Fourier Transform Analysis of Signals and Systems

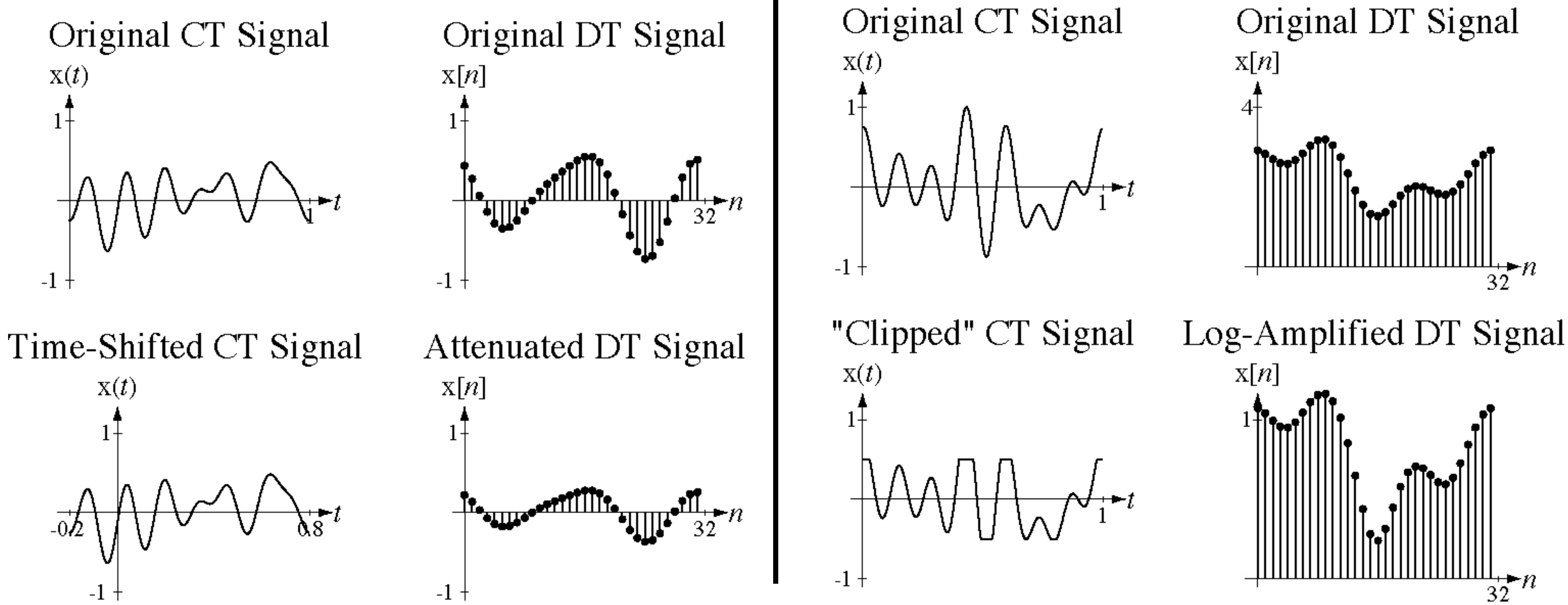
Ideal Filters

- Filters separate what is desired from what is not desired
- In the signals and systems context a filter separates signals in one frequency range from signals in another frequency range
- An *ideal filter* passes all signal power in its *passband* without distortion and completely blocks signal power outside its passband

Distortion

- *Distortion* is construed in signal analysis to mean “changing the shape” of a signal
- Multiplication of a signal by a constant (even a negative one) or shifting it in time do not change its shape

No Distortion | Distortion



Distortion

Since a system can multiply by a constant or shift in time without distortion, a distortionless system would have an impulse response of the form,

$$\mathrm{h}(t) = A\delta(t - t_0)$$

or

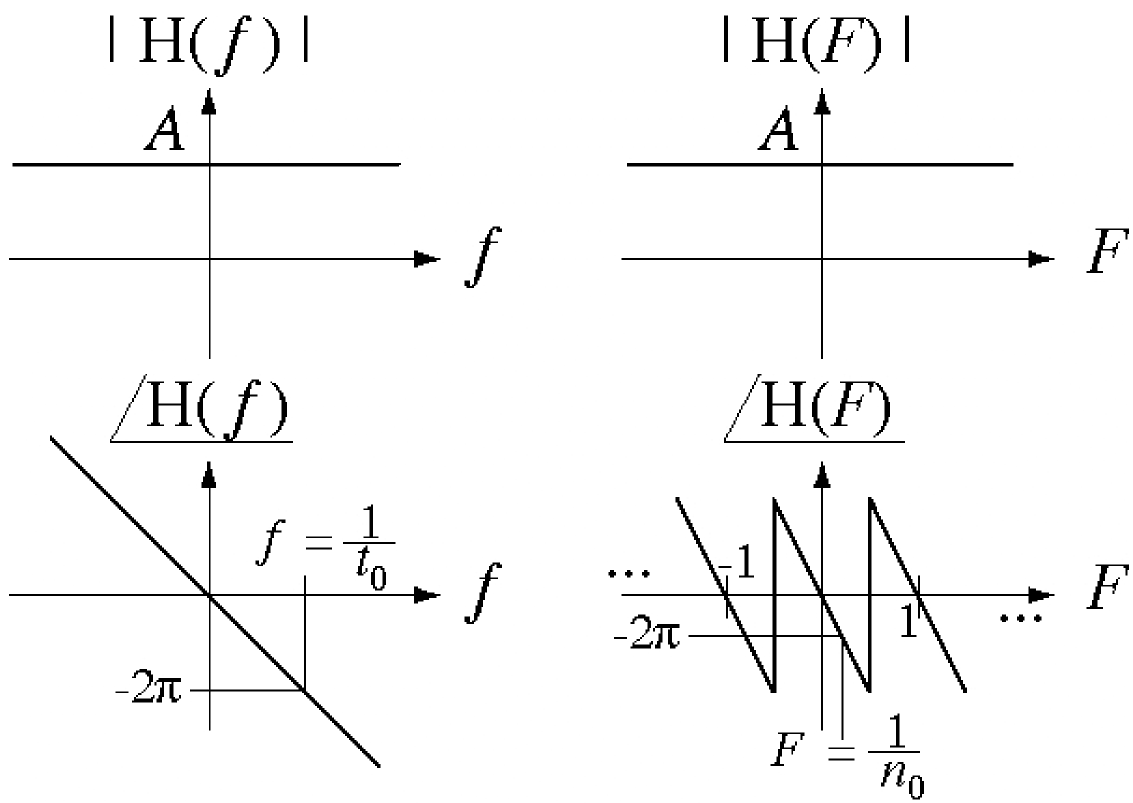
$$\mathrm{h}[n] = A\delta[n - n_0]$$

The corresponding transfer functions are

$$\mathrm{H}(f) = Ae^{-j2\pi f t_0}$$

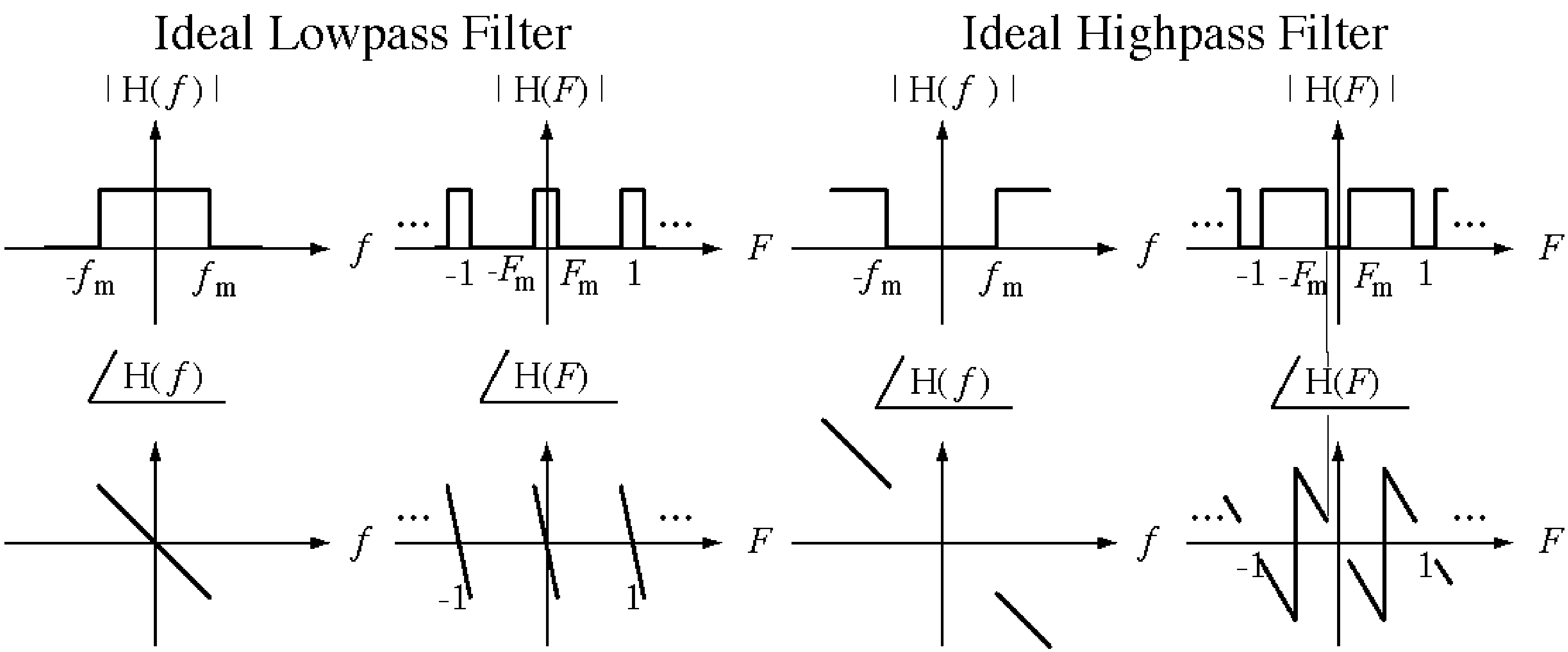
or

$$\mathrm{H}(F) = Ae^{-j2\pi F n_0}$$

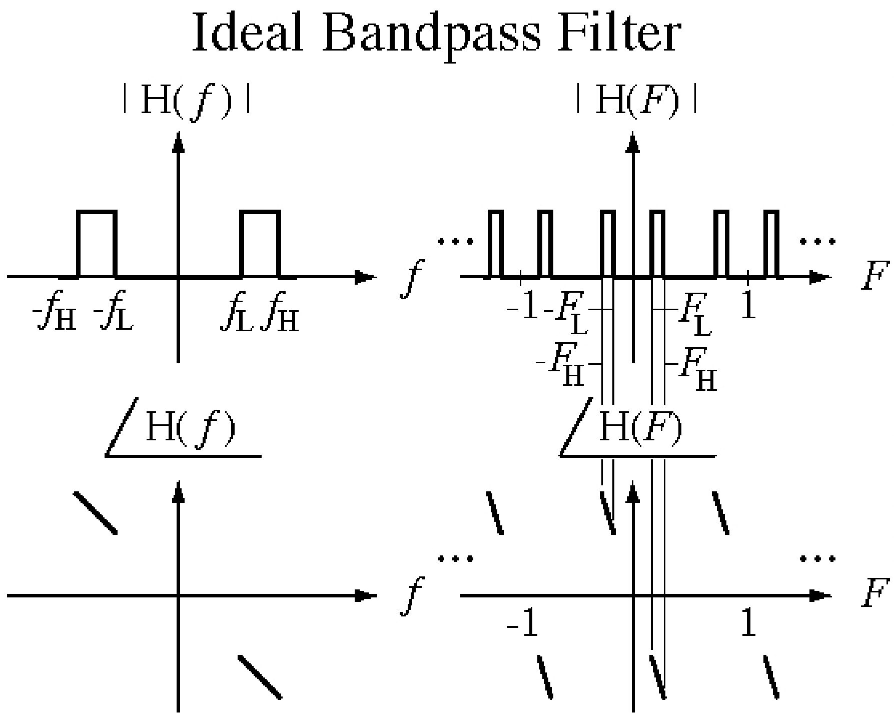


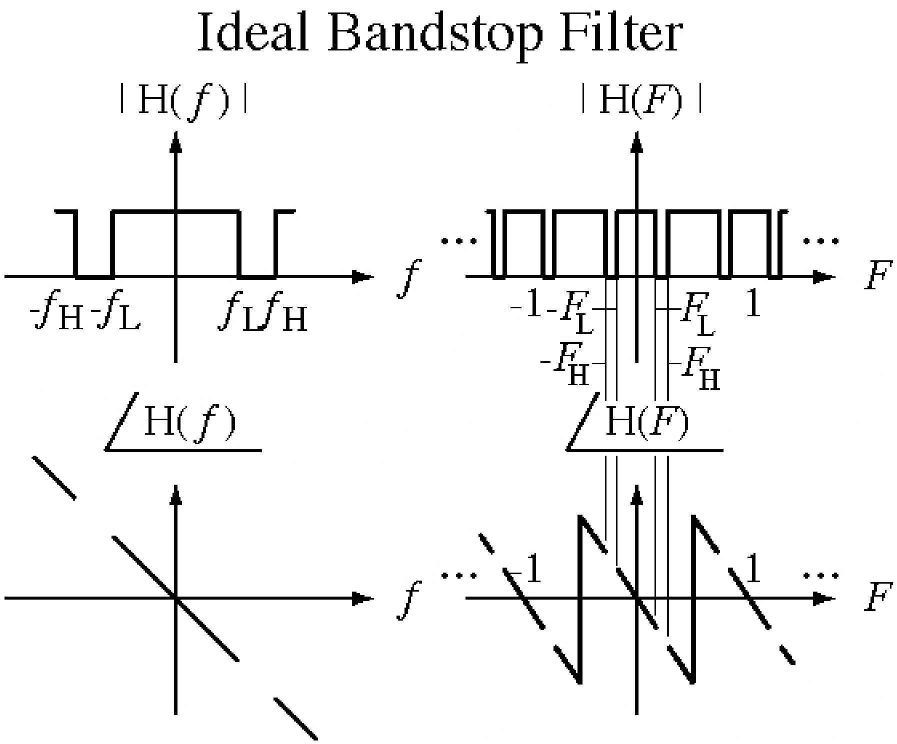
Filter Classifications

There are four commonly-used classification of filters, *lowpass*, *highpass*, *bandpass* and *bandstop*.



