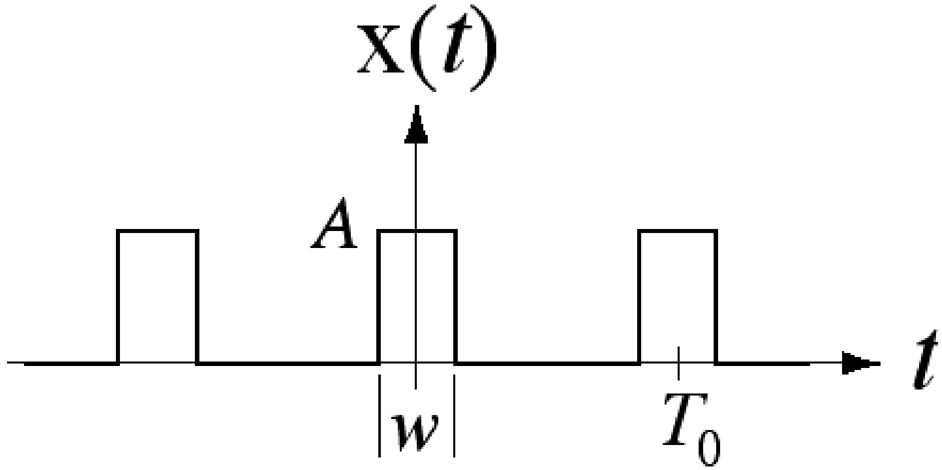
The Fourier Transform

Extending the CTFS

- The CTFS is a good analysis tool for systems with periodic excitation but the CTFS cannot represent an aperiodic signal for all time
- The continuous-time Fourier transform (CTFT) *can* represent an aperiodic signal for all time

CTFS-to-CTFT Transition

Consider a periodic pulse-train signal, x(t), with duty cycle, $\frac{w}{T_0}$

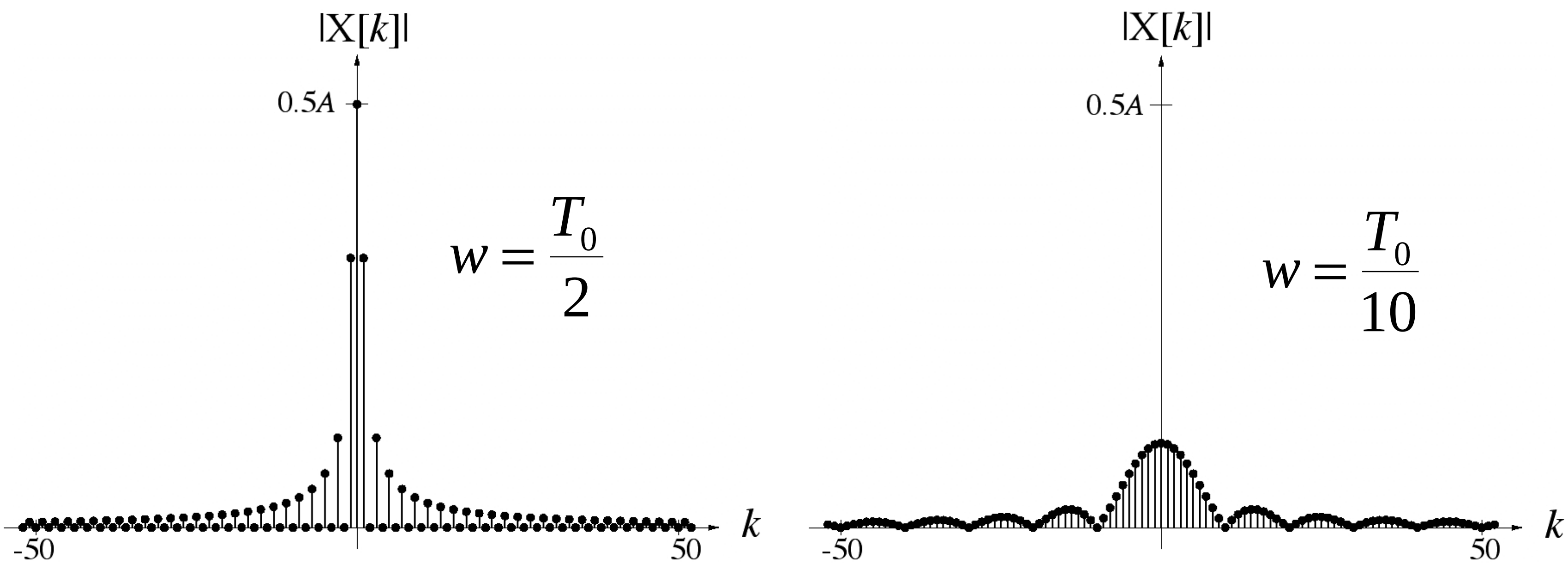


Its CTFS harmonic function is $\mathrm{X}[k] = \frac{Aw}{T_0}\operatorname{sinc}\left(\frac{kw}{T_0}\right)$

As the period, T_0 , is increased, holding w constant, the duty cycle is decreased. When the period becomes infinite (and the duty cycle becomes zero) x(t) is no longer periodic.

CTFS-to-CTFT Transition

Below are plots of the magnitude of X[k] for 50% and 10% duty cycles. As the period increases the sinc function widens and its magnitude falls. As the period approaches infinity, the CTFS harmonic function becomes an infinitely-wide sinc function with zero amplitude.

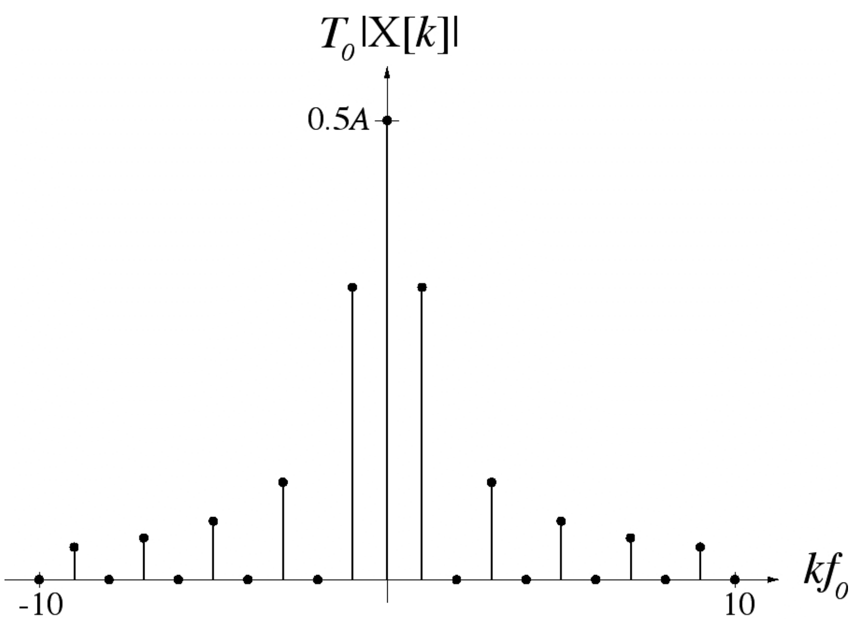


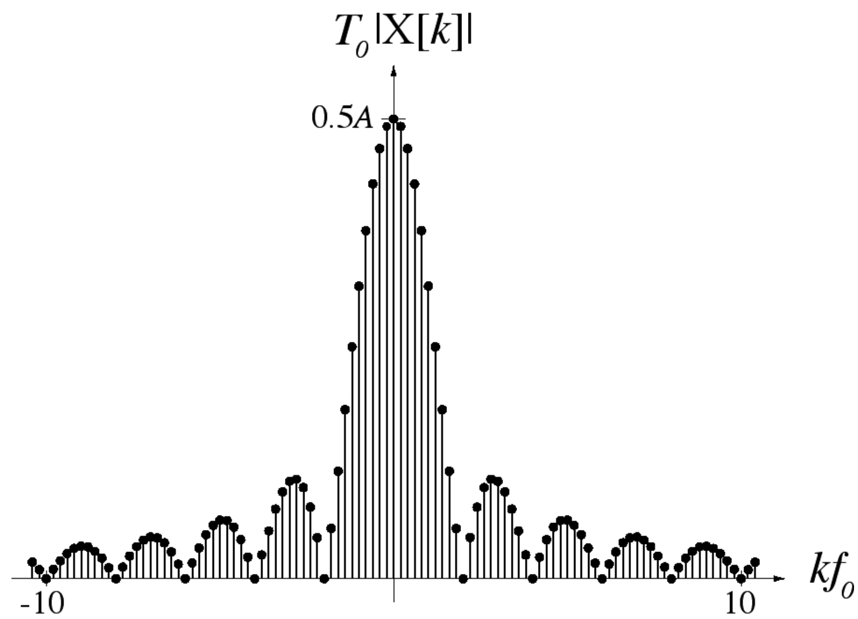
CTFS-to-CTFT Transition

This infinity-and-zero problem can be solved by normalizing the CTFS harmonic function. Define a new “modified” CTFS harmonic function,

$$T_0 \, \mathrm{X}[k] = Aw \, \mathrm{sinc}\left(w\left(kf_0\right)\right)$$

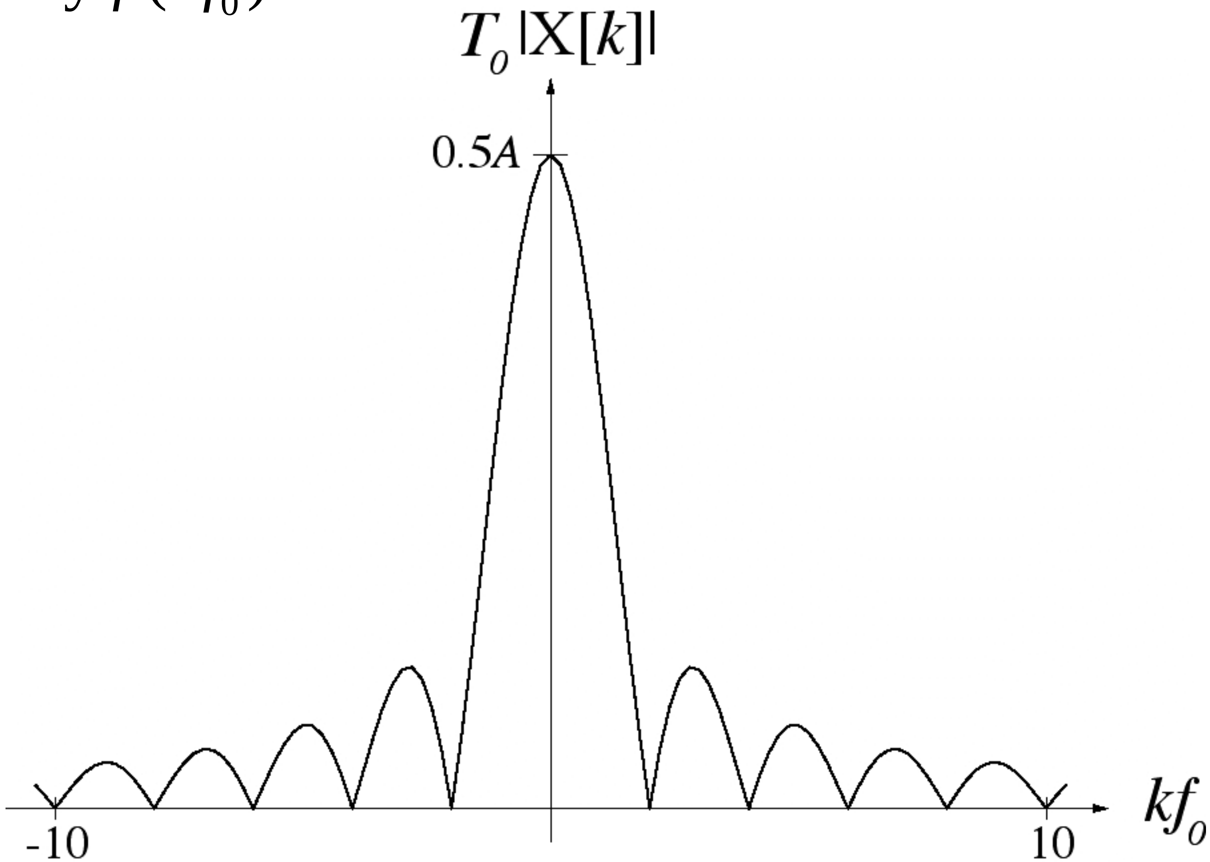
and graph it versus kf_0 instead of versus k.





CTFS-to-CTFT Transition

In the limit as the period approaches infinity, the modified CTFS harmonic function approaches a function of continuous frequency $f\,(kf_0)$.



Definition of the CTFT

Forward	f form	Inverse
$$X(f) = \mathcal{F}(x(t)) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$		$$x(t) = \mathcal{F}^{-1}(X(f)) = \int_{-\infty}^{\infty} X(f) e^{+j2\pi ft} df$$

Forward	ω form	Inverse
$$X(j\omega) = \mathcal{F}(x(t)) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$		$$x(t) = \mathcal{F}^{-1}(X(j\omega)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{+j\omega t} d\omega$$

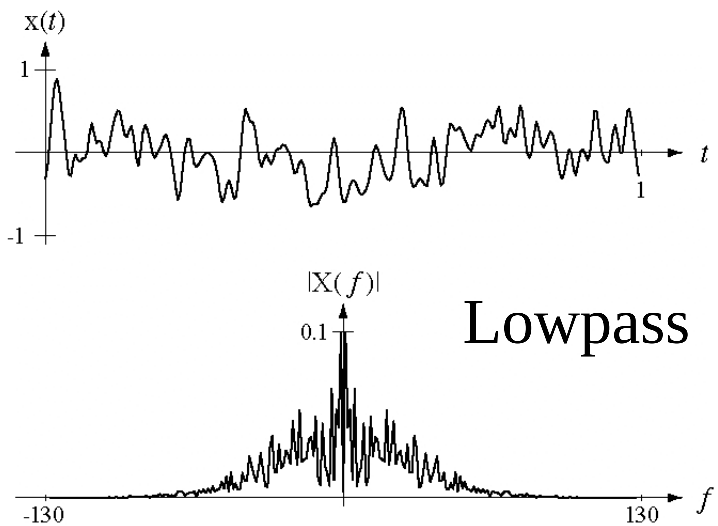
Commonly-used notation:

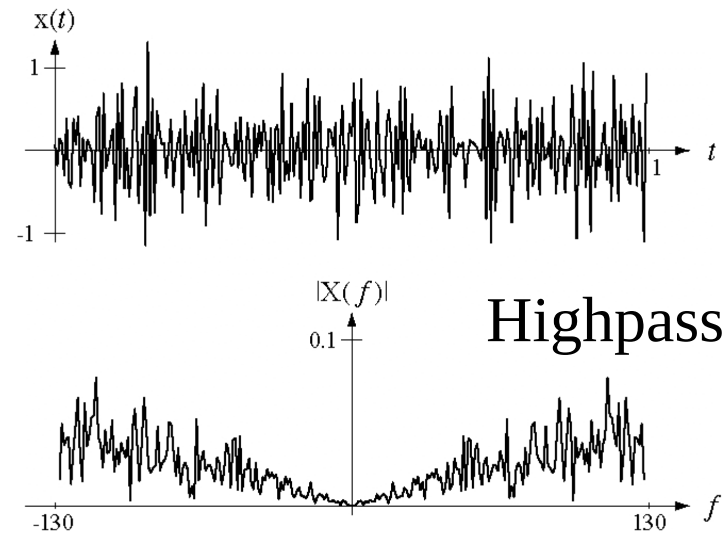
$$x(t) \xleftrightarrow{\mathcal{F}} X(f) \quad \text{or} \quad x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

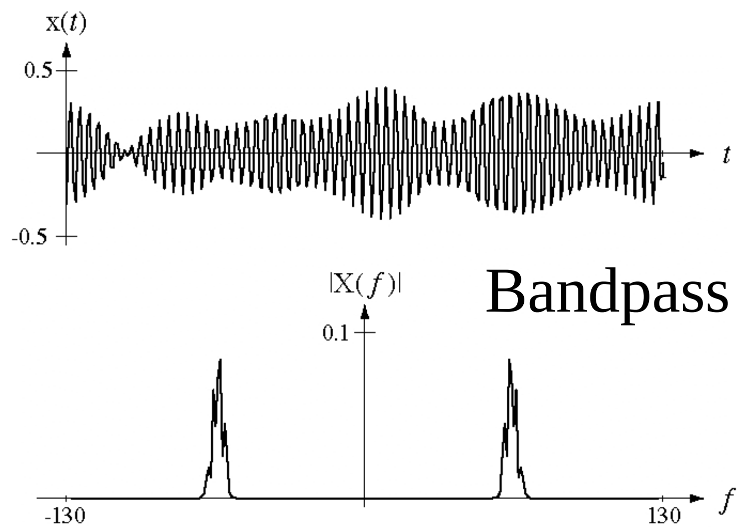
Some Remarkable Implications of the Fourier Transform

The CTFT expresses a finite-amplitude, real-valued, aperiodic signal which can also, in general, be time-limited, as a summation (an integral) of an infinite continuum of weighted, infinitesimal-amplitude, complex sinusoids, each of which is unlimited in time. (Time limited means “having non-zero values only for a finite time.”)

