signal. (The notation,

$$\sum_{n=\langle N \rangle}$$

means the sum over any set of consecutive n's exactly N in length.)

Signal Energy and Power

A signal with finite signal energy is called an *energy signal*.

A signal with infinite signal energy and finite average signal power is called a *power signal*.