Filter Classifications



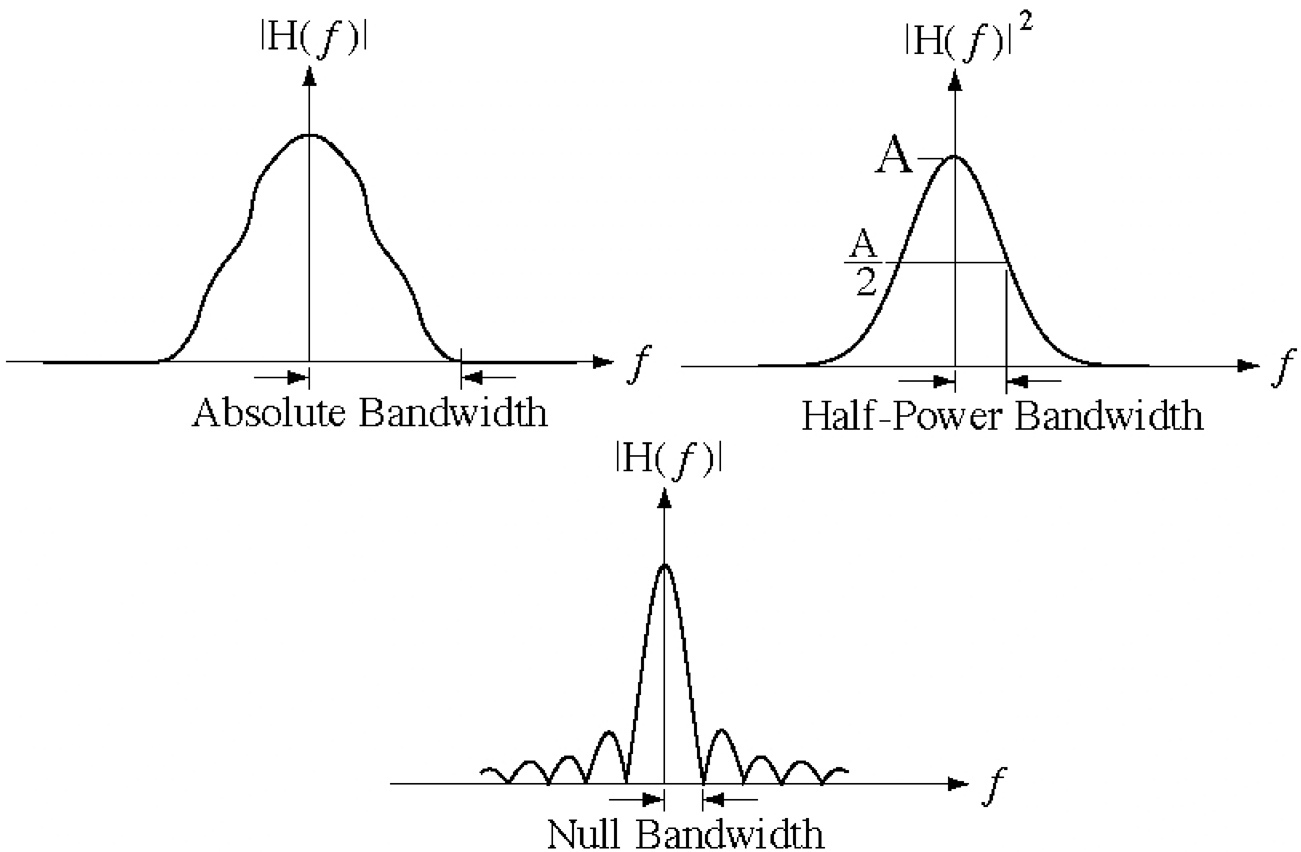


Bandwidth

- *Bandwidth* generally means “a range of frequencies”
- This range could be the range of frequencies a filter passes or the range of frequencies present in a signal
- Bandwidth is traditionally construed to be range of frequencies in *positive* frequency space

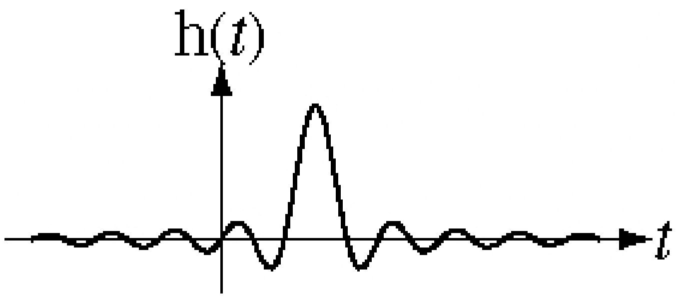
Bandwidth

Common Bandwidth Definitions

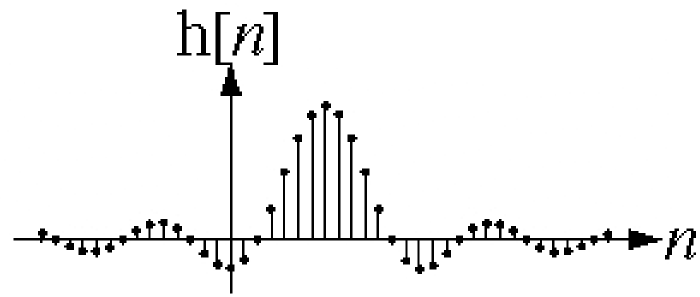


Impulse Responses of Ideal Filters

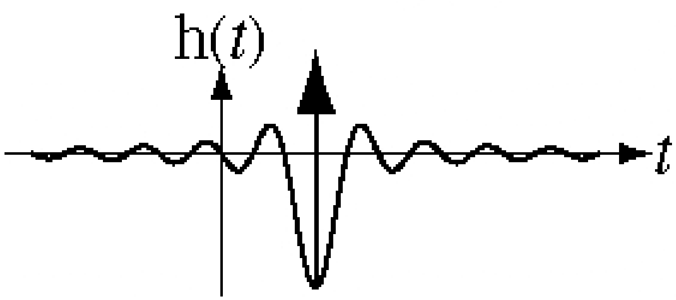
Ideal CT Lowpass



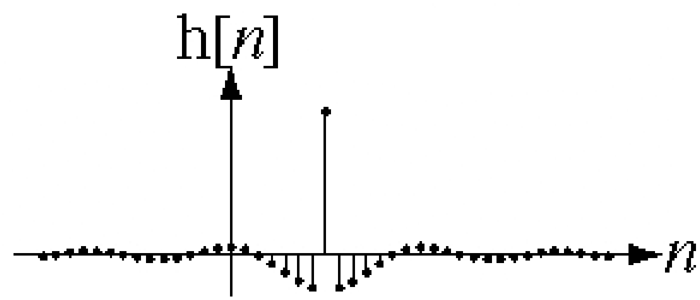
Ideal DT Lowpass



Ideal CT Highpass

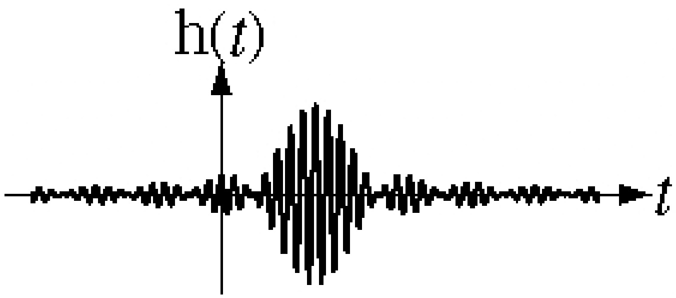


Ideal DT Highpass

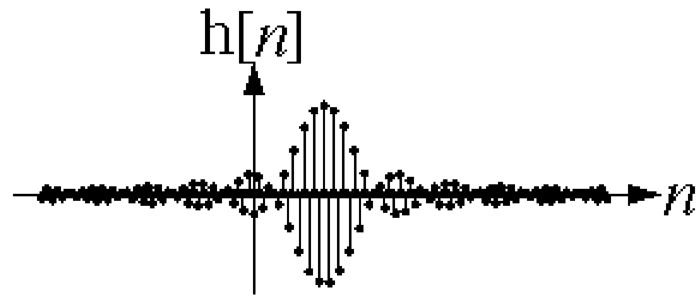


Impulse Responses of Ideal Filters

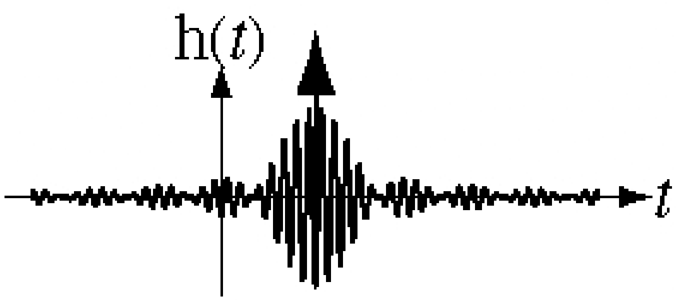
Ideal CT Bandpass



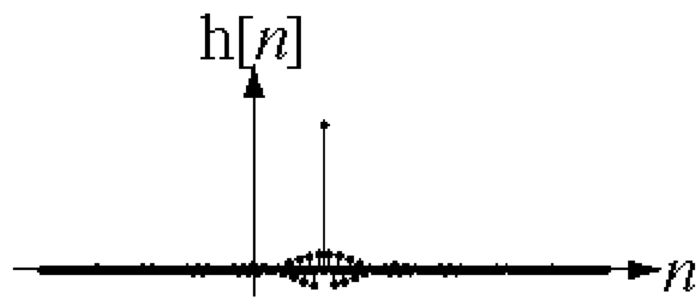
Ideal DT Bandpass



Ideal CT Bandstop



Ideal DT Bandstop

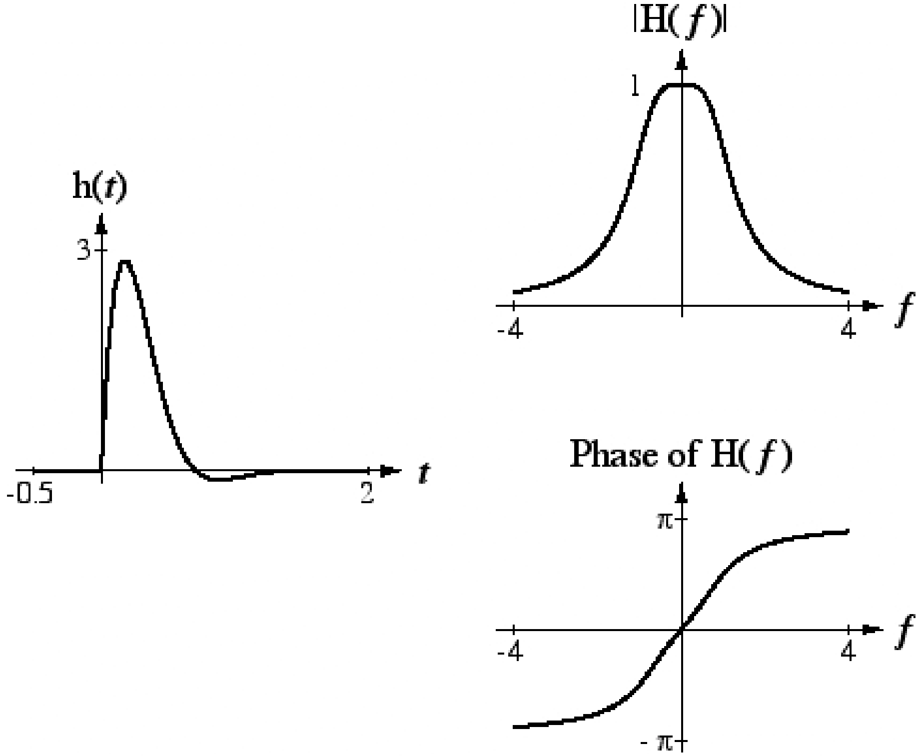


Impulse Response and Causality

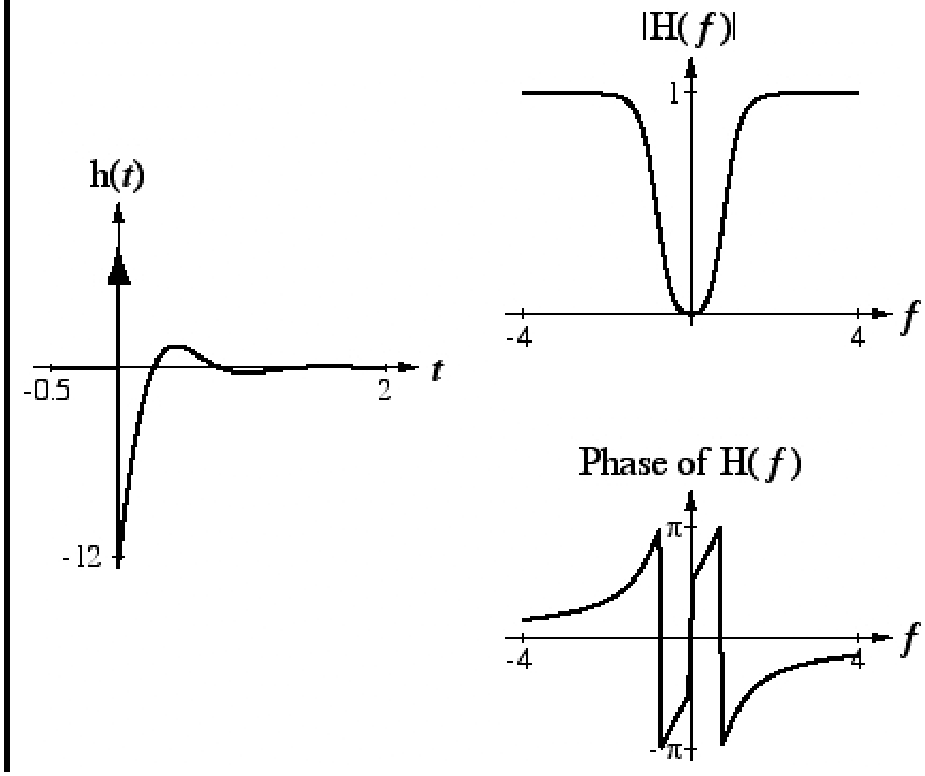
- All the impulse responses of ideal filters contain sinc functions, alone or in combinations, which are infinite in extent
- Therefore all ideal filter impulse responses begin before time, $t = 0$
- This makes ideal filters *non-causal*
- Ideal filters cannot be physically realized, but they can be closely approximated

