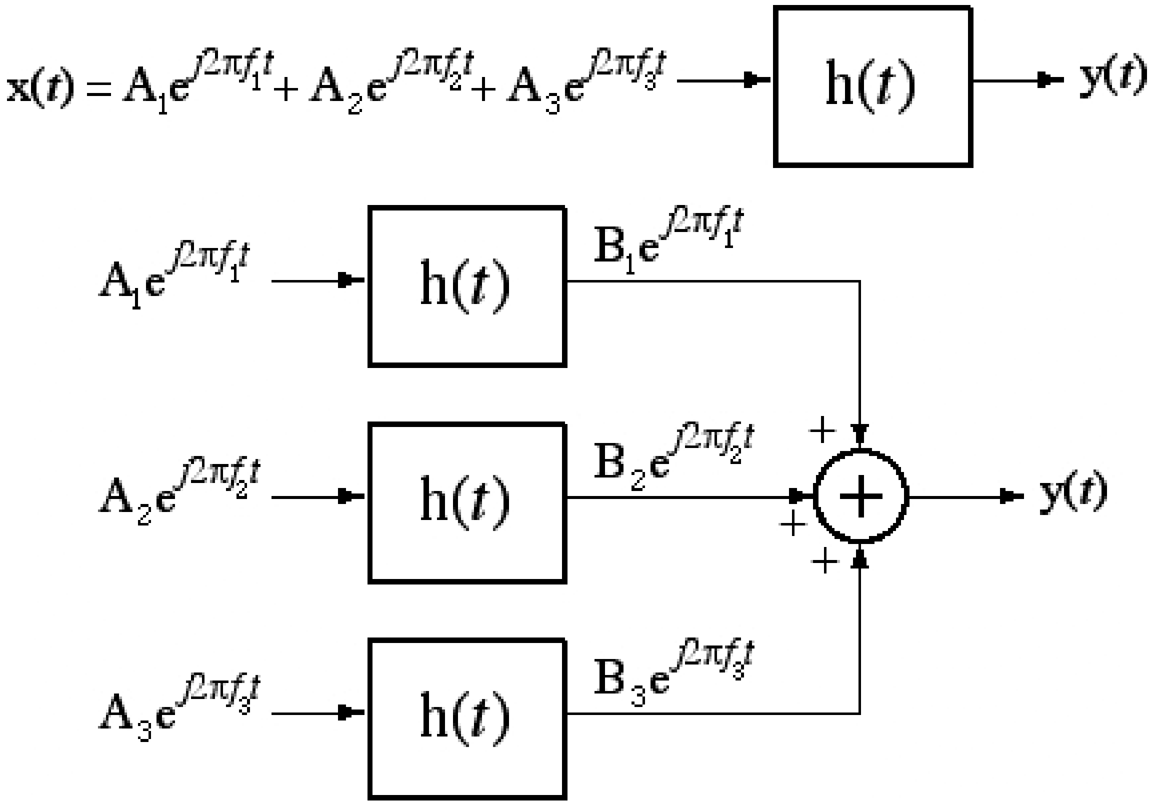
The Fourier Series

Representing a Signal

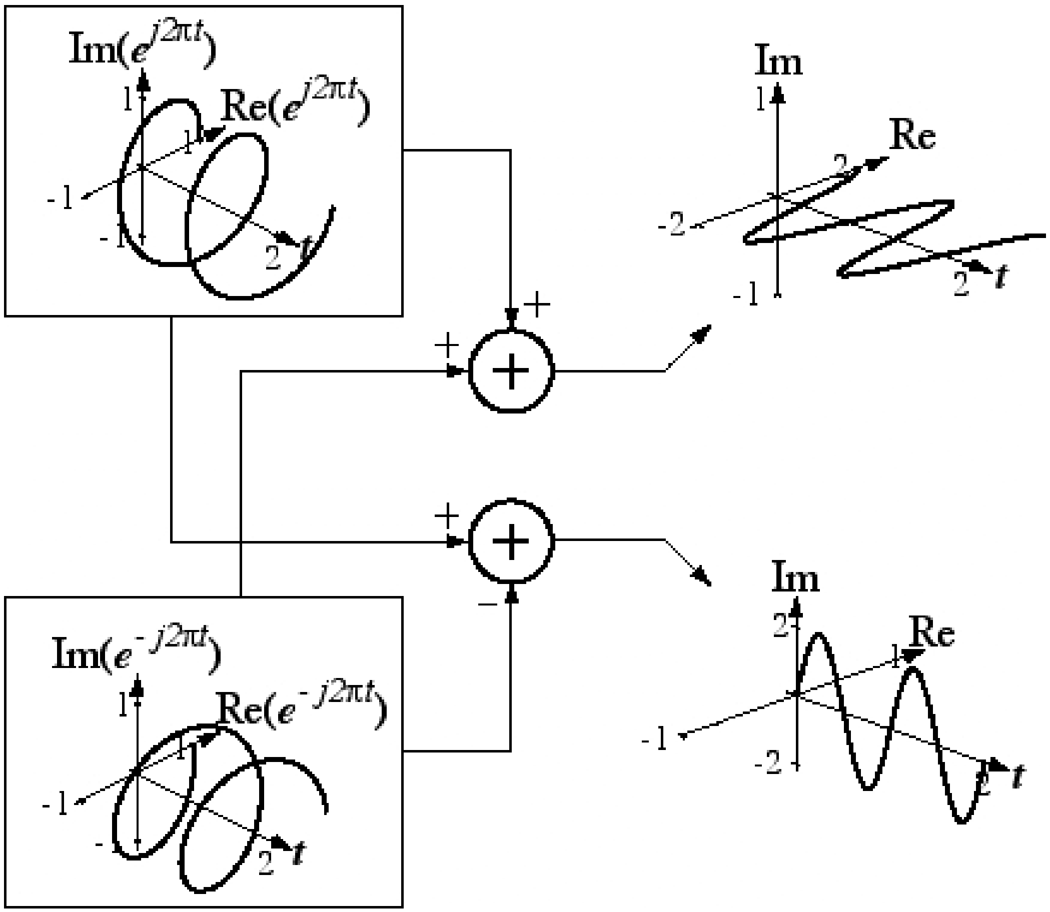
- The convolution method for finding the response of a system to an excitation takes advantage of the linearity and time-invariance of the system and represents the excitation as a linear combination of *impulses* and the response as a linear combination of *impulse responses*
- The Fourier series represents a signal as a linear combination of *complex sinusoids*

Linearity and Superposition

If an excitation can be expressed as a sum of complex sinusoids the response can be expressed as the sum of responses to complex sinusoids.



Real and Complex Sinusoids



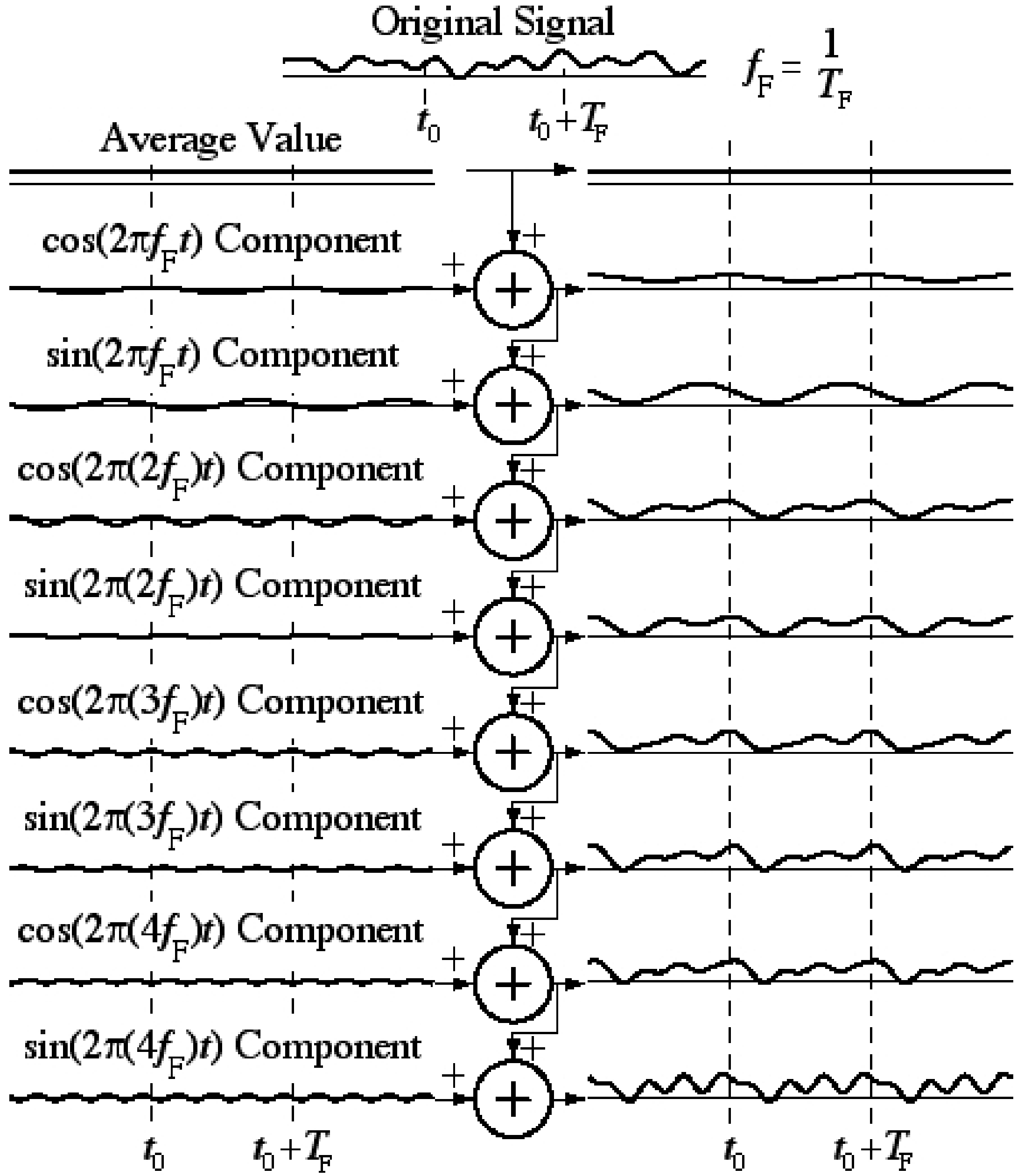
$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{j2}$$

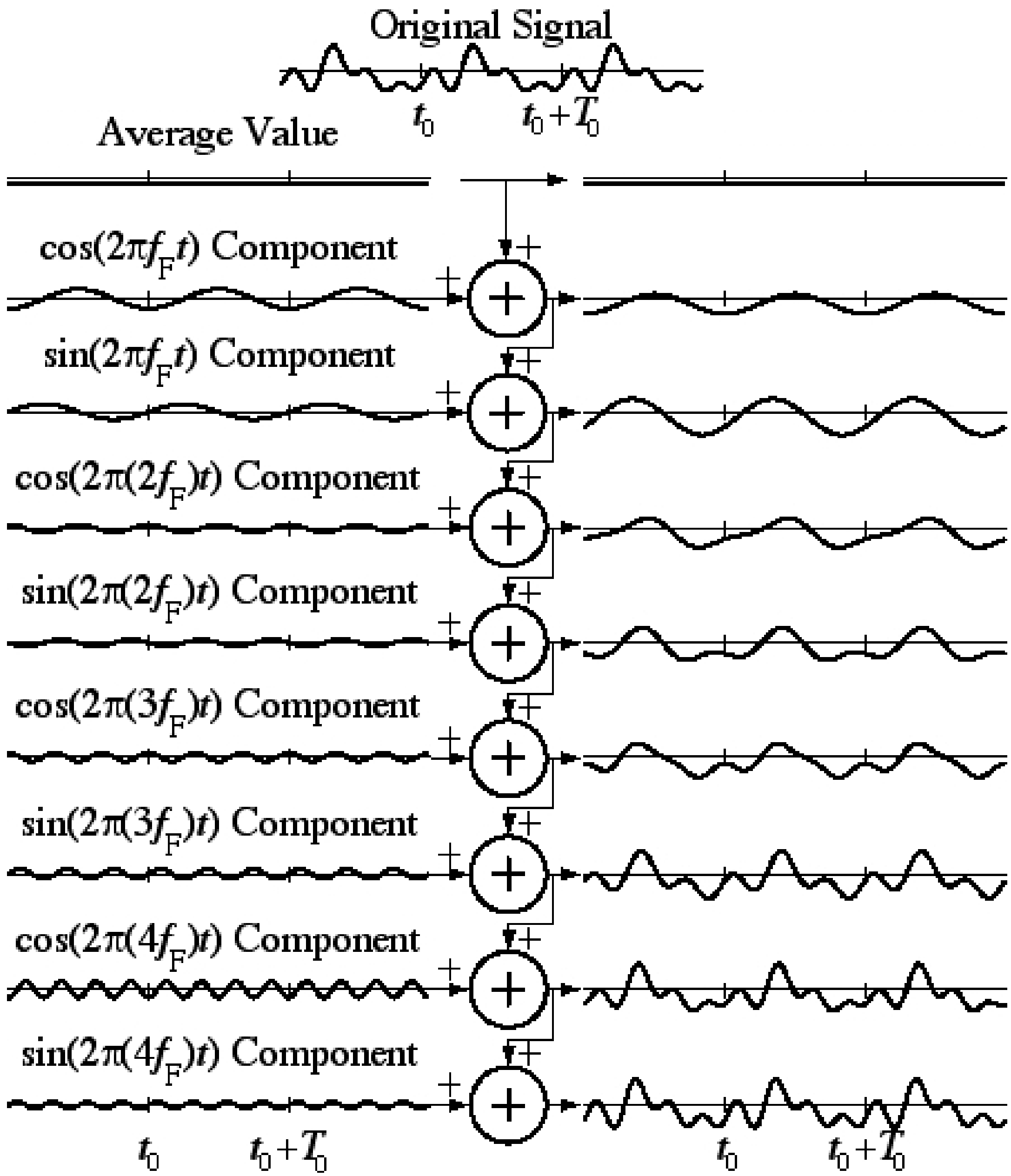
Jean Baptiste Joseph Fourier

3/21/1768 - 5/16/1830

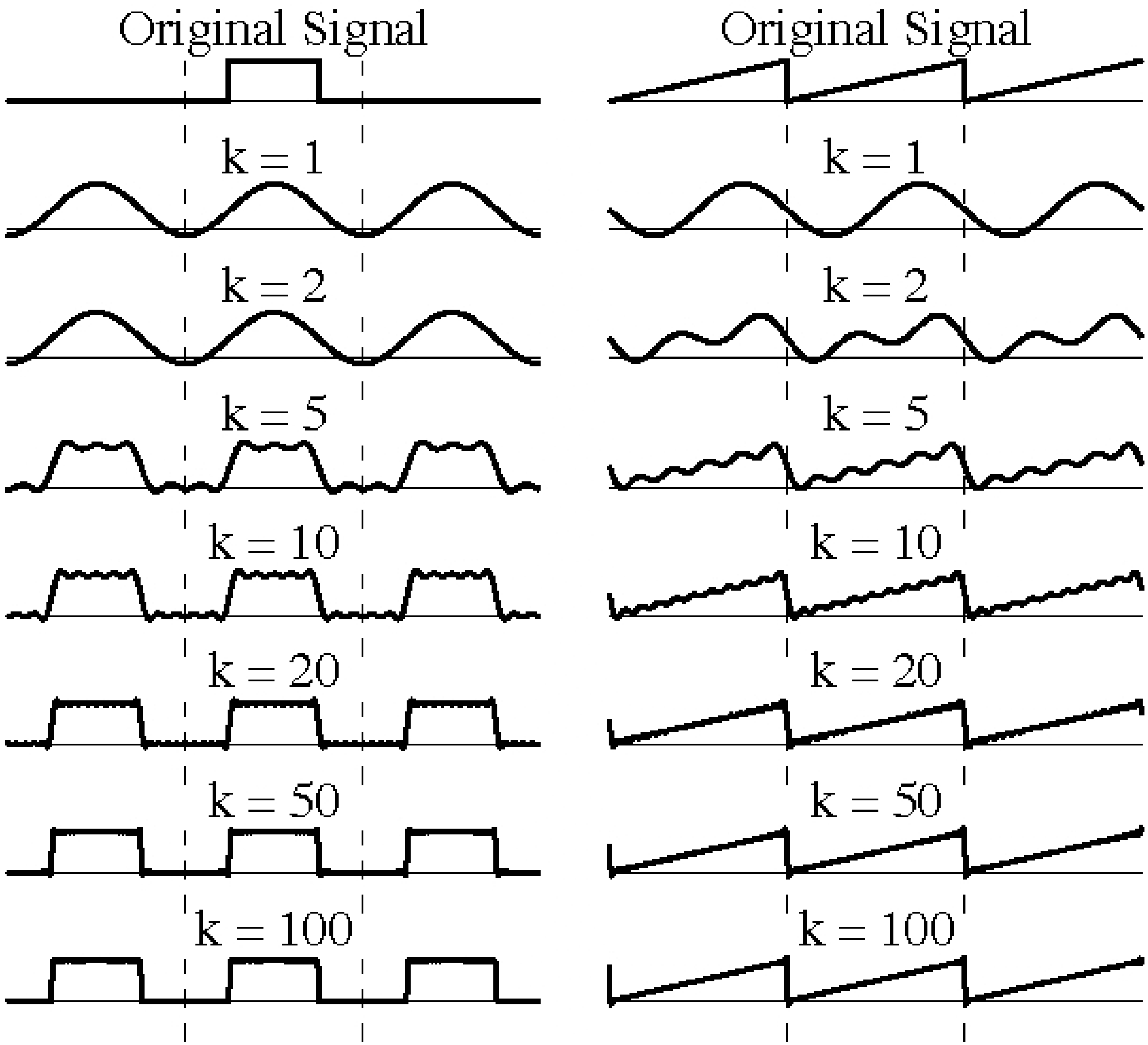
Continuous-Time Fourier Series Concept



Continuous-Time Fourier Series Concept



Continuous-Time Fourier Series Concept



CT Fourier Series Definition

The Fourier series representation, $x_F(t)$, of a signal, x(t), over a time, $t_0 < t < t_0 + T_F$, is

$$x_F(t) = \sum_{k=-\infty}^{\infty} X[k] e^{j2\pi(kf_F)t}$$

where X[k] is the *harmonic function*, k is the *harmonic number* and $f_F = 1/T_F$ (pp. 212-215). The harmonic function can be found from the signal as

$$X[k] = \frac{1}{T_F} \int_{t_0}^{t_0+T_F} x(t) e^{-j2\pi(kf_F)t} dt$$

The signal and its harmonic function form a *Fourier series pair* indicated by the notation, $x(t) \xleftrightarrow{\mathcal{FS}} X[k]$.

CTFS of a Real Function

It can be shown (pp. 216-217) that the continuous-time Fourier series (CTFS) harmonic function of any real-valued function, $\mathrm{x}(t)$, has the property that

$$\mathrm{X}[k] = \mathrm{X}^*[-k]$$

One implication of this fact is that, for real-valued functions, the magnitude of the harmonic function is an even function and the phase is an odd function.