Frequency Content



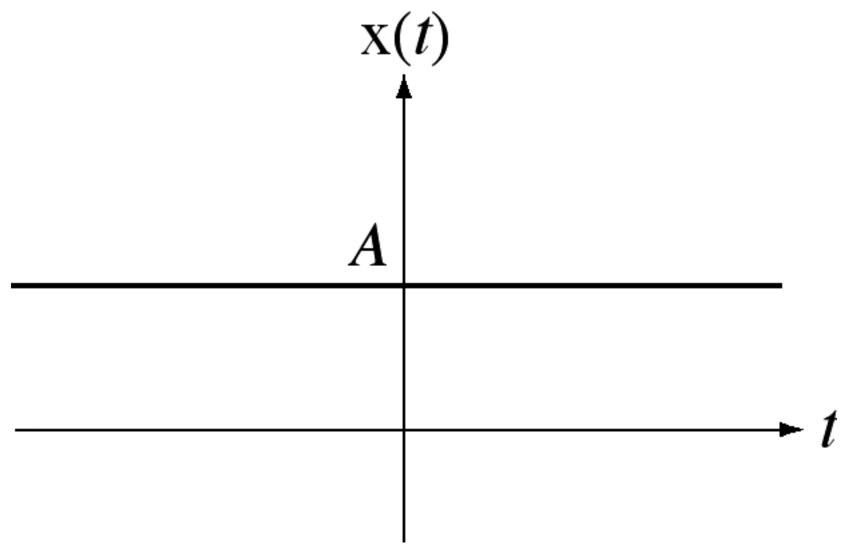




Convergence and the Generalized Fourier Transform

Let $x(t) = A$. Then from the definition of the CTFT,

$$X(f) = \int_{-\infty}^{\infty} A e^{-j2\pi ft} dt = A \int_{-\infty}^{\infty} e^{-j2\pi ft} dt$$



This integral does not converge so, strictly speaking, the CTFT does not exist.

Convergence and the Generalized Fourier Transform

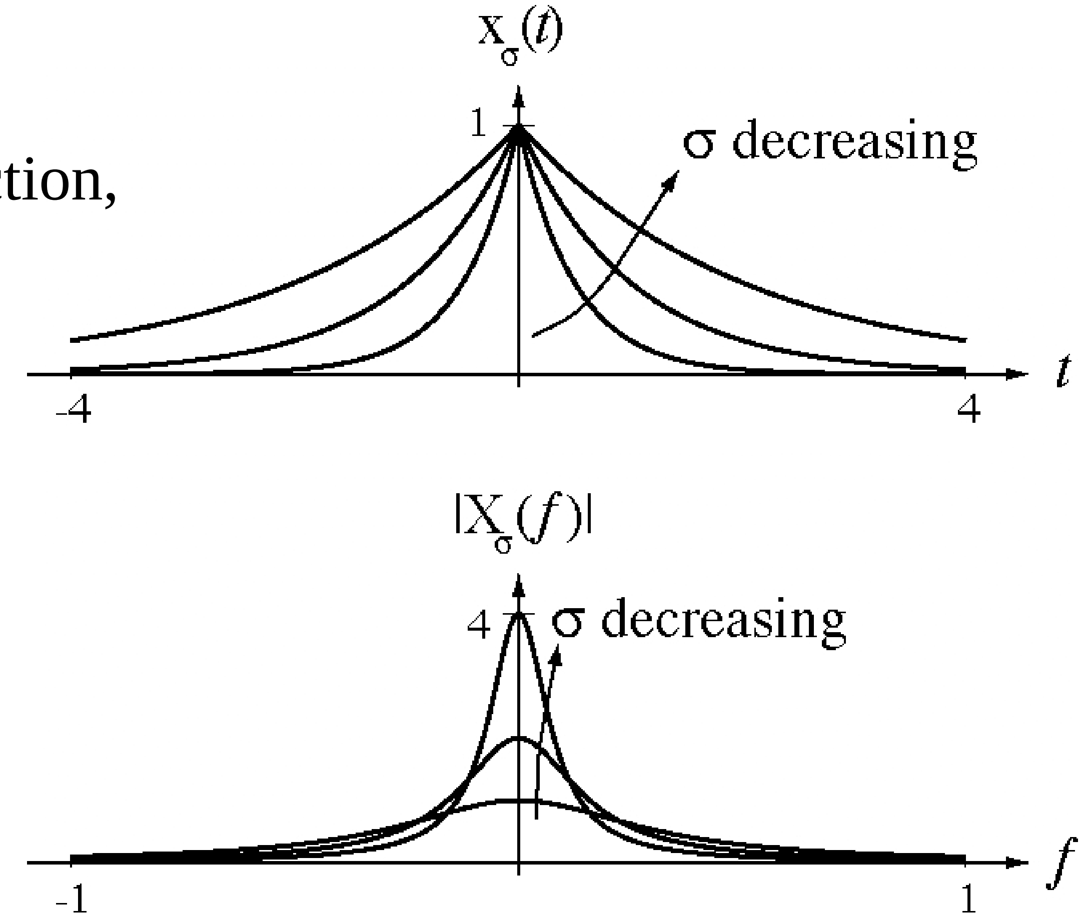
But consider a similar function,

$$x_\sigma(t) = Ae^{-\sigma|t|} \quad , \quad \sigma > 0$$

Its CTFT integral,

$$X_\sigma(f) = \int_{-\infty}^{\infty} Ae^{-\sigma|t|} e^{-j2\pi ft} dt$$

does converge.



Convergence and the Generalized Fourier Transform

Carrying out the integral, $\mathrm{X}_\sigma(f) = A\dfrac{2\sigma}{\sigma^2 + (2\pi f)^2}$.

Now let σ approach zero.

If $f \neq 0$ then $\lim_{\sigma \to 0} A\dfrac{2\sigma}{\sigma^2 + (2\pi f)^2} = 0$. The area under this function is

$$\text{Area} = A\int_{-\infty}^{\infty} \frac{2\sigma}{\sigma^2 + (2\pi f)^2}\, df$$

which is *A*, independent of the value of σ. So, in the limit as σ approaches zero, the CTFT has an area of *A* and is zero unless $f = 0$. This exactly defines an impulse of strength, *A*. Therefore

$$A \xleftrightarrow{\mathcal{F}} A\delta(f)$$

Convergence and the Generalized Fourier Transform

By a similar process it can be shown that

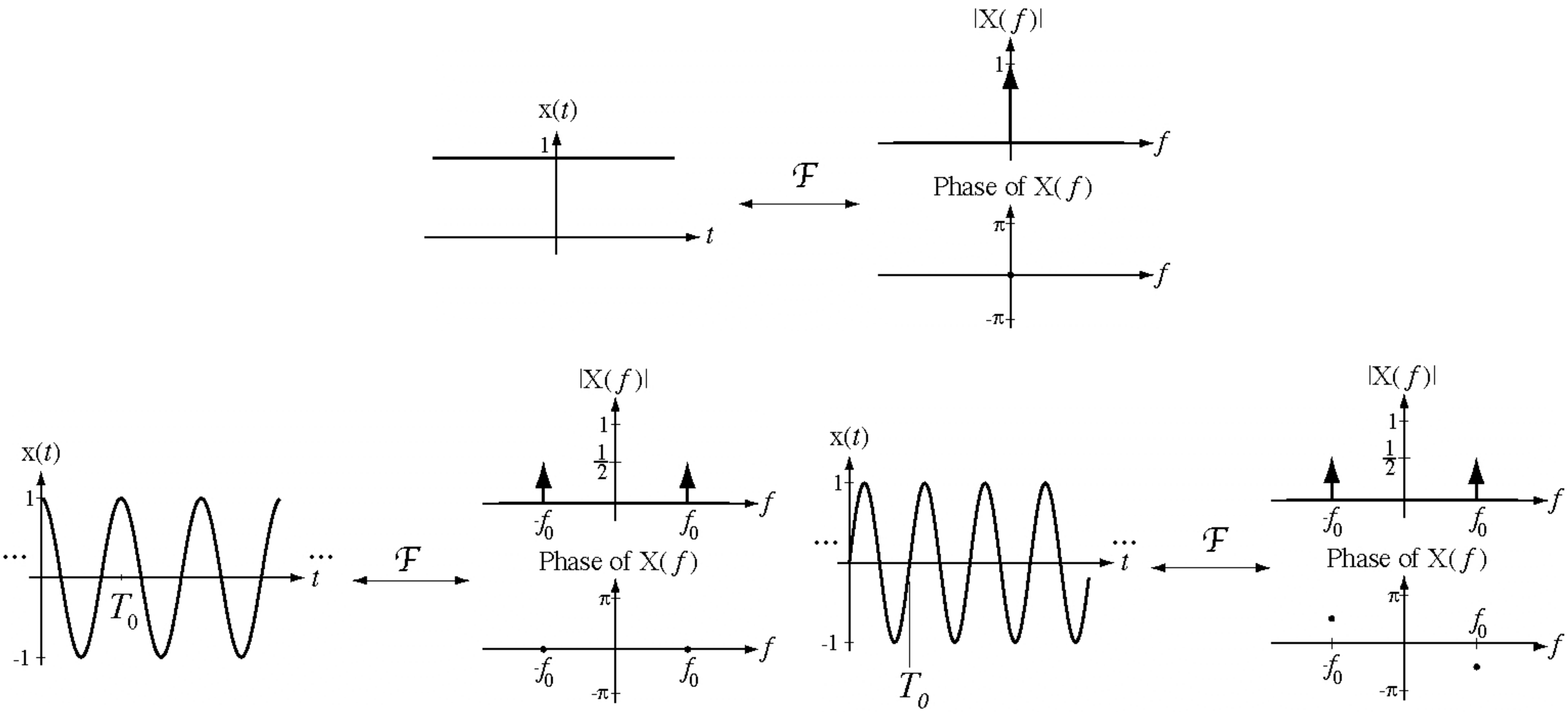
$$\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$

and

$$\sin(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{j}{2}\left[\delta(f + f_0) - \delta(f - f_0)\right]$$

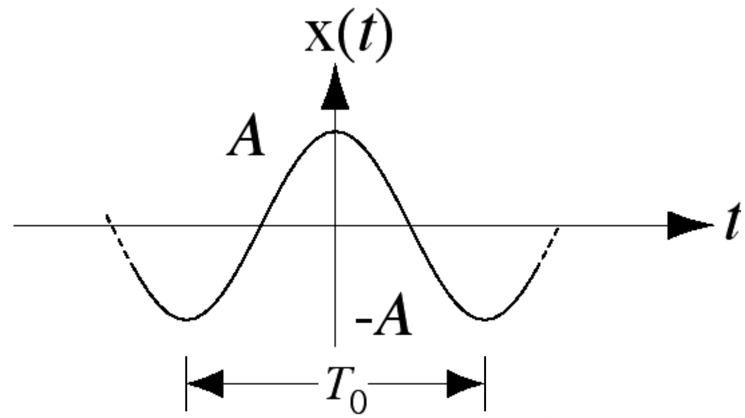
These CTFT's which involve impulses are called *generalized* Fourier transforms (probably because the impulse is a *generalized* function).

Convergence and the Generalized Fourier Transform



Negative Frequency

This signal is obviously a sinusoid. How is it described mathematically?



It could be described by

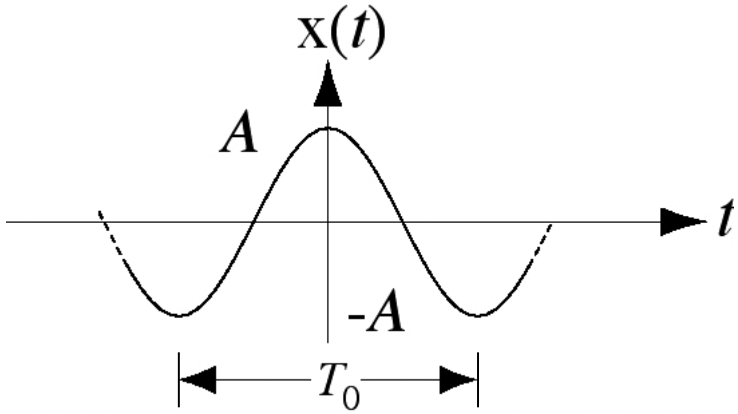
$$\mathrm{x}(t) = A\cos\left(\frac{2\pi t}{T_0}\right) = A\cos(2\pi f_0 t)$$

But it could also be described by

$$\mathrm{x}(t) = A\cos(2\pi(-f_0)t)$$

Negative Frequency

x(t) could also be described by



$$\mathrm{x}(t) = A\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}$$

or

$$\mathrm{x}(t) = A_1 \cos(2\pi f_0 t) + A_2 \cos(2\pi(-f_0)t) \quad , \quad A_1 + A_2 = A$$

and probably in a few other different-looking ways. So who is to say whether the frequency is positive or negative? For the purposes of signal analysis, it does not matter.