Examples of Impulse Responses and Frequency Responses of Real Causal Filters

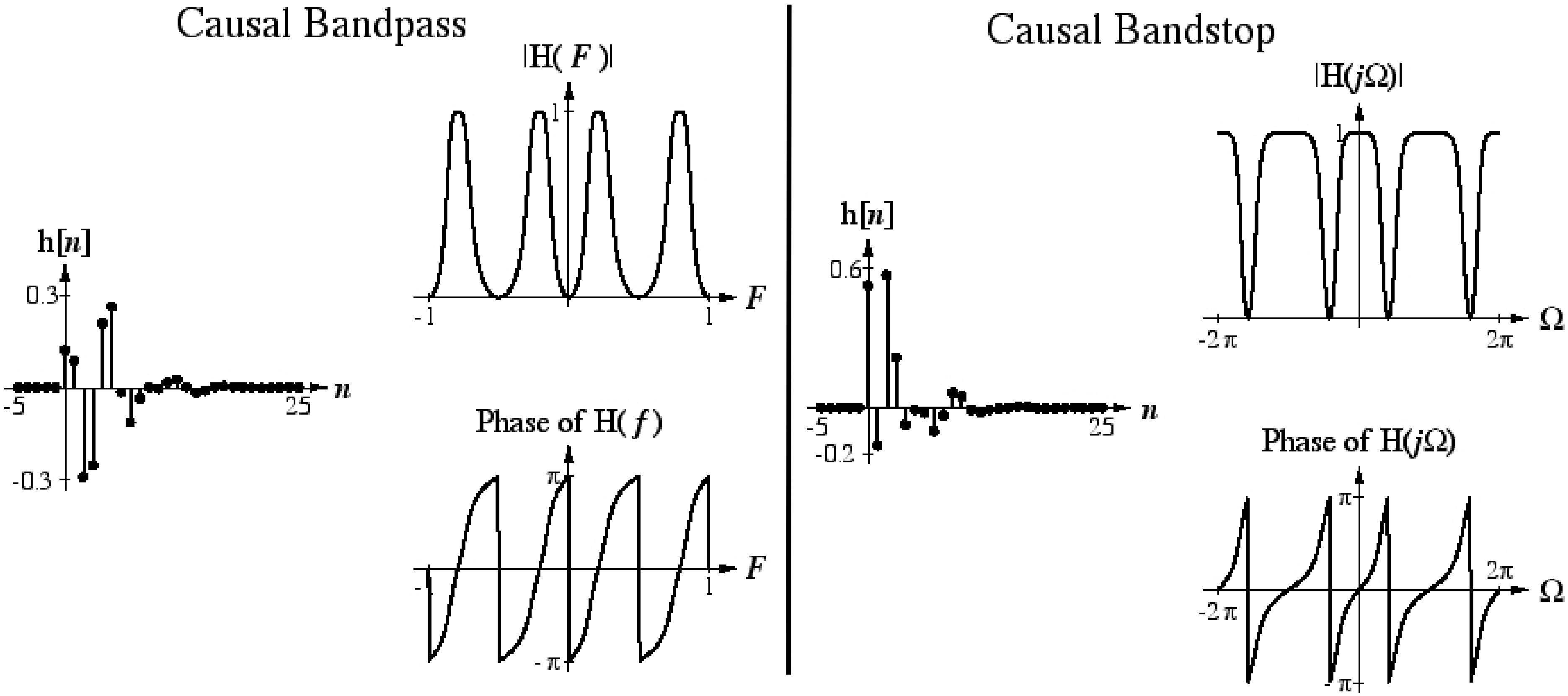
Causal Lowpass



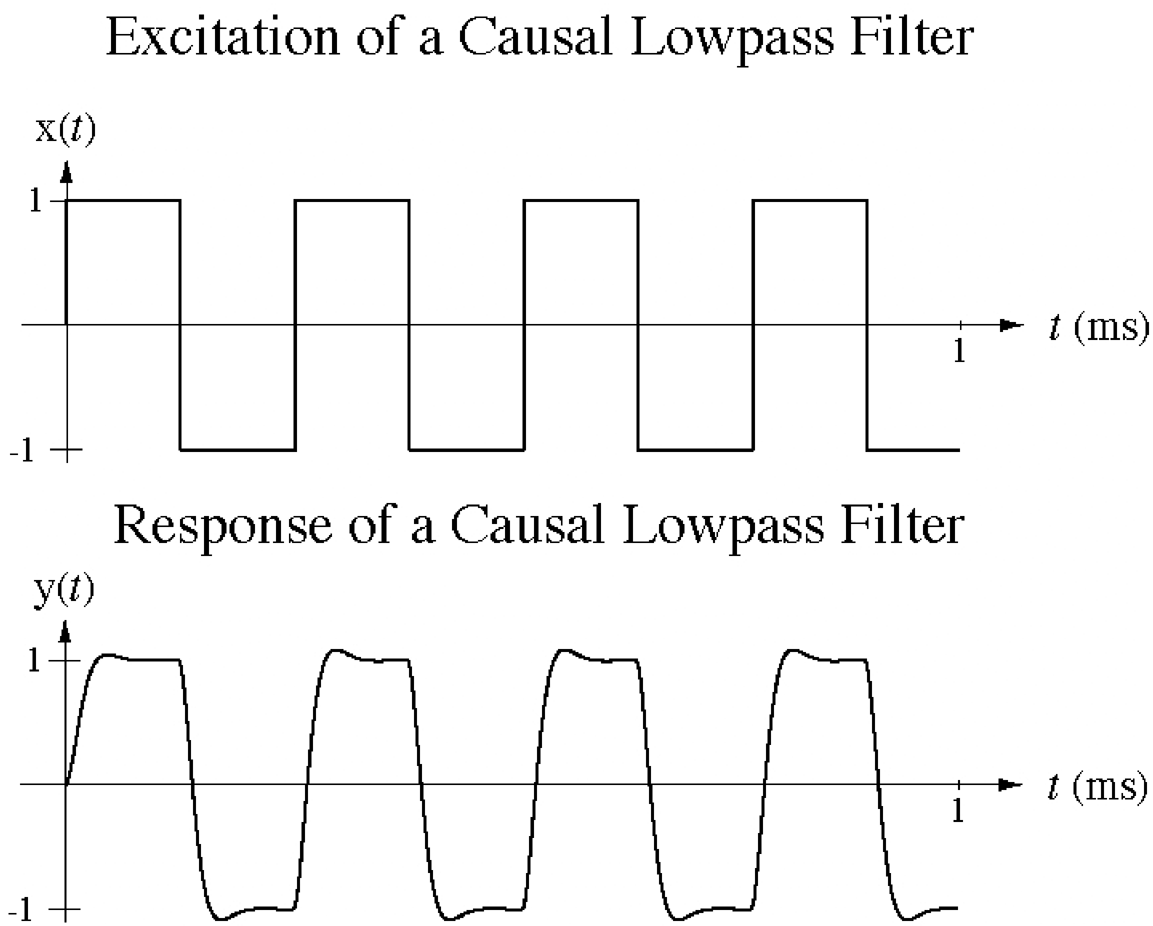
Causal Highpass



Examples of Impulse Responses and Frequency Responses of Real Causal Filters

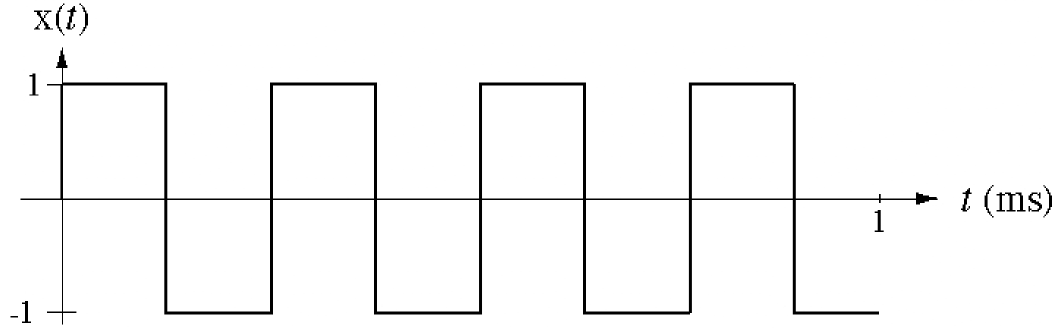


Examples of Causal Filter Effects on Signals

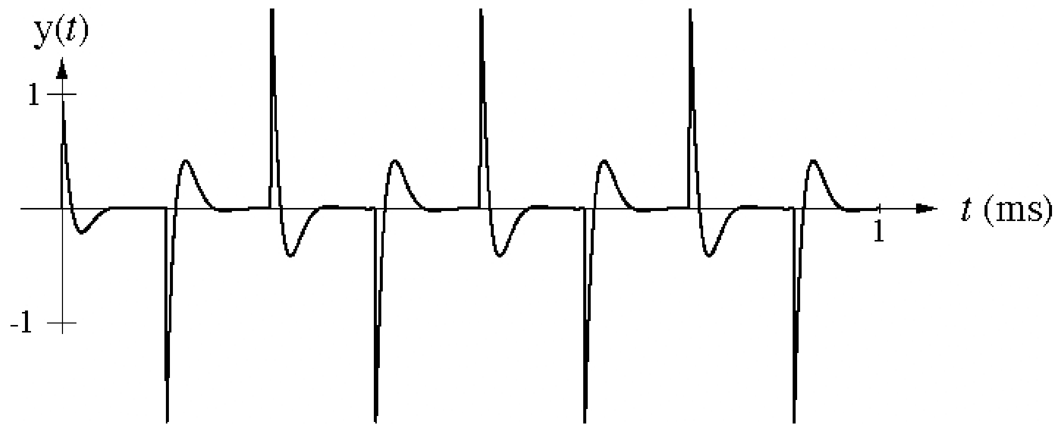


Examples of Causal Filter Effects on Signals

Excitation of a Causal Highpass Filter

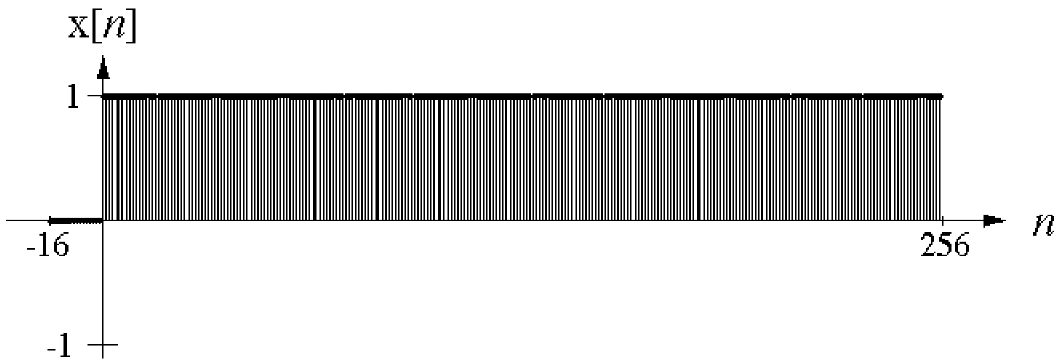


Response of a Causal Highpass Filter

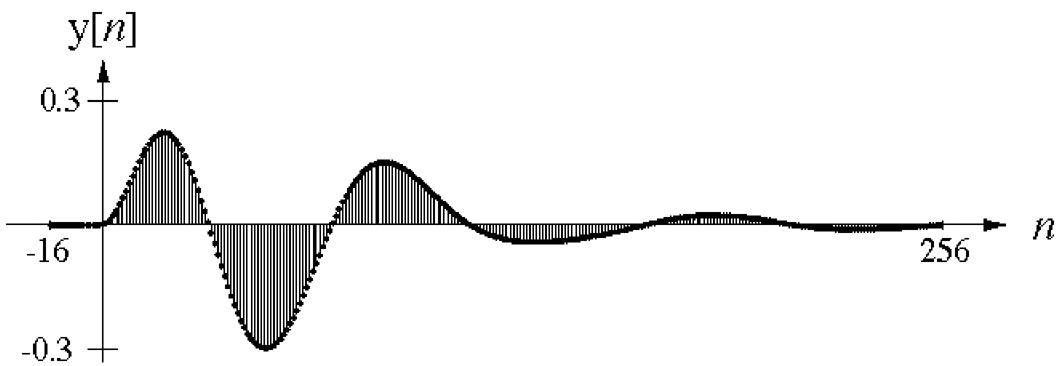


Examples of Causal Filter Effects on Signals

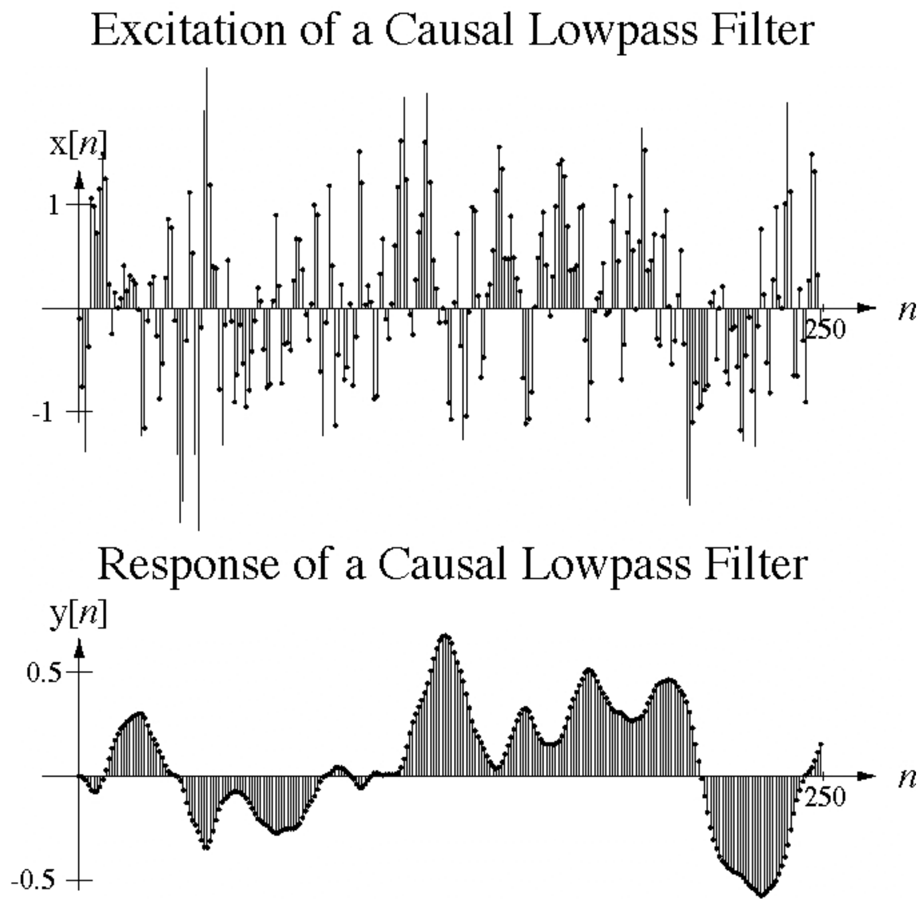
Excitation of a Causal Bandpass Filter



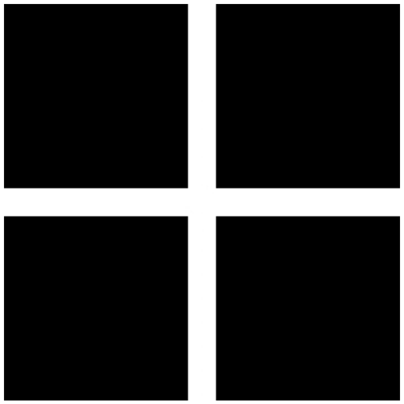
Response of a Causal Bandpass Filter



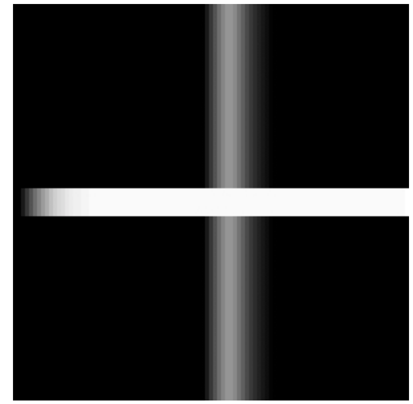
Examples of Causal Filter Effects on Signals

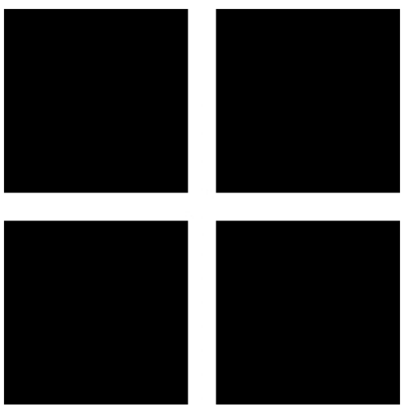


Two-Dimensional Filtering of Images

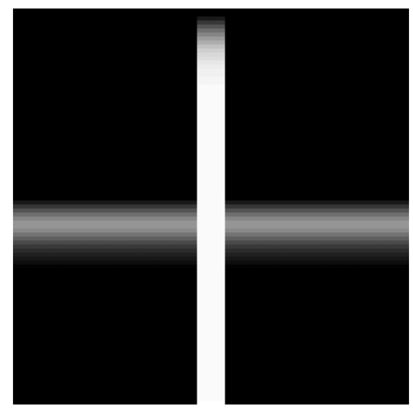


Causal Lowpass
Filtering
of Rows in
an Image

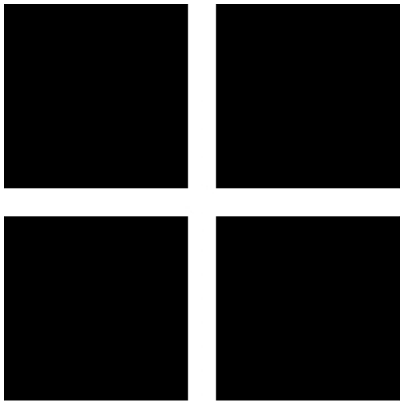




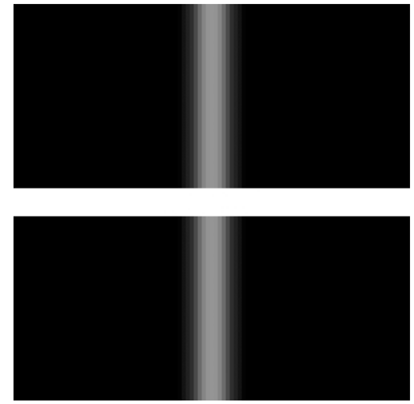
Causal Lowpass
Filtering
of Columns in
an Image

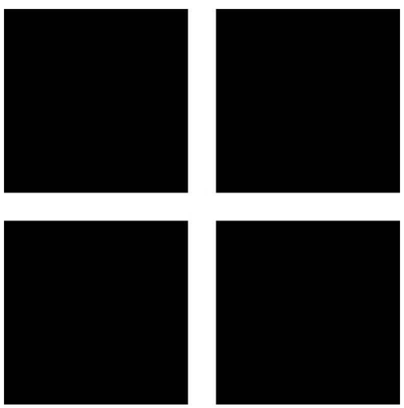


Two-Dimensional Filtering of Images

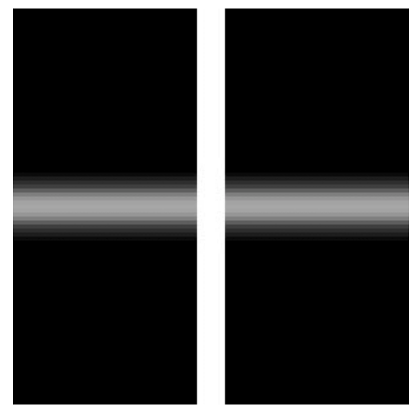


"Non-Causal" Lowpass Filtering of Rows in an Image

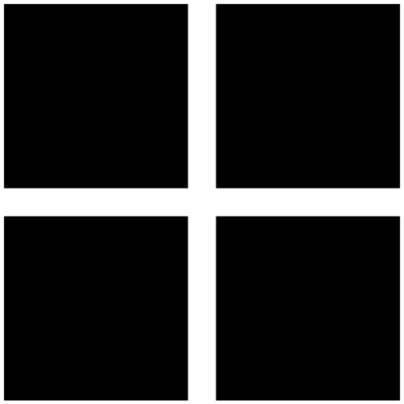




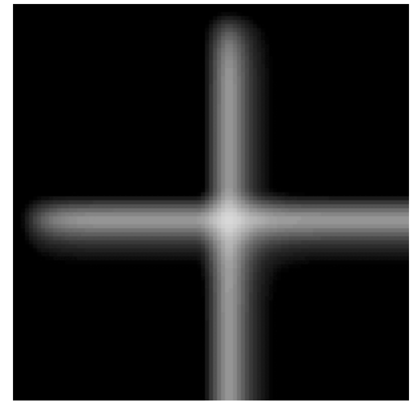
"Non-Causal" Lowpass Filtering of Columns in an Image