The Trigonometric CTFS

The fact that, for a real-valued function, x(t),

$$\mathrm{X}[k] = \mathrm{X}^*[-k]$$

also leads to the definition of an alternate form of the CTFS, the so-called *trigonometric* form.

$$\mathrm{x}_F(t) = \mathrm{X}_c[0] + \sum_{k=1}^{\infty} \left\{ \mathrm{X}_c[k]\cos\left(2\pi\left(kf_F\right)t\right) + \mathrm{X}_s[k]\sin\left(2\pi\left(kf_F\right)t\right) \right\}$$

where

$$\mathrm{X}_c[k] = \frac{2}{T_F}\int_{t_0}^{t_0+T_F} \mathrm{x}(t)\cos\left(2\pi\left(kf_F\right)t\right)dt$$

$$\mathrm{X}_s[k] = \frac{2}{T_F}\int_{t_0}^{t_0+T_F} \mathrm{x}(t)\sin\left(2\pi\left(kf_F\right)t\right)dt$$

The Trigonometric CTFS

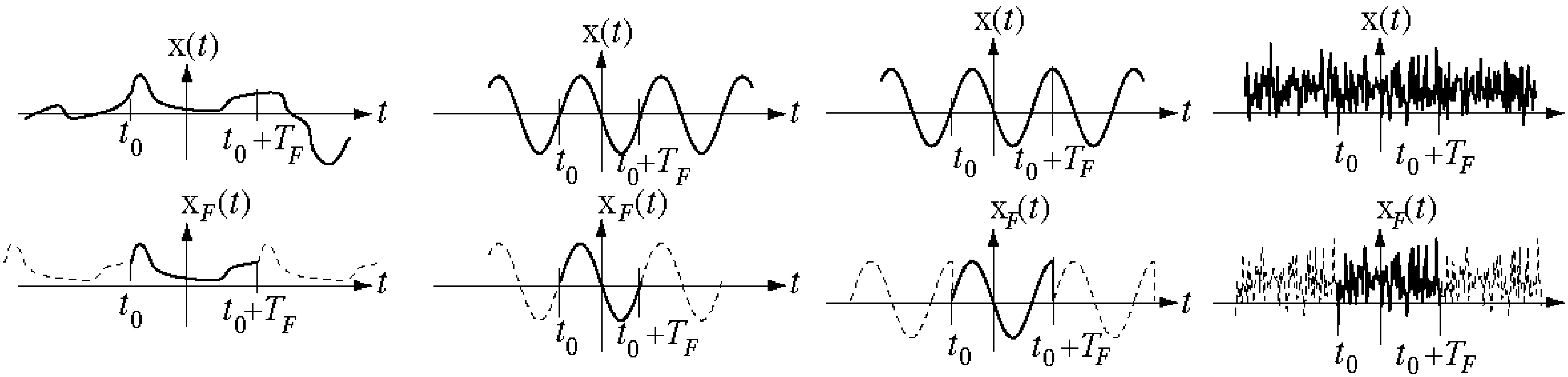
Since both the complex and trigonometric forms of the CTFS represent a signal, there must be relationships between the harmonic functions. Those relationships are

$$\left\{\begin{array}{c} X_c[0] = X[0] \\ X_s[0] = 0 \\ X_c[k] = X[k] + X^*[k] \\ X_s[k] = j\left(X[k] - X^*[k]\right) \end{array}\right\} \;, \quad k = 1, 2, 3, \ldots$$

$$\left\{\begin{array}{c} X[0] = X_c[0] \\ X[k] = \dfrac{X_c[k] - j\, X_s[k]}{2} \\ X[-k] = X^*[k] = \dfrac{X_c[k] + j\, X_s[k]}{2} \end{array}\right\} \;, \quad k = 1, 2, 3, \ldots$$

Periodicity of the CTFS

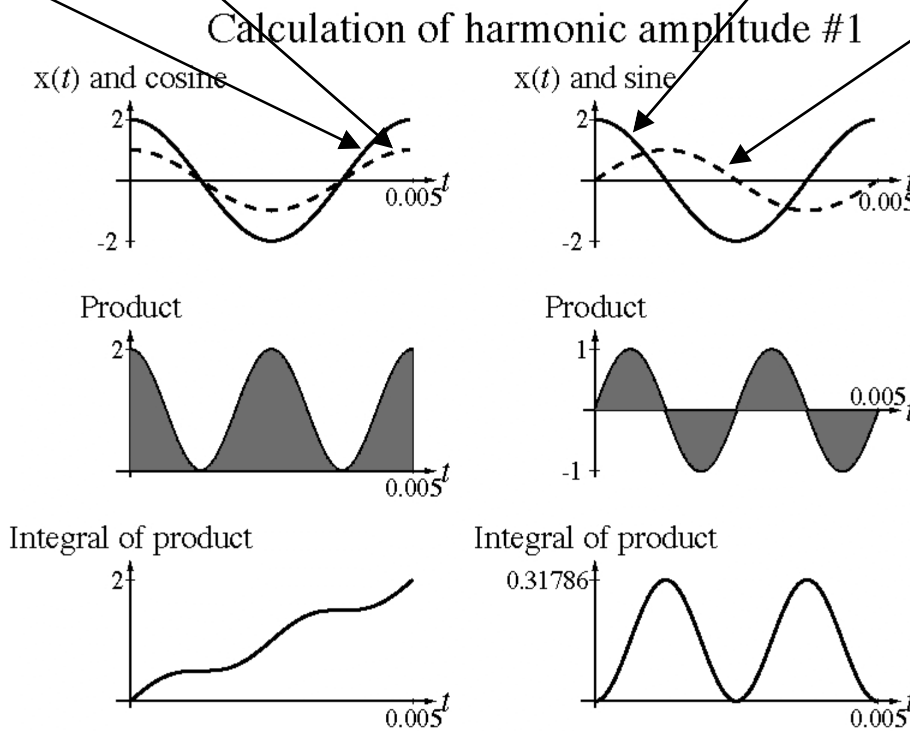
It can be shown (pg. 218) that the CTFS representation, $x_F(t)$ of a function, x(t), is periodic with fundamental period, T_F . Therefore, if x(t) is also periodic with fundamental period, T_0 and if T_F is an integer multiple of T_0 then the two functions are equal for all t, not just in the interval, $t_0 < t < t_0 + T_F$.



CTFS Example #1

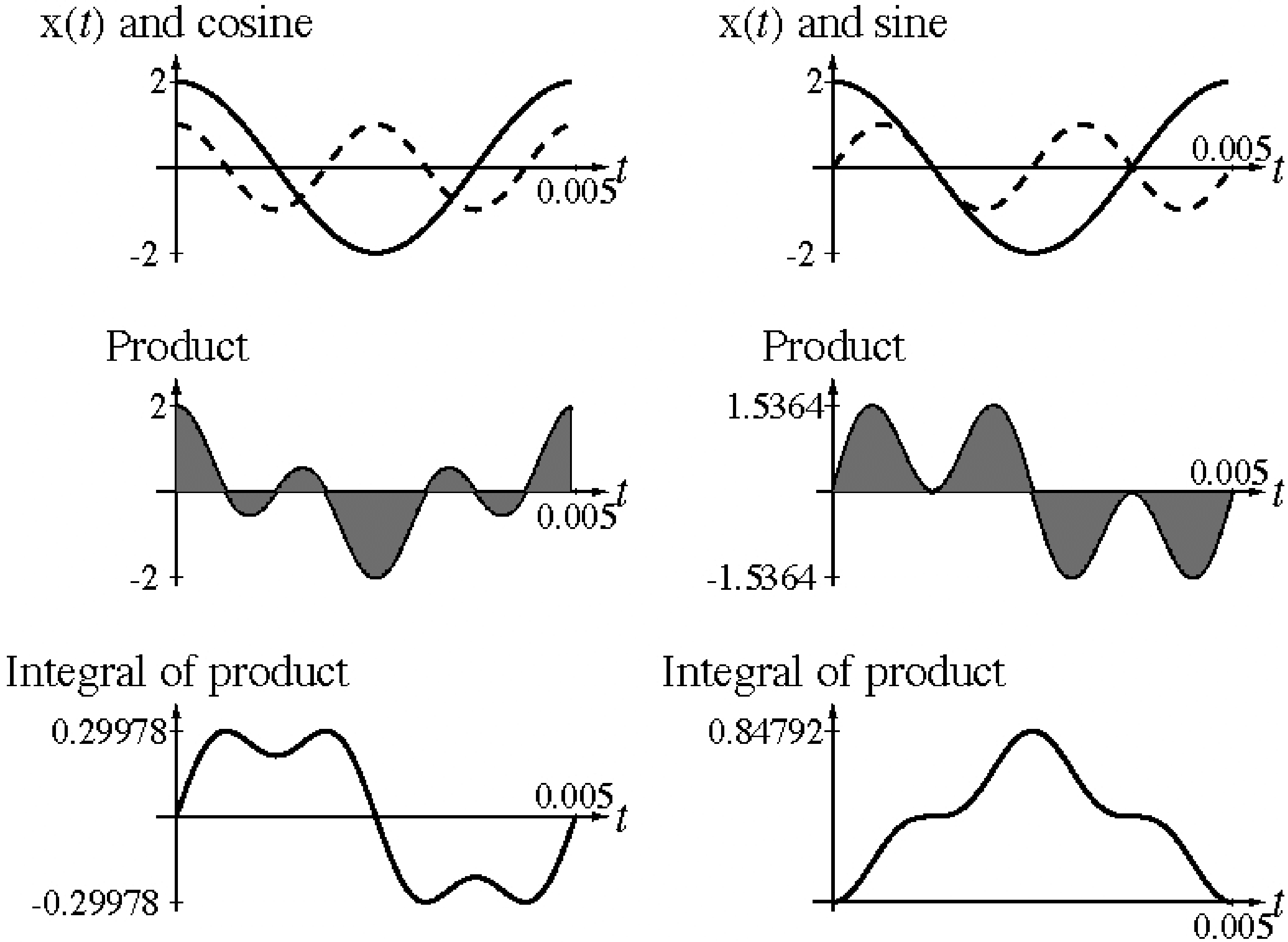
Let a signal be defined by $\mathrm{x}(t) = 2\cos(400\pi t)$ and let $T_F = 5$ ms which is the same as T_0

$$X_c[k] = \frac{2}{T_F}\int_{t_0}^{t_0+T_F} \underbrace{\mathrm{x}(t)}\,\underbrace{\cos(2\pi(kf_F)t)}\,dt \qquad X_s[k] = \frac{2}{T_F}\int_{t_0}^{t_0+T_F} \underbrace{\mathrm{x}(t)}\,\underbrace{\sin(2\pi(kf_F)t)}\,dt$$



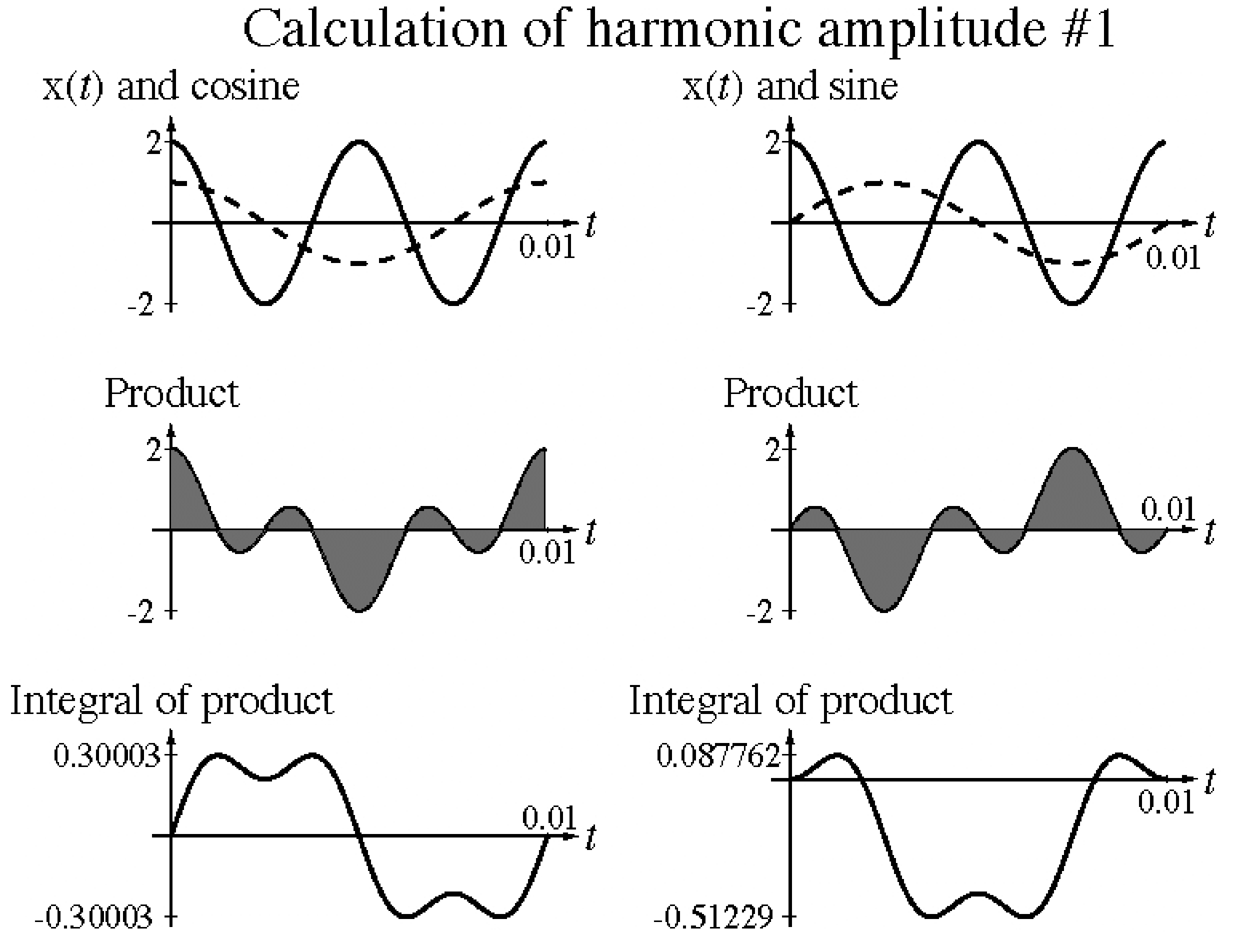
CTFS Example #1

Calculation of harmonic amplitude #2

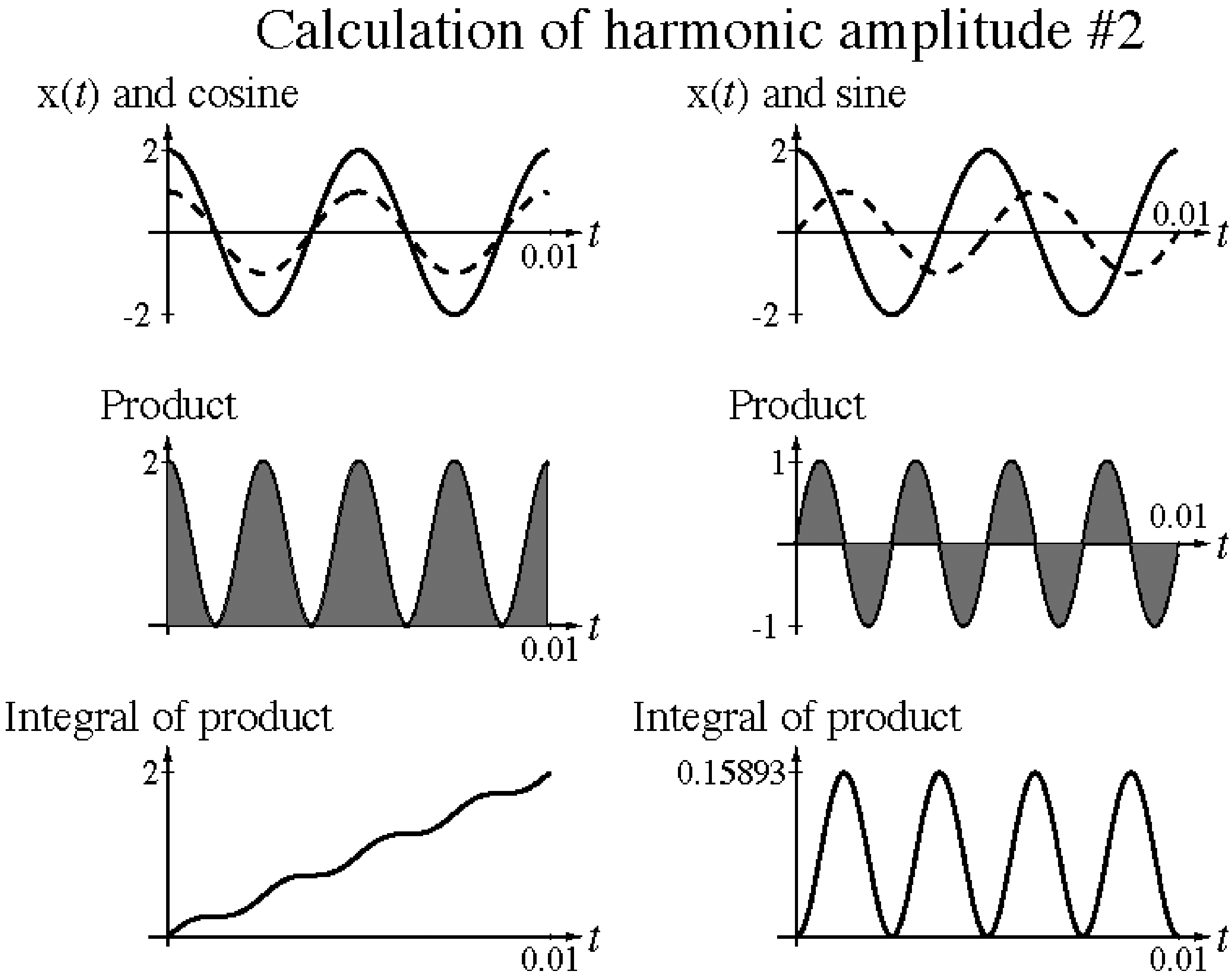


CTFS Example #2

Let a signal be defined by $\mathrm{x}(t) = 2\cos(400\pi t)$ and let $T_F = 10$ ms which is $2T_0$

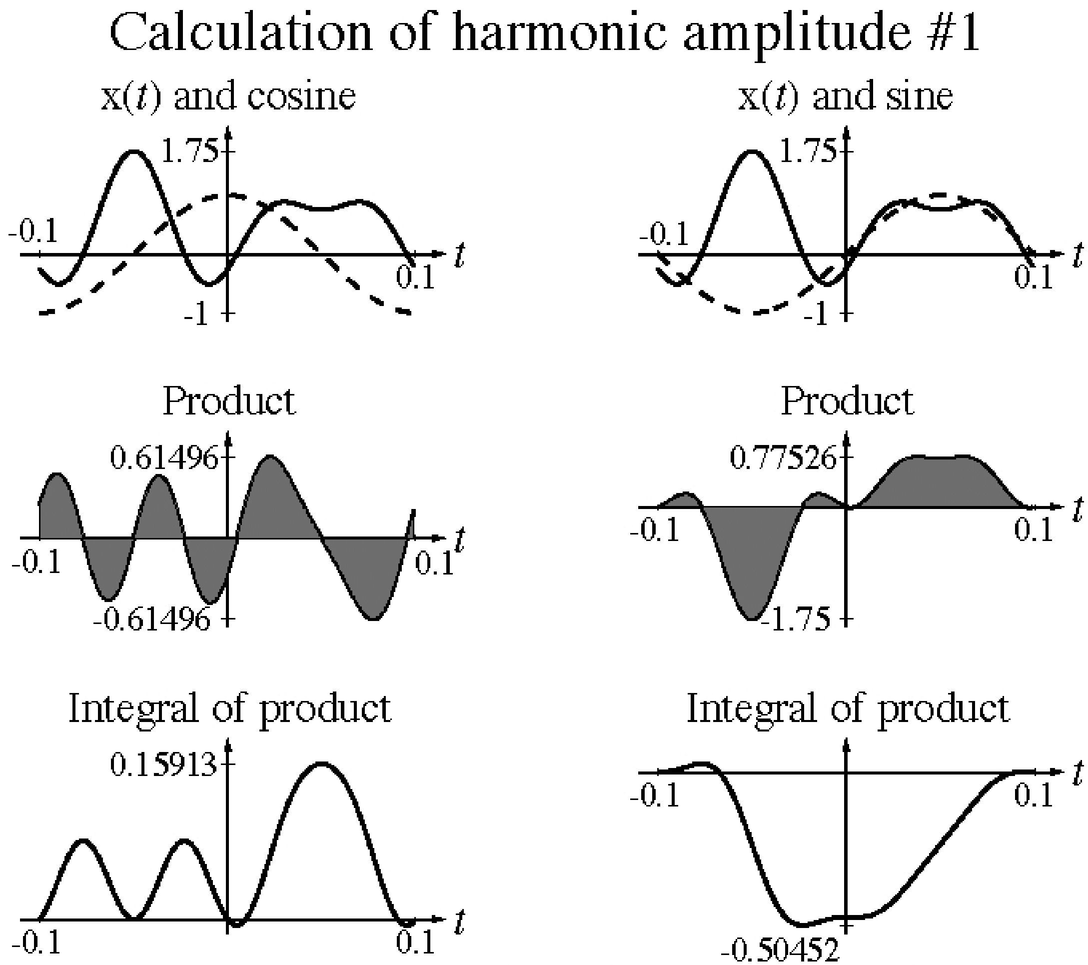


CTFS Example #2

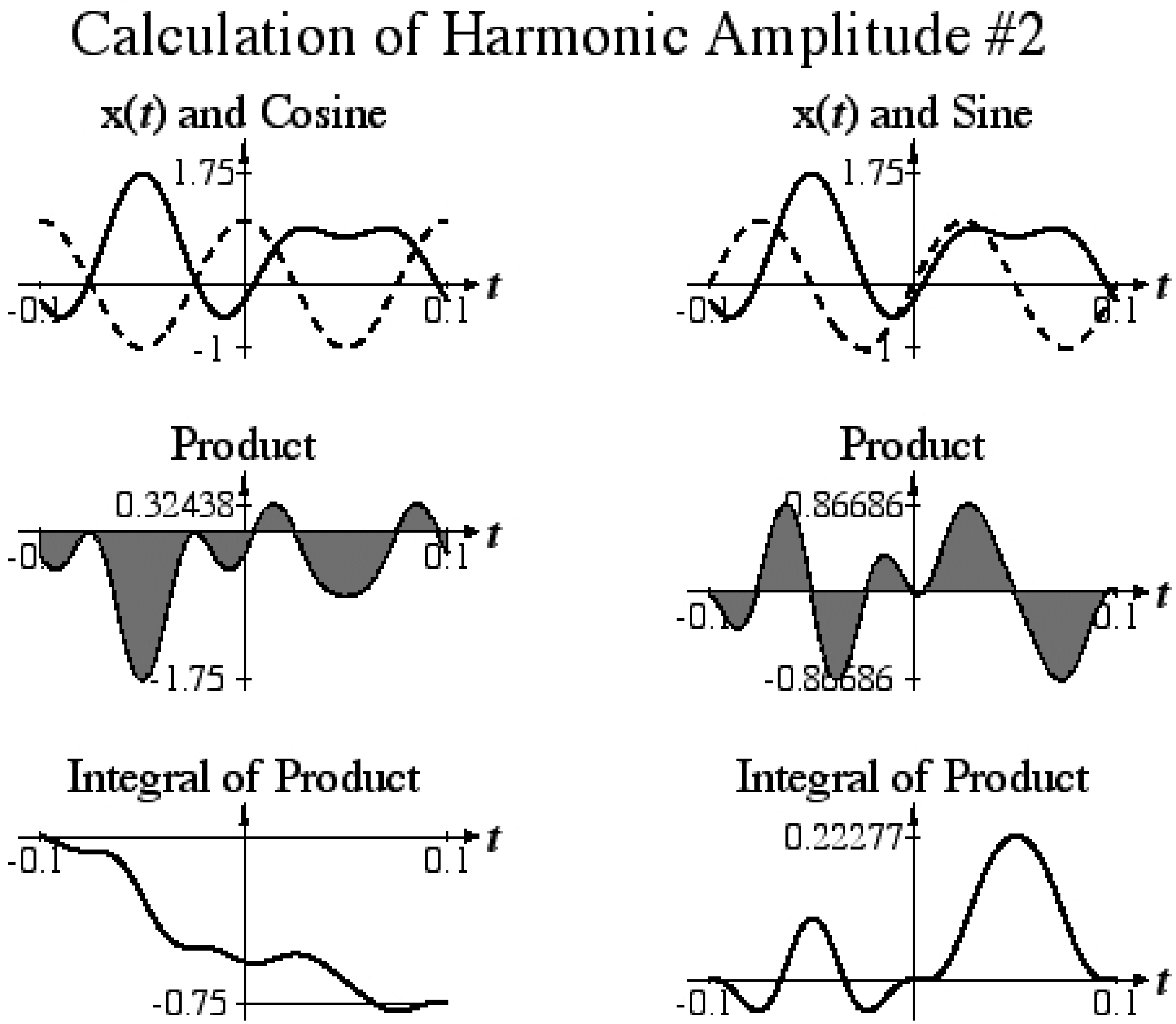


CTFS Example #3

Let $x(t) = \frac{1}{2} - \frac{3}{4}\cos(20\pi t) + \frac{1}{2}\sin(30\pi t)$ and let $T_F = 200$ ms