CTFT Properties

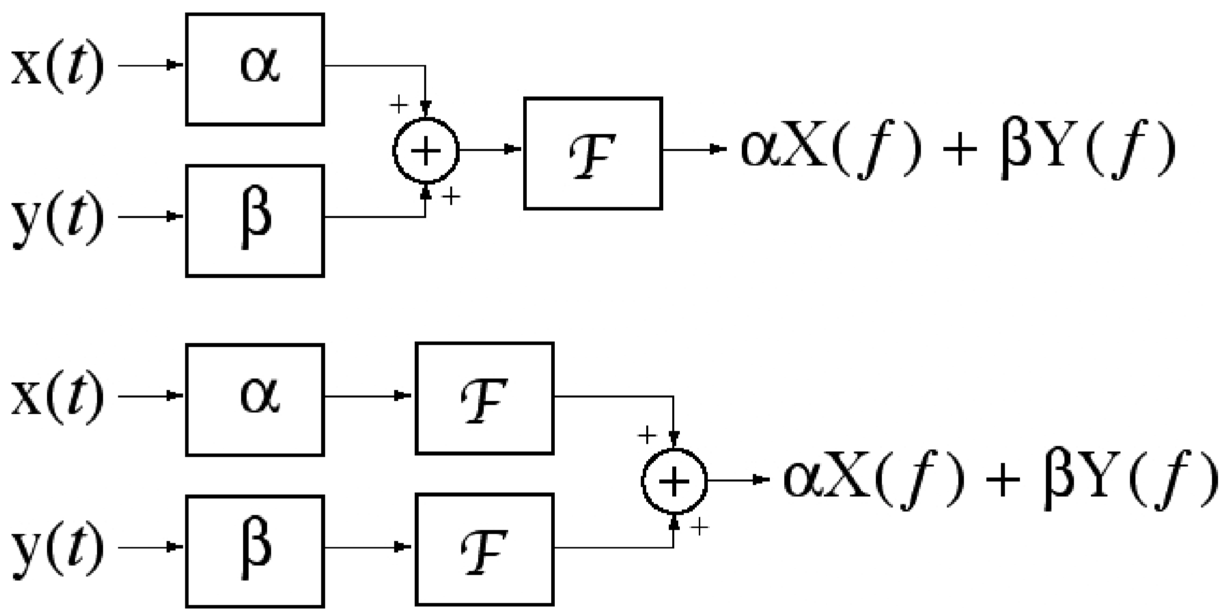
If $\mathcal{F}(\mathrm{x}(t)) = \mathrm{X}(f)$ or $\mathrm{X}(j\omega)$ and $\mathcal{F}(\mathrm{y}(t)) = \mathrm{Y}(f)$ or $\mathrm{Y}(j\omega)$

then the following properties can be proven.

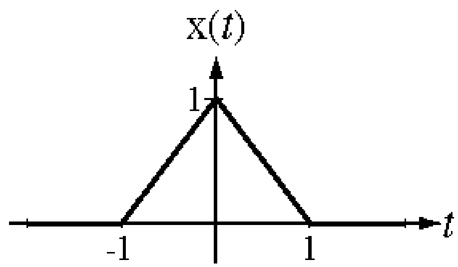
Linearity

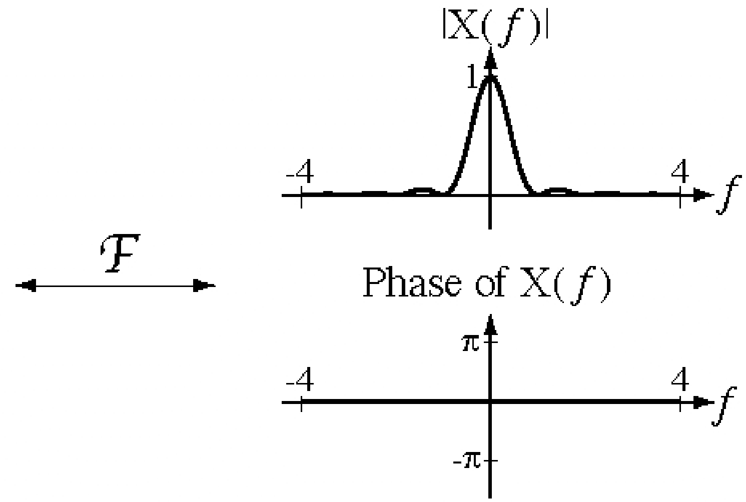
$$\alpha\,\mathrm{x}(t) + \beta\,\mathrm{y}(t) \xleftrightarrow{\mathcal{F}} \alpha\,\mathrm{X}(f) + \beta\,\mathrm{Y}(f)$$

$$\alpha\,\mathrm{x}(t) + \beta\,\mathrm{y}(t) \xleftrightarrow{\mathcal{F}} \alpha\,\mathrm{X}(j\omega) + \beta\,\mathrm{Y}(j\omega)$$



CTFT Properties

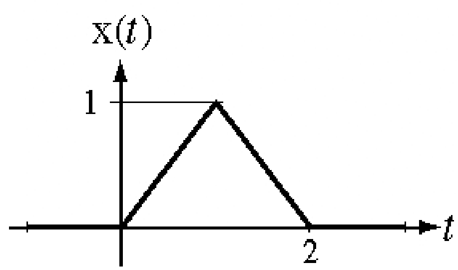


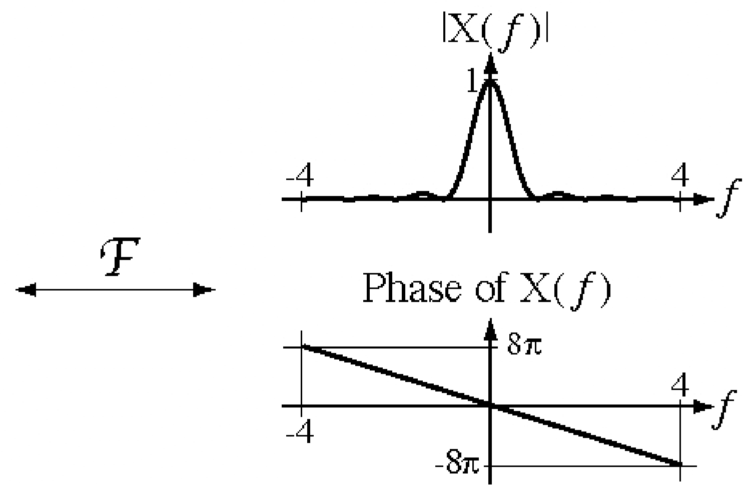


Time Shifting

$$\mathrm{x}(t-t_0) \xleftrightarrow{\mathcal{F}} \mathrm{X}(f)e^{-j2\pi f t_0}$$

$$\mathrm{x}(t-t_0) \xleftrightarrow{\mathcal{F}} \mathrm{X}(j\omega)e^{-j\omega t_0}$$



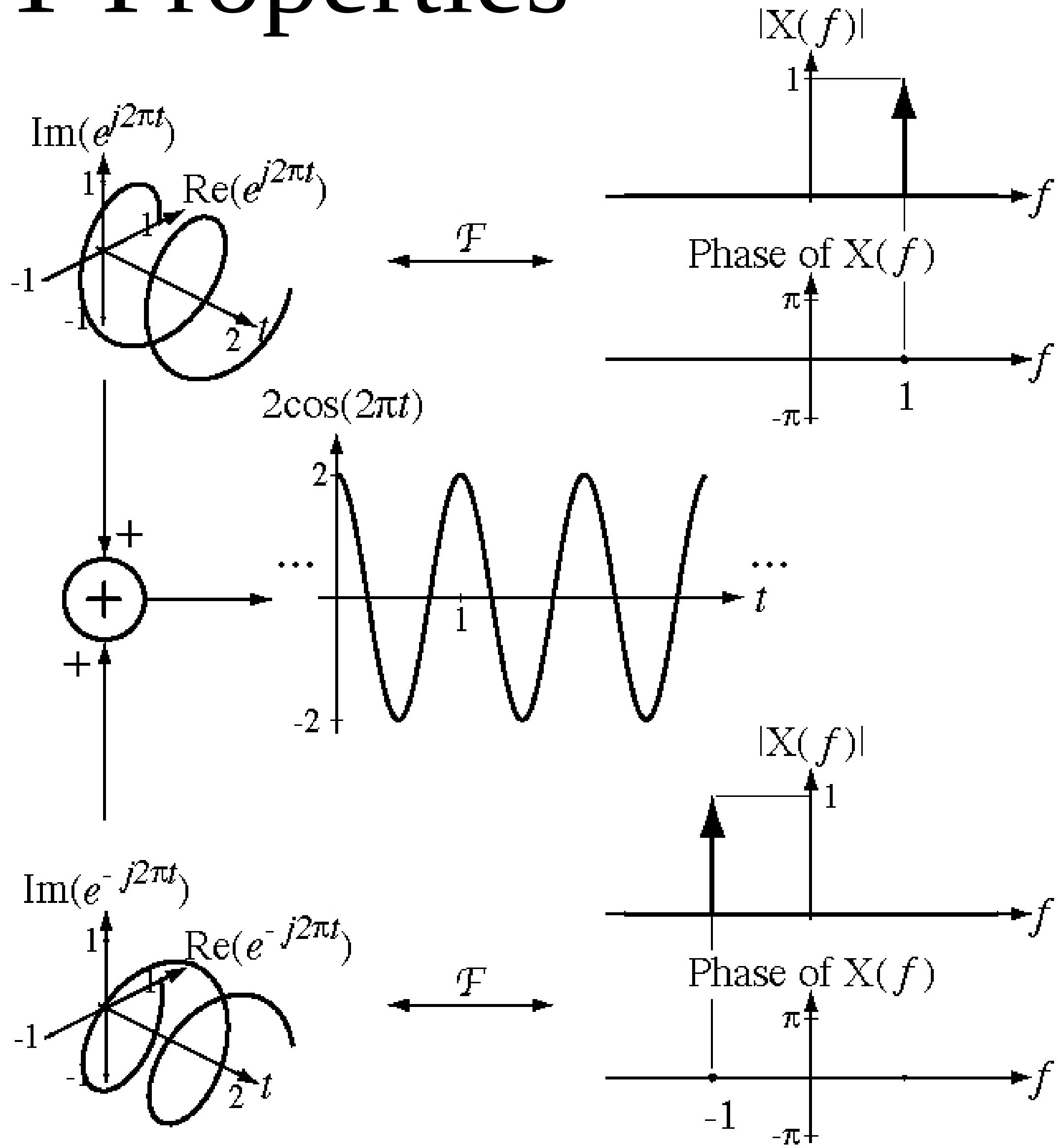


CTFT Properties

Frequency Shifting

$$x(t)e^{+j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} X(f - f_0)$$

$$x(t)e^{+j\omega_0 t} \xleftrightarrow{\mathcal{F}} X(\omega - \omega_0)$$



CTFT Properties

Time Scaling

$$\mathrm{x}(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|}\mathrm{X}\left(\frac{f}{a}\right)$$

$$\mathrm{x}(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|}\mathrm{X}\left(j\frac{\omega}{a}\right)$$

Frequency Scaling

$$\frac{1}{|a|}\mathrm{x}\left(\frac{t}{a}\right) \xleftrightarrow{\mathcal{F}} \mathrm{X}(af)$$

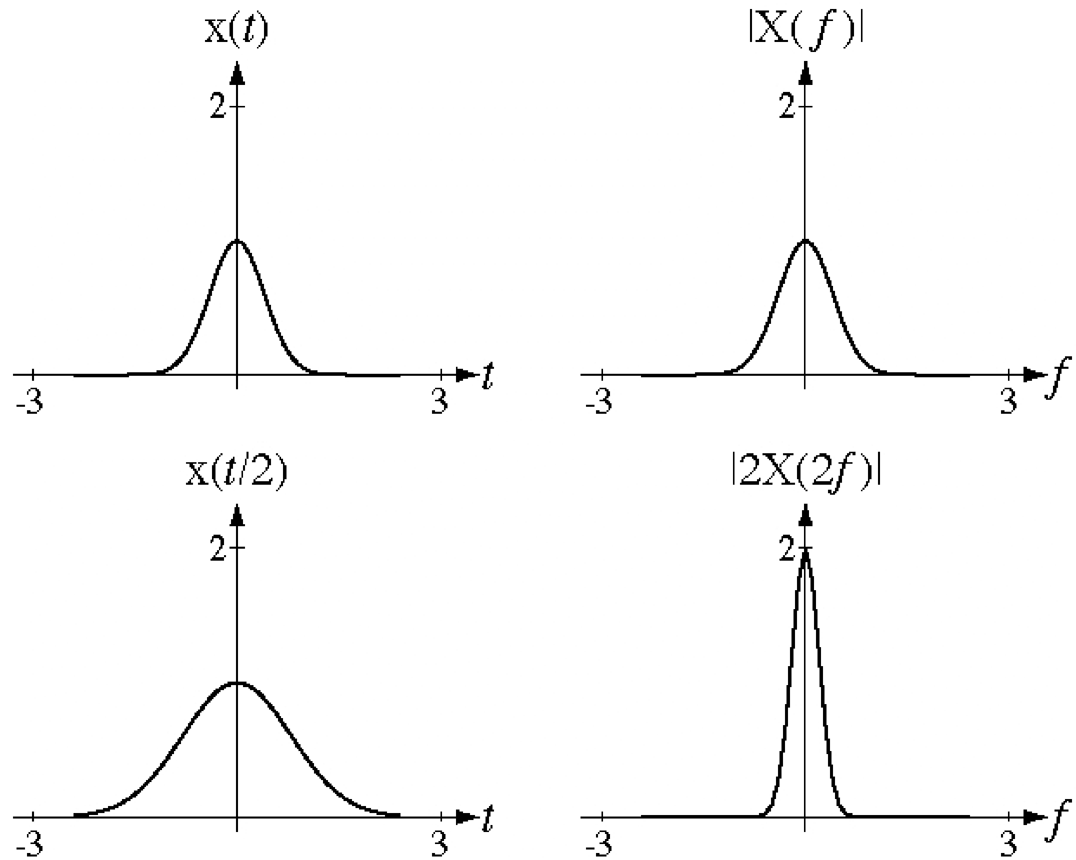
$$\frac{1}{|a|}\mathrm{x}\left(\frac{t}{a}\right) \xleftrightarrow{\mathcal{F}} \mathrm{X}(ja\omega)$$

The "Uncertainty" Principle

The time and frequency scaling properties indicate that if a signal is expanded in one domain it is compressed in the other domain. This is called the "uncertainty principle" of Fourier analysis.

$$e^{-\pi t^2} \xleftrightarrow{\mathcal{F}} e^{-\pi f^2}$$

$$e^{-\pi\left(\frac{t}{2}\right)^2} \xleftrightarrow{\mathcal{F}} 2e^{-\pi(2f)^2}$$