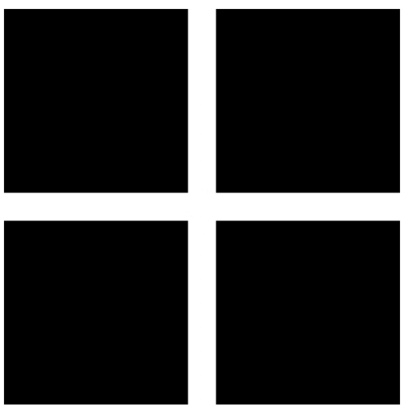


Two-Dimensional Filtering of Images



Causal
Lowpass
Filtering
of Rows and
Columns in
an Image





"Non-Causal"
Lowpass
Filtering
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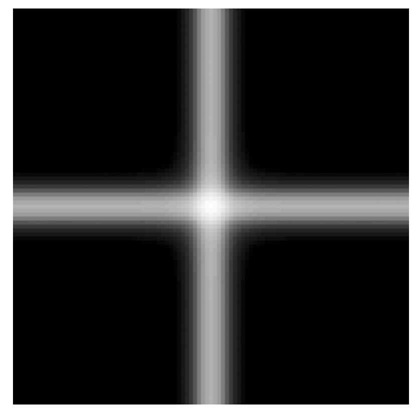
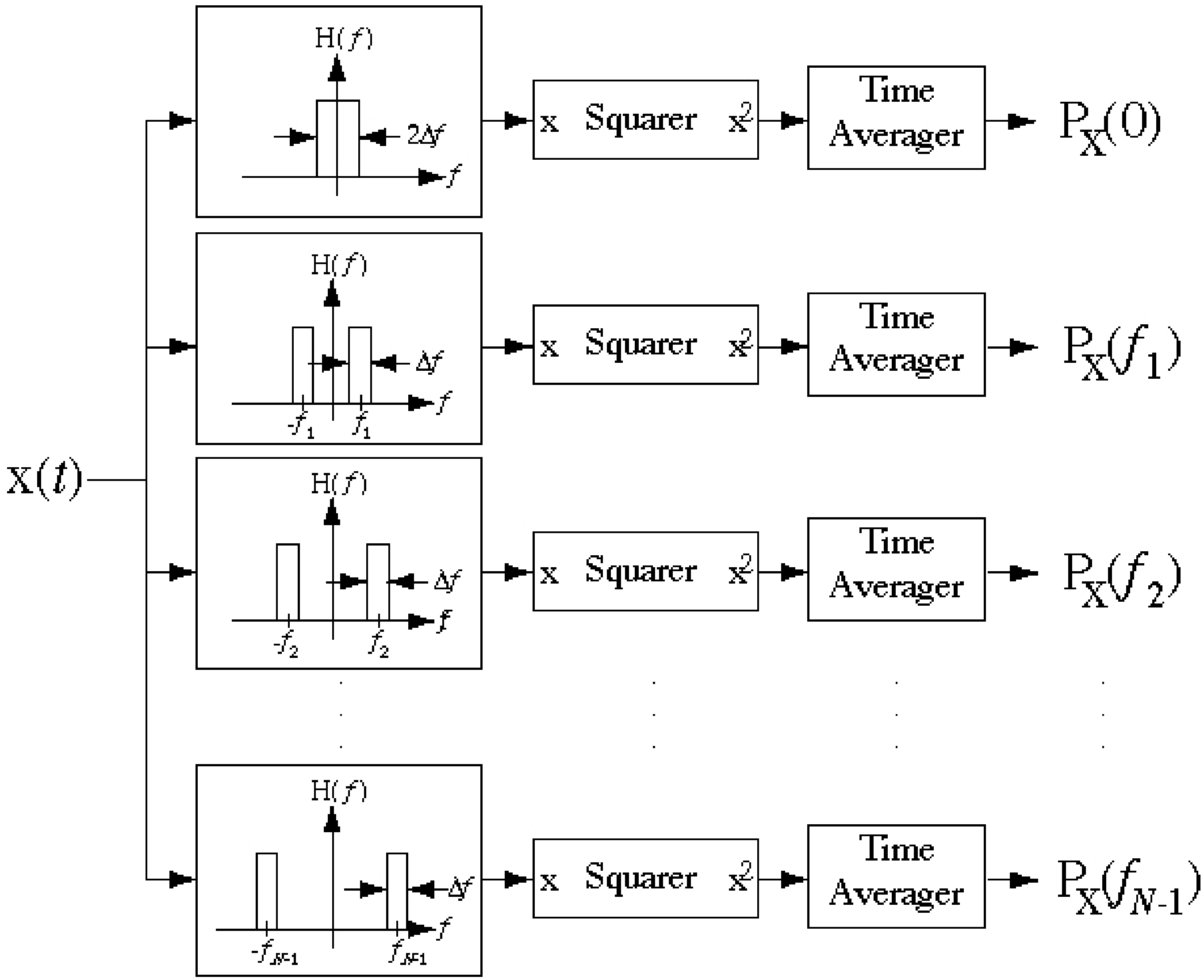
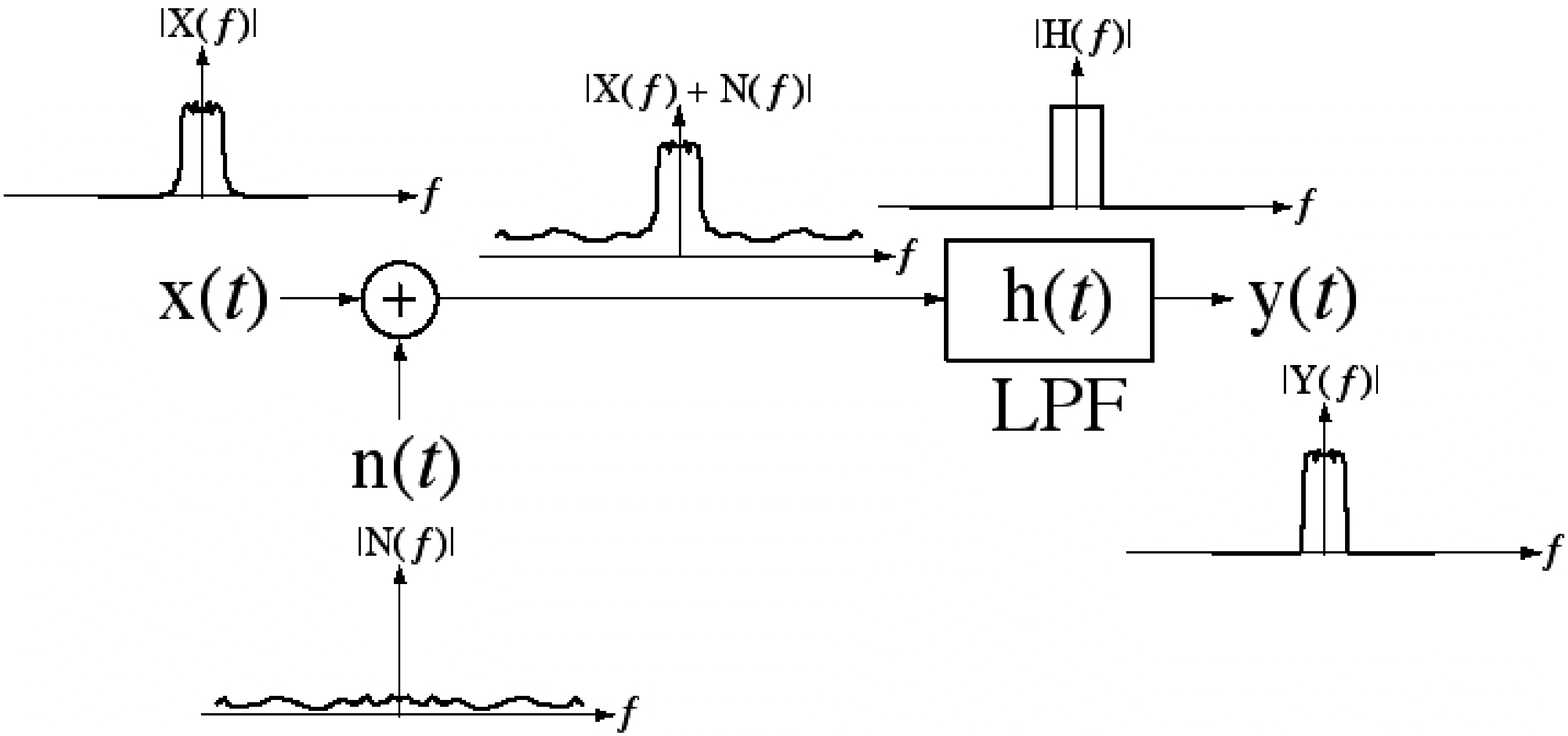


The Power Spectrum

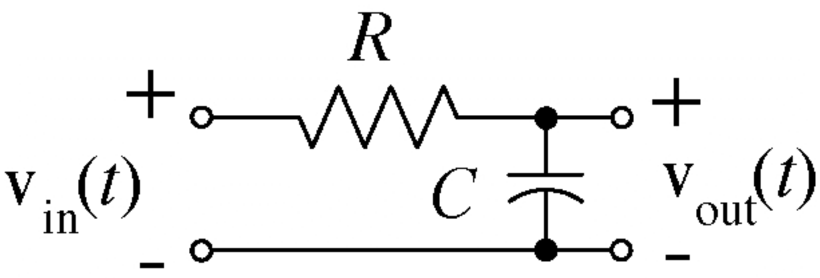


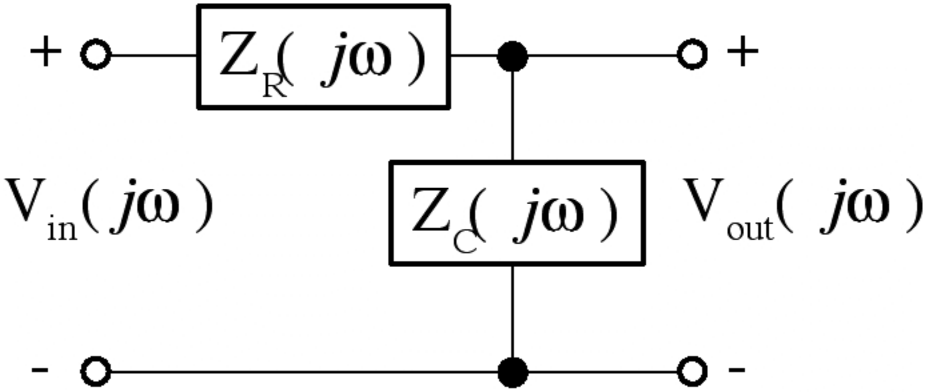
Noise Removal

A very common use of filters is to remove noise from a signal. If the noise bandwidth is much greater than the signal bandwidth a large improvement in signal fidelity is possible.



Practical Passive Filters

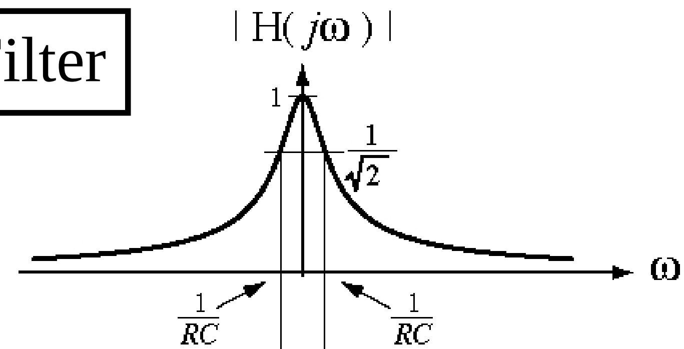


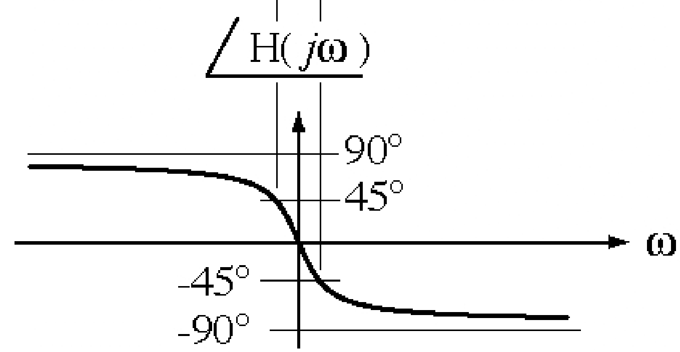


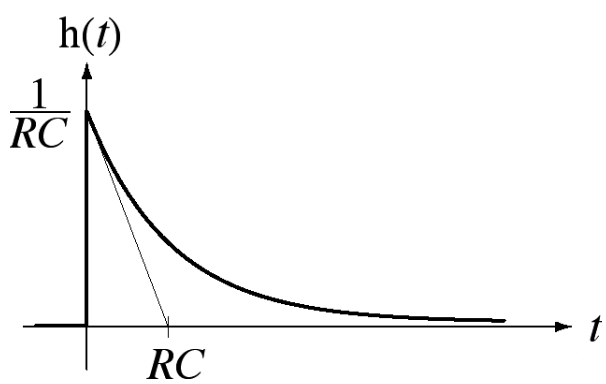
$$H(j\omega) = \frac{V_{out}(j\omega)}{V_{in}(j\omega)}$$

$$= \frac{Z_c(j\omega)}{Z_c(j\omega) + Z_R(j\omega)} = \frac{1}{j\omega RC + 1}$$