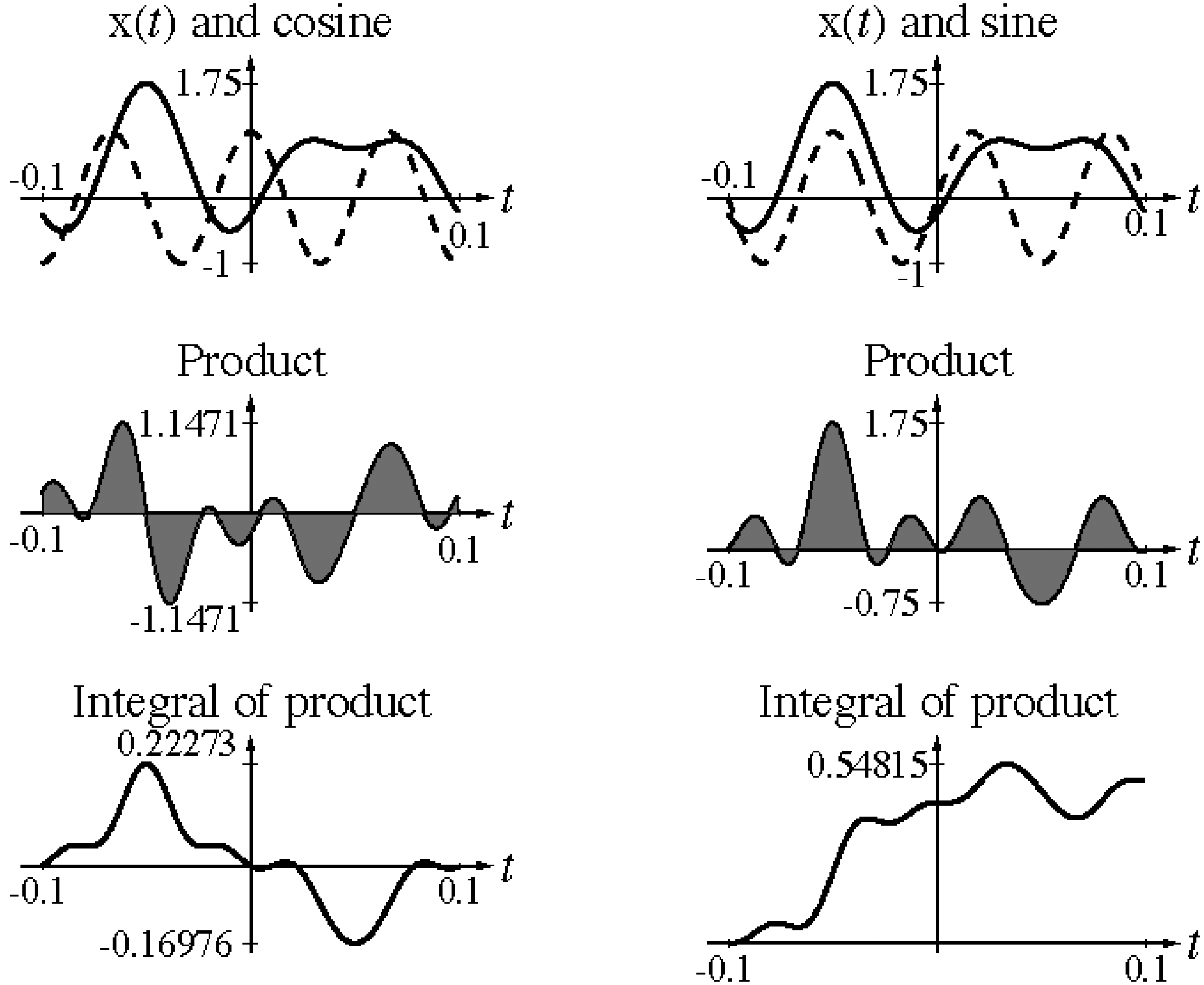


CTFS Example #3

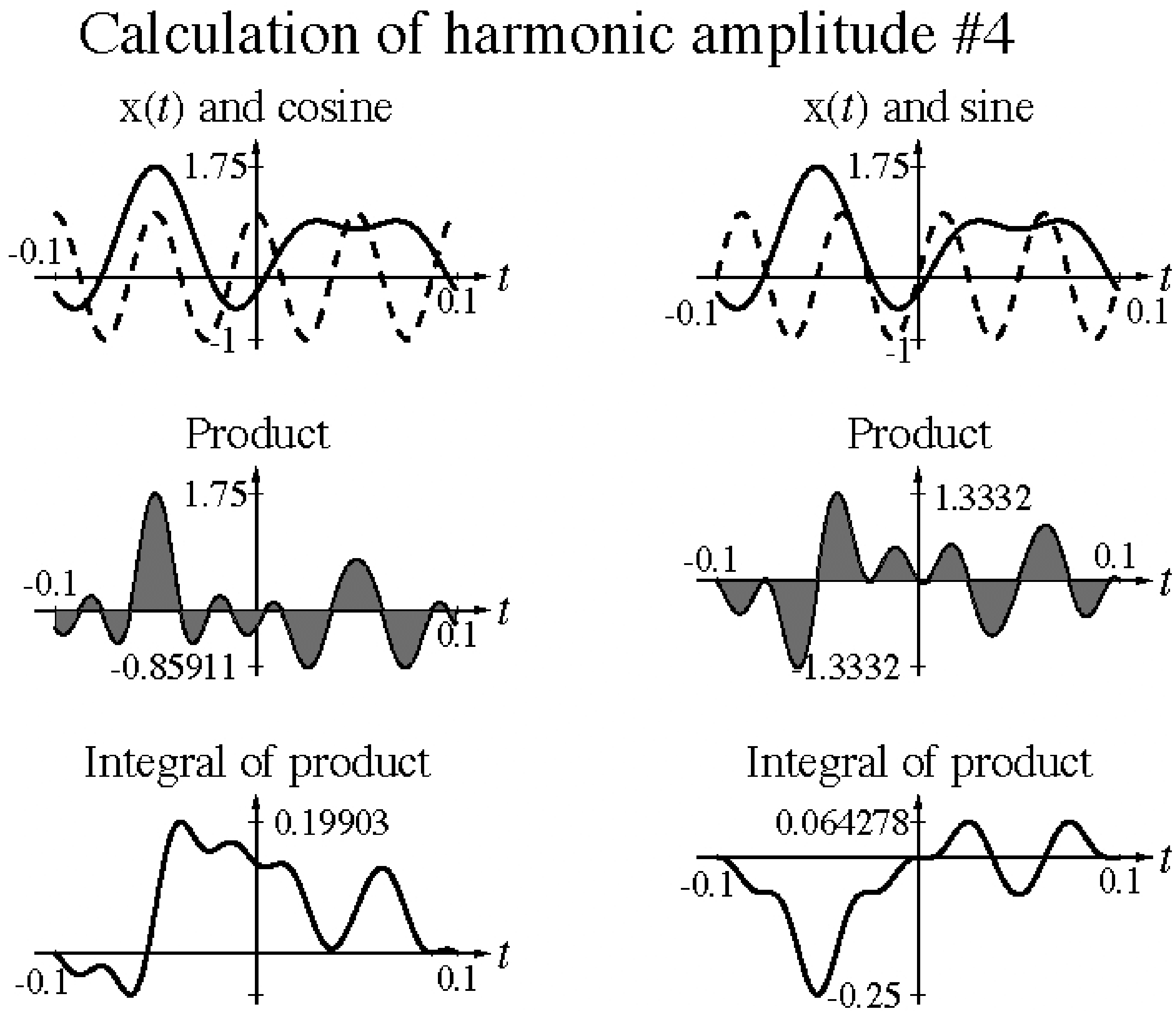


CTFS Example #3

Calculation of harmonic amplitude #3



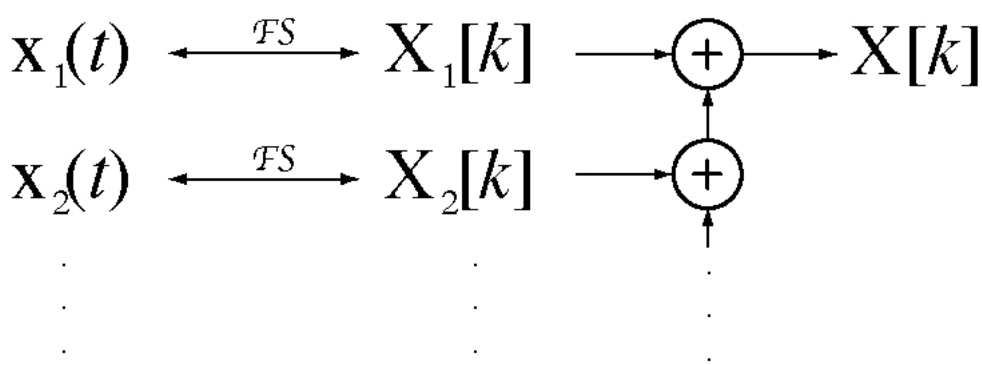
CTFS Example #3



Linearity of the CTFS

$$\mathrm{x}(t) = \mathrm{x}_1(t) + \mathrm{x}_2(t) + \ldots$$

$$\mathrm{x}(t) \xleftrightarrow{\mathcal{FS}} \mathrm{X}[k]$$

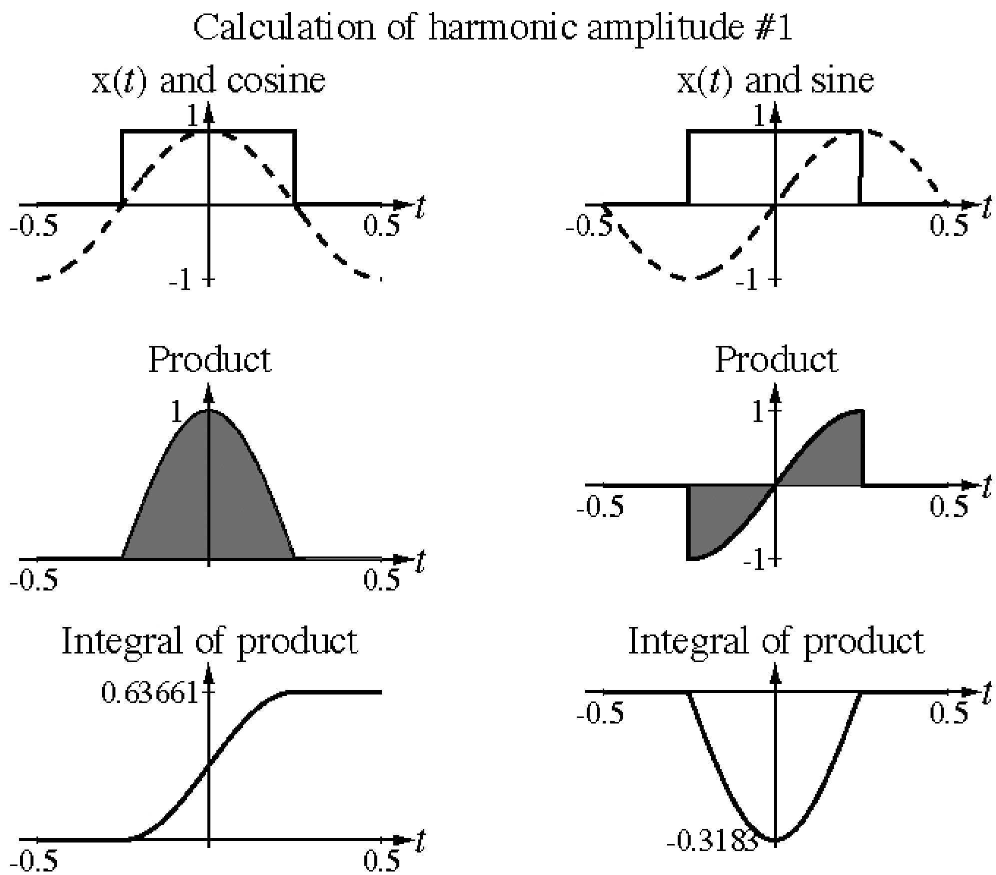


These relations hold *only if* the harmonic functions, X, of all the component functions, x, are based on the same representation time.

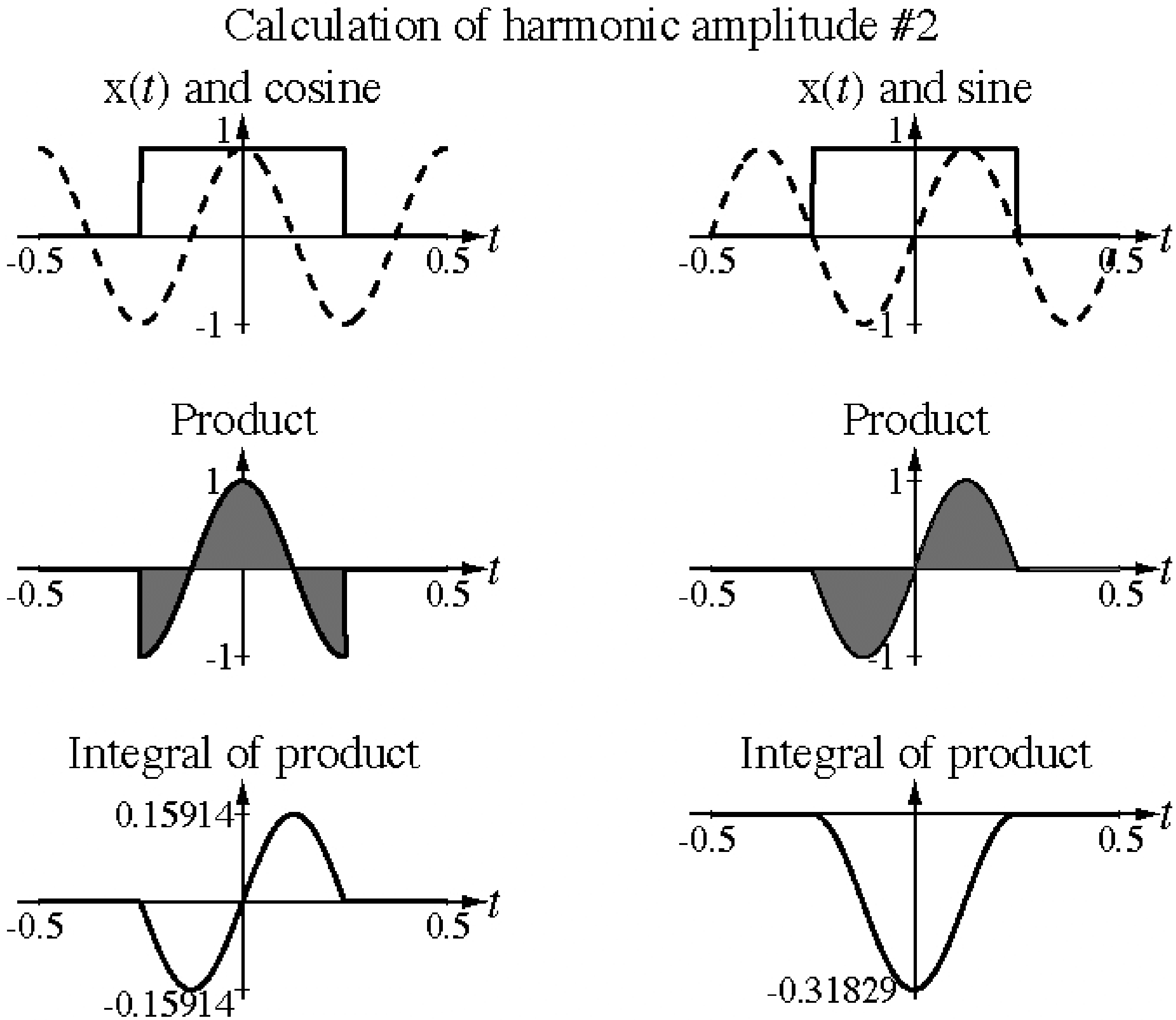
CTFS Example #4

Let the signal be a 50% duty-cycle square wave with an amplitude of one and a fundamental period , $T_0 = 1$

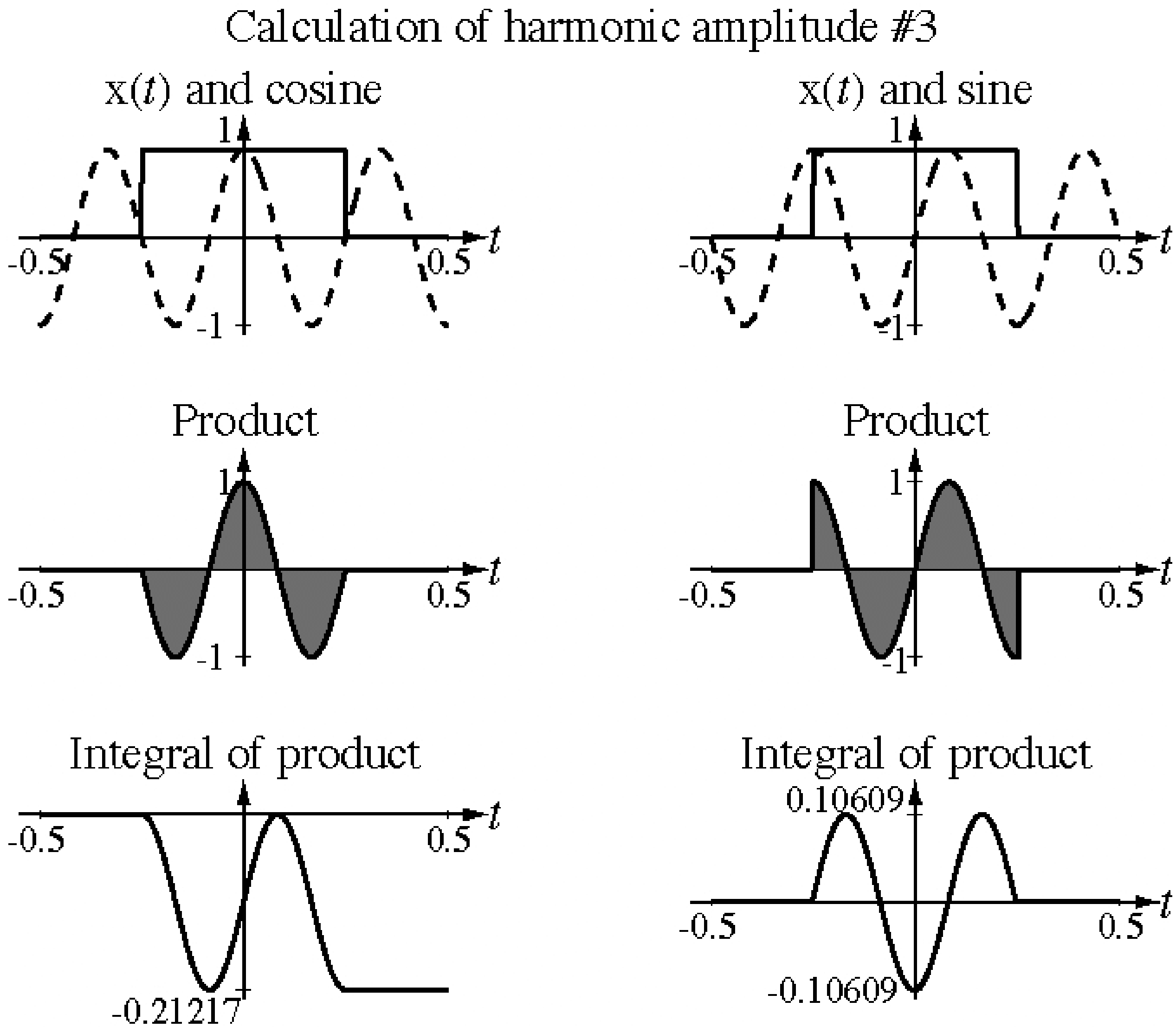
$$\mathrm{x}(t) = \mathrm{rect}(2t) * \mathrm{comb}(t)$$