CTFT Properties

Transform of a Conjugate

$$x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-f)$$

$$x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-j\omega)$$

Multiplication-Convolution Duality

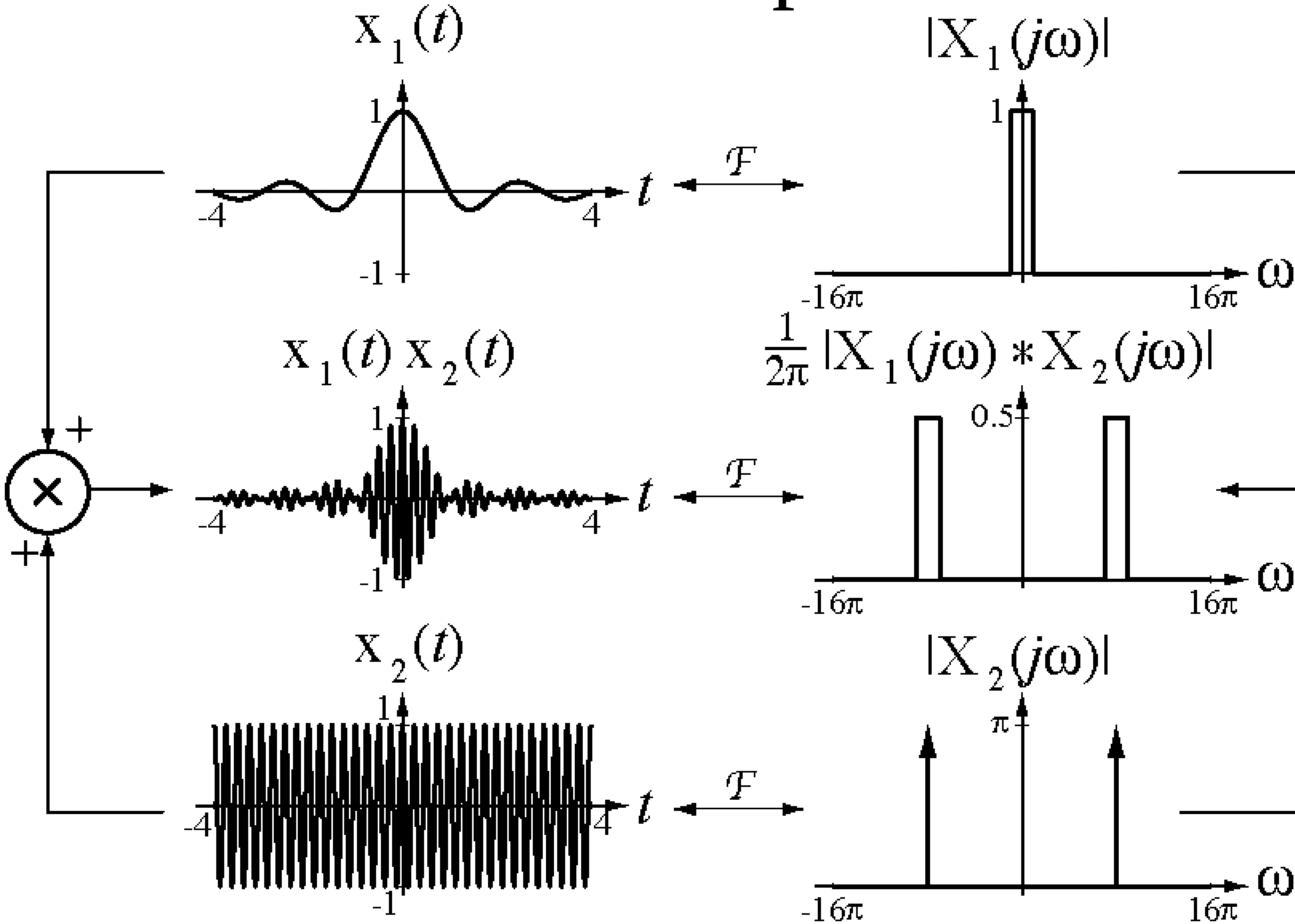
$$x(t) * y(t) \xleftrightarrow{\mathcal{F}} X(f)Y(f)$$

$$x(t) * y(t) \xleftrightarrow{\mathcal{F}} X(j\omega)Y(j\omega)$$

$$x(t)y(t) \xleftrightarrow{\mathcal{F}} X(f) * Y(f)$$

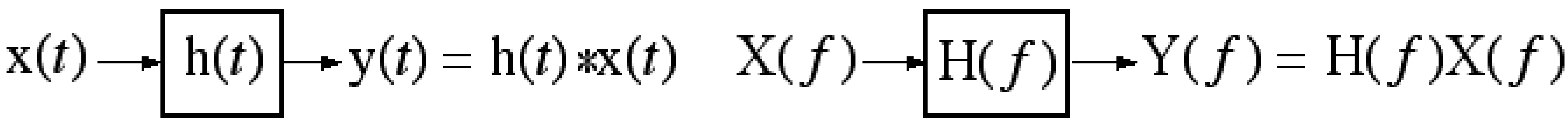
$$x(t)y(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} X(j\omega) * Y(j\omega)$$

CTFT Properties

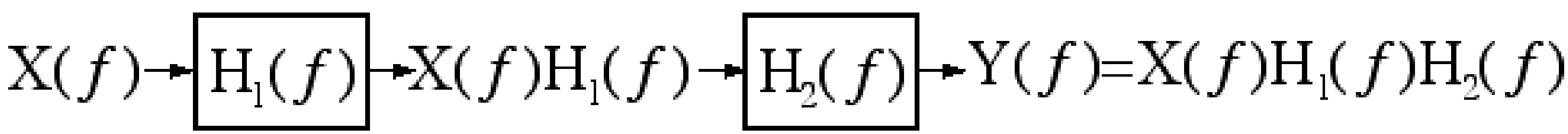


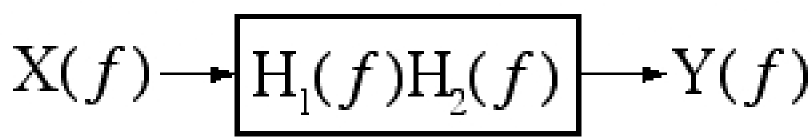
CTFT Properties

An important consequence of multiplication-convolution duality is the concept of the *transfer function*.



In the frequency domain, the cascade connection multiplies the transfer functions instead of convolving the impulse responses.





CTFT Properties

Time Differentiation

$$\frac{d}{dt}(\mathrm{x}(t)) \xleftrightarrow{\mathcal{F}} j2\pi f\,\mathrm{X}(f)$$

$$\frac{d}{dt}(\mathrm{x}(t)) \xleftrightarrow{\mathcal{F}} j\omega\,\mathrm{X}(j\omega)$$

Modulation

$$\mathrm{x}(t)\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\mathrm{X}(f - f_0) + \mathrm{X}(f + f_0)\right]$$

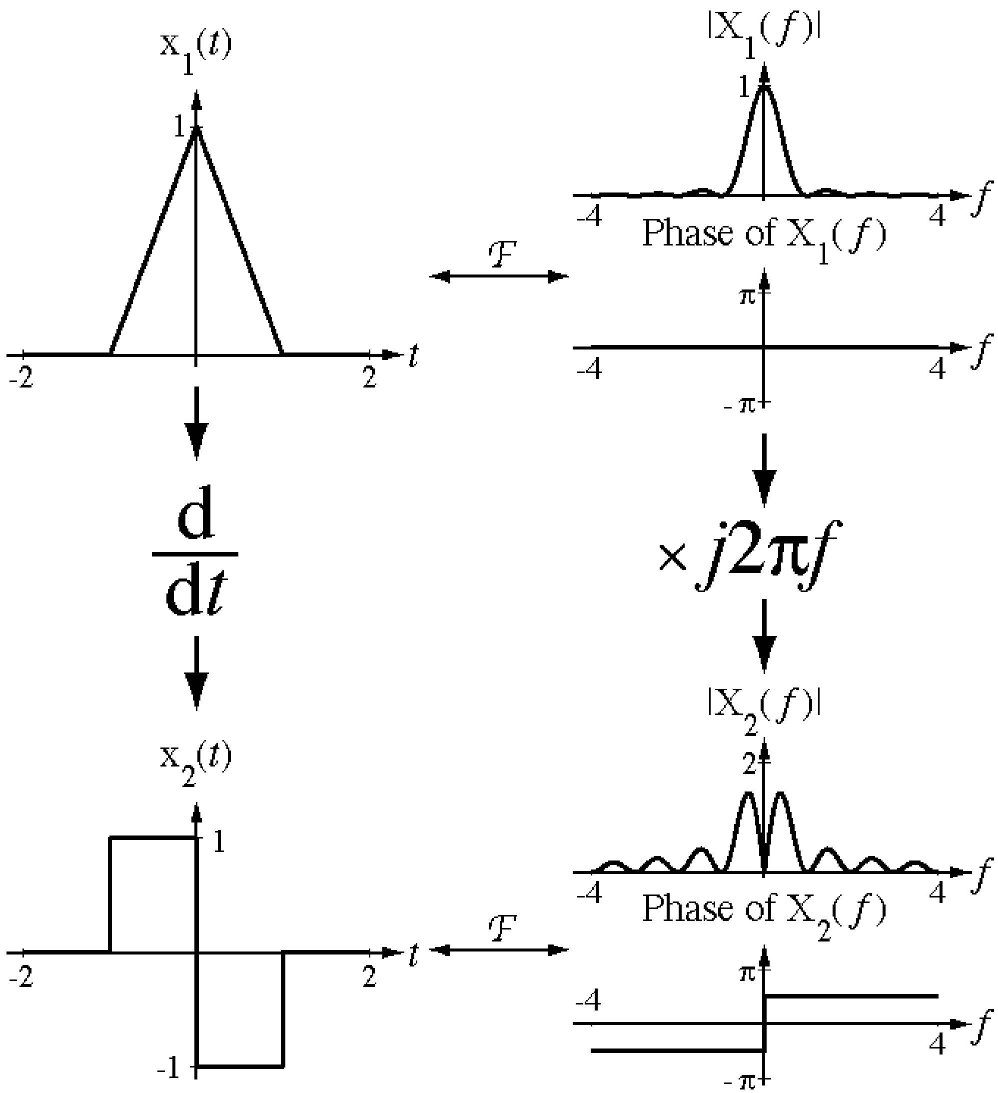
$$\mathrm{x}(t)\cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\mathrm{X}(j(\omega - \omega_0)) + \mathrm{X}(j(\omega + \omega_0))\right]$$

Transforms of Periodic Signals

$$\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \mathrm{X}[k] e^{-j2\pi(kf_F)t} \xleftrightarrow{\mathcal{F}} \mathrm{X}(f) = \sum_{k=-\infty}^{\infty} \mathrm{X}[k]\delta(f - kf_0)$$

$$\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \mathrm{X}[k] e^{-j(k\omega_F)t} \xleftrightarrow{\mathcal{F}} \mathrm{X}(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} \mathrm{X}[k]\delta(\omega - k\omega_0)$$

CTFT Properties



CTFT Properties

Parseval's Theorem

$$\int_{-\infty}^{\infty} |\mathrm{x}(t)|^2 \, dt = \int_{-\infty}^{\infty} |\mathrm{X}(f)|^2 \, df$$

$$\int_{-\infty}^{\infty} |\mathrm{x}(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathrm{X}(j\omega)|^2 \, df$$

Integral Definition of an Impulse

$$\int_{-\infty}^{\infty} e^{-j2\pi xy} dy = \delta(x)$$

Duality

$$\mathrm{X}(t) \xleftrightarrow{\mathcal{F}} \mathrm{x}(-f) \quad \text{and} \quad \mathrm{X}(-t) \xleftrightarrow{\mathcal{F}} \mathrm{x}(f)$$

$$\mathrm{X}(jt) \xleftrightarrow{\mathcal{F}} 2\pi\,\mathrm{x}(-\omega) \quad \text{and} \quad \mathrm{X}(-jt) \xleftrightarrow{\mathcal{F}} 2\pi\,\mathrm{x}(\omega)$$

CTFT Properties

Total-Area Integral

$$\mathrm{X}(0) = \left[\int_{-\infty}^{\infty} \mathrm{x}(t) e^{-j2\pi ft} dt \right]_{f \to 0} = \int_{-\infty}^{\infty} \mathrm{x}(t) dt$$

$$\mathrm{x}(0) = \left[\int_{-\infty}^{\infty} \mathrm{X}(f) e^{+j2\pi ft} df \right]_{t \to 0} = \int_{-\infty}^{\infty} \mathrm{X}(f) df$$

$$\mathrm{X}(0) = \left[\int_{-\infty}^{\infty} \mathrm{x}(t) e^{-j\omega t} dt \right]_{\omega \to 0} = \int_{-\infty}^{\infty} \mathrm{x}(t) dt$$

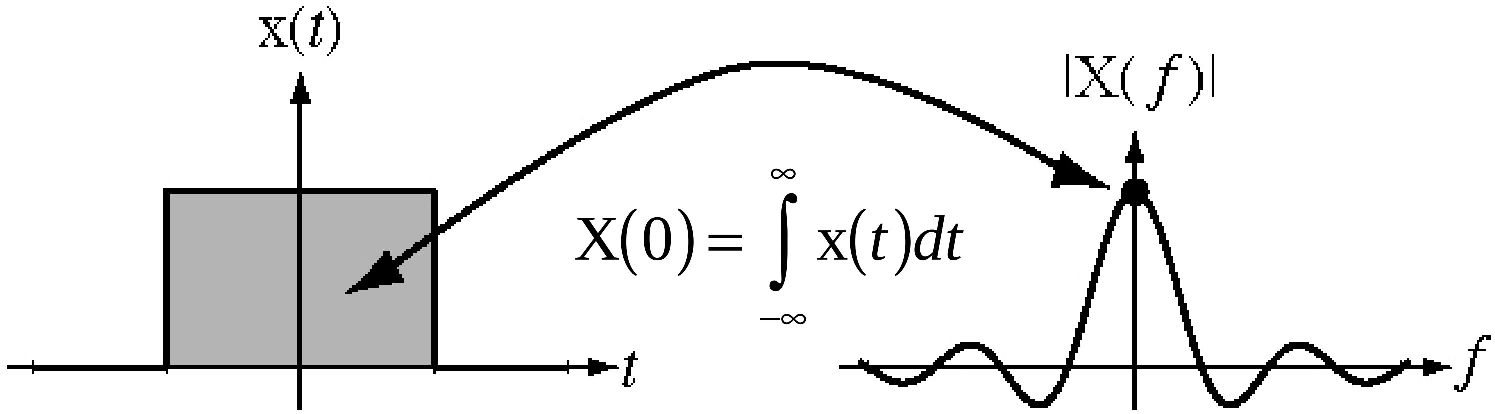
$$\mathrm{x}(0) = \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{X}(j\omega) e^{+j\omega t} d\omega \right]_{t \to 0} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{X}(j\omega) d\omega$$