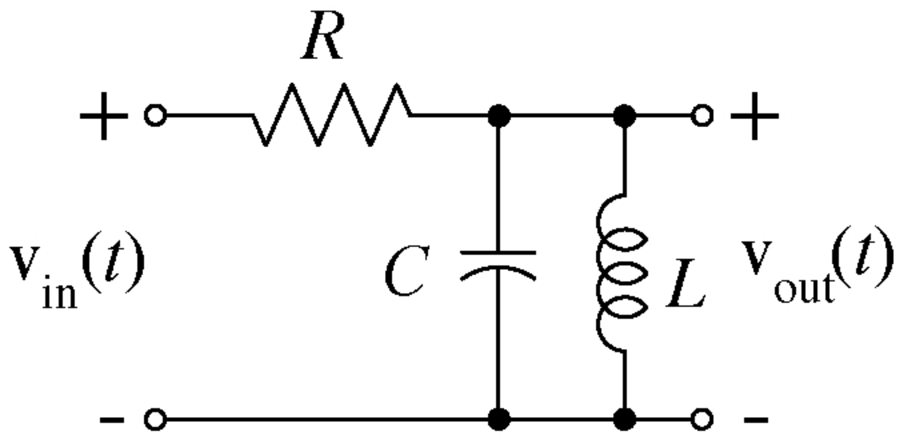
RC Lowpass Filter

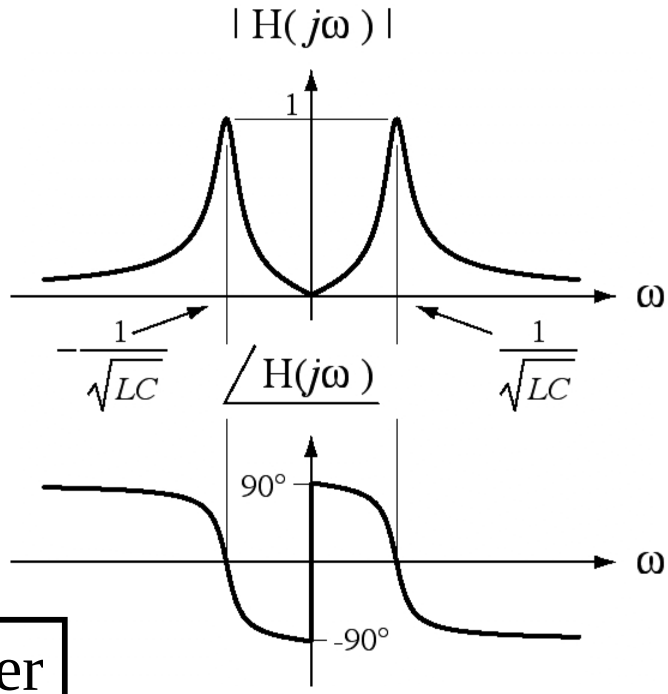




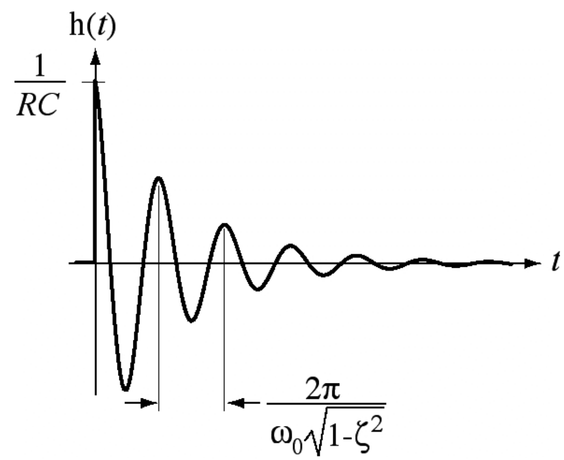


Practical Passive Filters





RLC Bandpass Filter

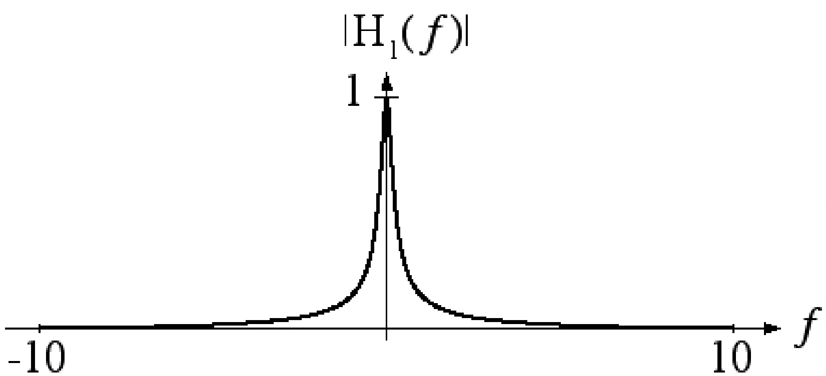


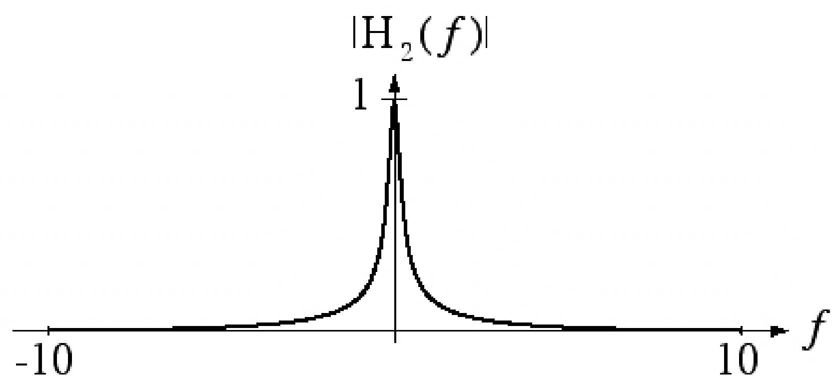
$$\mathrm{H}(f)=\frac{\mathrm{V}_{out}(f)}{\mathrm{V}_{in}(f)}=\frac{j\dfrac{2\pi f}{RC}}{(j2\pi f)^2+j\dfrac{2\pi f}{RC}+\dfrac{1}{LC}}$$

Log-Magnitude Frequency-Response Plots

Consider the two (different) transfer functions,

$$\mathrm{H}_1(f) = \frac{1}{j2\pi f + 1} \quad \text{and} \quad \mathrm{H}_2(f) = \frac{30}{30 - 4\pi^2 f^2 + j62\pi f}$$

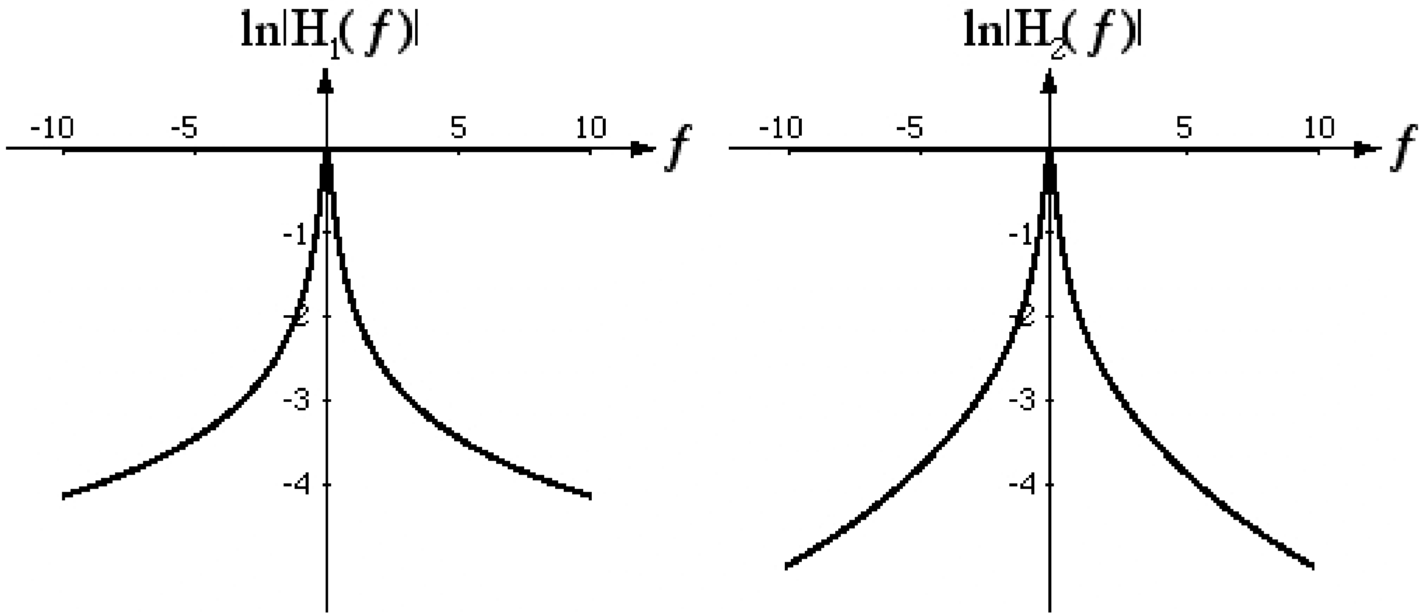




When plotted on this scale, these magnitude frequency response plots are indistinguishable.

Log-Magnitude Frequency-Response Plots

When the magnitude frequency responses are plotted on a logarithmic scale the difference is visible.



Bode Diagrams

A Bode diagram is a plot of a frequency response in *decibels* versus frequency on a logarithmic scale.

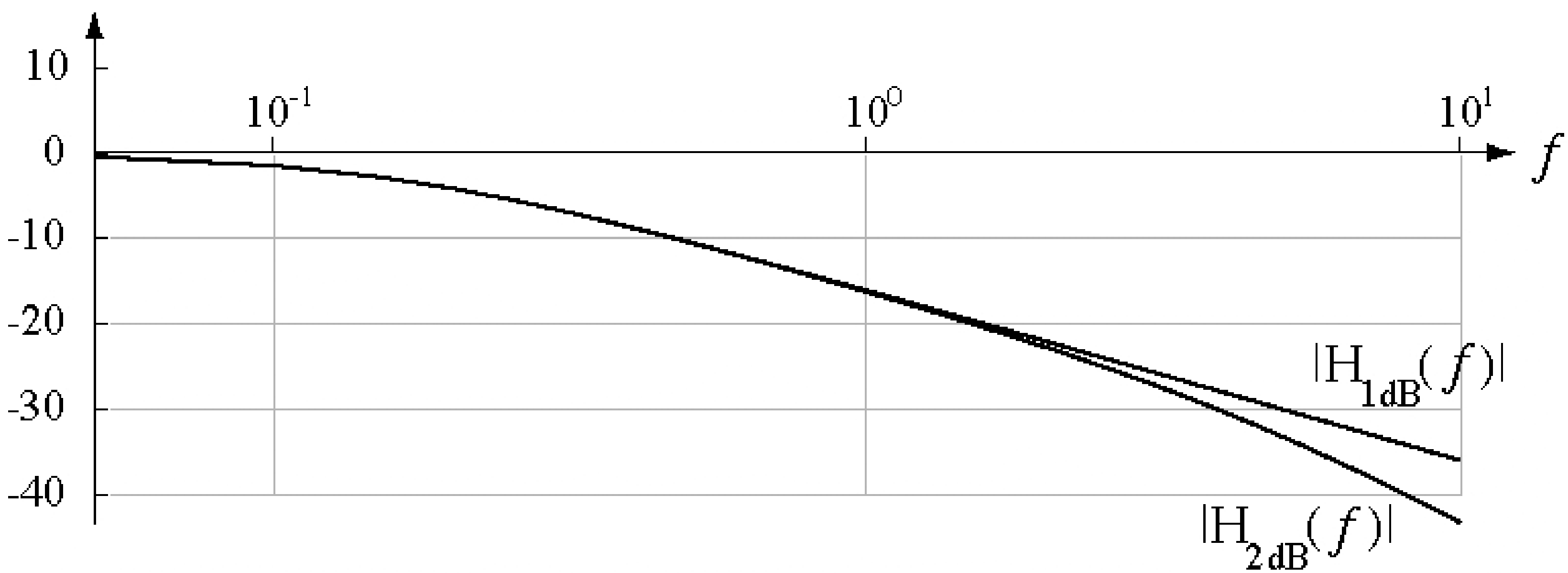
The *Bel* (B) is the common (base 10) logarithm of a power ratio and a decibel (dB) is one-tenth of a Bel.

The Bel is named in honor of Alexander Graham Bell.

A signal ratio, expressed in decibels, is 20 times the common logarithm of the signal ratio because signal power is proportional to the square of the signal.

Bode Diagrams

$$H_1(f) = \frac{1}{j2\pi f + 1} \quad \text{and} \quad H_2(f) = \frac{30}{30 - 4\pi^2 f^2 + j62\pi f}$$



Bode Diagrams

Continuous-time LTI systems are described by equations of the general form,

$$\sum_{k=0}^{D} a_k \frac{d^k}{dt^k} \mathrm{y}(t) = \sum_{k=0}^{N} b_k \frac{d^k}{dt^k} \mathrm{x}(t)$$

Fourier transforming, the transfer function is of the general form,

$$\mathrm{H}(j\omega) = \frac{\mathrm{Y}(j\omega)}{\mathrm{X}(j\omega)} = \frac{\sum_{k=0}^{N} b_k (j\omega)^k}{\sum_{k=0}^{D} a_k (j\omega)^k}$$

Bode Diagrams

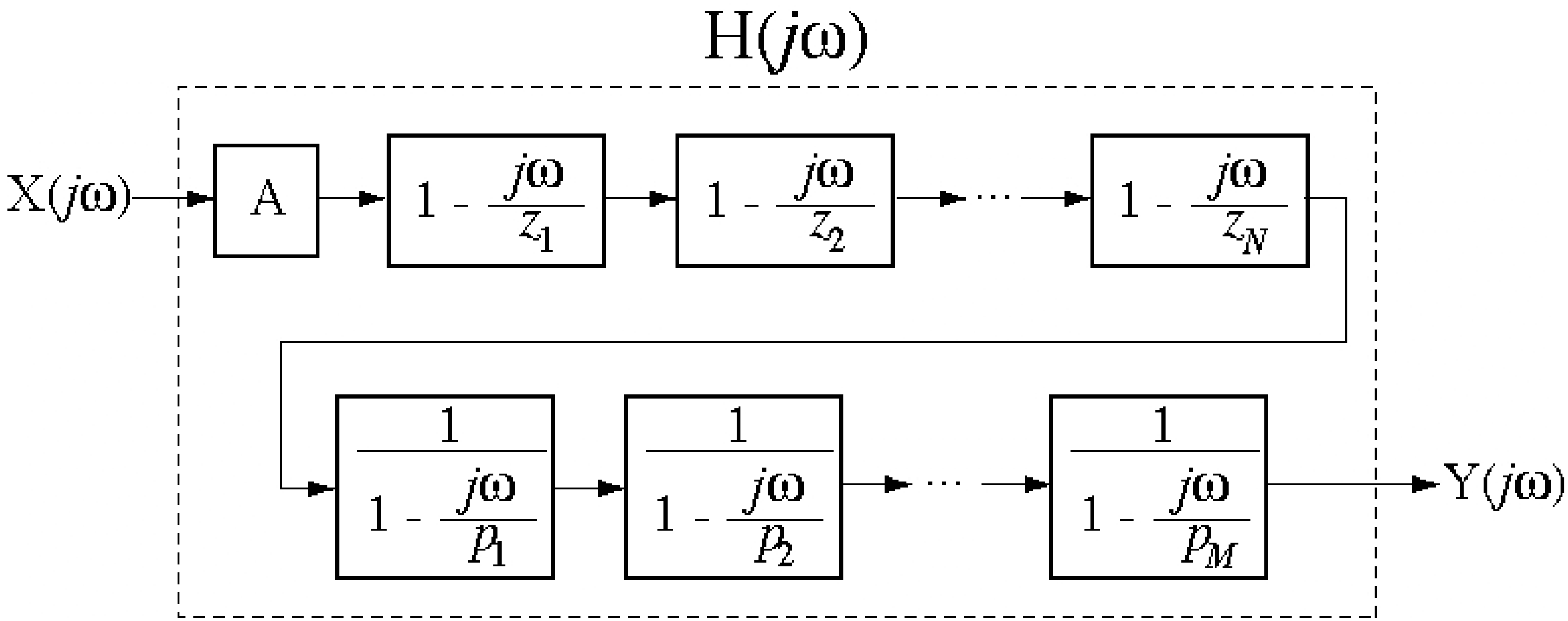
A transfer function can be written in the form,

$$H(j\omega) = A \frac{\left(1 - \frac{j\omega}{z_1}\right)\left(1 - \frac{j\omega}{z_2}\right)\cdots\left(1 - \frac{j\omega}{z_N}\right)}{\left(1 - \frac{j\omega}{p_1}\right)\left(1 - \frac{j\omega}{p_2}\right)\cdots\left(1 - \frac{j\omega}{p_D}\right)}$$

where the "*z*'s" are the values of $j\omega$ (not ω) at which the transfer function goes to zero and the "*p*'s" are the values of $j\omega$ at which the transfer function goes to infinity. These *z*'s and *p*'s are commonly referred to as the "zeros" and "poles" of the system.

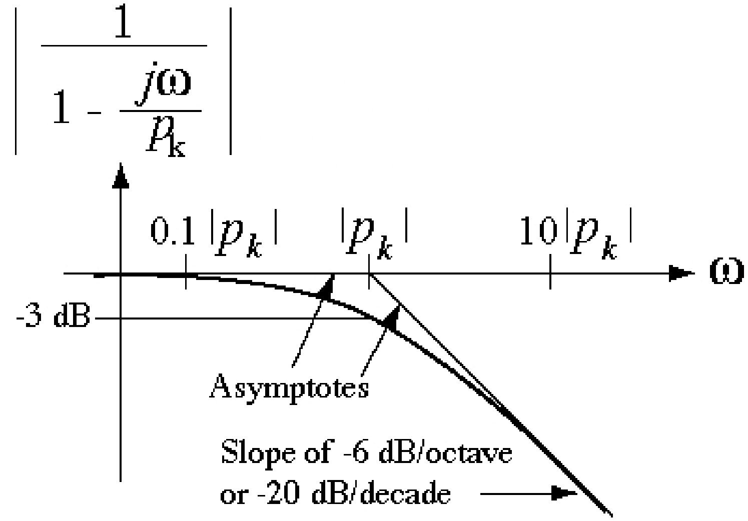
Bode Diagrams

From the factored form of the transfer function a system can be conceived as the cascade of simple systems, each of which has only one numerator factor or one denominator factor. Since the Bode diagram is logarithmic, multiplied transfer functions add when expressed in dB.



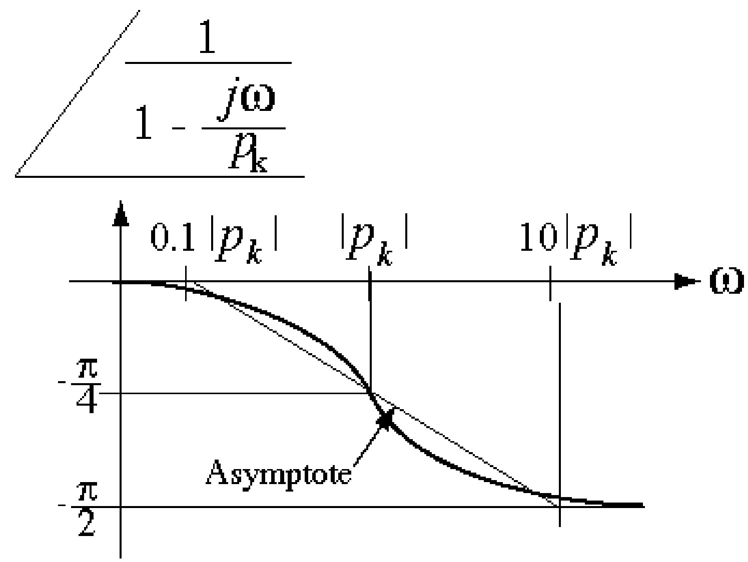
Bode Diagrams

System Bode diagrams are formed by adding the Bode diagrams of the simple systems which are in cascade. Each simple-system diagram is called a *component diagram*.