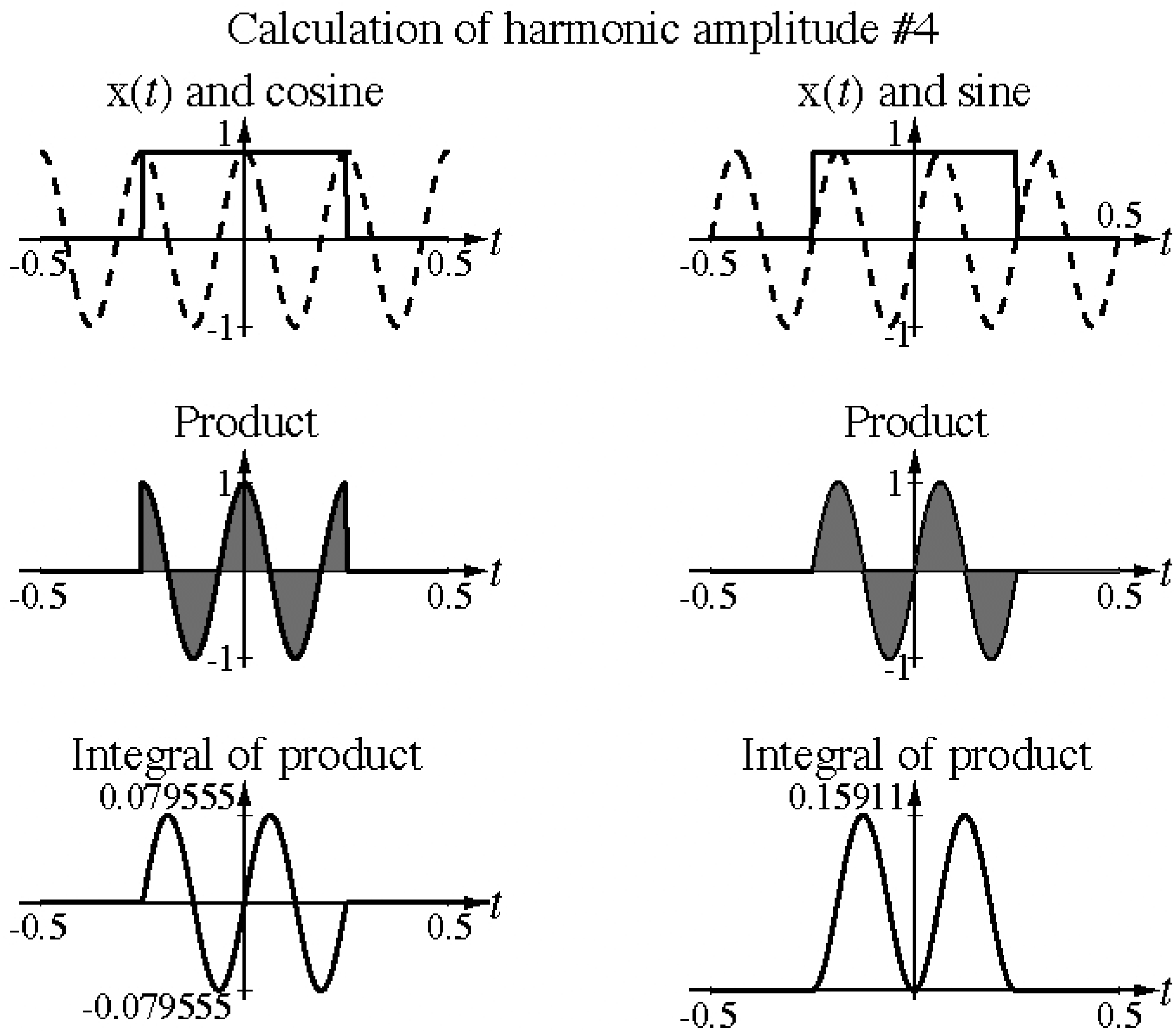
CTFS Example #4



CTFS Example #4

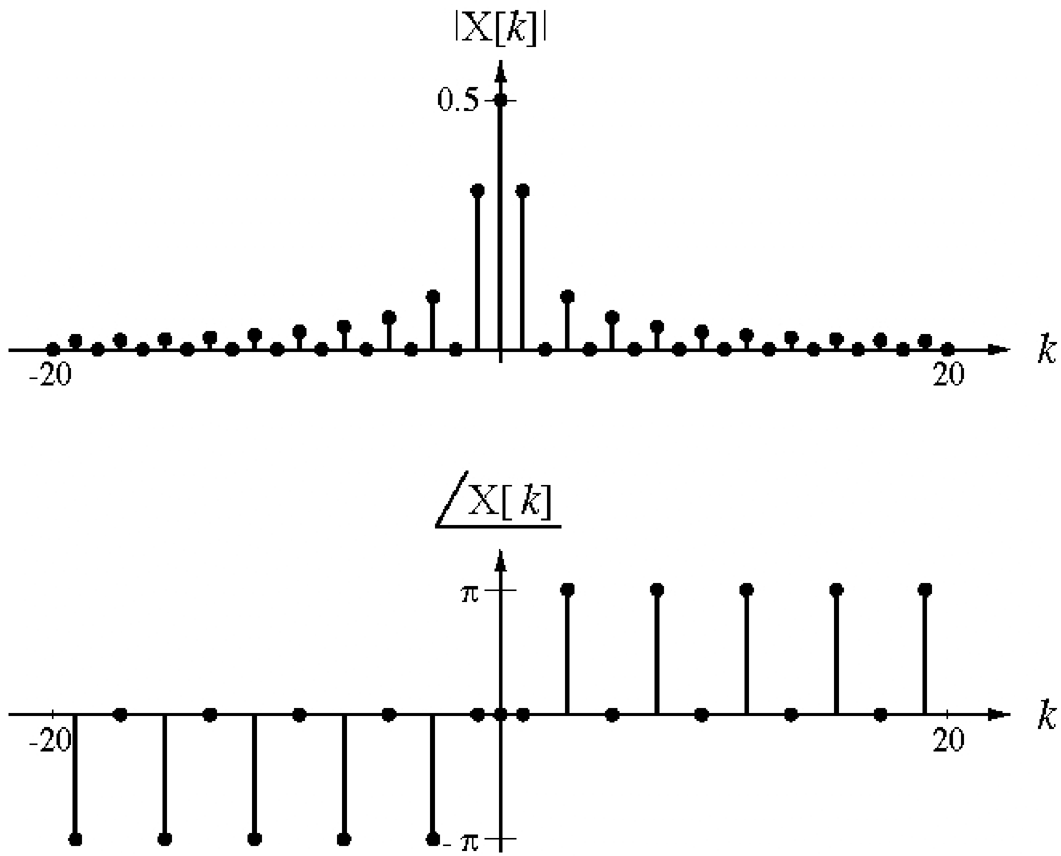


CTFS Example #4



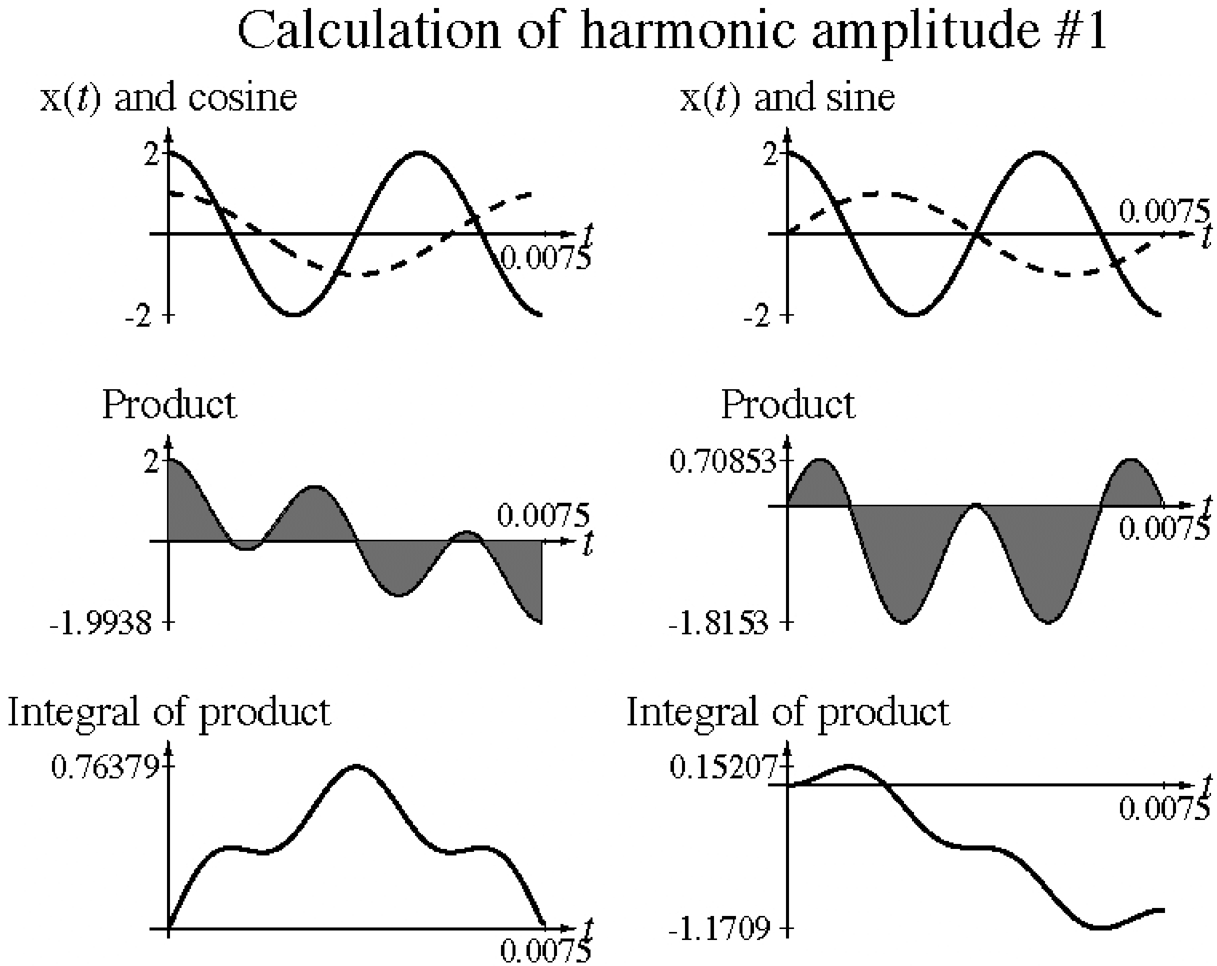
CTFS Example #4

A graph of the magnitude and phase of the harmonic function as a function of harmonic number is a good way of illustrating it.

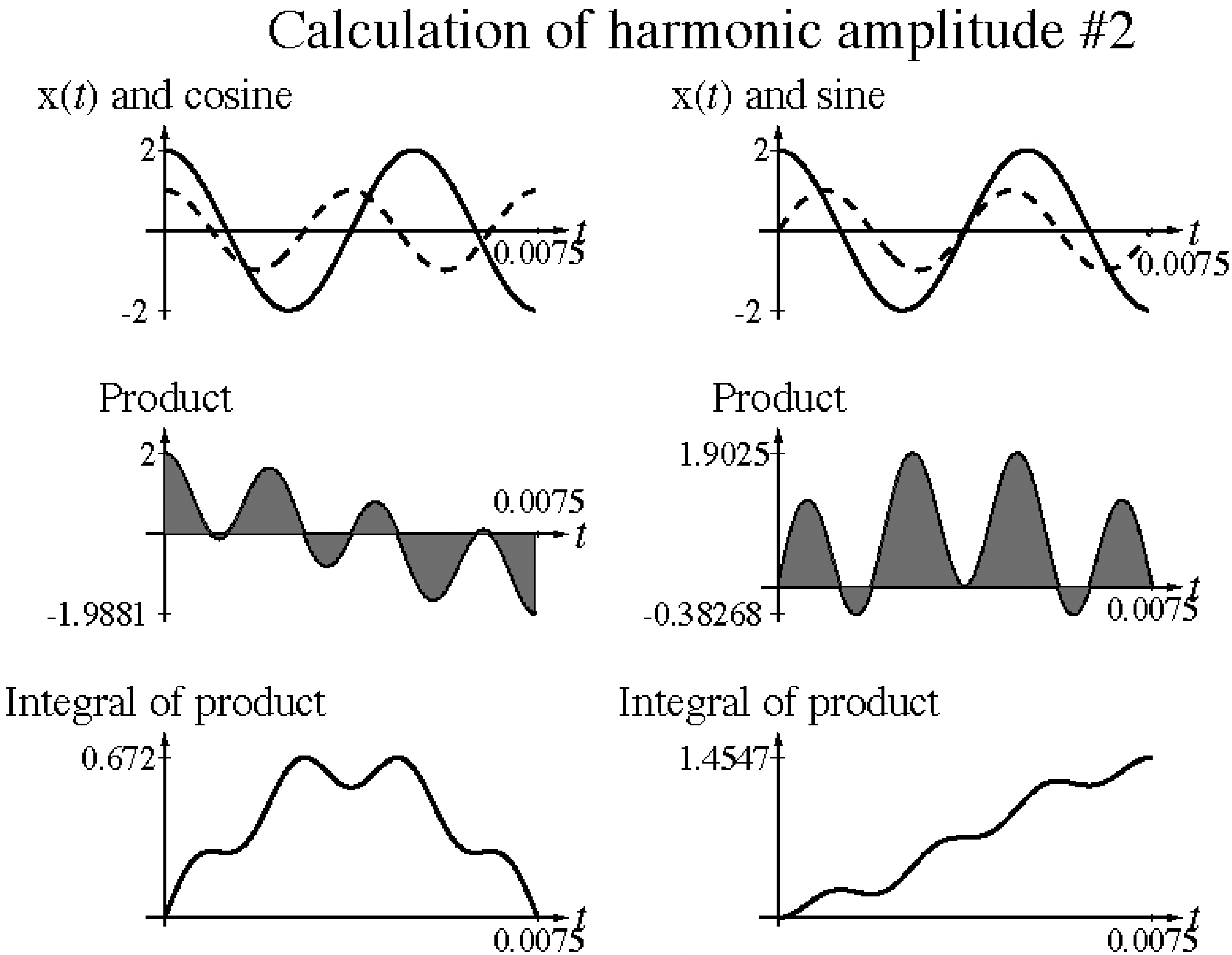


CTFS Example #5

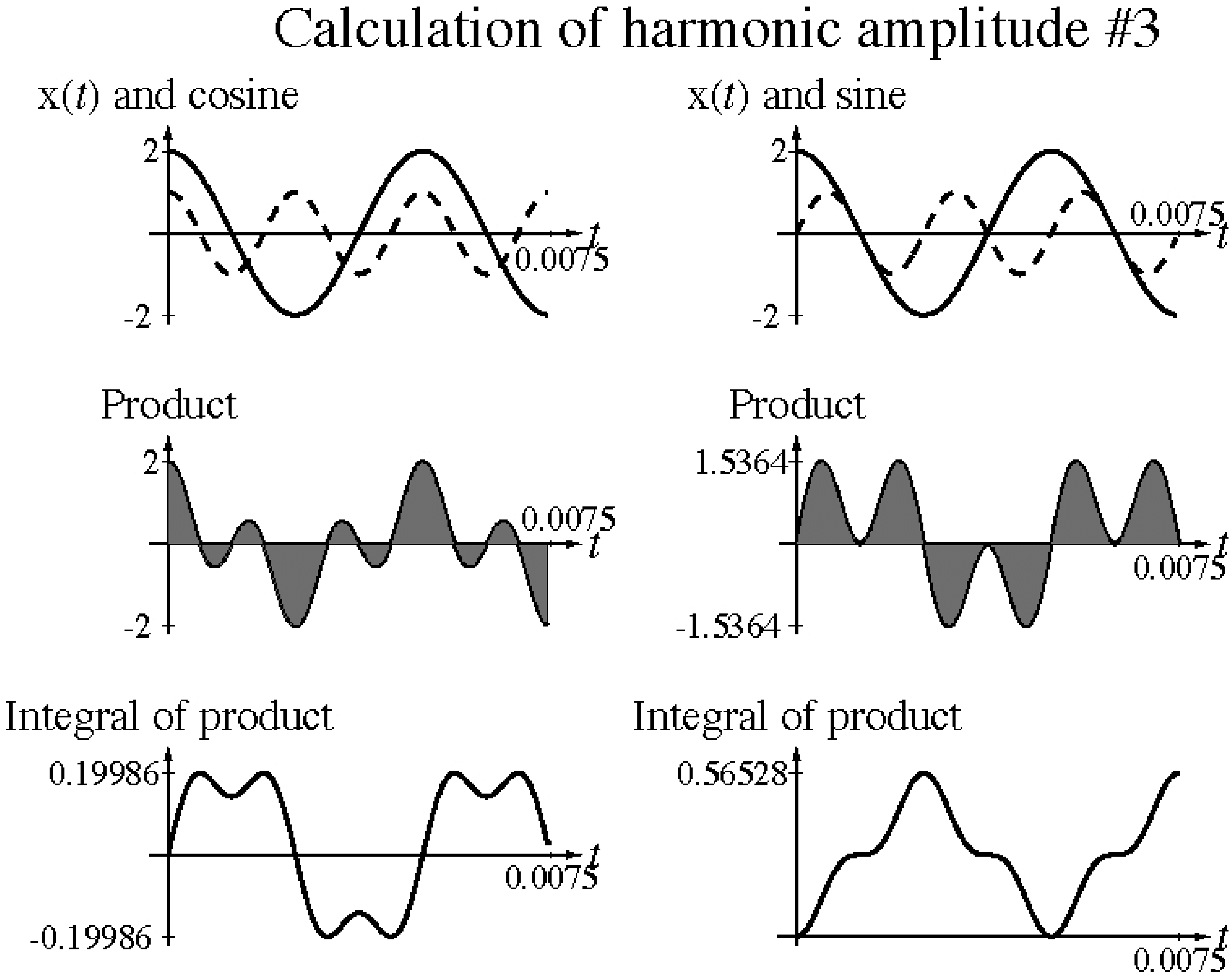
Let $x(t) = 2\cos(400\pi t)$ and let $T_F = 7.5$ ms which is 1.5 periods of this signal.



CTFS Example #5

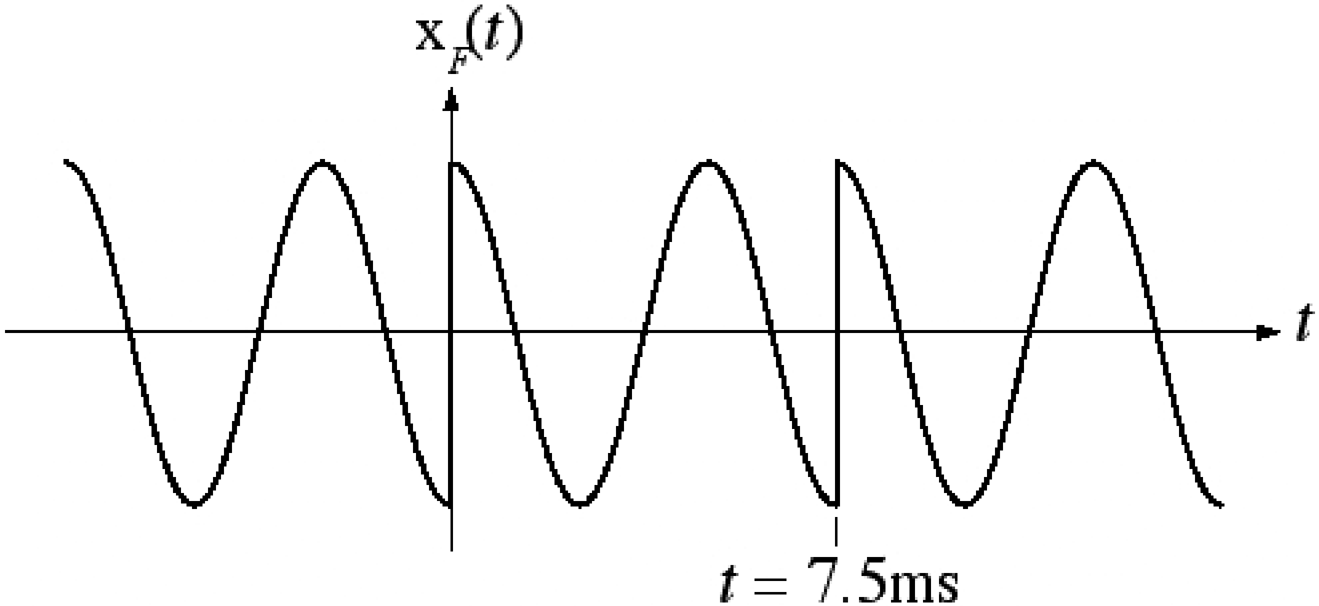


CTFS Example #5



CTFS Example #5

The CTFS representation of this cosine is the signal below, which is an odd function, and the discontinuities make the representation have significant higher harmonic content. This is a very inelegant representation.