Integration

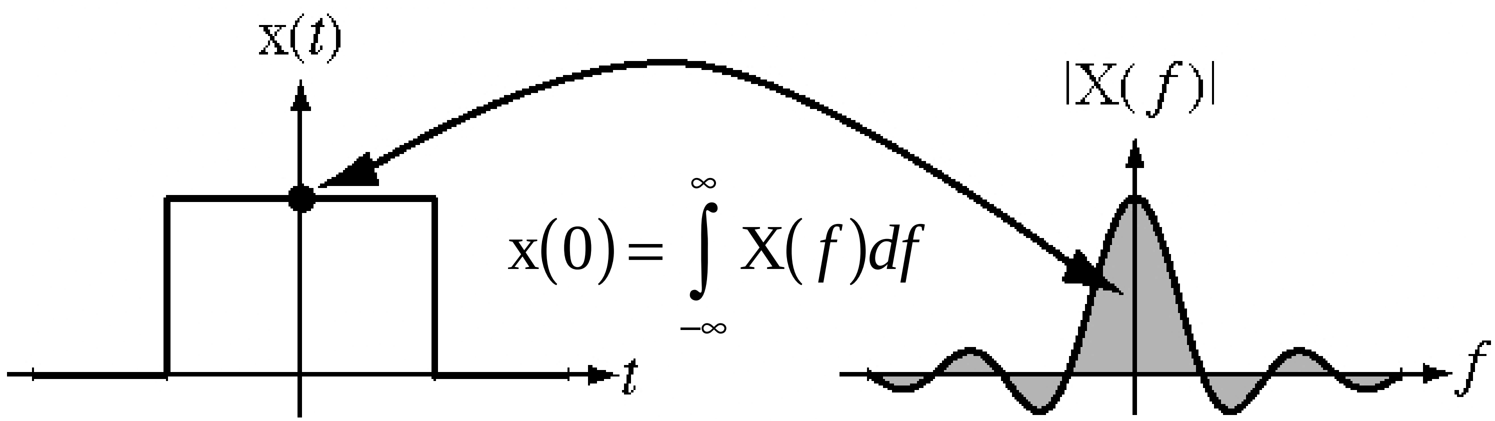
$$\int_{-\infty}^{t} \mathrm{x}(\lambda) d\lambda \xleftrightarrow{\mathcal{F}} \frac{\mathrm{X}(f)}{j2\pi f} + \frac{1}{2} \mathrm{X}(0) \delta(f)$$

$$\int_{-\infty}^{t} \mathrm{x}(\lambda) d\lambda \xleftrightarrow{\mathcal{F}} \frac{\mathrm{X}(j\omega)}{j\omega} + \pi \mathrm{X}(0) \delta(\omega)$$

CTFT Properties

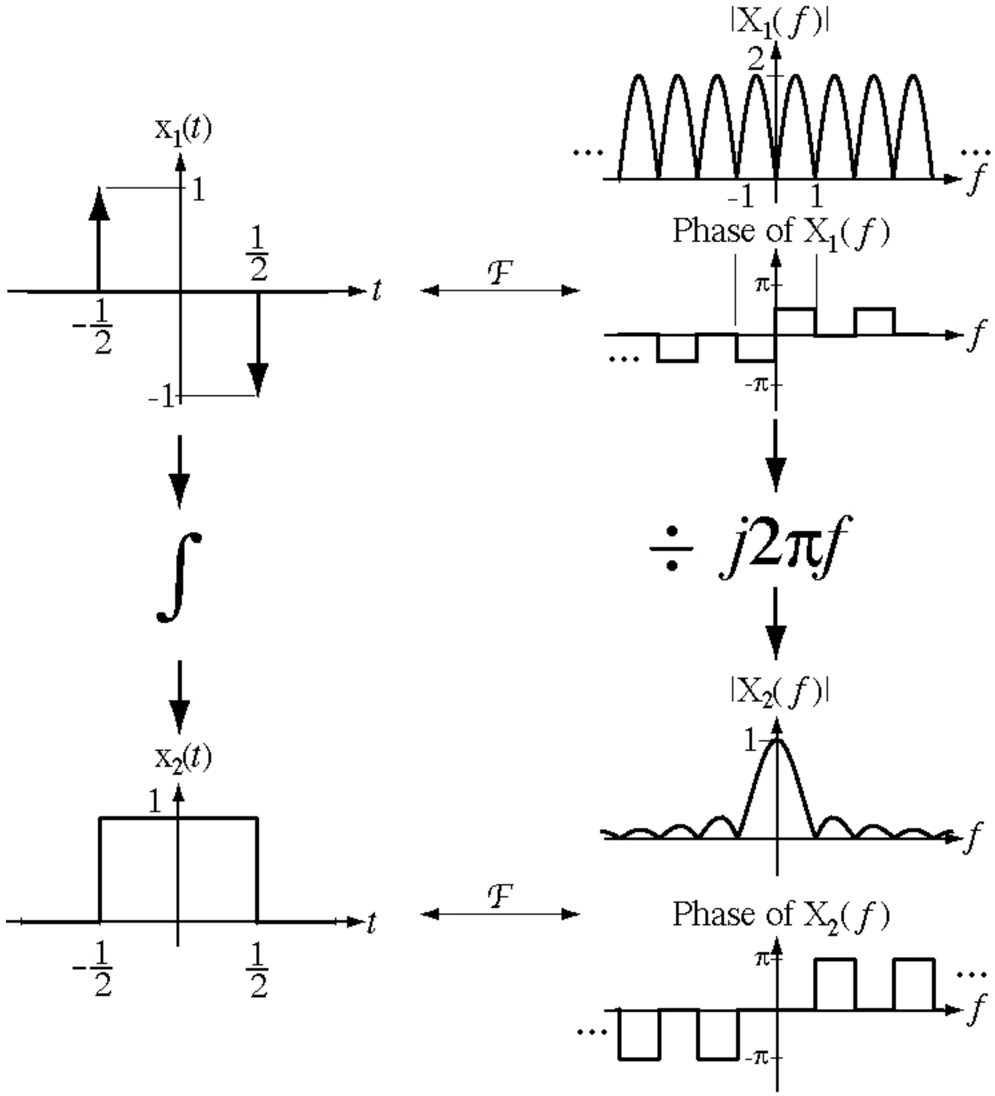


$$X(0) = \int_{-\infty}^{\infty} x(t)\,dt$$



$$x(0) = \int_{-\infty}^{\infty} X(f)\,df$$

CTFT Properties

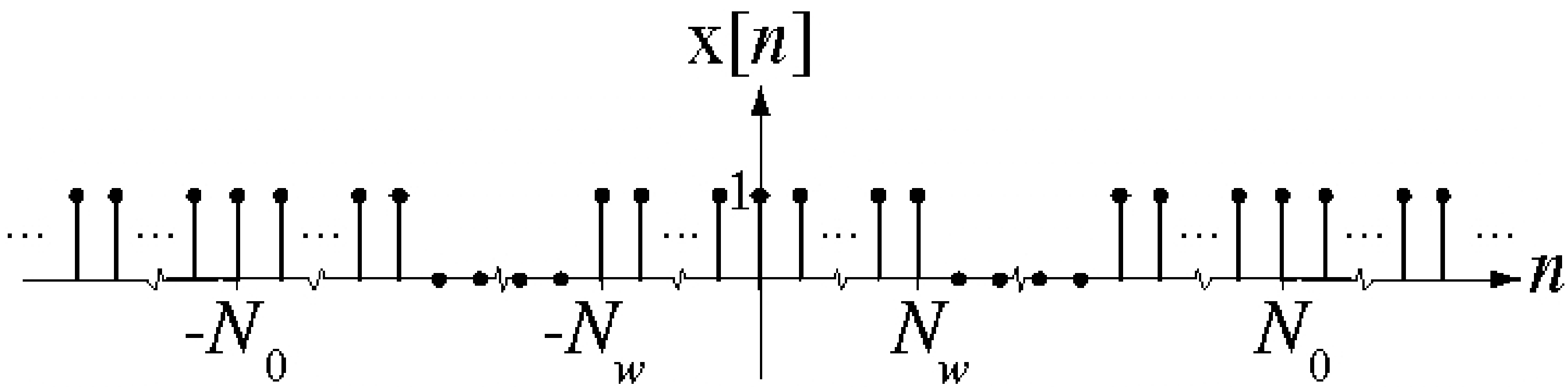


Extending the DTFS

- Analogous to the CTFS, the DTFS is a good analysis tool for systems with periodic excitation but cannot represent an aperiodic DT signal for all time
- The discrete-time Fourier transform (DTFT) can represent an aperiodic DT signal for all time

DTFS-to-DTFT Transition

DT Pulse Train

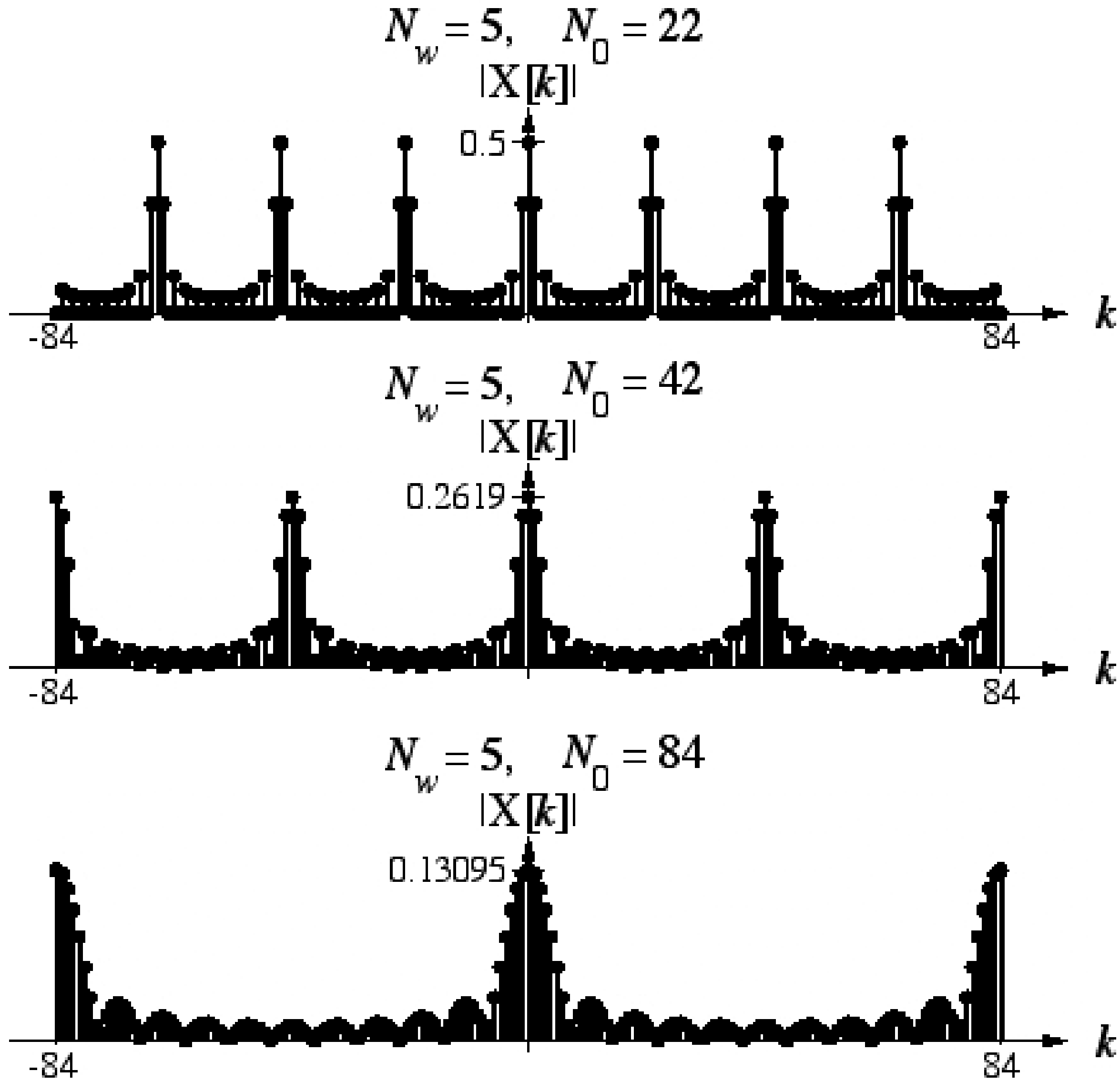


This DT periodic rectangular-wave signal is analogous to the CT periodic rectangular-wave signal used to illustrate the transition from the CTFS to the CTFT.

DTFS-to-DTFT Transition

DTFS of DT Pulse Train

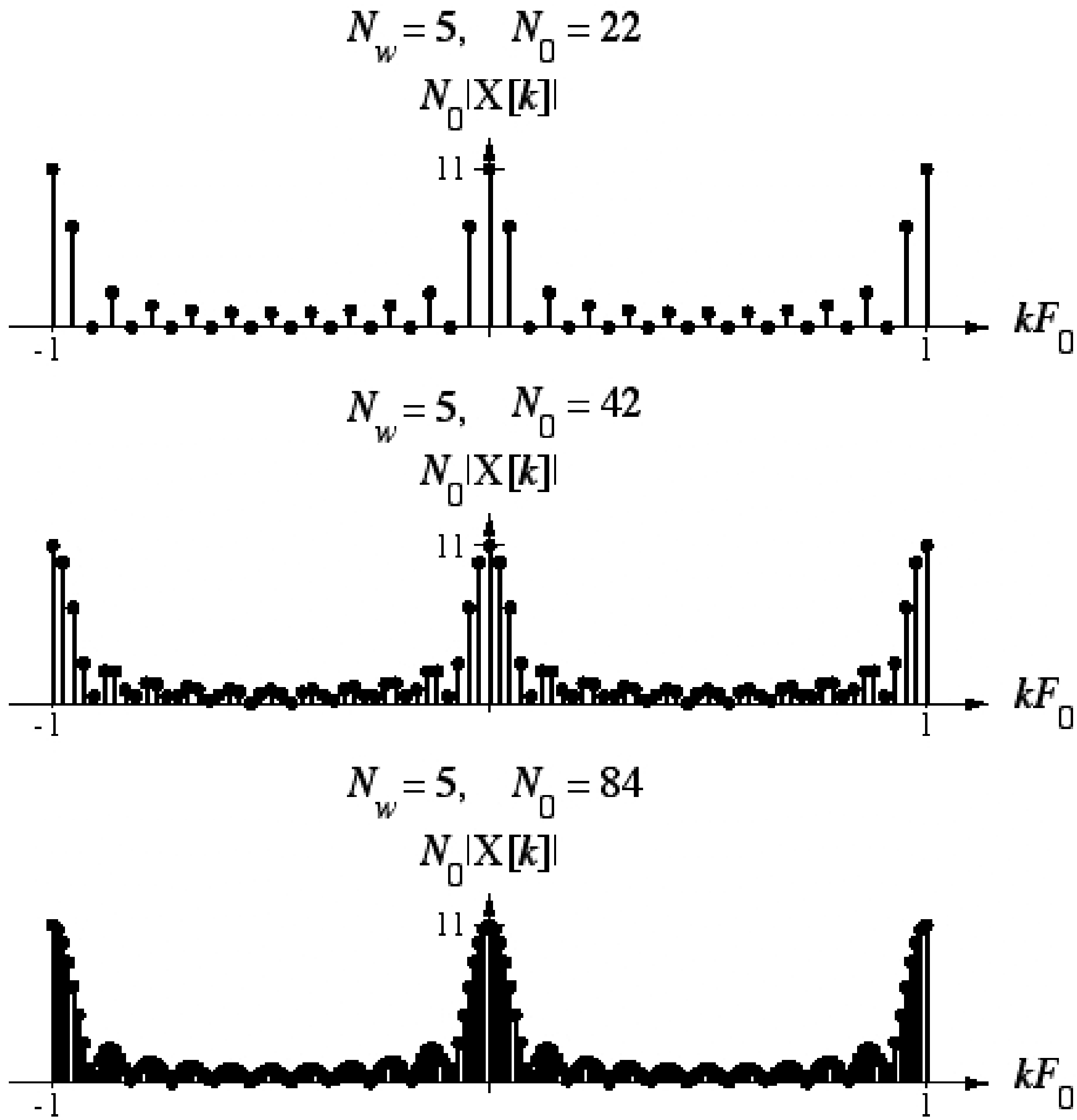
As the period of the rectangular wave increases, the period of the DTFS increases and the amplitude of the DTFS decreases.