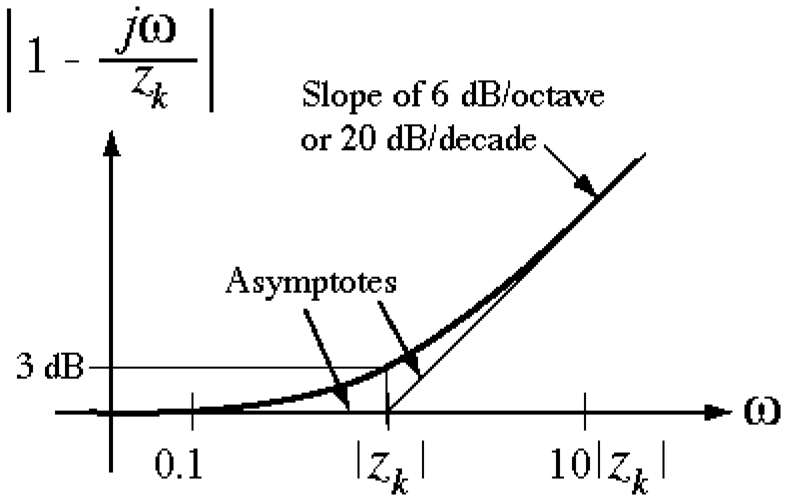
One Real Pole

$$H(j\omega) = \frac{1}{1 - \frac{j\omega}{p_k}}$$

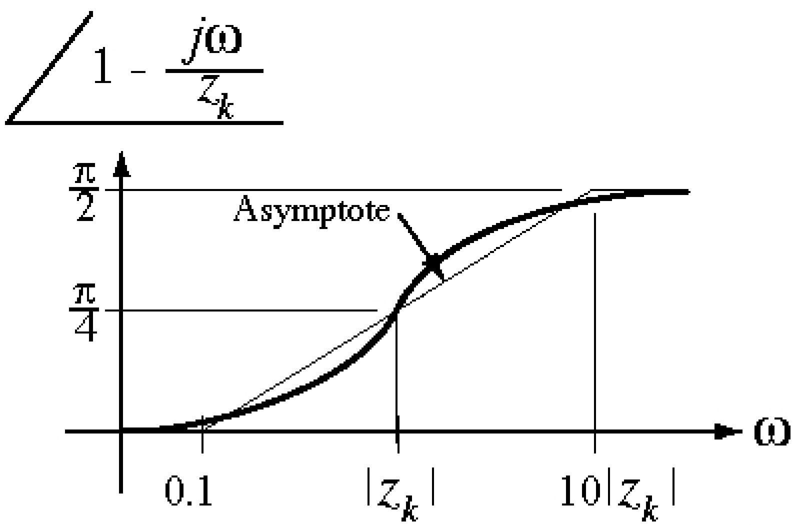


Bode Diagrams

One real zero



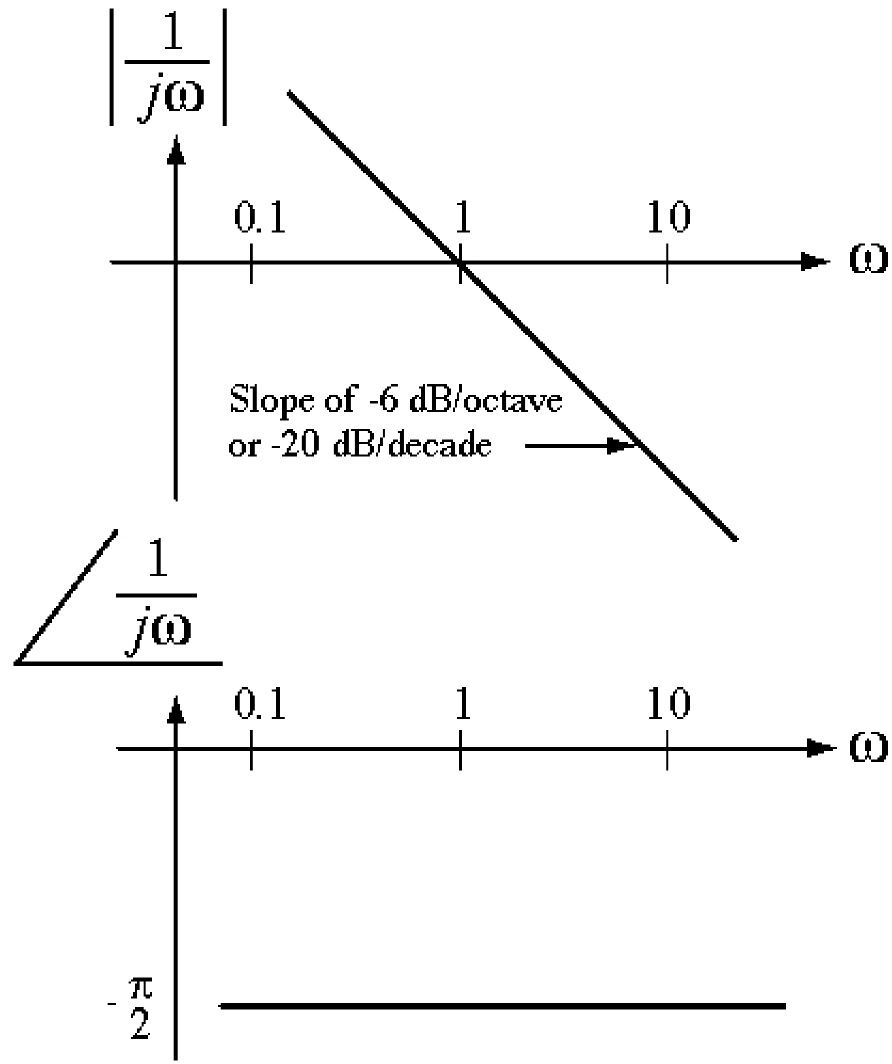
$$H(j\omega) = 1 - \frac{j\omega}{z_k}$$



Bode Diagrams

Integrator
(Pole at zero)

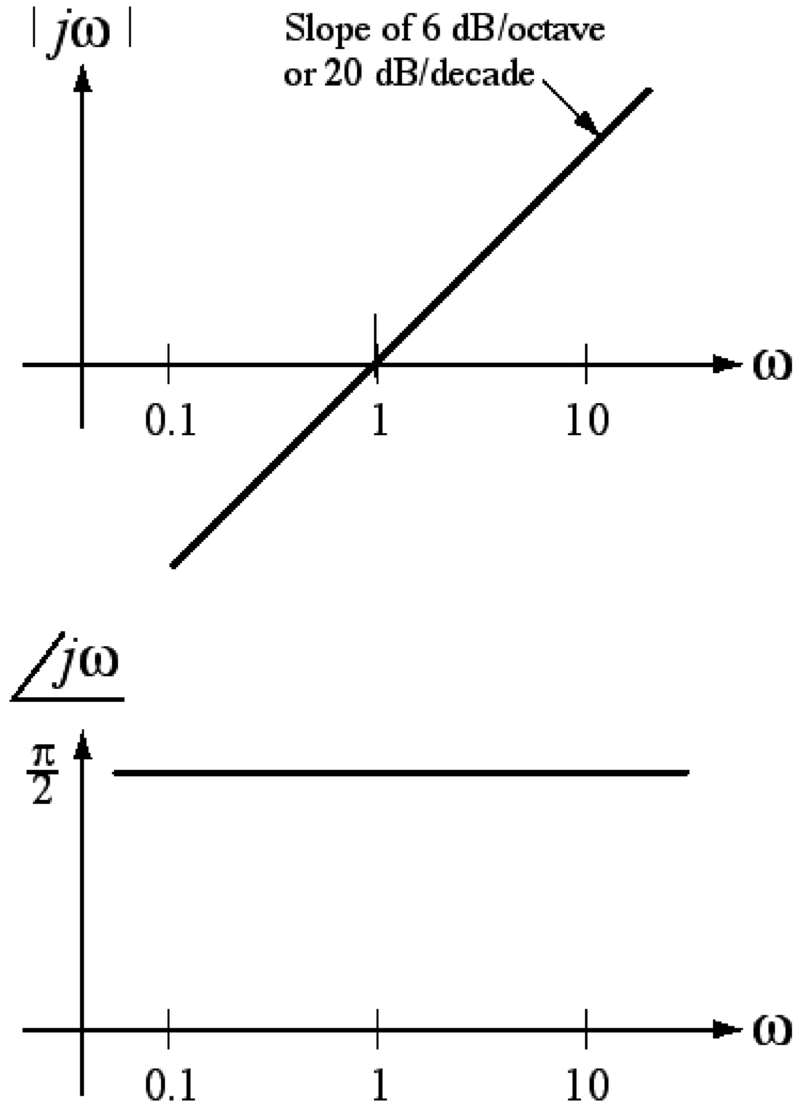
$$\mathrm{H}(j\omega) = \frac{1}{j\omega}$$



Bode Diagrams

Differentiator
(Zero at zero)

$$H(j\omega) = j\omega$$

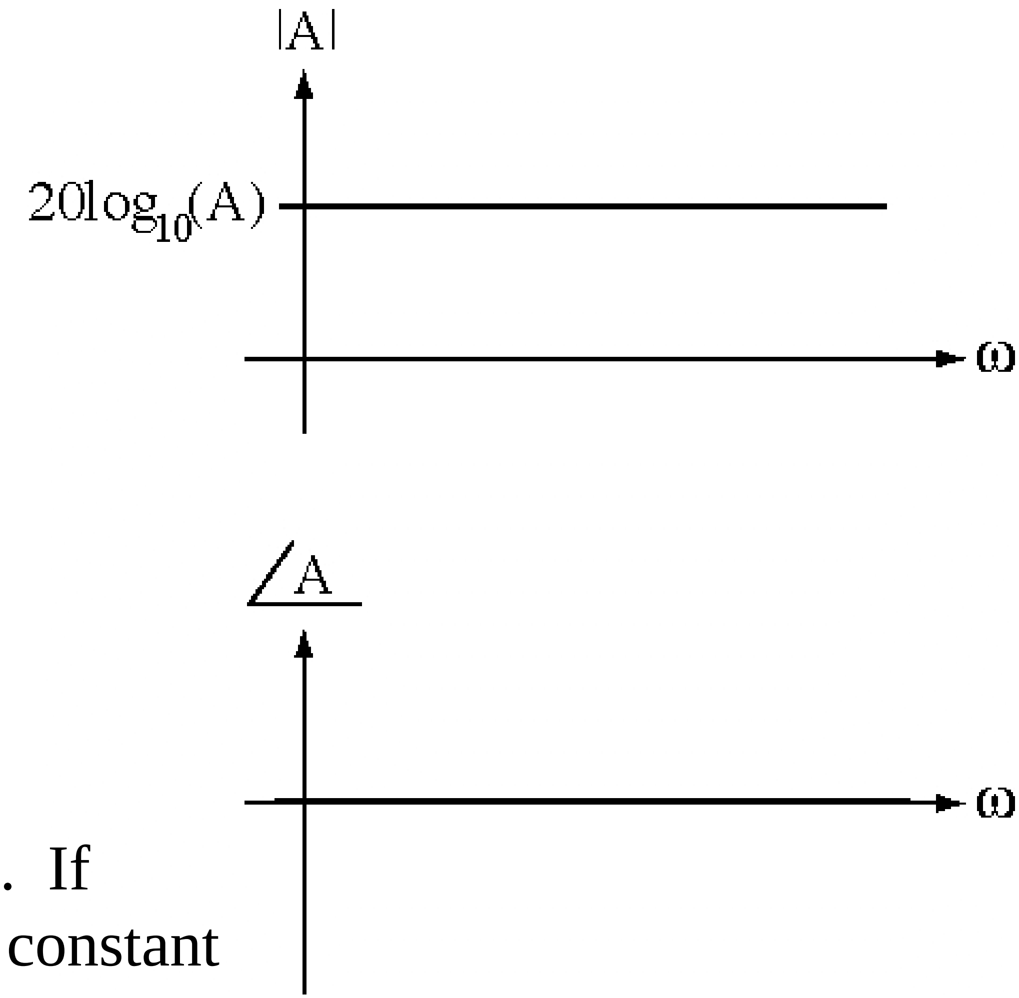


Bode Diagrams

Frequency-Independent
Gain

$$\mathrm{H}(j\omega) = A$$

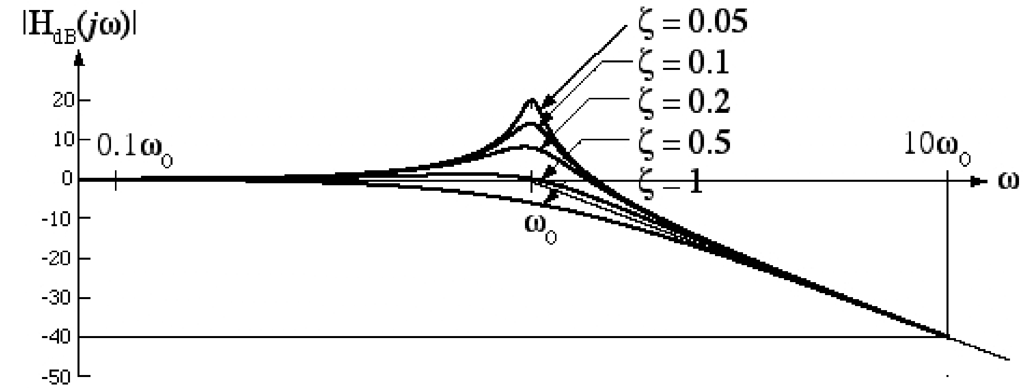
(This phase plot is for $A > 0$. If $A < 0$, the phase would be a constant π or - π radians.)

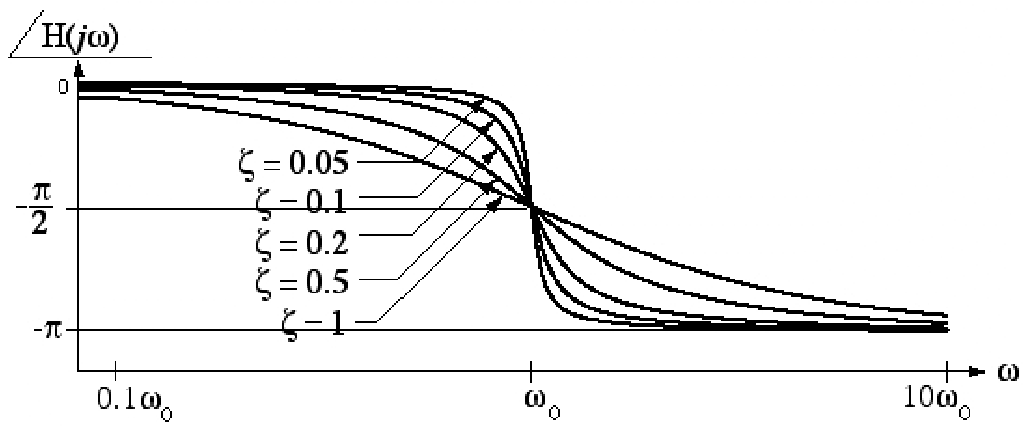


Bode Diagrams

Complex
Pole Pair

$$\mathrm{H}(j\omega)=\frac{1}{\left(1-\frac{j\omega}{p_1}\right)\left(1-\frac{j\omega}{p_2}\right)}=\frac{1}{1-j\omega\frac{2\,\mathrm{Re}(p_1)}{|p_1|^2}+\frac{(j\omega)^2}{|p_1|^2}}$$