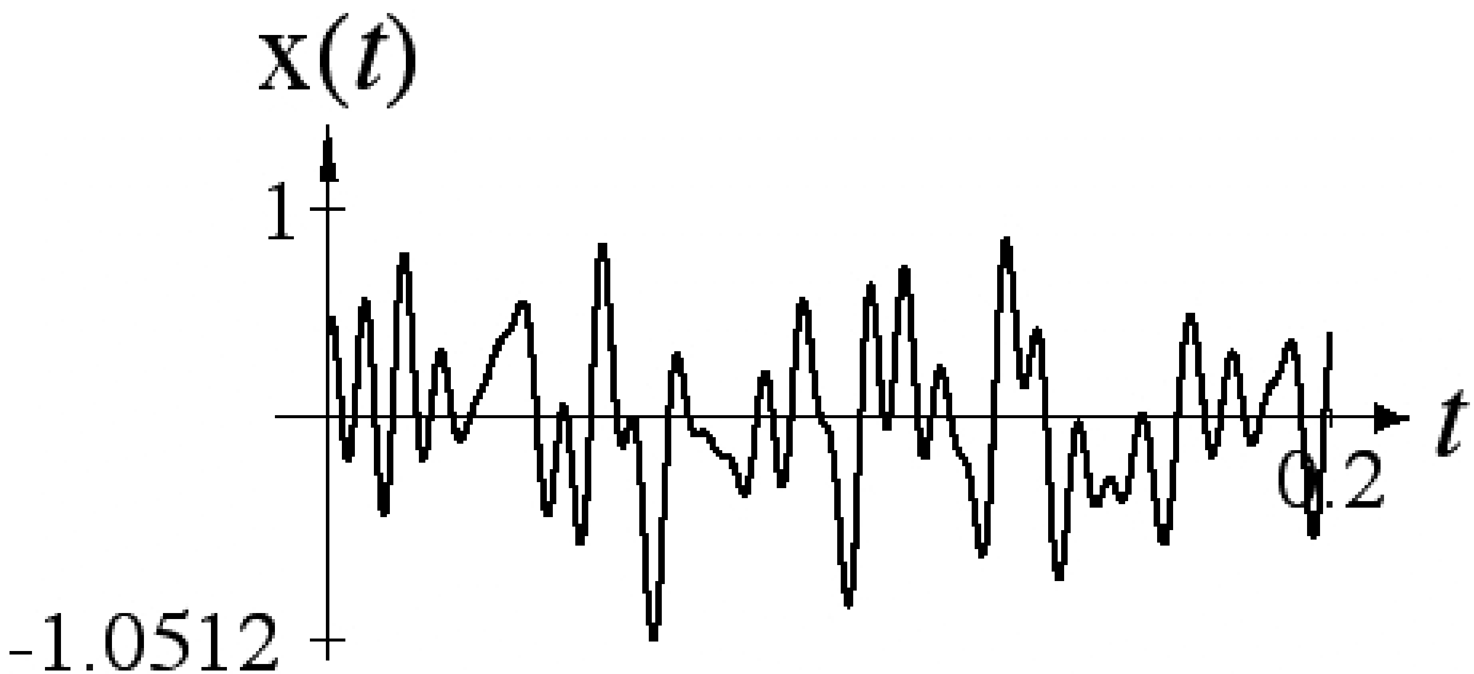


CTFS of Even and Odd Functions

- For an even function
  - The complex CTFS harmonic function, $\mathrm{X}[k]$, is purely real
  - The sine harmonic function, $\mathrm{X}_s[k]$, is zero
- For an odd function
  - The complex CTFS harmonic function, $\mathrm{X}[k]$, is purely imaginary
  - The cosine harmonic function, $\mathrm{X}_c[k]$, is zero

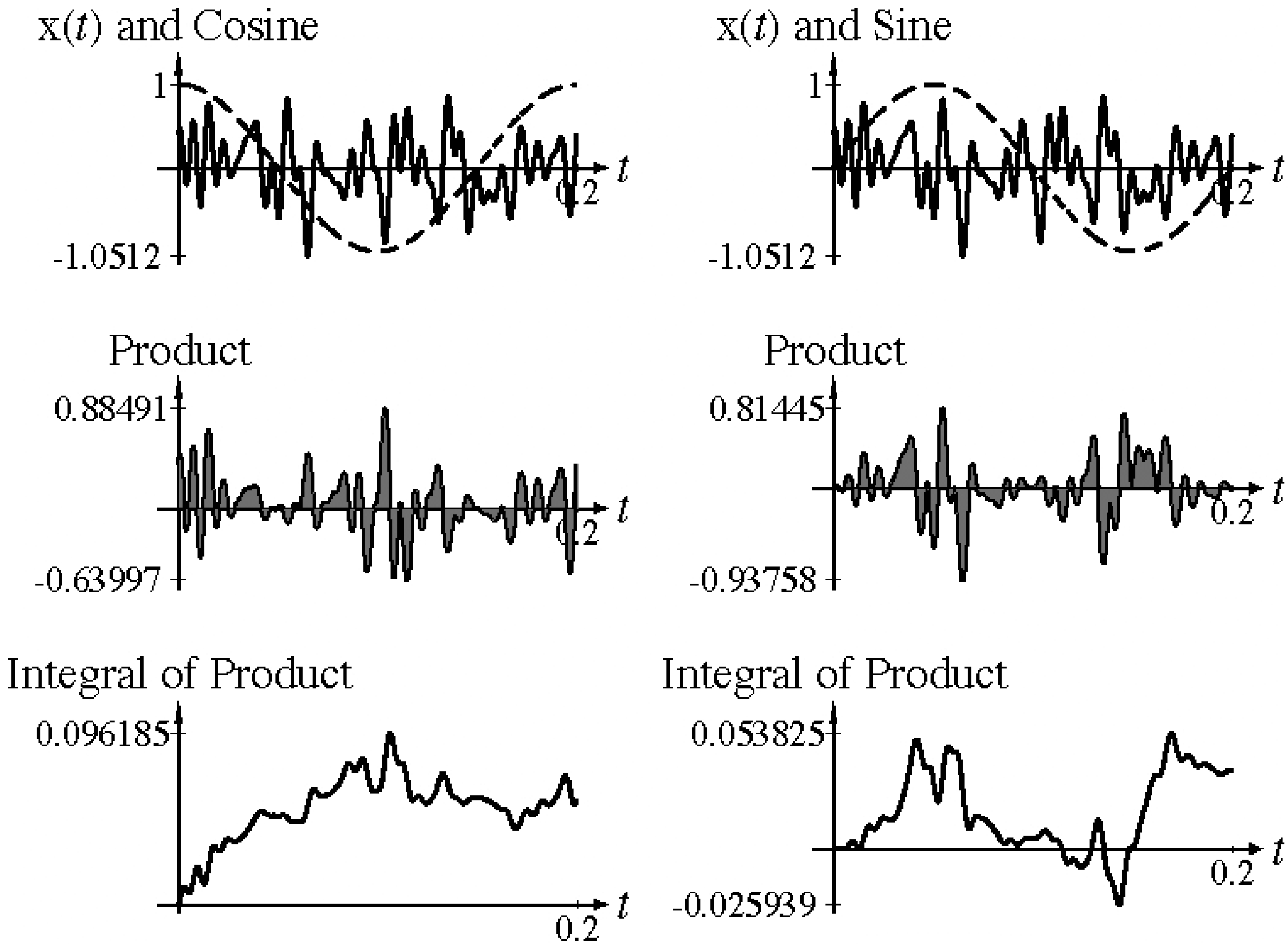
CTFS Example #6

This signal has no known functional description but it can still be represented by a CTFS.



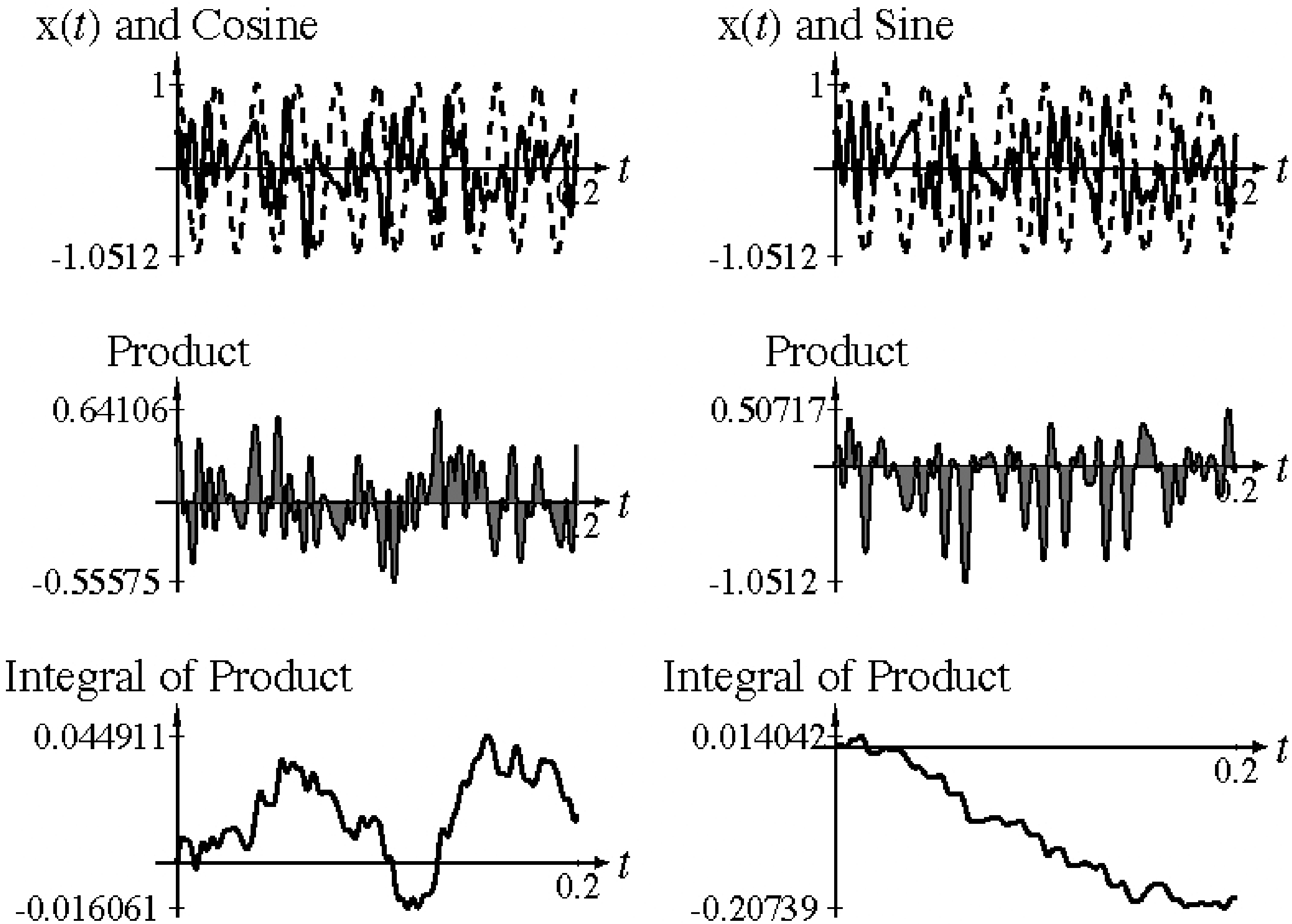
CTFS Example #6

Calculation of Harmonic Amplitude #1

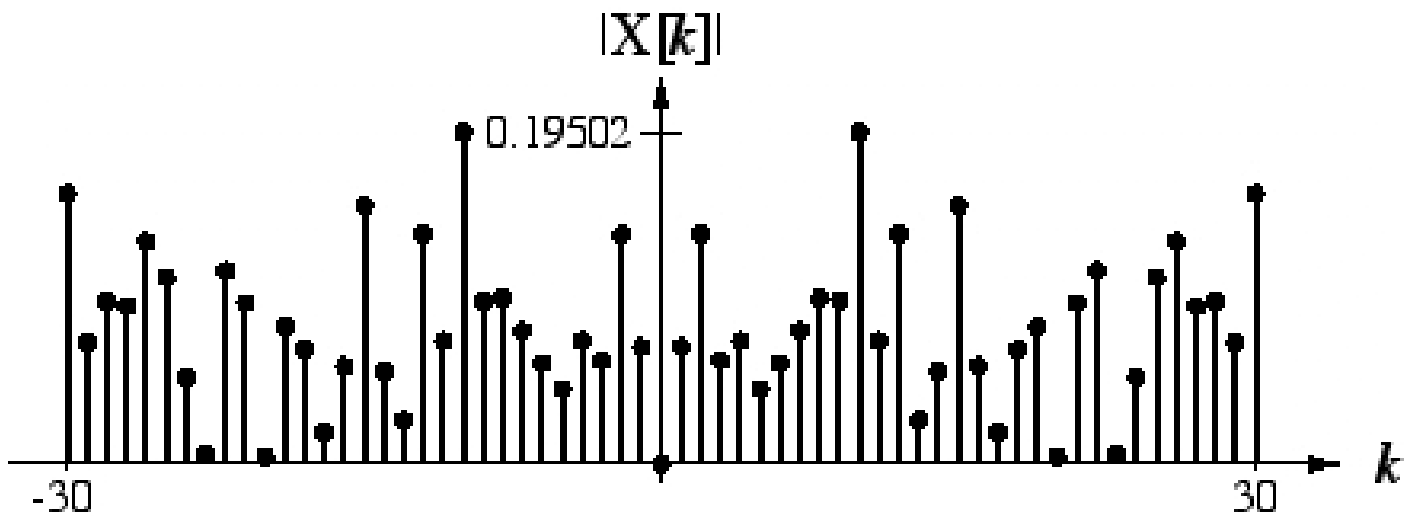


CTFS Example #6

Calculation of Harmonic Amplitude #10



CTFS Example #6

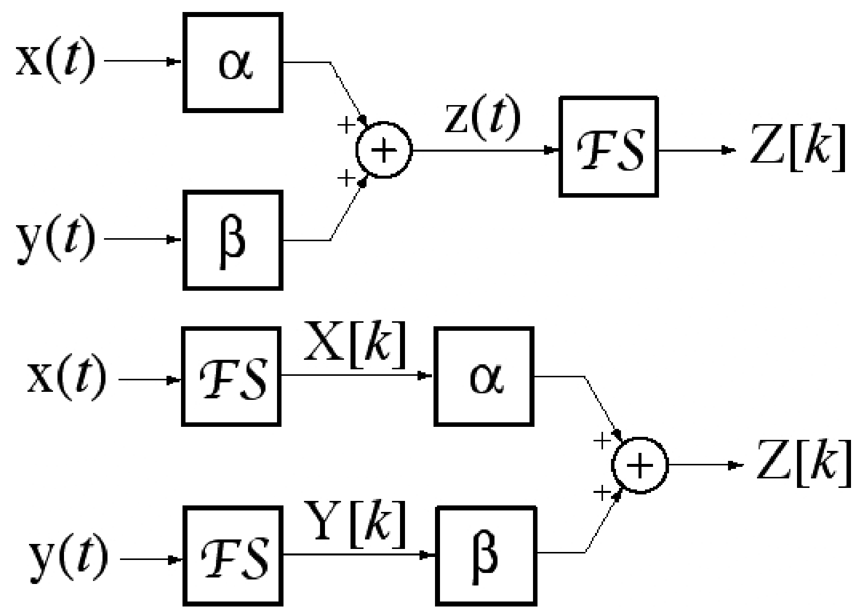


CTFS Properties

Let a signal, x(t), have a fundamental period, T_{0x} and let a signal, y(t), have a fundamental period, T_{0y}. Let the CTFS harmonic functions, each using the fundamental period as the representation time, T_F, be X[k] and Y[k]. In the properties which follow the two fundamental periods are the same unless otherwise stated.

Linearity

$$\alpha\,\mathrm{x}(t)+\beta\,\mathrm{y}(t)\xleftrightarrow{\mathcal{FS}}\alpha\,\mathrm{X}[k]+\beta\,\mathrm{Y}[k]$$

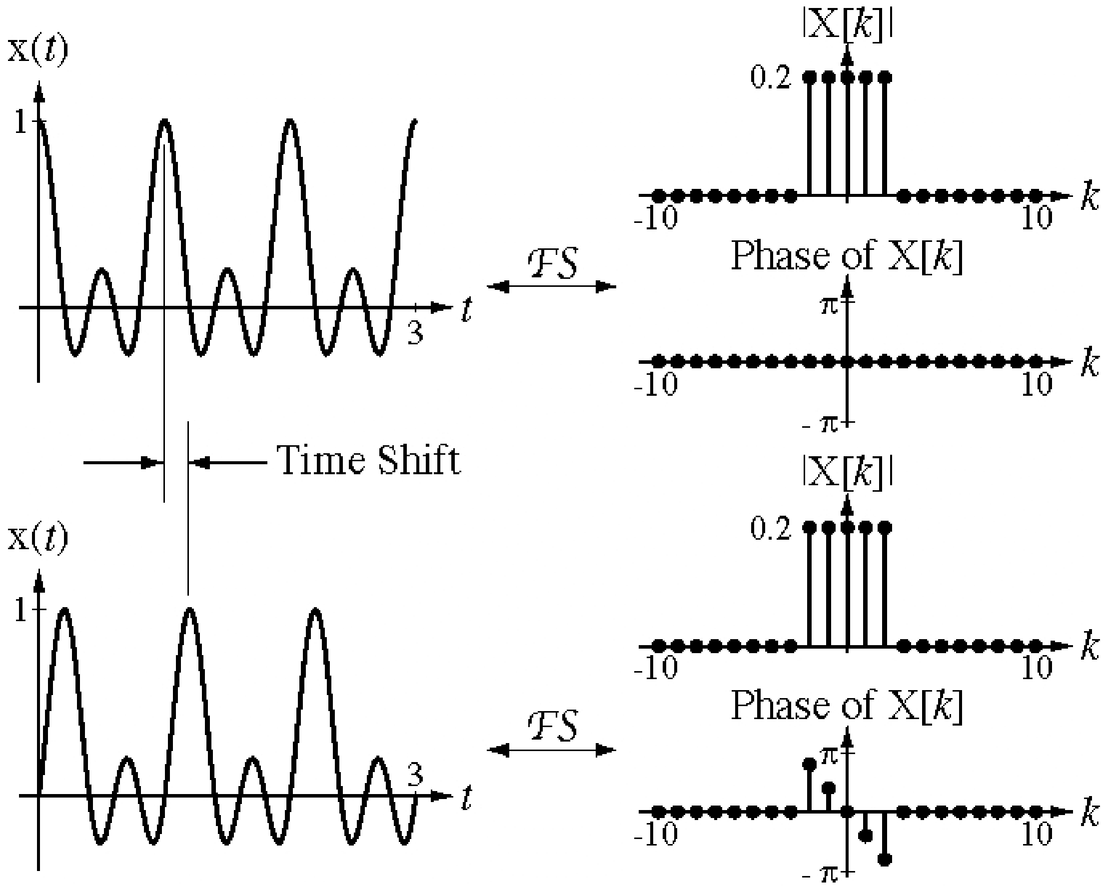


CTFS Properties

Time Shifting

$$x(t - t_0) \xleftrightarrow{\mathcal{FS}} e^{-j2\pi(kf_0)t_0}\, X[k]$$

$$x(t - t_0) \xleftrightarrow{\mathcal{FS}} e^{-j(k\omega_0)t_0}\, X[k]$$



CTFS Properties

Frequency Shifting (Harmonic Number Shifting)

$$e^{j2\pi(k_0 f_0)t}\,\mathrm{x}(t) \xleftrightarrow{\mathcal{FS}} \mathrm{X}[k-k_0]$$

$$e^{j(k_0\omega_0)t}\,\mathrm{x}(t) \xleftrightarrow{\mathcal{FS}} \mathrm{X}[k-k_0]$$

A shift in frequency (harmonic number) corresponds to multiplication of the time function by a complex exponential.

Time Reversal $\mathrm{x}(-t) \xleftrightarrow{\mathcal{FS}} \mathrm{X}[-k]$

CTFS Properties

Time Scaling

Let $z(t) = x(at)$, $a > 0$

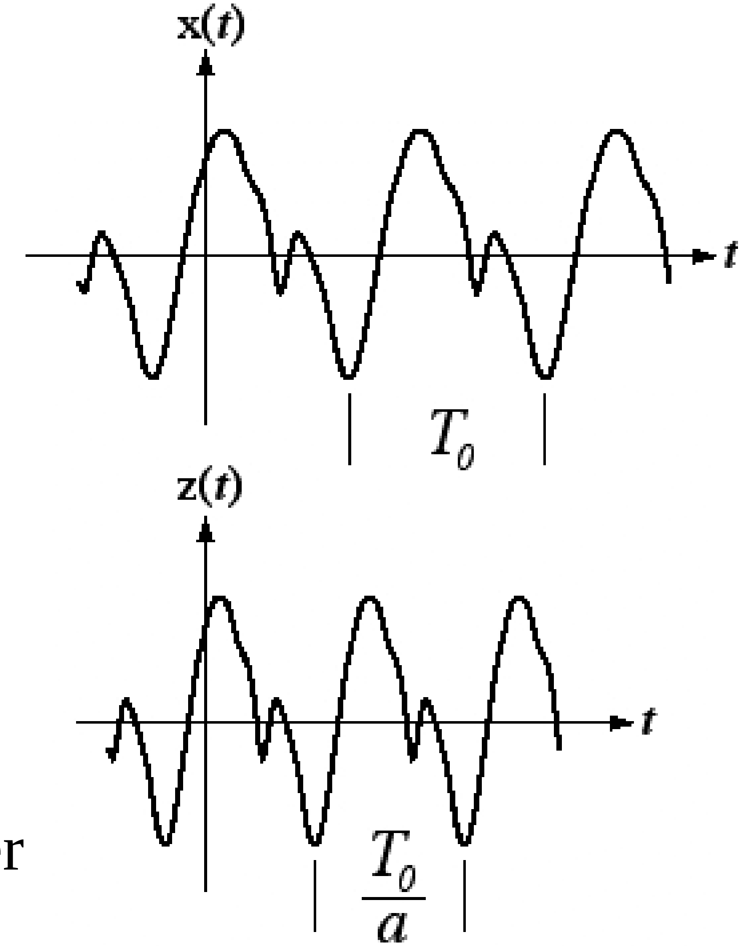
Case 1. $T_F = \dfrac{T_{0x}}{a} = T_{0z}$ for $z(t)$

$$Z[k] = X[k]$$

Case 2. $T_F = T_{0x}$ for $z(t)$

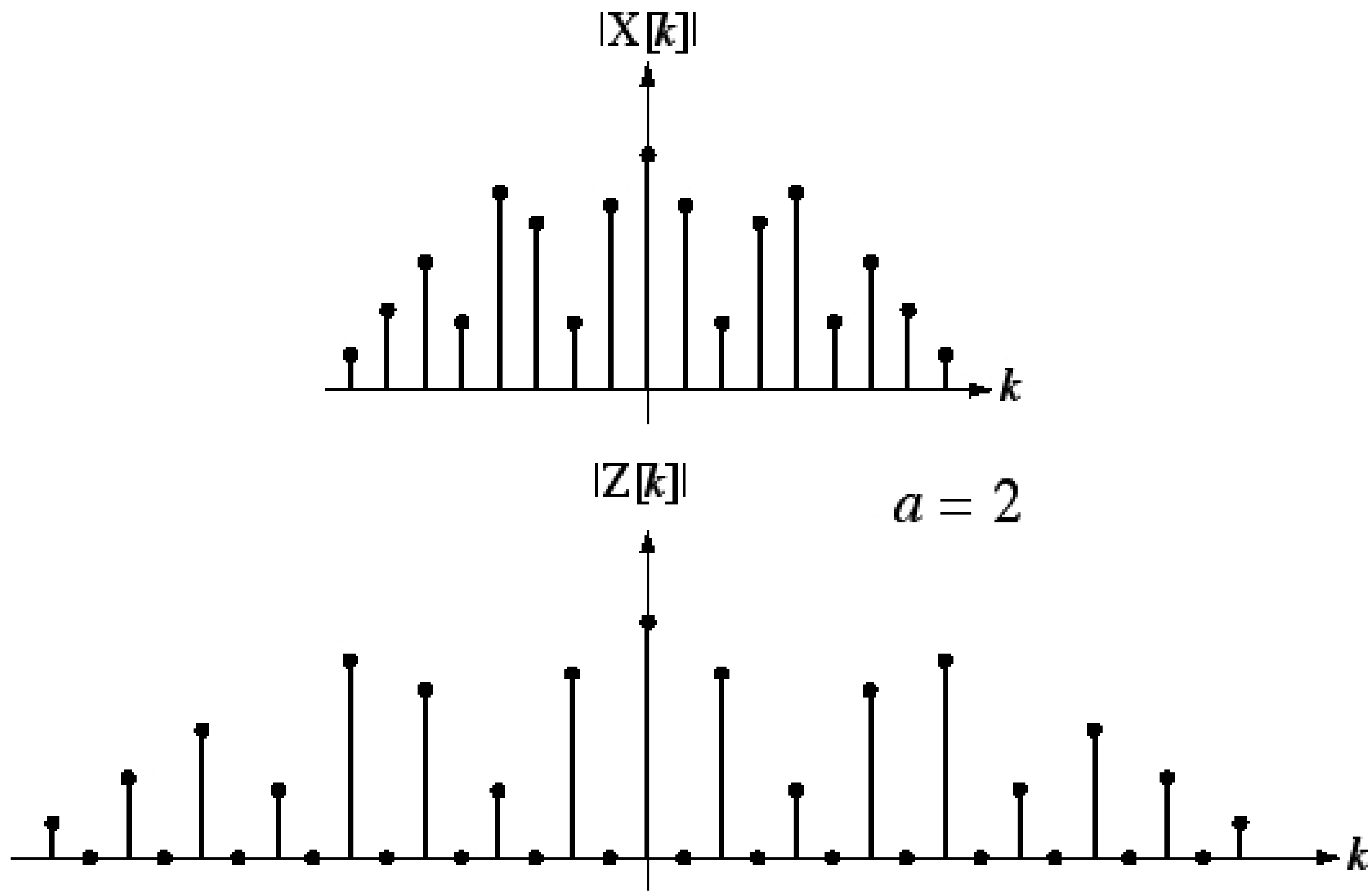
If a is an integer,

$$Z[k] = \begin{cases} X\left[\dfrac{k}{a}\right] , & \dfrac{k}{a} \text{ an integer} \\ 0 , & \text{otherwise} \end{cases}$$