DTFS-to-DTFT Transition

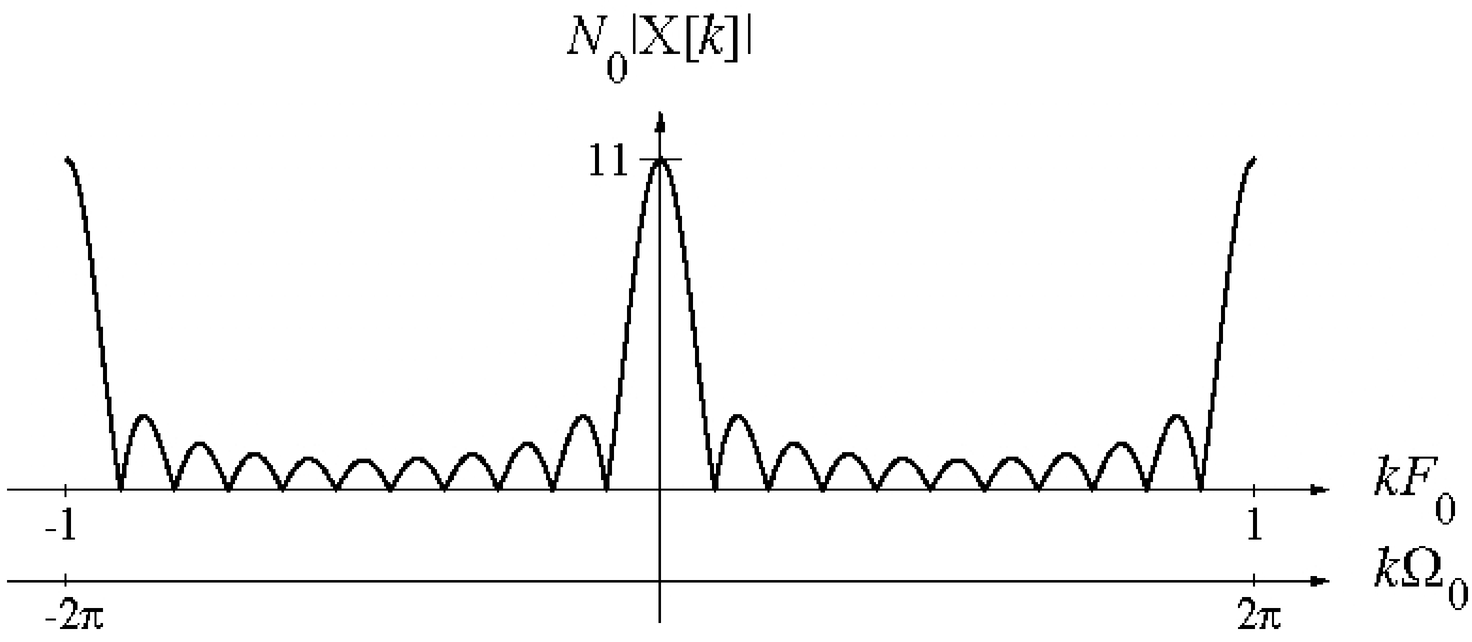
Normalized DTFS of DT Pulse Train

By multiplying the DTFS by its period and plotting versus kF_0 instead of k, the amplitude of the DTFS stays the same as the period increases and the period of the normalized DTFS stays at one.



DTFS-to-DTFT Transition

The normalized DTFS approaches this limit as the DT period approaches infinity.



Definition of the DTFT

F Form

Inverse | Forward

$$\mathrm{x}[n]=\int_1 \mathrm{X}(F)e^{j2\pi Fn}dF \xleftrightarrow{\mathcal{F}} \mathrm{X}(F)=\sum_{n=-\infty}^{\infty}\mathrm{x}[n]e^{-j2\pi Fn}$$

Ω Form

Inverse | Forward

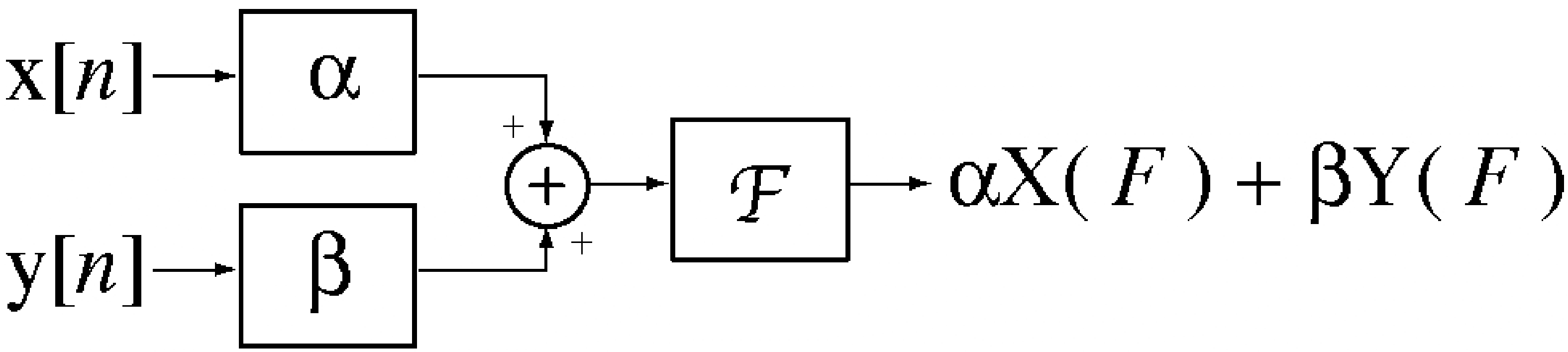
$$\mathrm{x}[n]=\frac{1}{2\pi}\int_{2\pi} \mathrm{X}(j\Omega)e^{j\Omega n}d\Omega \xleftrightarrow{\mathcal{F}} \mathrm{X}(j\Omega)=\sum_{n=-\infty}^{\infty}\mathrm{x}[n]e^{-j\Omega n}$$

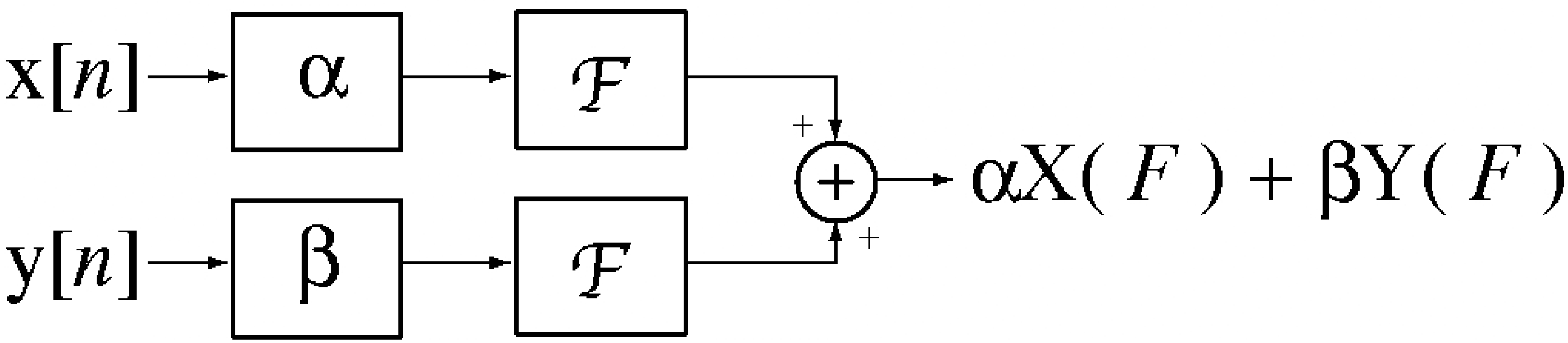
DTFT Properties

Linearity

$$\alpha\,\mathrm{x}[n]+\beta\,\mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \alpha\,\mathrm{X}(F)+\beta\,\mathrm{Y}(F)$$

$$\alpha\,\mathrm{x}[n]+\beta\,\mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \alpha\,\mathrm{X}(j\Omega)+\beta\,\mathrm{Y}(j\Omega)$$



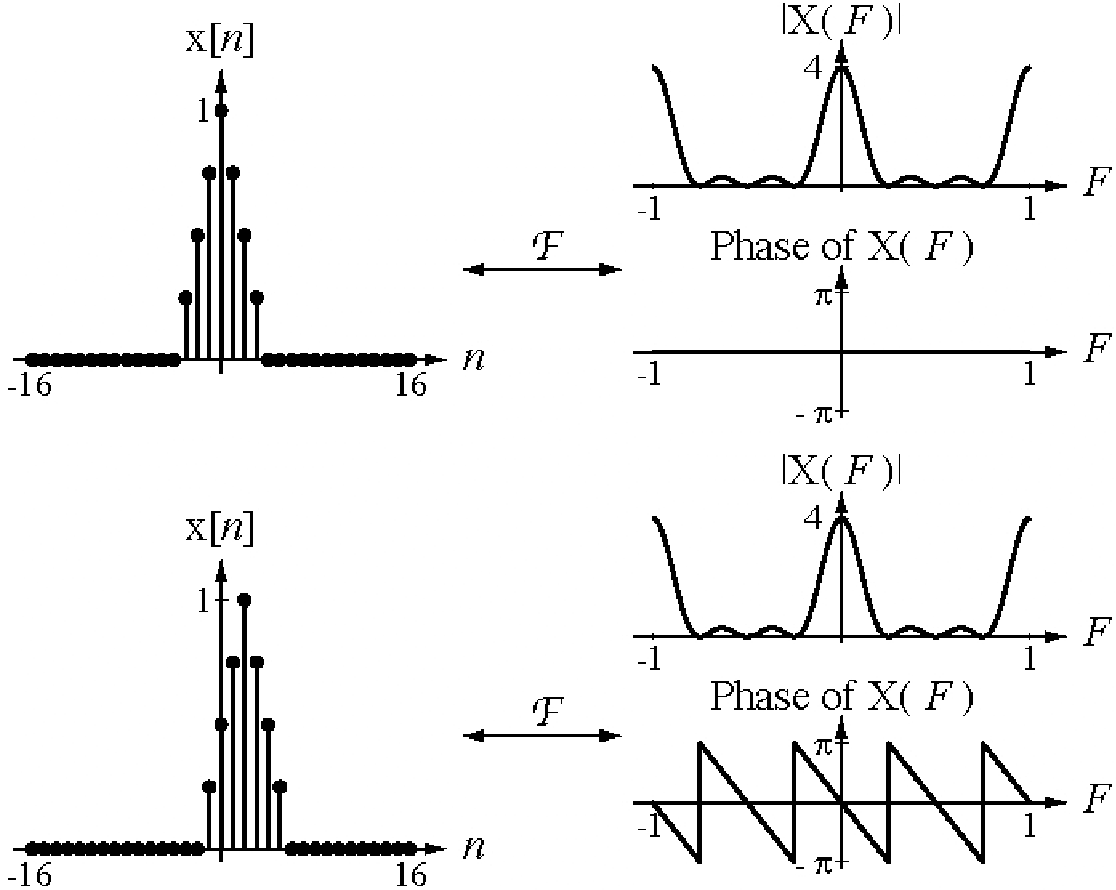


DTFT Properties

Time Shifting

$$\mathrm{x}[n-n_0] \xleftrightarrow{\mathcal{F}} e^{-j2\pi F n_0}\,\mathrm{X}(F)$$

$$\mathrm{x}[n-n_0] \xleftrightarrow{\mathcal{F}} e^{-j\Omega n_0}\,\mathrm{X}(j\Omega)$$



DTFT Properties

Frequency Shifting

$$e^{j2\pi F_0 n}\,\mathrm{x}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(F - F_0)$$

$$e^{j\Omega_0 n}\,\mathrm{x}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(j(\Omega - \Omega_0))$$

Time Reversal

$$\mathrm{x}[-n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(-F)$$

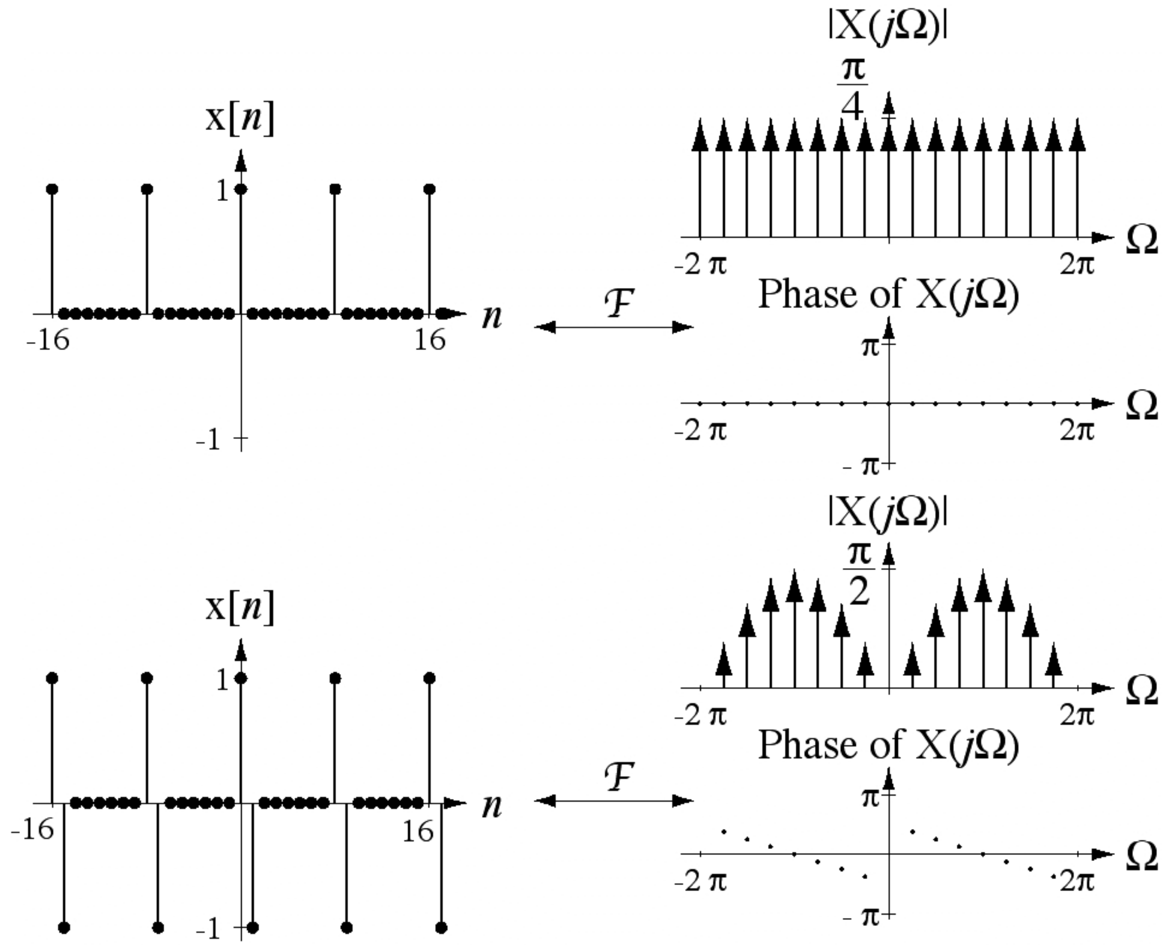
$$\mathrm{x}[-n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(-j\Omega)$$