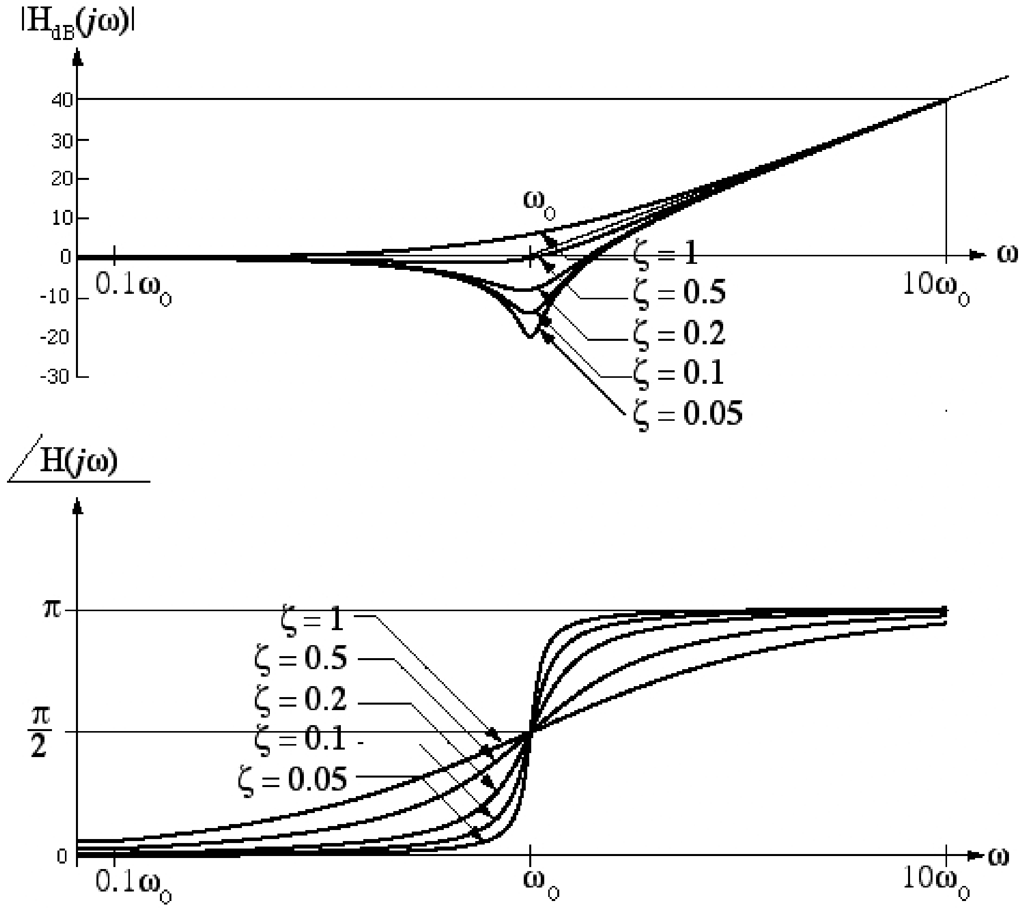




Bode Diagrams

Complex Zero Pair

$$\mathrm{H}(j\omega)=\left(1-\frac{j\omega}{z_1}\right)\left(1-\frac{j\omega}{z_2}\right)=1-j\omega\frac{2\,\mathrm{Re}(z_1)}{|z_1|^2}+\frac{(j\omega)^2}{|z_1|^2}$$

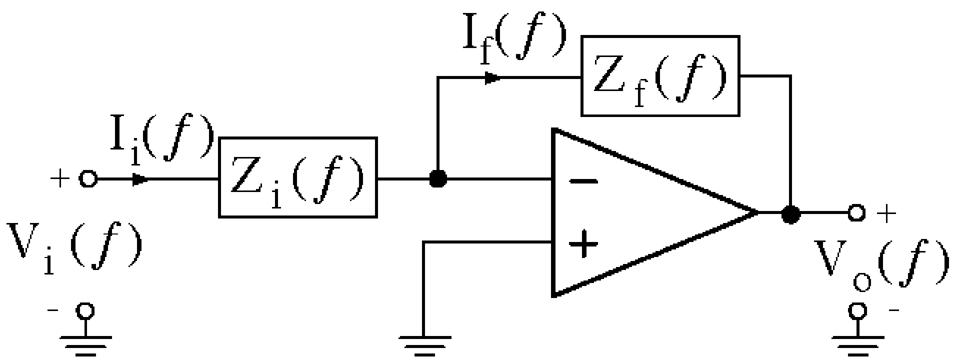


Practical Active Filters

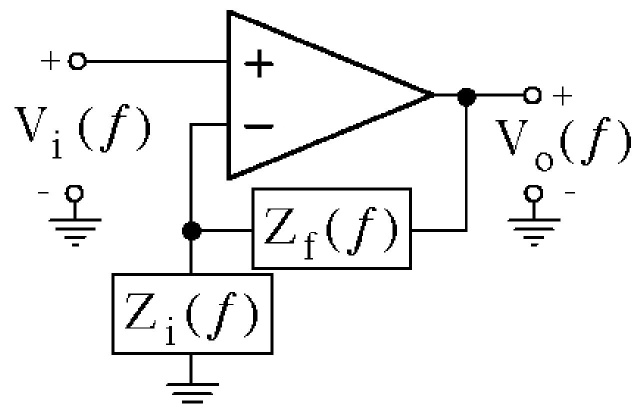
Operational Amplifiers

The ideal operational amplifier has infinite input impedance, zero output impedance, infinite gain and infinite bandwidth.

Inverting Amplifier



Non-Inverting Amplifier

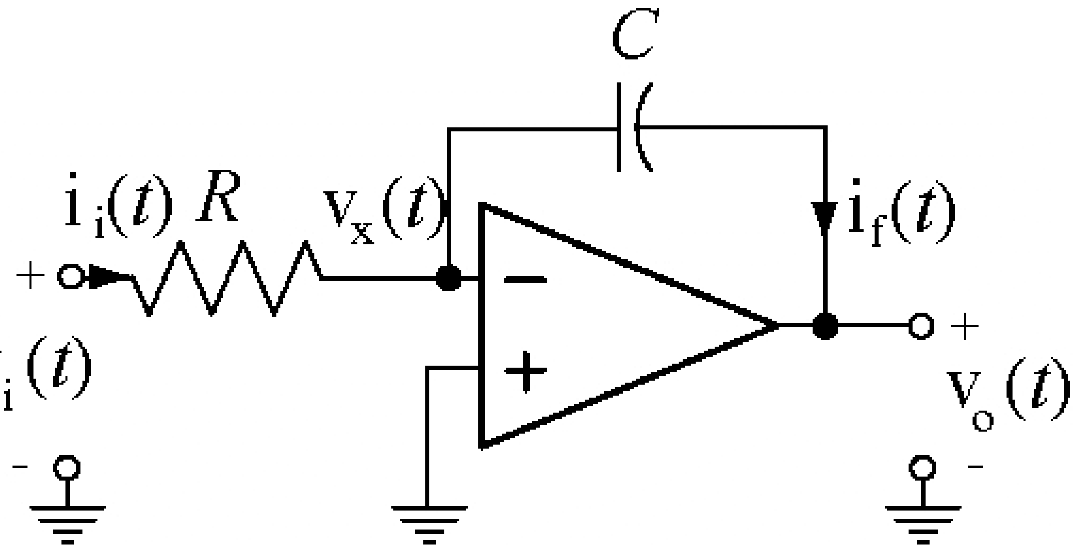


$$H(f) = \frac{V_o(f)}{V_i(f)} = -\frac{Z_f(f)}{Z_i(f)}$$

$$H(f) = \frac{Z_f(f) + Z_i(f)}{Z_i(f)}$$

Practical Active Filters

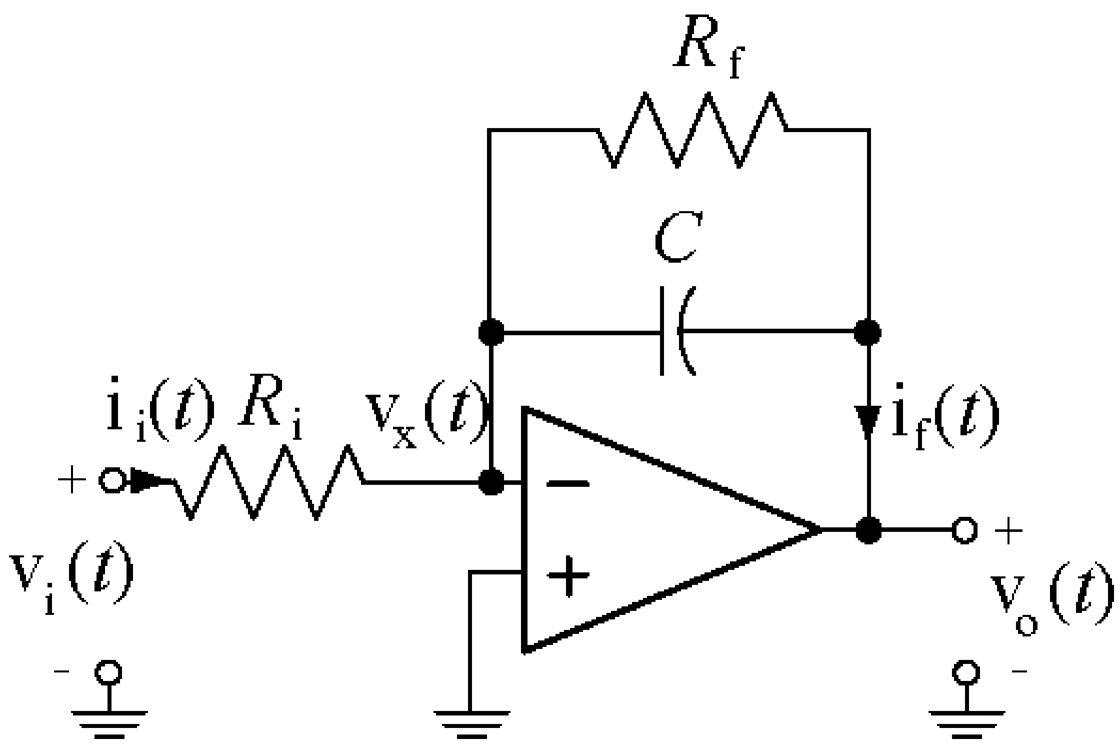
Active Integrator



$$\mathrm{V}_o(f) = -\frac{1}{RC}\underbrace{\frac{\mathrm{V}_i(f)}{j2\pi f}}_{\substack{\text{integral} \\ \text{of } \mathrm{V}_i(f)}}$$

Practical Active Filters

Active RC Lowpass Filter

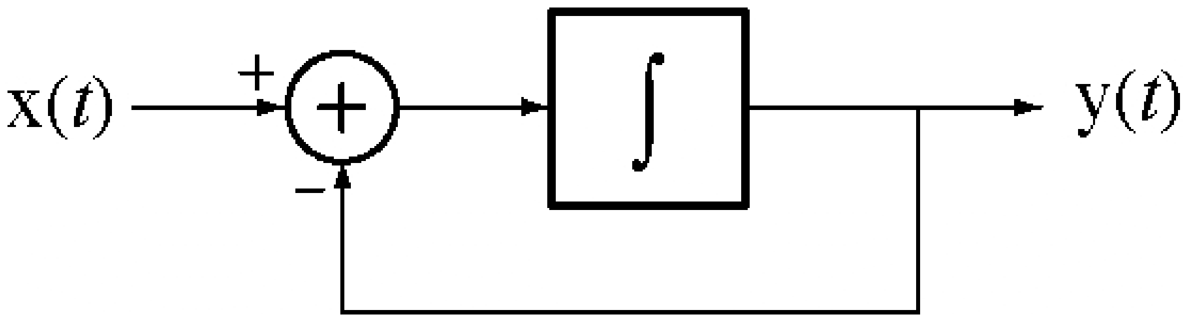


$$\frac{V_0(f)}{V_i(f)} = -\frac{R_f}{R_s}\frac{1}{j2\pi fCR_f + 1}$$

Practical Active Filters

Lowpass Filter

An integrator with feedback is a lowpass filter.

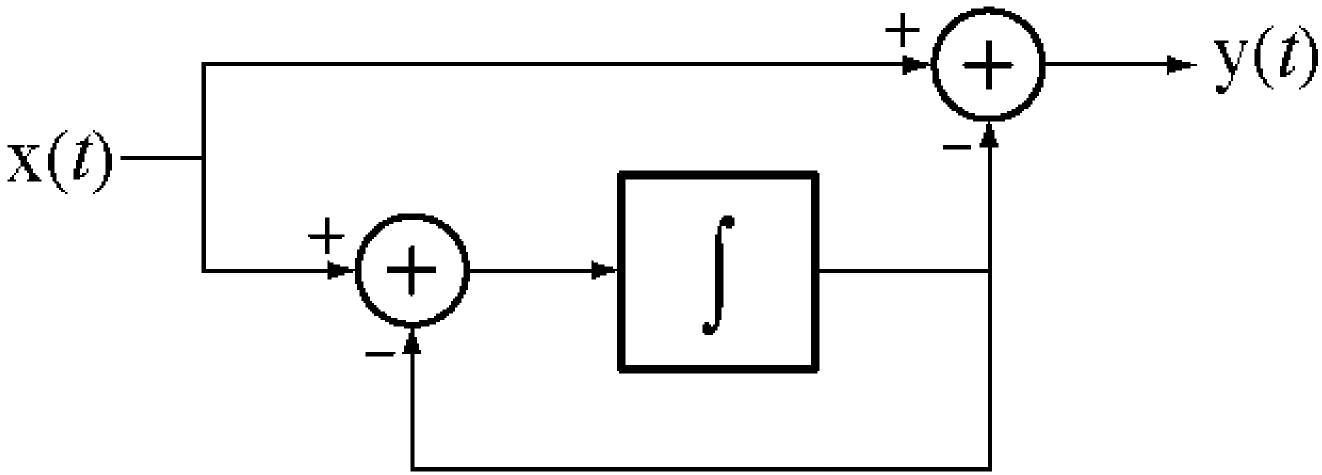


$$y'(t) + y(t) = x(t)$$

$$H(j\omega) = \frac{1}{j\omega + 1}$$

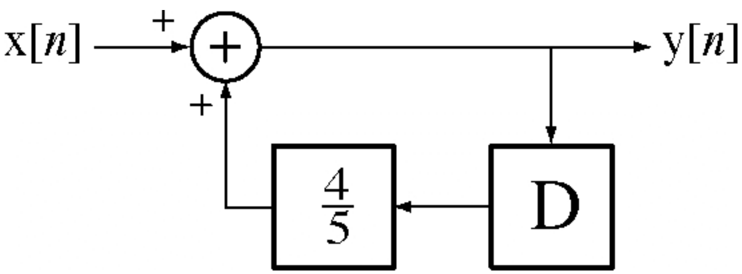
Practical Active Filters

Highpass Filter

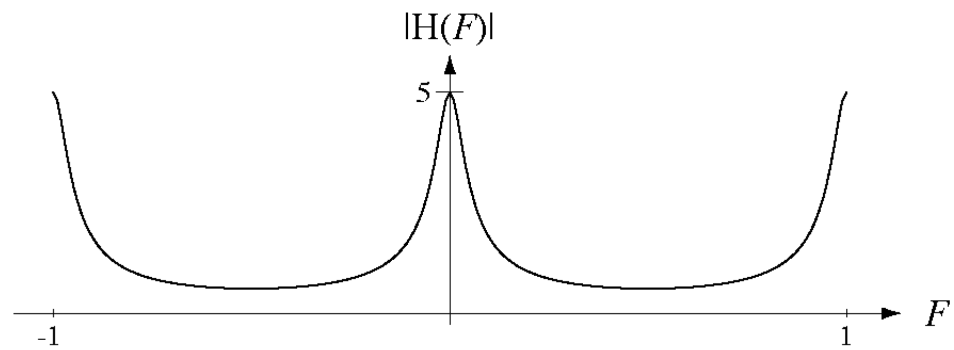


Discrete-Time Filters

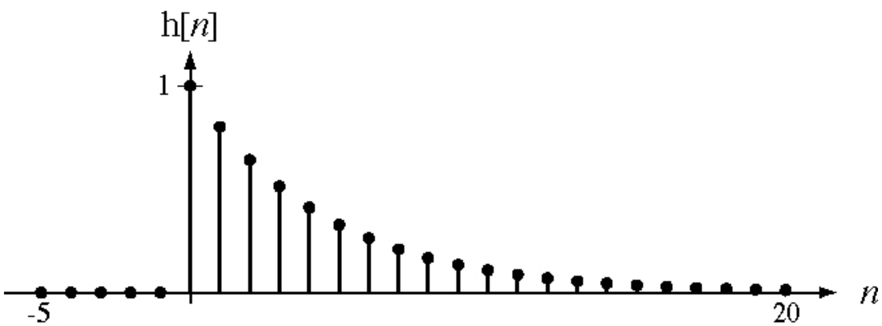
DT Lowpass Filter

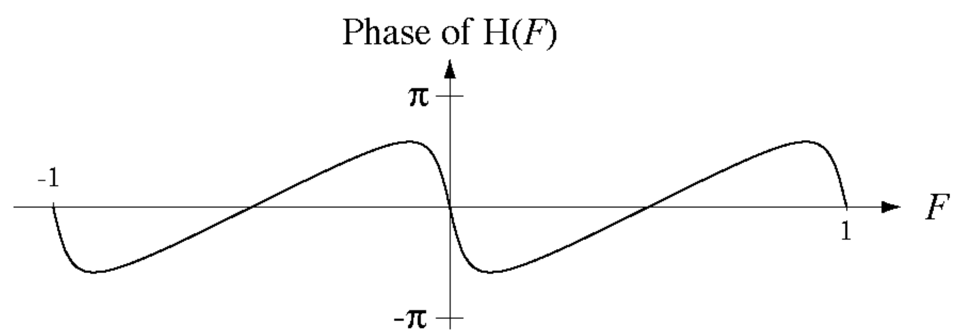


$$\mathrm{H}(F) = \frac{1}{1 - \frac{4}{5} e^{-j2\pi F}}$$



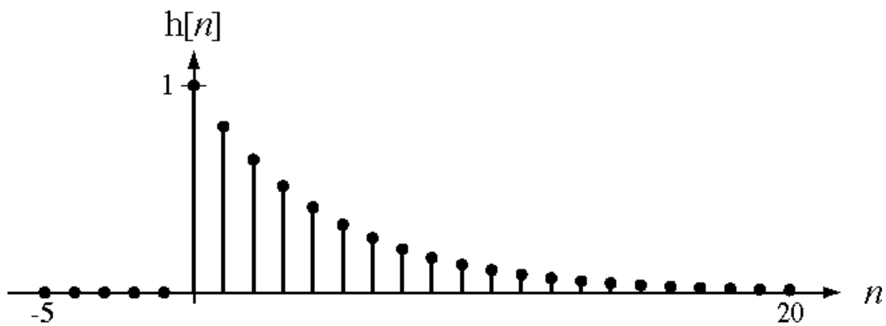
$$\mathrm{h}[n] = \left(\frac{4}{5}\right)^n \mathrm{u}[n]$$