CTFS Properties

Time Scaling (continued)



CTFS Properties

Change of Representation Time

$$\text{With } T_F = T_{0x},\ \ \mathrm{x}(t) \xleftrightarrow{\mathcal{FS}} \mathrm{X}[k]$$

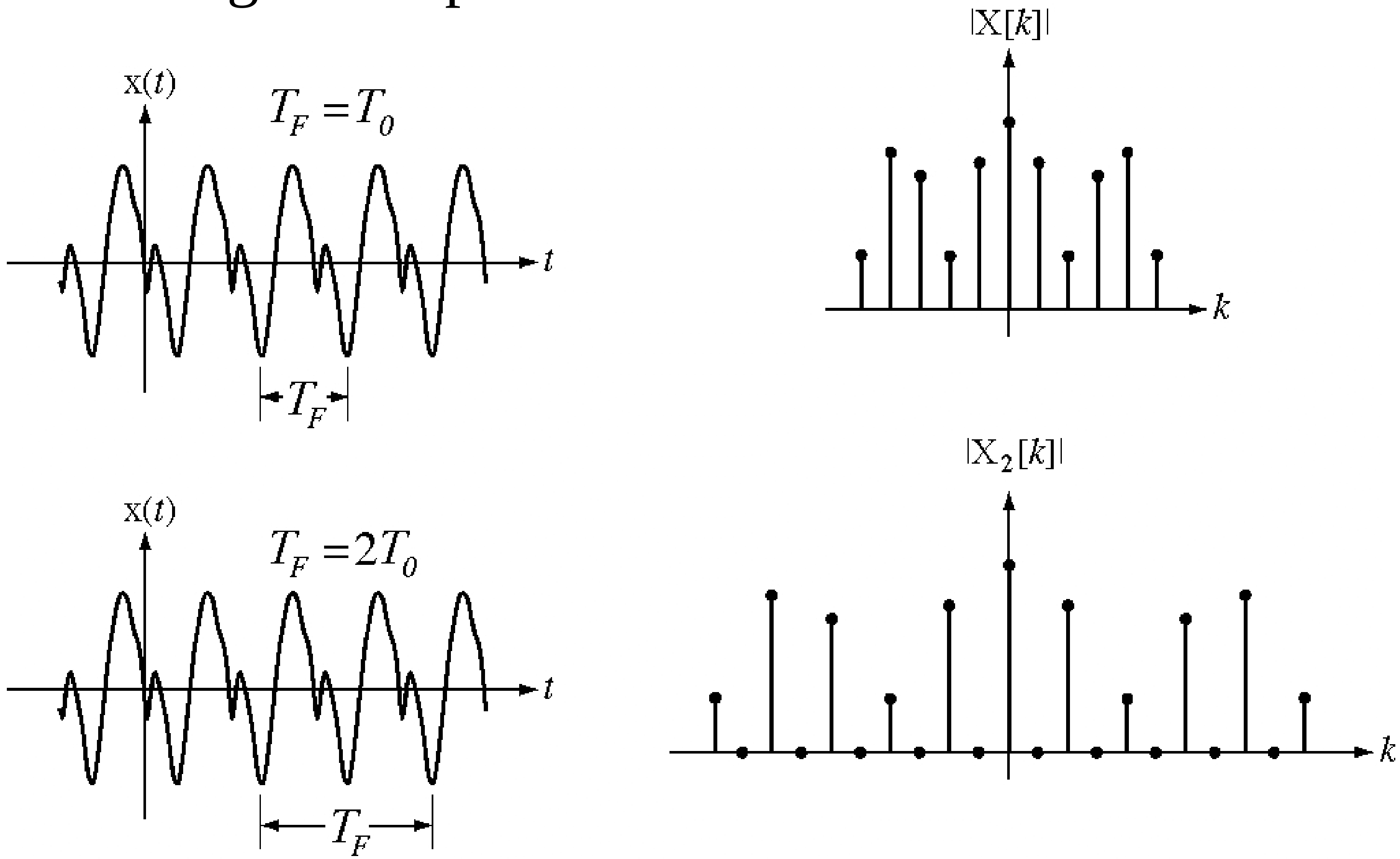
$$\text{With } T_F = mT_{0x},\ \ \mathrm{x}(t) \xleftrightarrow{\mathcal{FS}} \mathrm{X}_m[k]$$

$$\mathrm{X}_m[k] = \begin{cases} \mathrm{X}\left[\dfrac{k}{m}\right] & , \ \dfrac{k}{m} \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}$$

(m is any positive integer)

CTFS Properties

Change of Representation Time

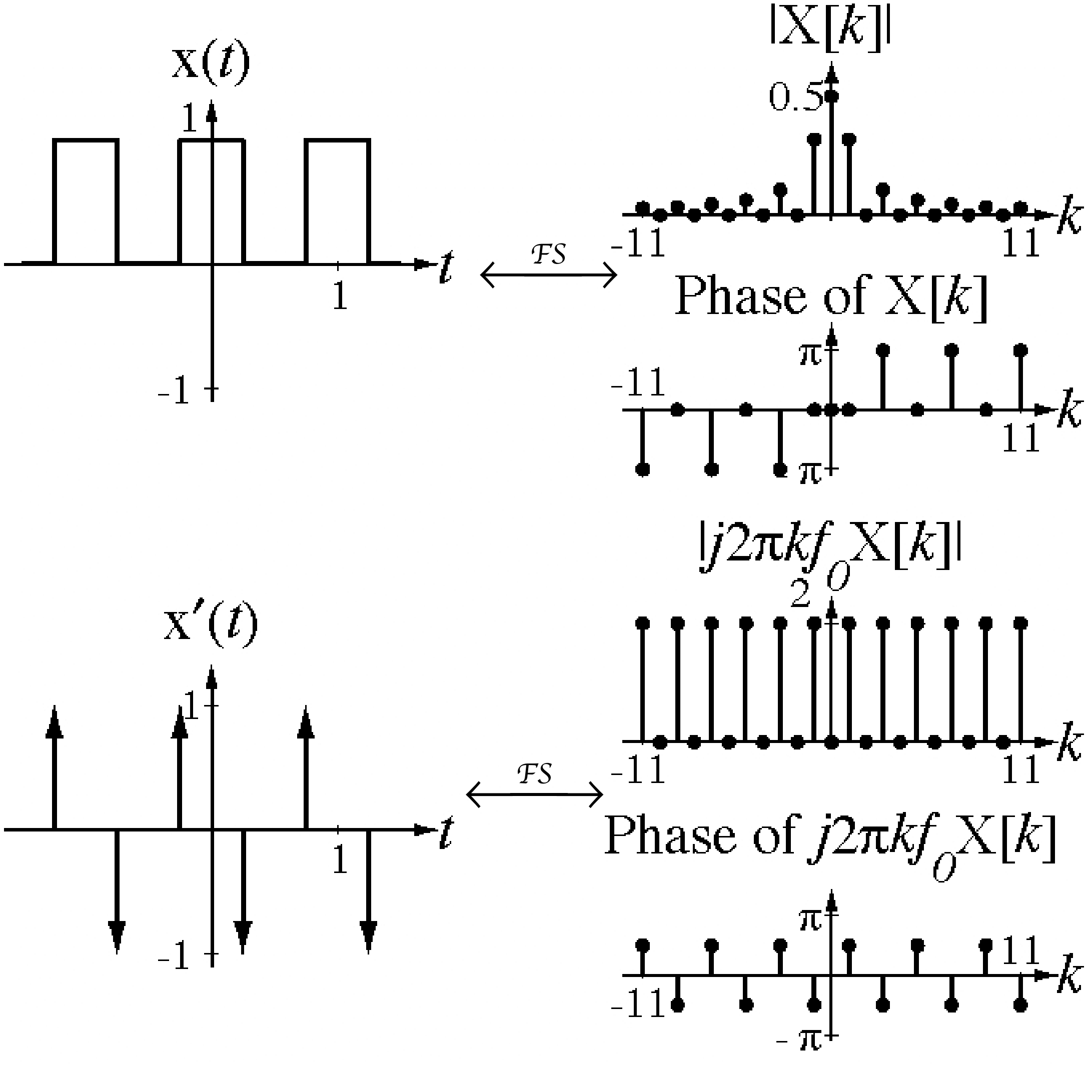


CTFS Properties

Time Differentiation

$$\frac{d}{dt}(\mathrm{x}(t)) \xleftrightarrow{\mathcal{FS}} j2\pi(kf_0)\,\mathrm{X}[k]$$

$$\frac{d}{dt}(\mathrm{x}(t)) \xleftrightarrow{\mathcal{FS}} j(k\omega_0)\,\mathrm{X}[k]$$



CTFS Properties

Time Integration

Case 1. $X[0]=0$

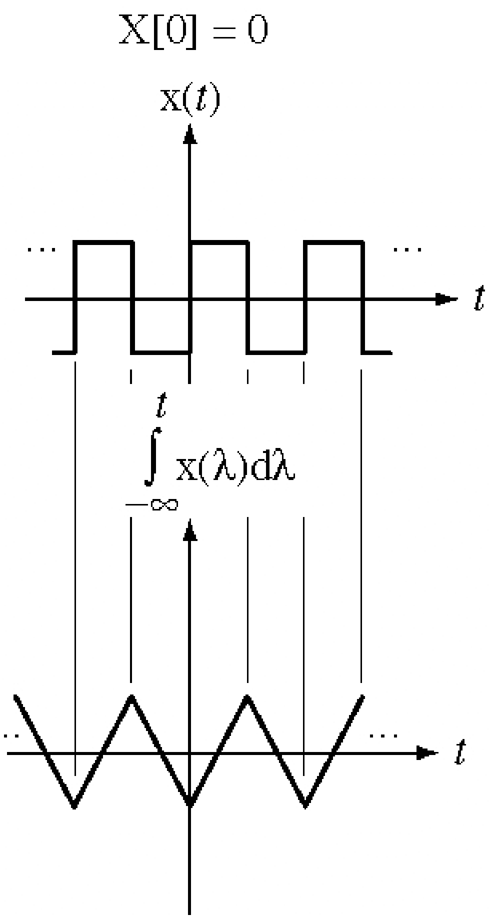
$$\int_{-\infty}^{t} x(\lambda)d\lambda \xleftrightarrow{\mathcal{FS}} \frac{X[k]}{j2\pi(kf_0)}$$

$$\int_{-\infty}^{t} x(\lambda)d\lambda \xleftrightarrow{\mathcal{FS}} \frac{X[k]}{j(k\omega_0)}$$

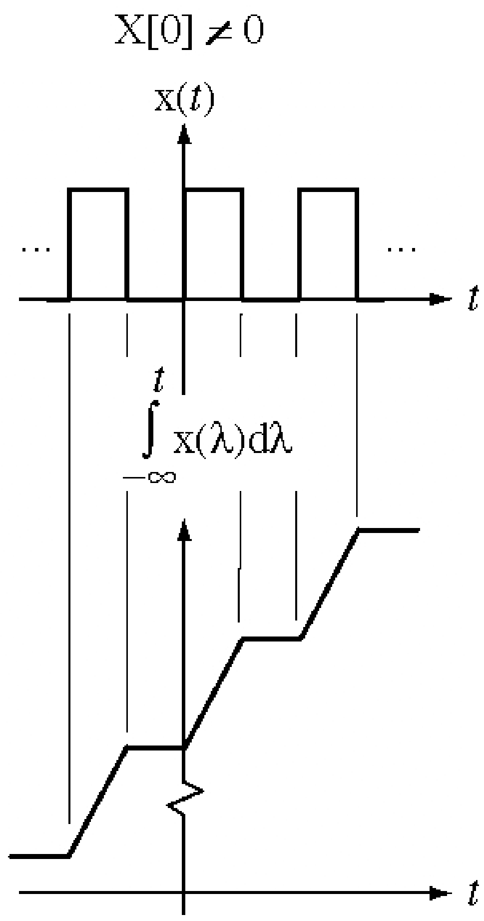
Case 2. $X[0]\neq 0$

$\int_{-\infty}^{t} x(\lambda)d\lambda$ is not periodic

Case 1



Case 2



CTFS Properties

Multiplication-Convolution Duality

$$\mathrm{x}(t)\,\mathrm{y}(t) \xleftrightarrow{\mathcal{FS}} \mathrm{X}[k] * \mathrm{Y}[k]$$

(The harmonic functions, X[k] and Y[k], must be based on the same representation period, T_F .)

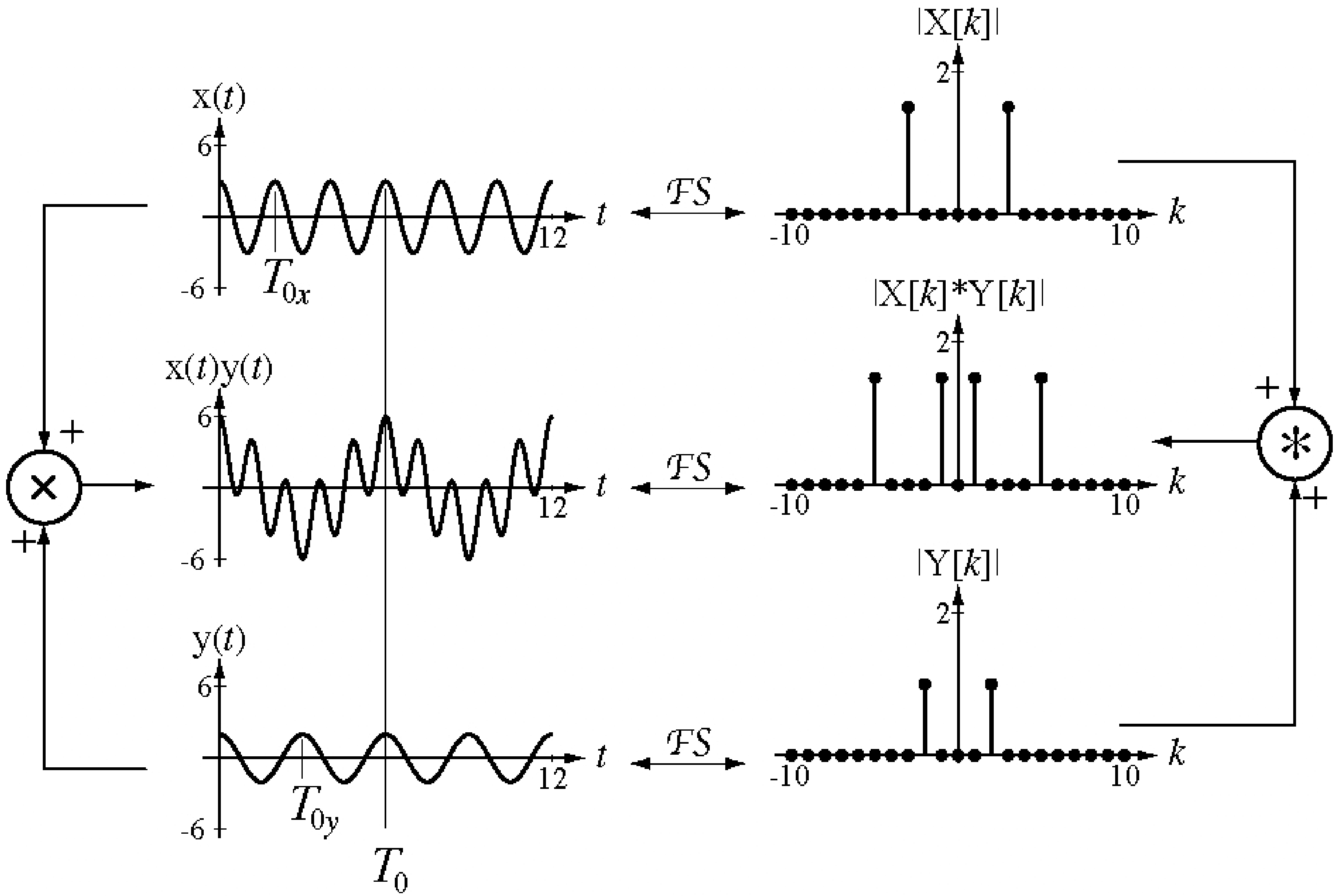
$$\mathrm{x}(t) \circledast \mathrm{y}(t) \xleftrightarrow{\mathcal{FS}} T_0\,\mathrm{X}[k]\,\mathrm{Y}[k]$$

The symbol, $\circledast$, indicates *periodic convolution*. Periodic convolution is defined mathematically by

$$\mathrm{x}(t) \circledast \mathrm{y}(t) = \int_{T_0} \mathrm{x}(\tau)\,\mathrm{y}(t-\tau)\,d\tau$$

$\mathrm{x}(t) \circledast \mathrm{y}(t) = \mathrm{x}_{ap}(t) * \mathrm{y}(t)$ where $\mathrm{x}_{ap}(t)$ is any single period of $\mathrm{x}(t)$

CTFS Properties



CTFS Properties

Conjugation

$$x^*(t) \xleftrightarrow{\mathcal{FS}} X^*[-k]$$

Parseval's Theorem

$$\frac{1}{T_0}\int_{T_0} |x(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |X[k]|^2$$

The average power of a periodic signal is the sum of the average powers in its harmonic components.

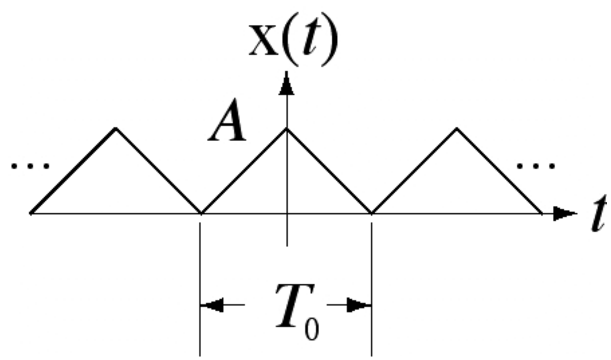
Convergence of the CTFS

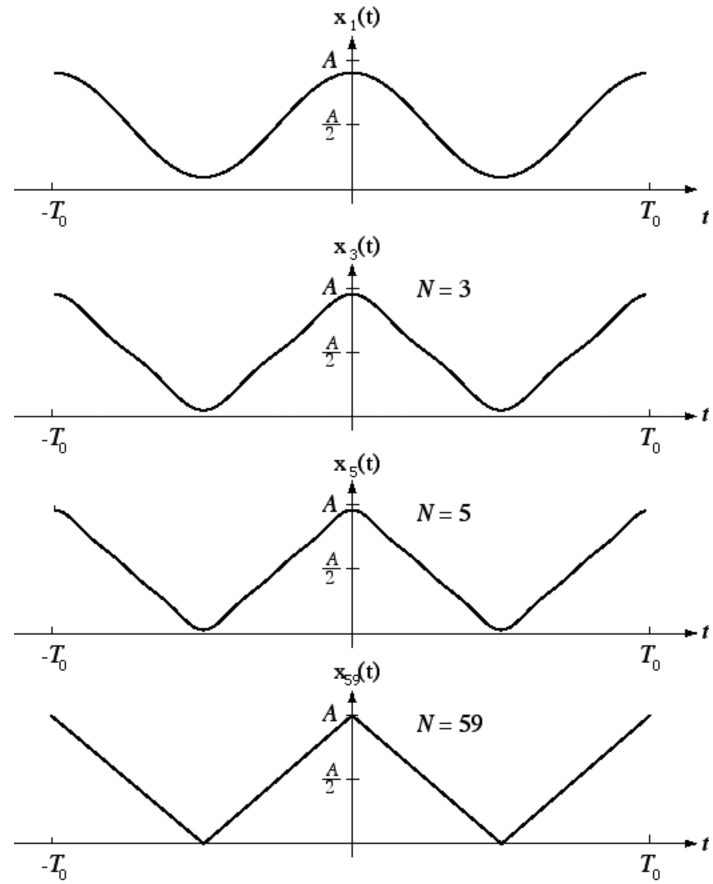
Partial CTFS Sums

$$\mathrm{x}_N(t) = \sum_{k=-N}^{N} \mathrm{X}[k] e^{j2\pi(kf_0)t}$$

For continuous signals, convergence is exact at every point.