DTFT Properties

Differencing

$$\mathrm{x}[n]-\mathrm{x}[n-1] \xleftrightarrow{\mathcal{F}} \left(1-e^{-j2\pi F}\right)\mathrm{X}(F)$$

$$\mathrm{x}[n]-\mathrm{x}[n-1] \xleftrightarrow{\mathcal{F}} \left(1-e^{-j\Omega}\right)\mathrm{X}(j\Omega)$$

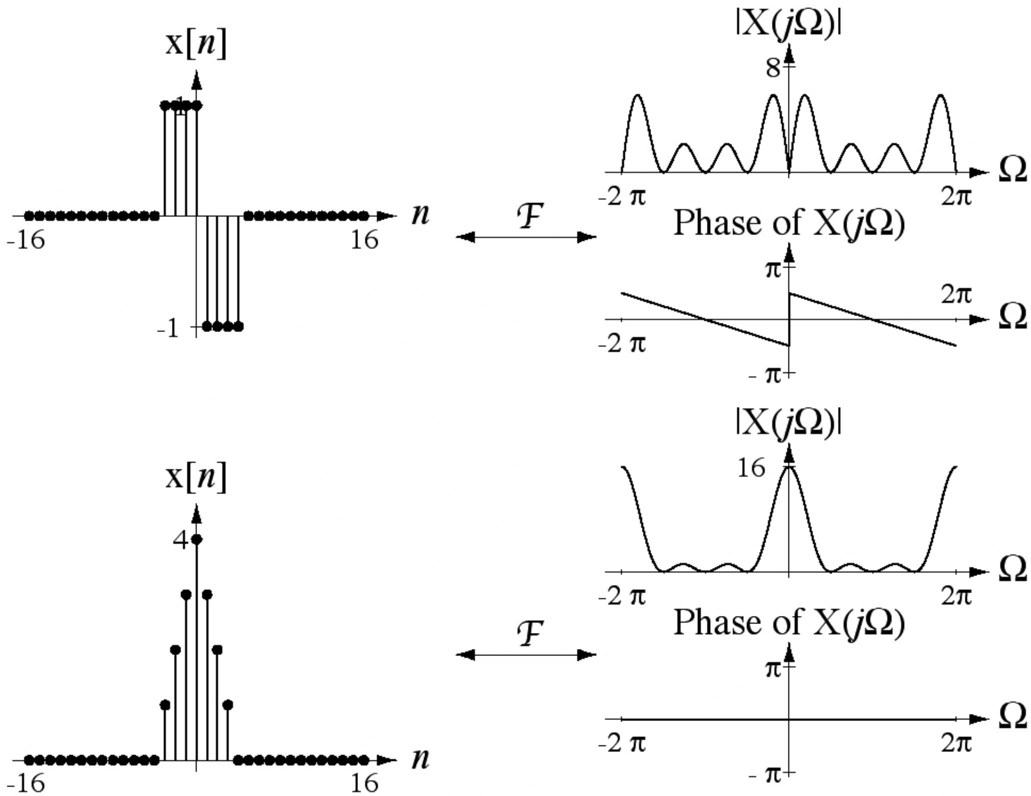


DTFT Properties

Accumulation

$$\sum_{m=-\infty}^{n} \mathrm{x}[m] \xleftrightarrow{\mathcal{F}} \frac{\mathrm{X}(F)}{1-e^{-j2\pi F}} + \frac{1}{2}\mathrm{X}(0)\,\mathrm{comb}(F)$$

$$\sum_{m=-\infty}^{n} \mathrm{x}[m] \xleftrightarrow{\mathcal{F}} \frac{\mathrm{X}(j\Omega)}{1-e^{-j\Omega}} + \frac{1}{2}\mathrm{X}(0)\,\mathrm{comb}\left(\frac{\Omega}{2\pi}\right)$$



DTFT Properties

Multiplication-Convolution Duality

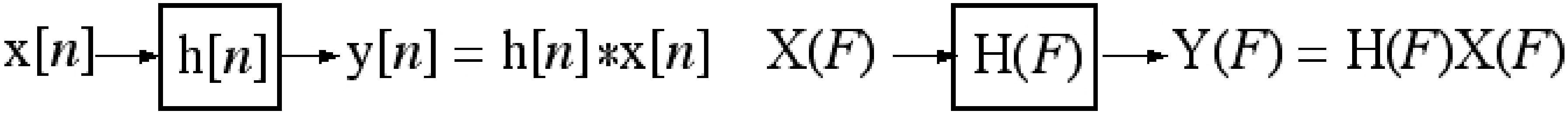
$$\mathrm{x}[n] * \mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(F)\,\mathrm{Y}(F)$$

$$\mathrm{x}[n] * \mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(j\Omega)\,\mathrm{Y}(j\Omega)$$

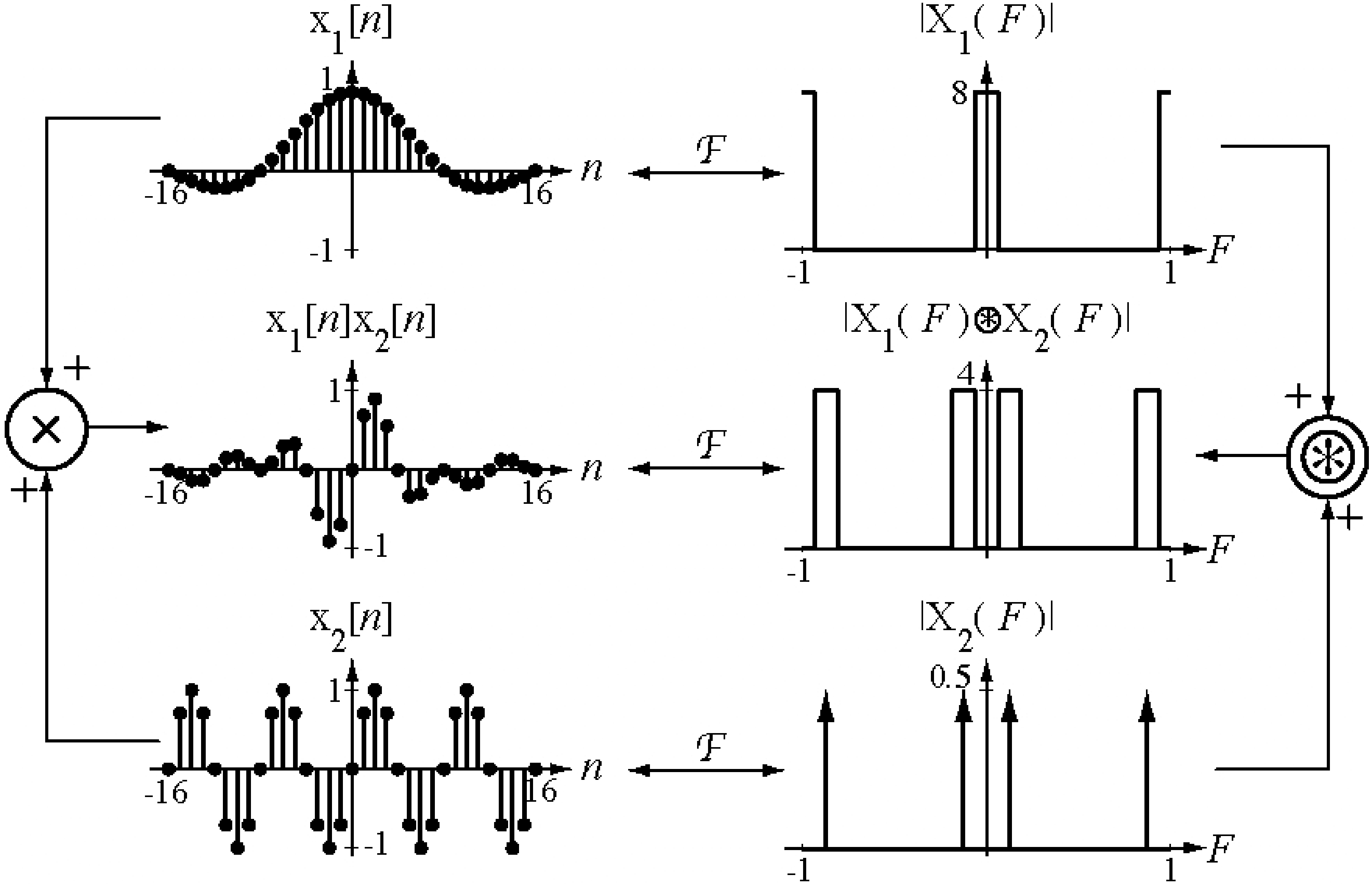
$$\mathrm{x}[n]\,\mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \mathrm{X}(F) \circledast \mathrm{Y}(F)$$

$$\mathrm{x}[n]\,\mathrm{y}[n] \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi}\,\mathrm{X}(j\Omega) \circledast \mathrm{Y}(j\Omega)$$

As is true for other transforms, convolution in the time domain is equivalent to multiplication in the frequency domain



DTFT Properties



DTFT Properties

Accumulation Definition of a Comb Function

$$\sum_{n=-\infty}^{\infty} e^{j2\pi Fn} = \mathrm{comb}(F)$$

Parseval's Theorem

$$\sum_{n=-\infty}^{\infty} \left|\mathrm{x}[n]\right|^2 = \int_{1} \left|\mathrm{X}(F)\right|^2 dF$$

$$\sum_{n=-\infty}^{\infty} \left|\mathrm{x}[n]\right|^2 = \frac{1}{2\pi}\int_{2\pi} \left|\mathrm{X}(j\Omega)\right|^2 d\Omega$$

The signal energy is proportional to the integral of the squared magnitude of the DTFT of the signal over one period.

The Four Fourier Methods

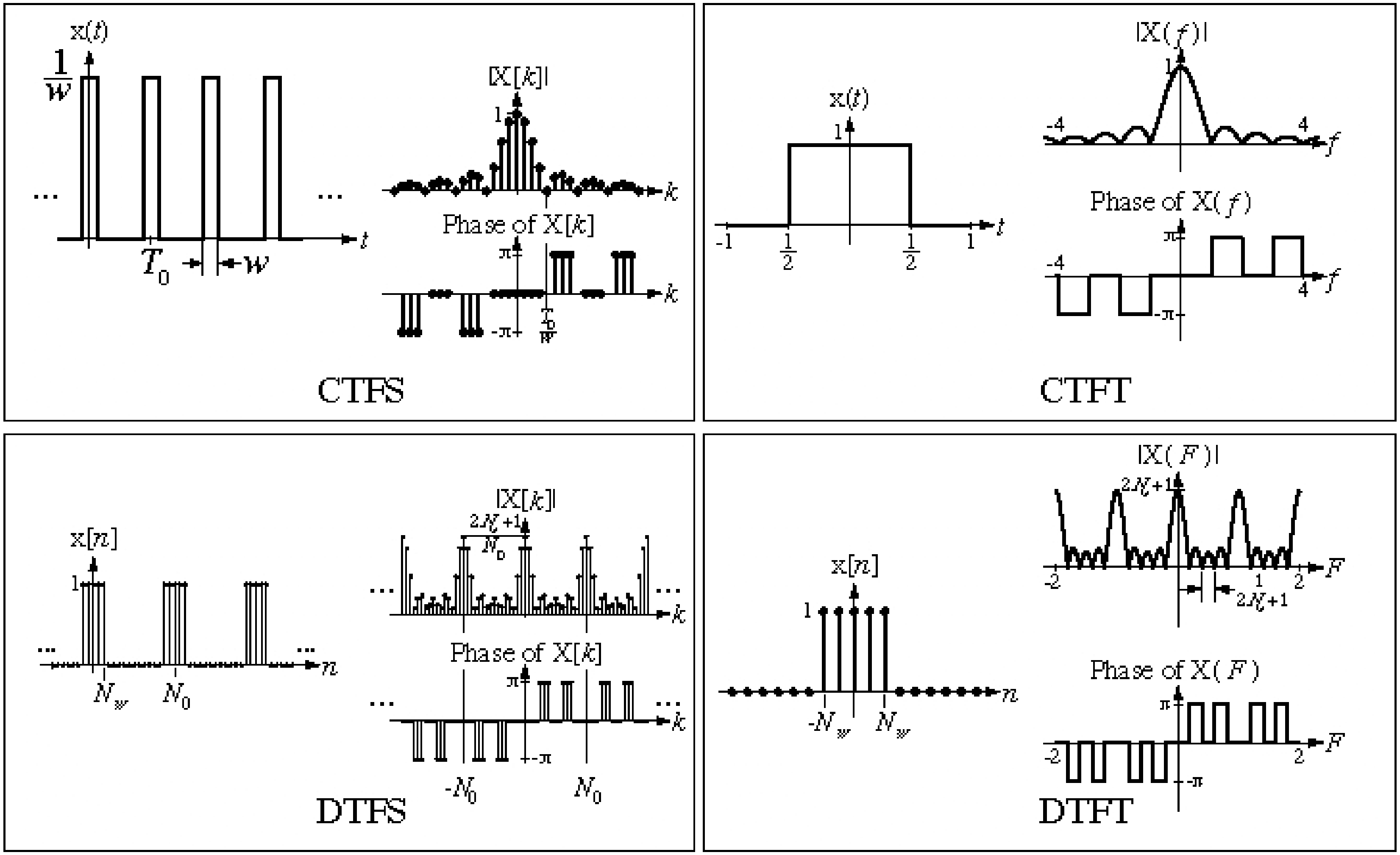
	Continuous Frequency	Discrete Frequency
Continuous Time	CTFT	CTFS
Discrete Time	DTFT	DTFS

Relations Among Fourier Methods

Discrete Frequency Continuous Frequency

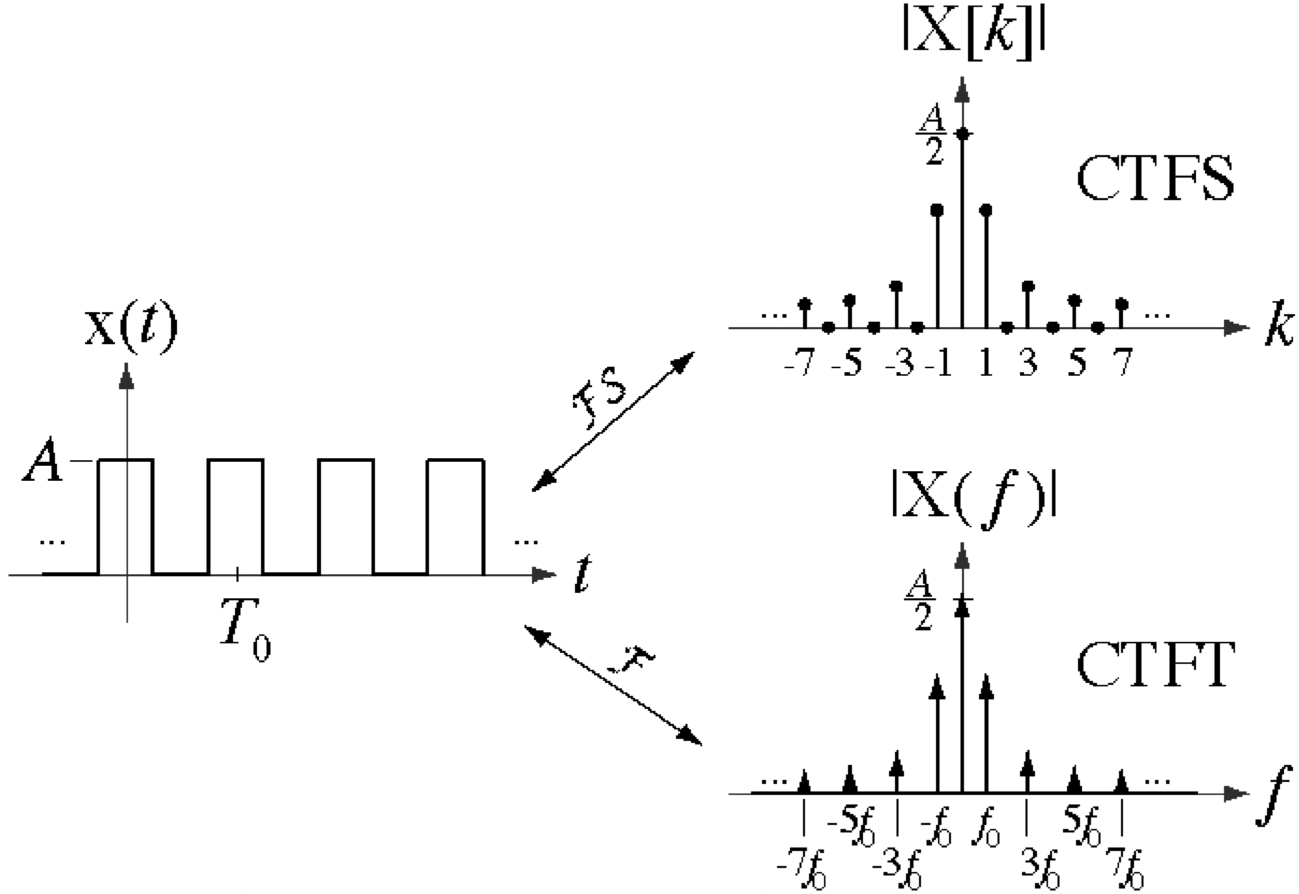
CT

DT



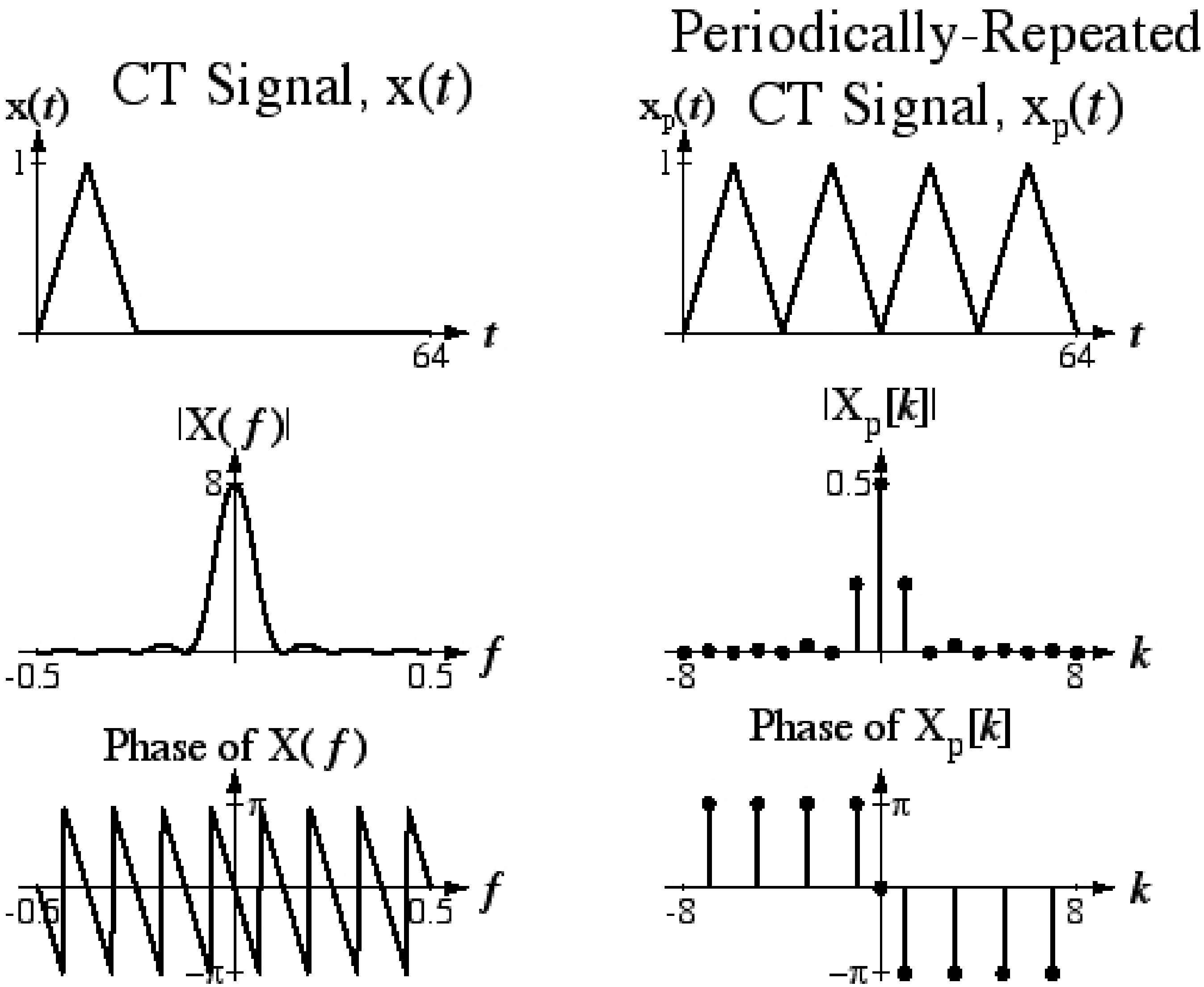
CTFT - CTFS Relationship

$$\mathrm{X}(f) = \sum_{k=-\infty}^{\infty} \mathrm{X}[k]\delta(f - kf_0)$$



CTFT - CTFS Relationship

$$X_p[k] = f_p \, X(k f_p)$$



CTFT - DTFT Relationship

Let $\mathrm{x}_\delta(t) = \mathrm{x}(t)\dfrac{1}{T_s}\mathrm{comb}\left(\dfrac{t}{T_s}\right) = \sum_{n=-\infty}^{\infty} \mathrm{x}(nT_s)\delta(t - nT_s)$

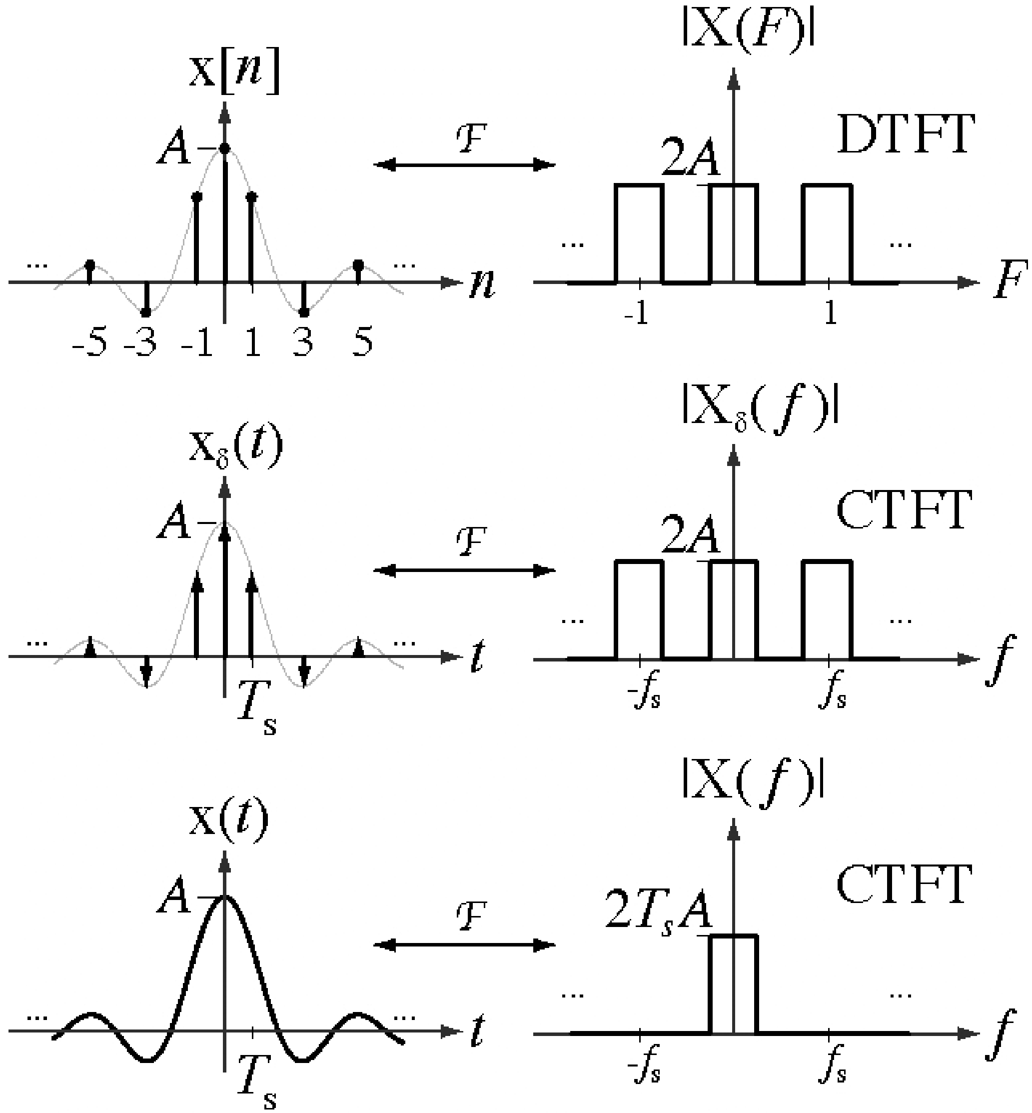
and let $\mathrm{x}[n] = \mathrm{x}(nT_s)$

There is an "information equivalence" between $\mathrm{x}_\delta(t)$ and $\mathrm{x}[n]$. They are both completely described by the same set of numbers.

$$\mathrm{X}_{DTFT}(F) = \mathrm{X}_\delta(f_s F) \qquad \mathrm{X}_\delta(f) = \mathrm{X}_{DTFT}\left(\frac{f}{f_s}\right)$$

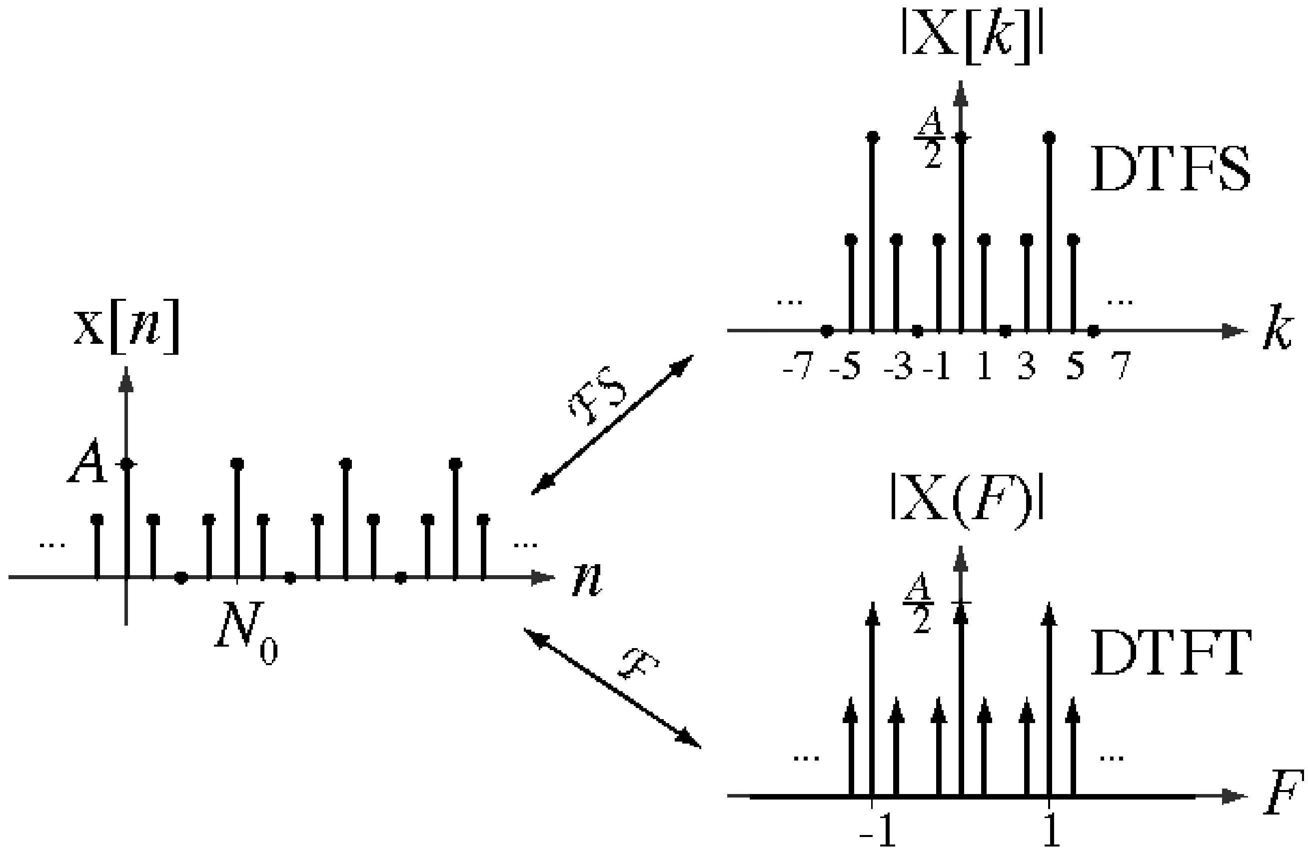
$$\mathrm{X}_{DTFT}(F) = f_s \sum_{k=-\infty}^{\infty} \mathrm{X}_{CTFT}(f_s(F - k))$$

CTFT - DTFT Relationship



DTFS - DTFT Relationship

$$\mathrm{X}(F) = \sum_{k=-\infty}^{\infty} \mathrm{X}[k]\delta(F - kF_0)$$



DTFS - DTFT Relationship

$$X_p[k] = \frac{1}{N_p} X(kF_p)$$