A Continuous Signal

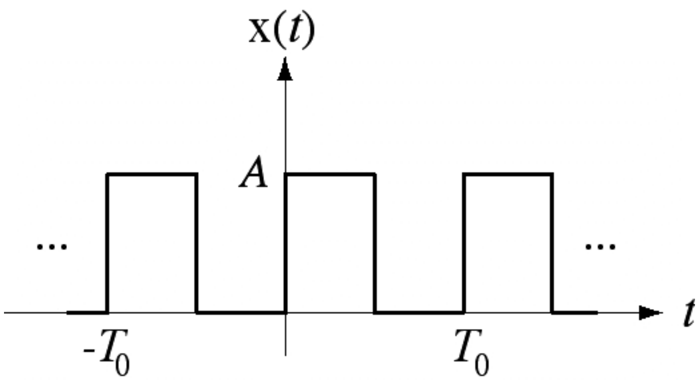




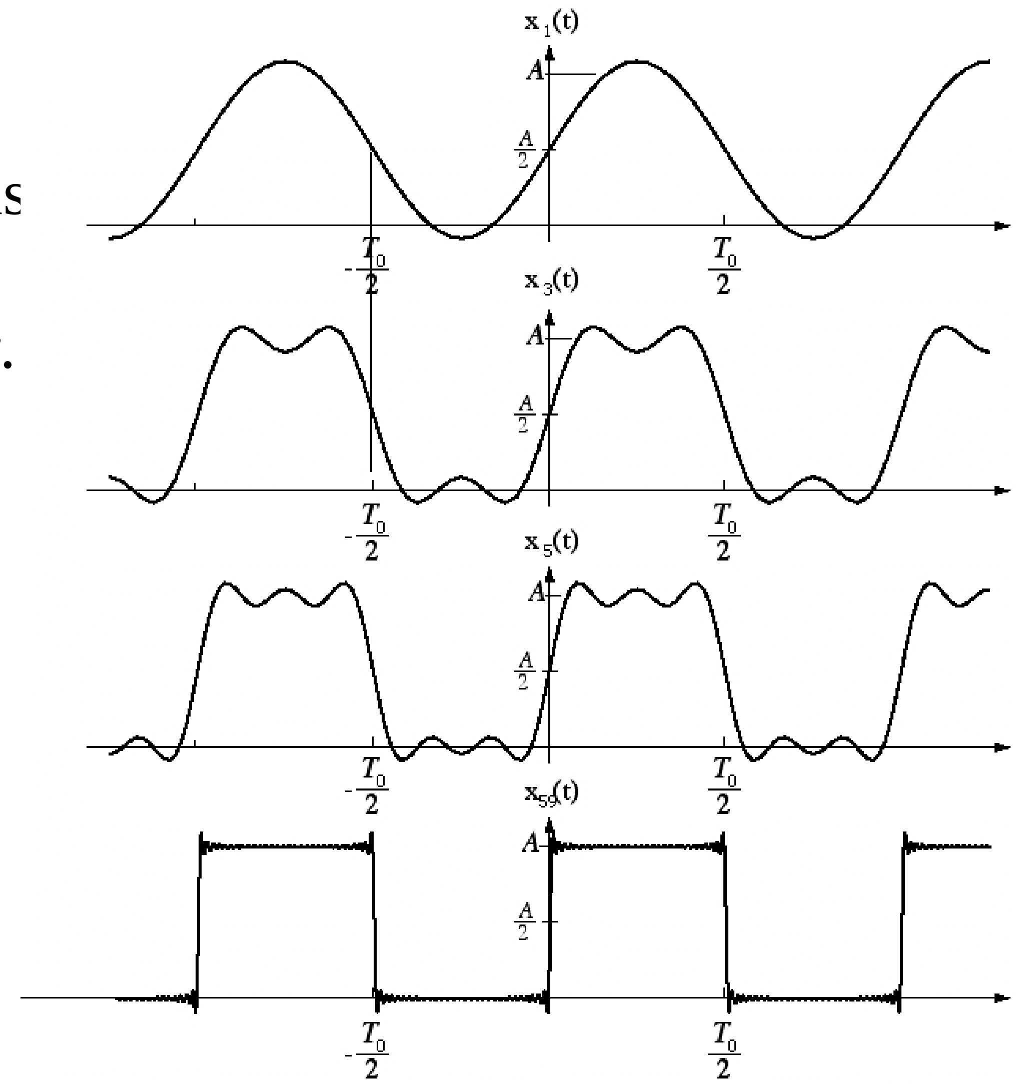
Convergence of the CTFS

For discontinuous signals convergence is exact at every point *of continuity*.

Discontinuous Signal

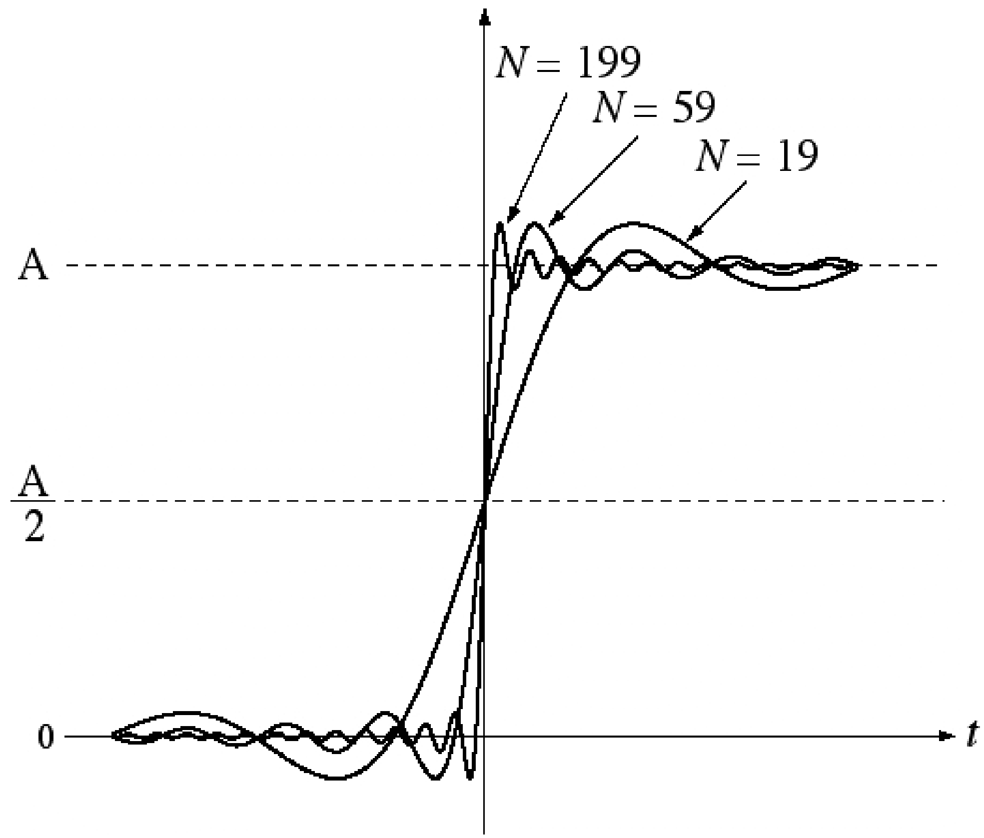


Partial CTFS Sums

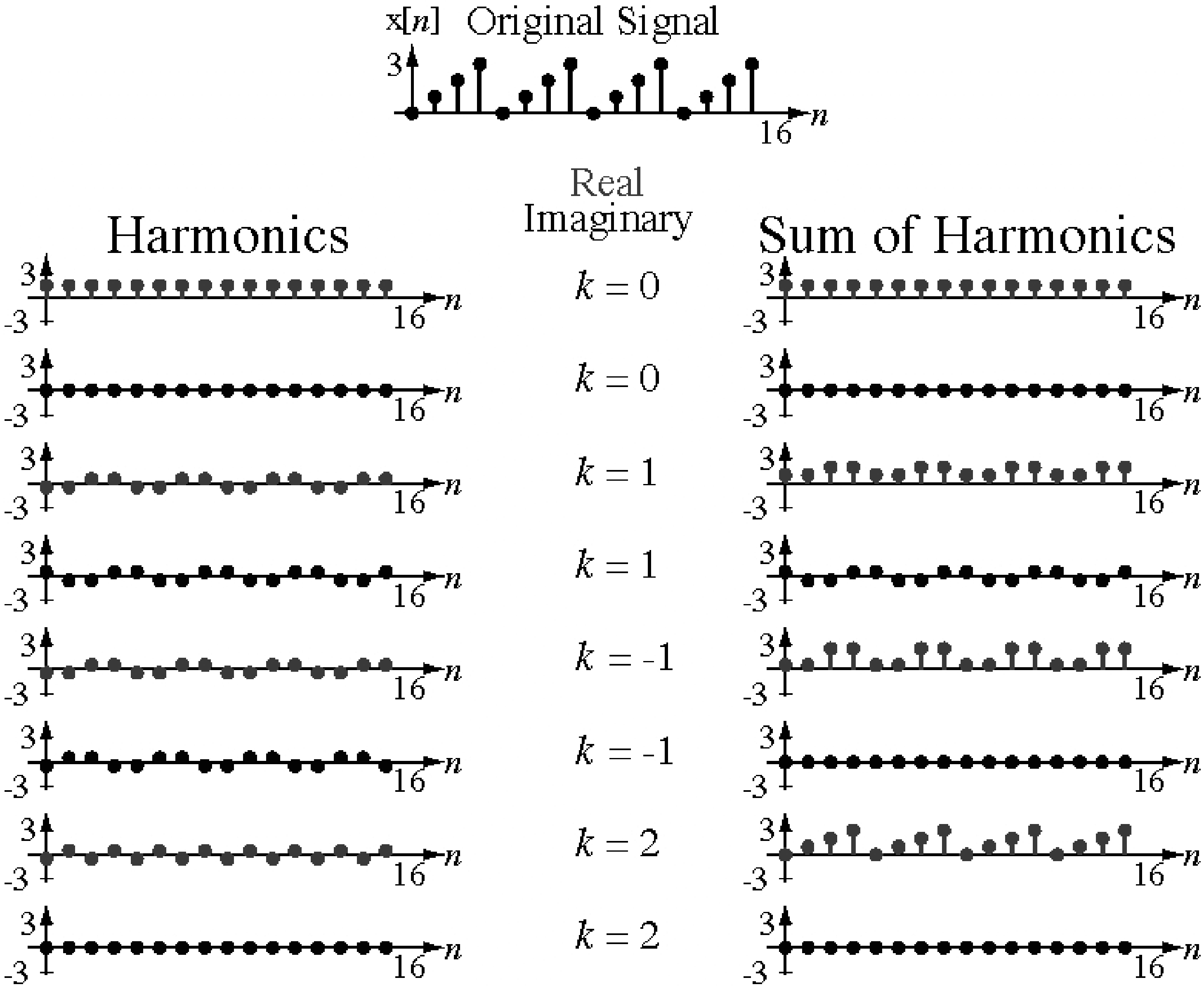


Convergence of the CTFS

The Gibbs Phenomenon



Discrete-Time Fourier Series Concept



The Discrete-Time Fourier Series

The formal derivation of the discrete-time Fourier series (DTFS) is on pages 259-262. The results are

$$x_F[n] = \sum_{k=\langle N_F \rangle} X[k] e^{j2\pi(kF_F)n} \qquad X[k] = \frac{1}{N_F} \sum_{n=n_0}^{n_0+N_F-1} x[n] e^{-j2\pi(kF_F)n}$$

where N_F is the representation time, $F_F = \dfrac{1}{N_F}$, and the notation,

$$\sum_{k=\langle N_F \rangle}$$

means a summation over any range of consecutive k's exactly N_F in length.

The Discrete-Time Fourier Series

Notice that in

$$x_F[n] = \sum_{k=\langle N_F \rangle} X[k] e^{j2\pi(kF_F)n}$$

the summation is over exactly one period, a finite summation. This is because of the periodicity of the complex sinusoid,

$$e^{-j2\pi(kF_F)n}$$

in harmonic number, *k*. That is, if *k* is increased by any integer multiple of N_F the complex sinusoid does not change.

$$e^{-j2\pi(kF_F)n} = e^{-j2\pi((k+mN_F)F_F)n} \qquad (m \text{ an integer})$$

This occurs because discrete time, *n*, is always an integer.

The Discrete-Time Fourier Series

In the very common case in which the representation time is taken as the fundamental period, N_0 , the DTFS is

$$\mathrm{x}[n] = \sum_{k=\langle N_0 \rangle} \mathrm{X}[k] e^{j2\pi(kF_0)n} \xleftrightarrow{\mathcal{FS}} \mathrm{X}[k] = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} \mathrm{x}[n] e^{-j2\pi(kF_0)n}$$

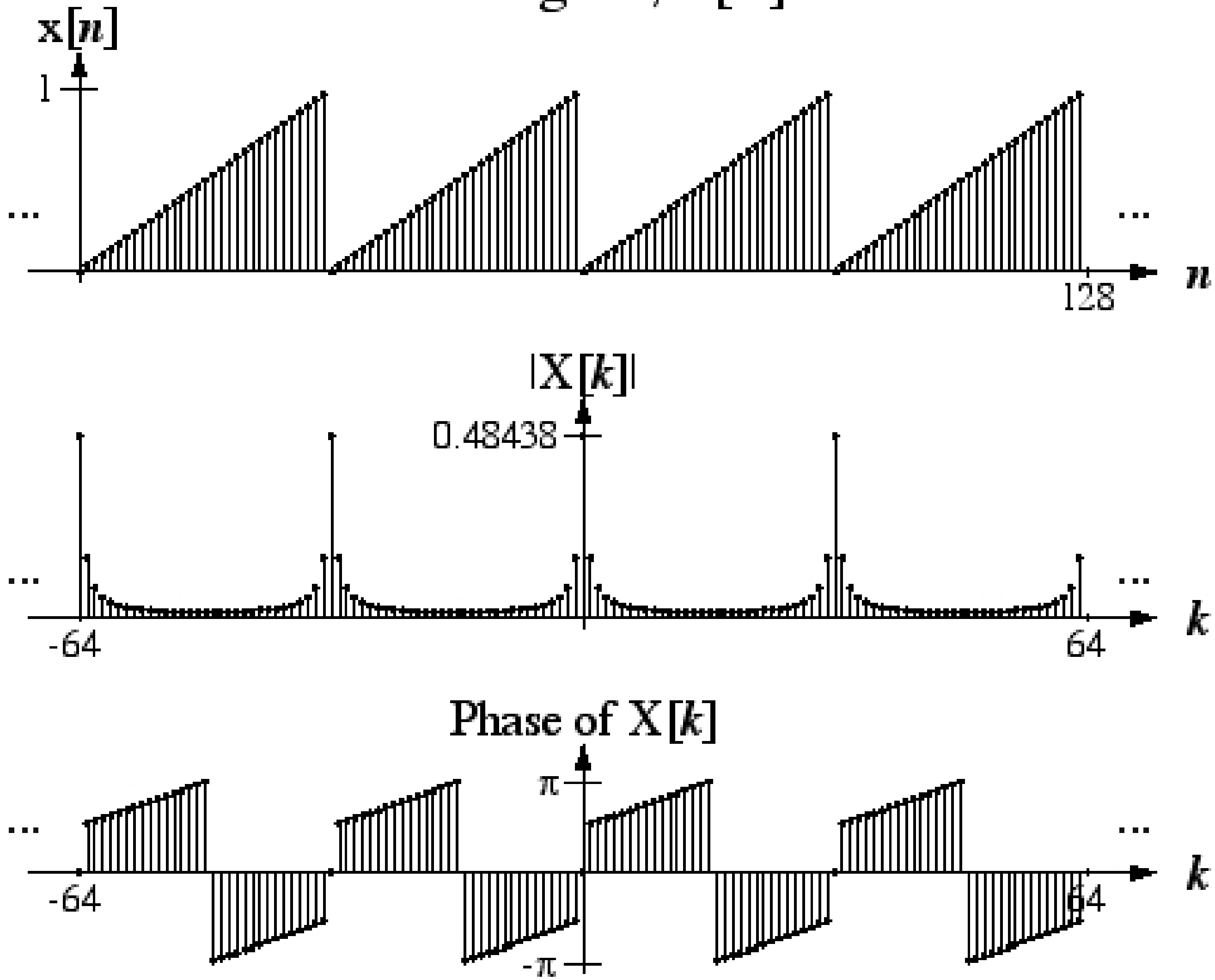
or in terms of radian frequency

$$\mathrm{x}[n] = \sum_{k=\langle N_0 \rangle} \mathrm{X}[k] e^{j(k\Omega_0)n} \xleftrightarrow{\mathcal{FS}} \mathrm{X}[k] = \frac{1}{N_0} \sum_{n=\langle N_0 \rangle} \mathrm{x}[n] e^{-j(k\Omega_0)n}$$

where $\Omega_0 = 2\pi F_0 = \frac{2\pi}{N_0}$

DTFS Example

DT Signal, x[n]

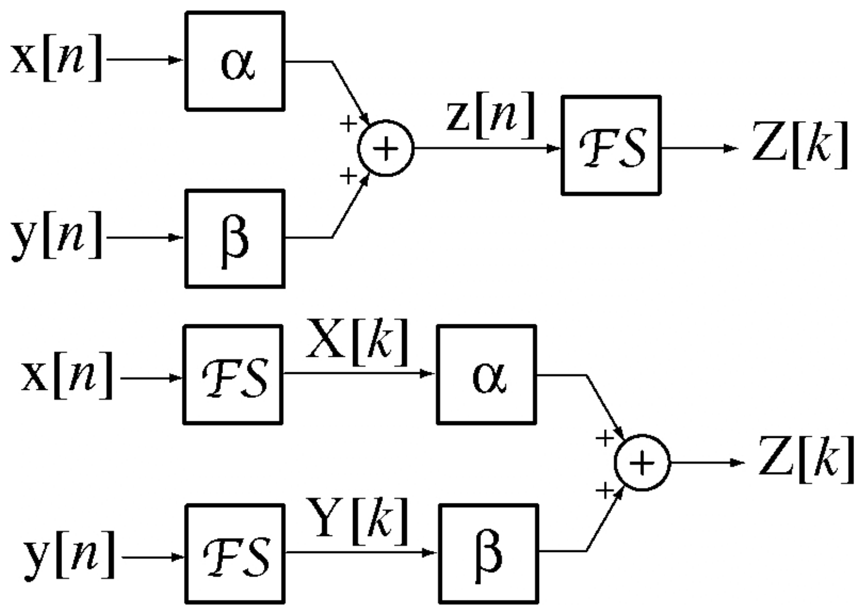


DTFS Properties

Let a signal, x[n], have a fundamental period, N_{0x}, and let a signal, y[n], have a fundamental period, N_{0y}. Let the DTFS harmonic functions, each using the fundamental period as the representation time, N_F, be X[k] and Y[k]. In the properties to follow the two fundamental periods are the same unless otherwise stated.

Linearity

$$\alpha\,\mathrm{x}(t)+\beta\,\mathrm{y}(t) \xleftrightarrow{\mathcal{FS}} \alpha\,\mathrm{X}[k]+\beta\,\mathrm{Y}[k]$$

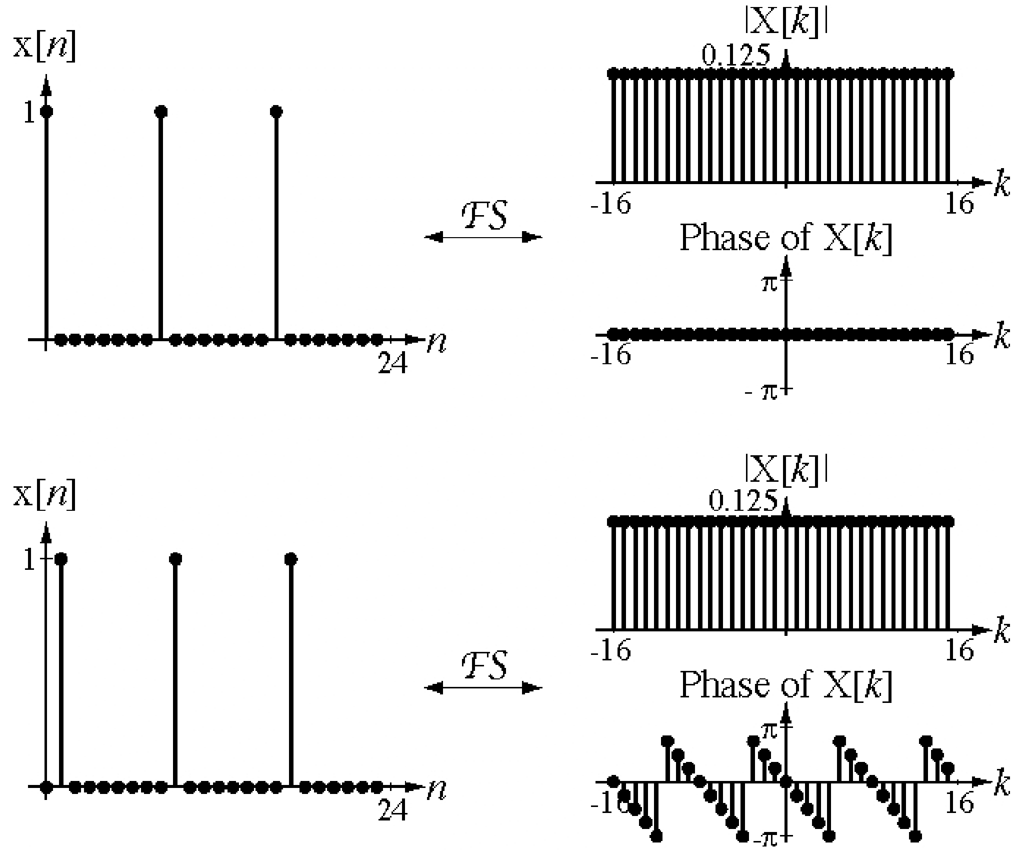


DTFS Properties

Time Shifting

$$\mathrm{x}[n-n_0] \xleftrightarrow{\mathcal{FS}} e^{-j2\pi(kF_0)n_0}\,\mathrm{X}[k]$$

$$\mathrm{x}[n-n_0] \xleftrightarrow{\mathcal{FS}} e^{-j(k\Omega_0)n_0}\,\mathrm{X}[k]$$



DTFS Properties

Frequency Shifting (Harmonic Number Shifting)

$$e^{j2\pi(k_0 F_0)n}\,\mathrm{x}[n] \xleftrightarrow{\mathcal{FS}} \mathrm{X}[k-k_0]$$

$$e^{j(k_0 \Omega_0)n}\,\mathrm{x}[n] \xleftrightarrow{\mathcal{FS}} \mathrm{X}[k-k_0]$$

Conjugation

$$\mathrm{x}^*[n] \xleftrightarrow{\mathcal{FS}} \mathrm{X}^*[-k]$$

Time Reversal

$$\mathrm{x}[-n] \xleftrightarrow{\mathcal{FS}} \mathrm{X}[-k]$$

DTFS Properties

Time Scaling

Let $\mathrm{z}[n] = \mathrm{x}[an]\,,\ a > 0$

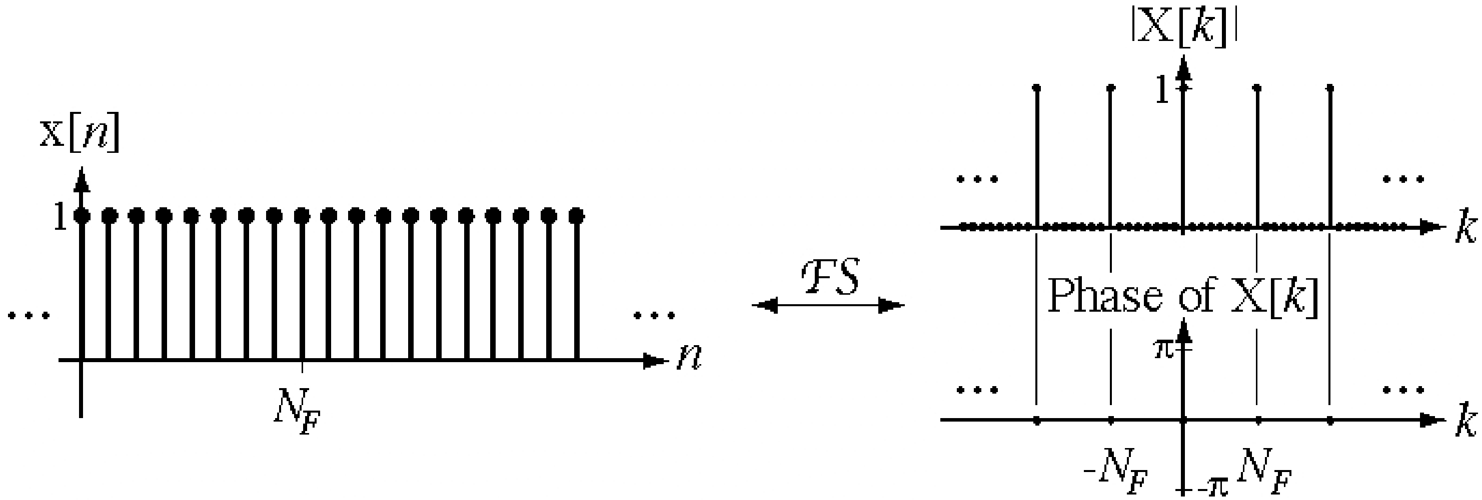
If a is not an integer, some values of z[n] are undefined and no DTFS can be found. If a is an integer (other than 1) then z[n] is a decimated version of x[n] with some values missing and there cannot be a unique relationship between their harmonic functions. However, if

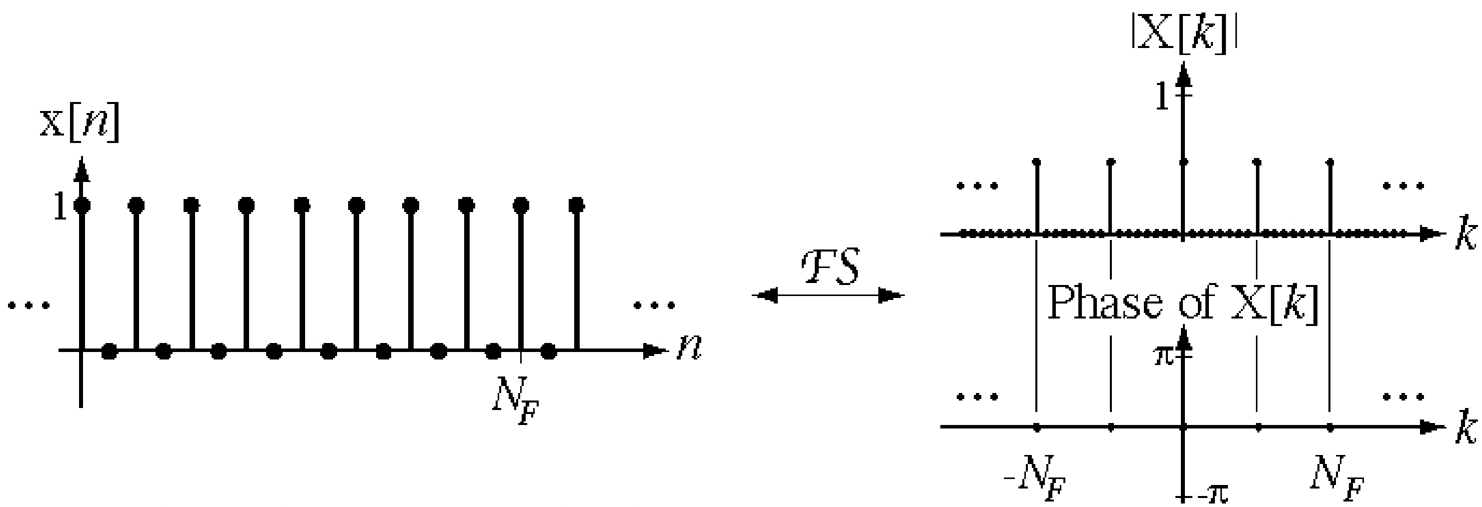
$$\mathrm{z}[n] = \begin{cases} \mathrm{x}\left[\dfrac{n}{m}\right] & , \ \dfrac{n}{m} \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}$$

then

$$\mathrm{Z}[k] = \frac{1}{m}\mathrm{X}[k]\ \ ,\ \ N_F = mN_0$$

DTFS Properties





DTFS Properties

Change of Representation Time

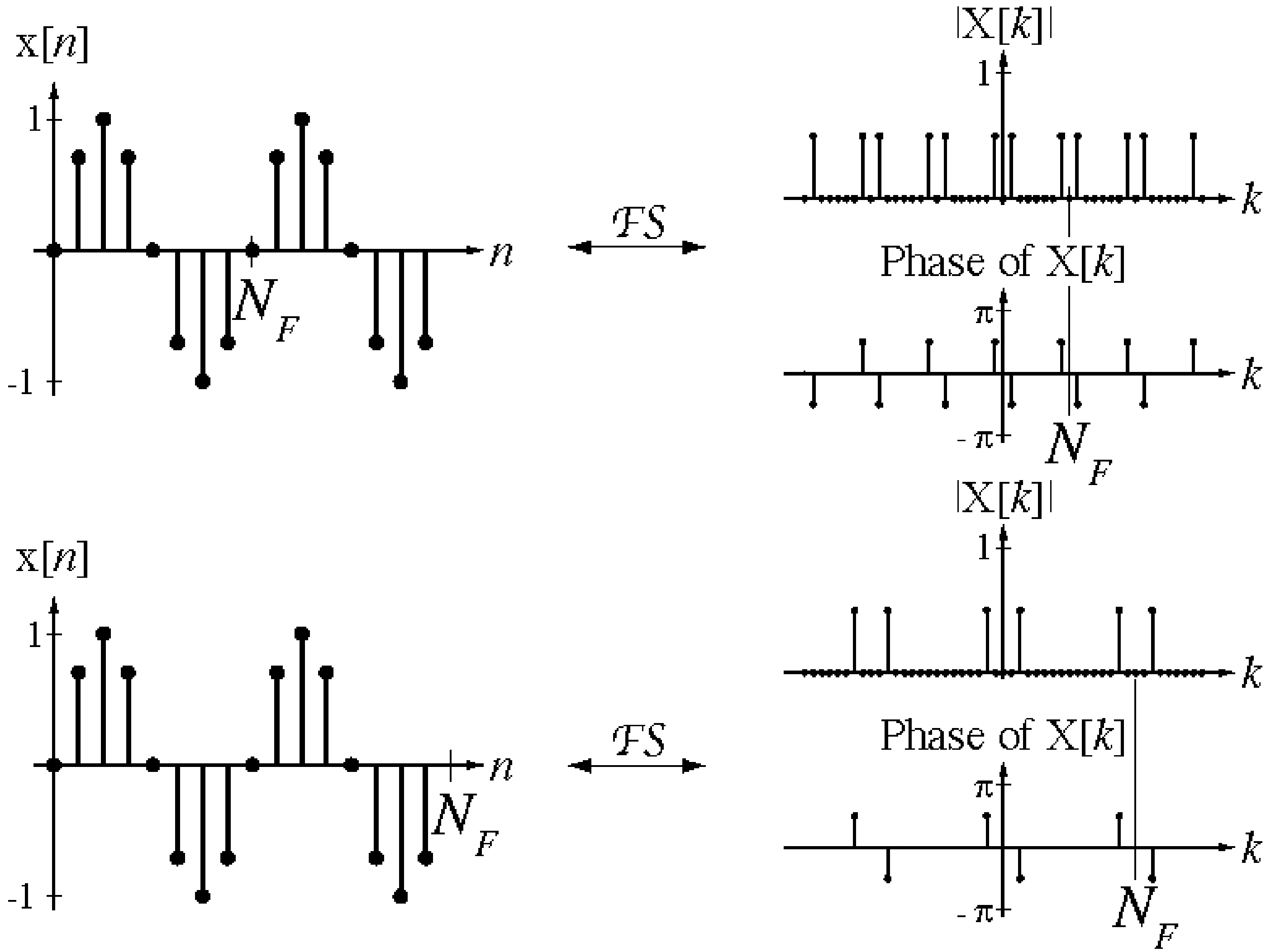
$$\text{With } N_F = N_{x0},\ \mathrm{x}[n] \xleftrightarrow{\mathcal{FS}} \mathrm{X}[k]$$

$$\text{With } N_F = qN_{x0},\ \mathrm{x}[n] \xleftrightarrow{\mathcal{FS}} \mathrm{X}_q[k]$$

$$\mathrm{X}_q[k] = \begin{cases} \mathrm{X}\left[\dfrac{k}{q}\right] & , \ \dfrac{k}{q} \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}$$

(q is any positive integer)

DTFS Properties



DTFS Properties

Accumulation

$$\sum_{m=-\infty}^{n} \mathrm{x}[m] \overset{\mathcal{FS}}{\longleftrightarrow} \frac{\mathrm{X}[k]}{1-e^{-j2\pi(kF_0)}} \quad , \quad k \neq 0$$

$$\sum_{m=-\infty}^{n} \mathrm{x}[m] \overset{\mathcal{FS}}{\longleftrightarrow} \frac{\mathrm{X}[k]}{1-e^{-j(k\Omega_0)}} \quad , \quad k \neq 0$$

Parseval's Theorem

$$\frac{1}{N_0} \sum_{n=\langle N_0 \rangle} |\mathrm{x}[n]|^2 = \sum_{k=\langle N_0 \rangle} |\mathrm{X}[k]|^2$$

DTFS Properties

Multiplication-Convolution Duality

$$\mathrm{x}[n]\mathrm{y}[n] \xleftrightarrow{\mathcal{FS}} \mathrm{Y}[k] \circledast \mathrm{X}[k] = \sum_{q=\langle N_0 \rangle} \mathrm{Y}[q]\,\mathrm{X}[k-q]$$

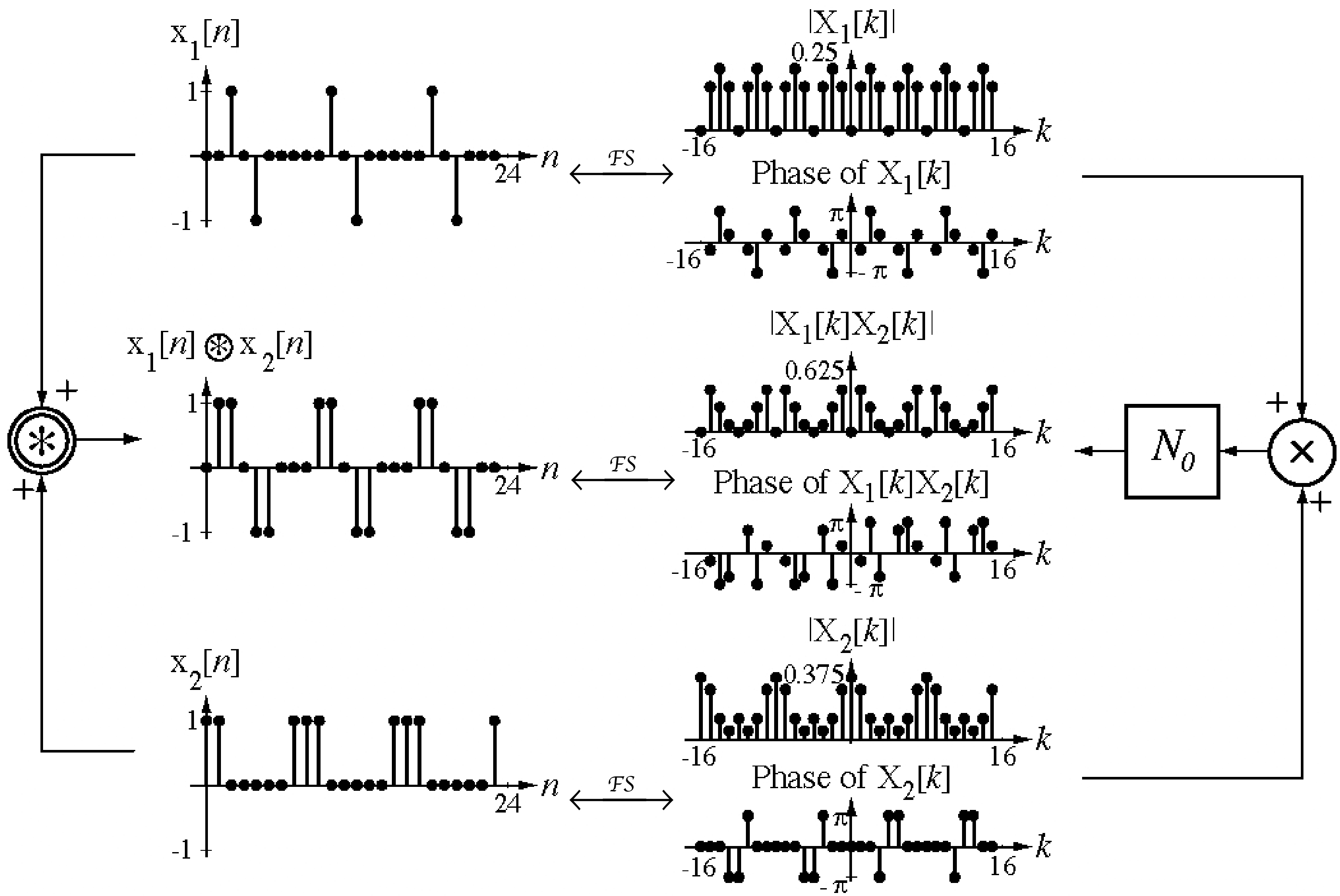
$$\mathrm{x}[n] \circledast \mathrm{y}[n] \xleftrightarrow{\mathcal{FS}} N_0\,\mathrm{Y}[k]\,\mathrm{X}[k]$$

First Backward Difference

$$\mathrm{x}[n]-\mathrm{x}[n-1] \xleftrightarrow{\mathcal{FS}} \left(1-e^{-j2\pi(kF_0)}\right)\mathrm{X}[k]$$

$$\mathrm{x}[n]-\mathrm{x}[n-1] \xleftrightarrow{\mathcal{FS}} \left(1-e^{-j(k\Omega_0)}\right)\mathrm{X}[k]$$

DTFS Properties



Convergence of the DTFS

- The DTFS converges exactly with a finite number of terms. It does not have a “Gibbs phenomenon” in the same sense that the CTFS does

LTI Systems with Periodic Excitation

The differential equation describing an RC lowpass filter is

$$RC\,\mathrm{v}'_{out}(t)+\mathrm{v}_{out}(t)=\mathrm{v}_{in}(t)$$

If the excitation, $\mathrm{v}_{in}(t)$, is periodic it can be expressed as a CTFS,

$$\mathrm{v}_{in}(t)=\sum_{k=-\infty}^{\infty}\mathrm{V}_{in}[k]e^{j2\pi(kf_0)t}$$

The equation for the *k*th harmonic alone is

$$RC\,\mathrm{v}'_{out,k}(t)+\mathrm{v}_{out,k}(t)=\mathrm{v}_{in,k}(t)=\mathrm{V}_{in}[k]e^{j2\pi(kf_0)t}$$

LTI Systems with Periodic Excitation

If the excitation is periodic, the response is also, with the same fundamental period. Therefore the response can be expressed as a CTFS also.

$$\mathrm{v}_{out,k}(t) = \mathrm{V}_{out}[k]e^{j2\pi(kf_0)t}$$

Then the equation for the *k*th harmonic becomes

$$j2k\pi f_0 RC\,\mathrm{V}_{out}[k]e^{j2\pi(kf_0)t} + \mathrm{V}_{out}[k]e^{j2\pi(kf_0)t} = \mathrm{V}_{in}[k]e^{j2\pi(kf_0)t}$$

Notice that what was once a *differential* equation is now an *algebraic* equation.

LTI Systems with Periodic Excitation

Solving the *k*th-harmonic equation,

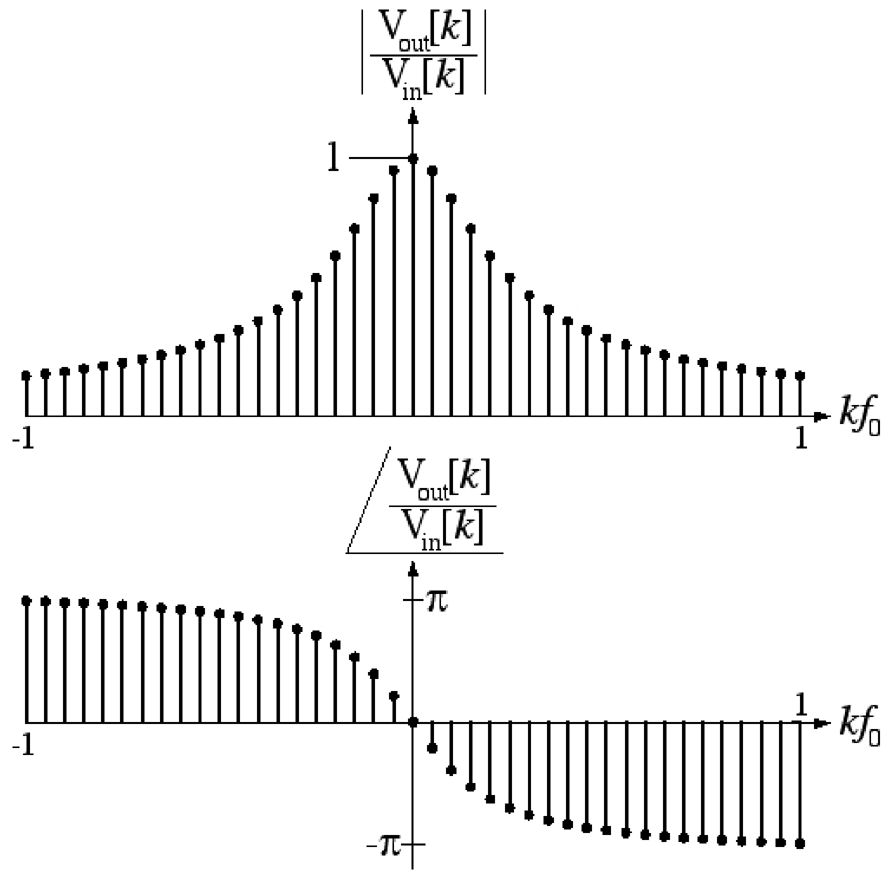
$$\mathrm{V}_{out}[k] = \frac{\mathrm{V}_{in}[k]}{j2k\pi f_0 RC + 1}$$

Then the response can be written as

$$\mathrm{v}_{out}(t) = \sum_{k=-\infty}^{\infty} \mathrm{V}_{out}[k] e^{j2\pi(kf_0)t} = \sum_{k=-\infty}^{\infty} \frac{\mathrm{V}_{in}[k]}{j2k\pi f_0 RC + 1} e^{j2\pi(kf_0)t}$$

LTI Systems with Periodic Excitation

The ratio, $\frac{V_{out}[k]}{V_{in}[k]}$, is the *frequency response* of the system.



$R = 1\Omega$, $C = 1$ F, $f_0 = 0.05$ Hz