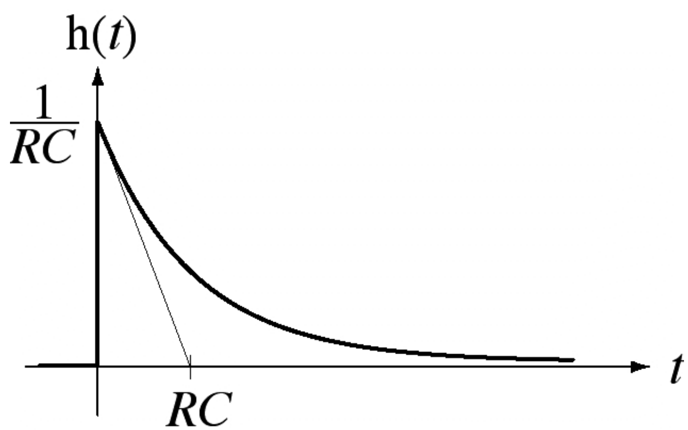




Discrete-Time Filters

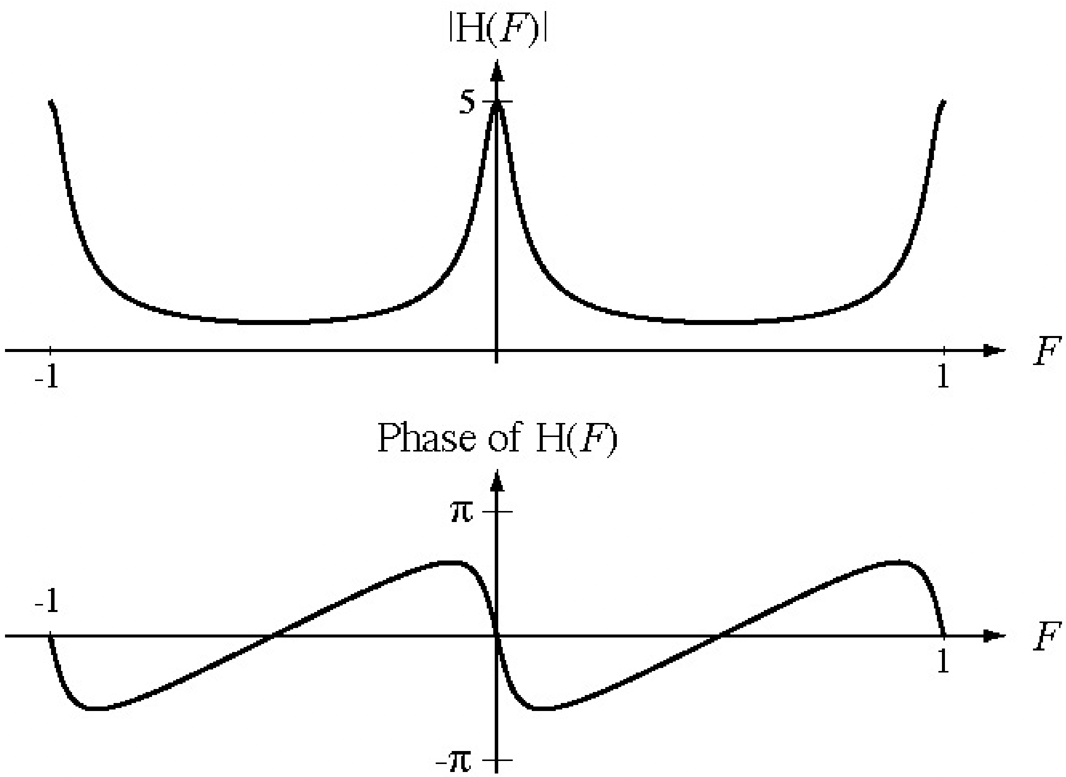
Comparison of a DT lowpass filter impulse response with an RC passive lowpass filter impulse response



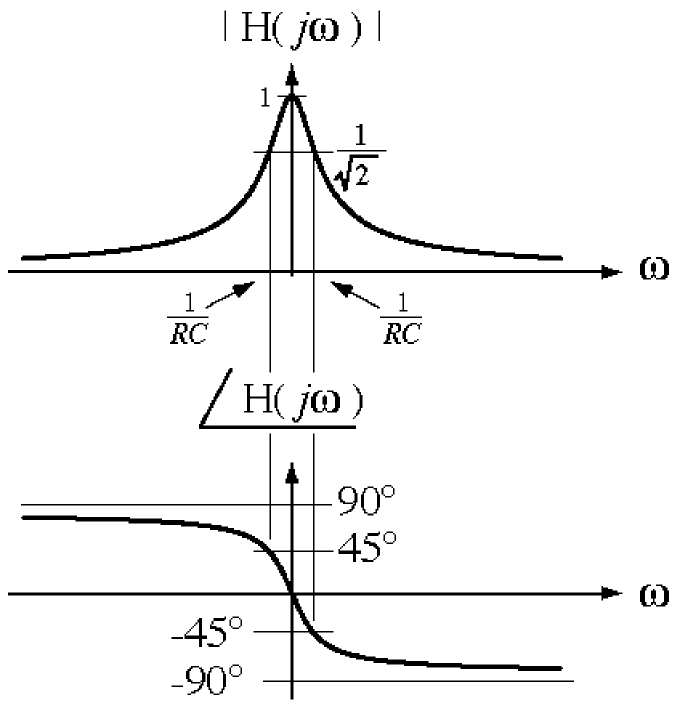


Discrete-Time Filters

DT Lowpass Filter
Frequency Response

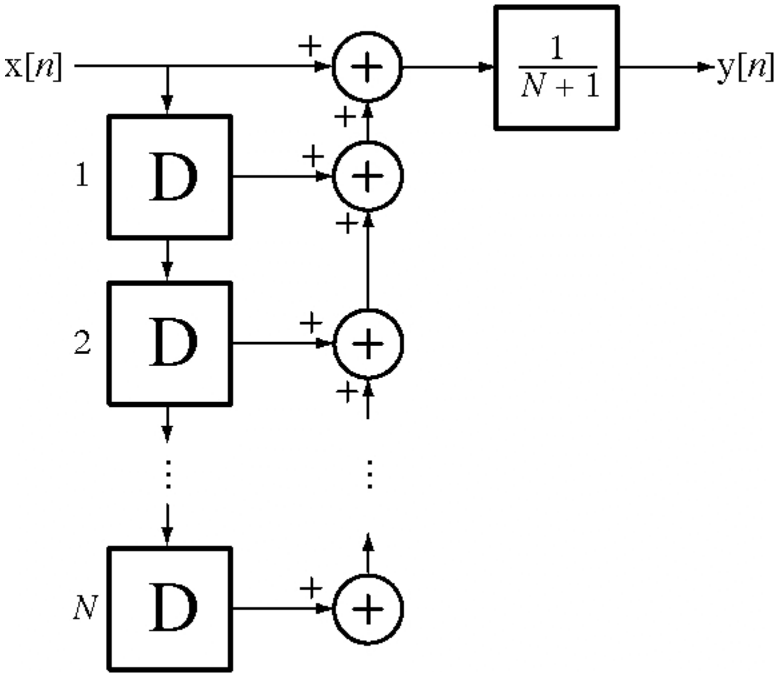


RC Lowpass Filter
Frequency Response

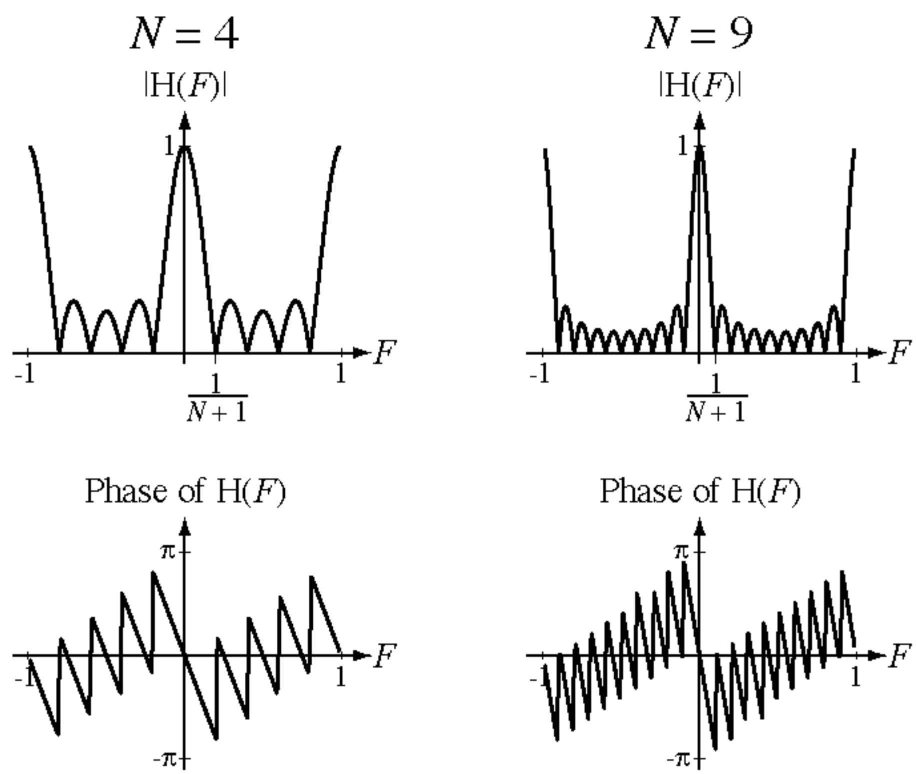


Discrete-Time Filters

Moving-Average Filter



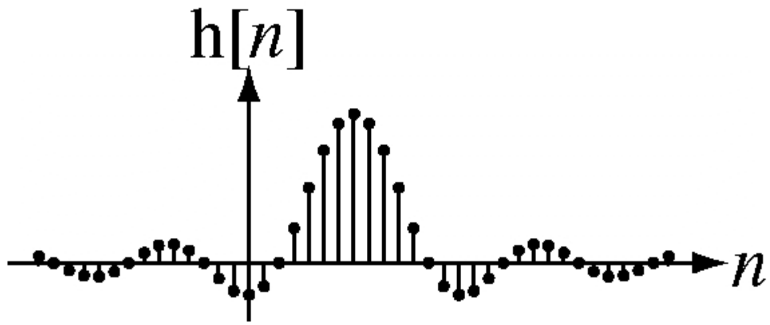
$$\mathrm{H}(F) = e^{-j\pi NF}\,\mathrm{drcl}(F, N+1)$$



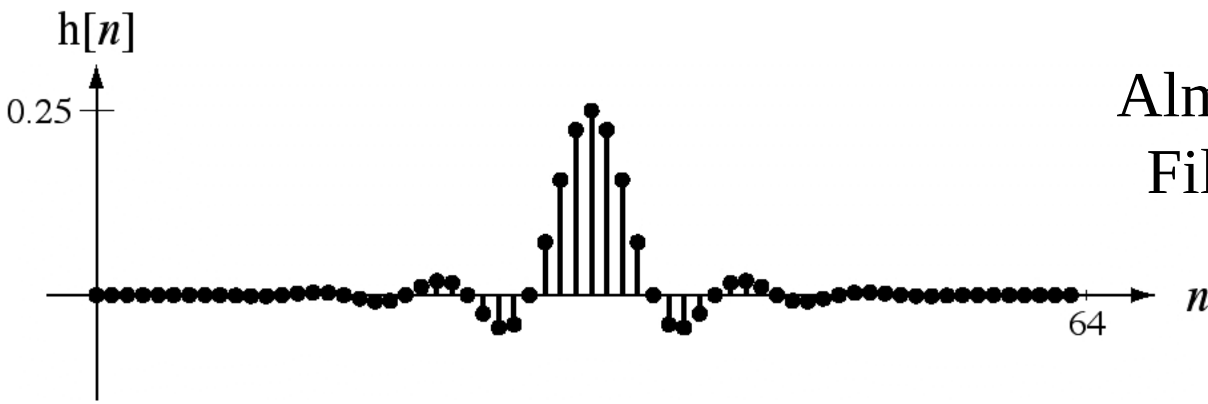
$$\mathrm{h}[n] = \frac{\delta[n] + \delta[n-1] + \delta[n-2] + \cdots + \delta[n-N]}{N+1}$$

Always Stable

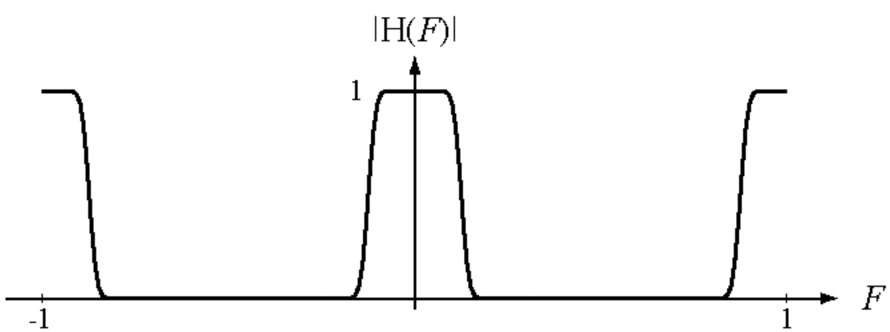
Discrete-Time Filters



Ideal DT Lowpass
Filter Impulse Response



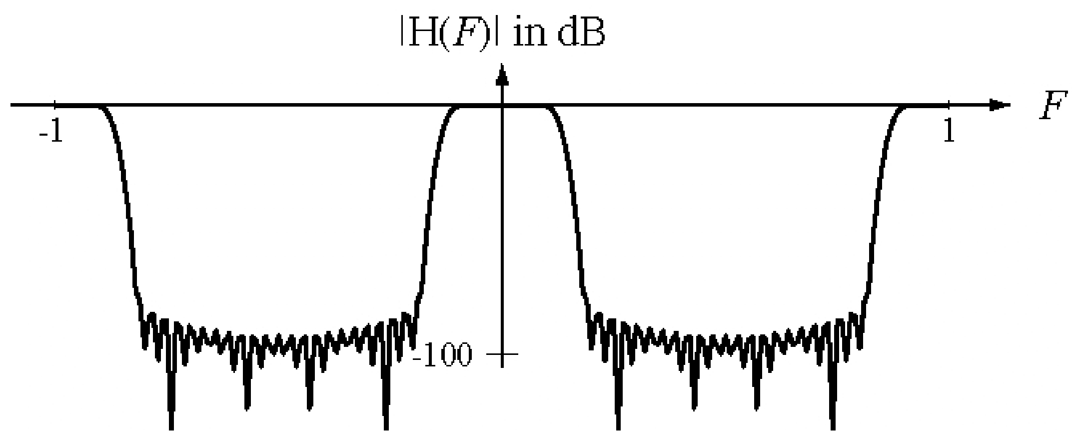
Almost-Ideal DT Lowpass
Filter Impulse Response



Almost-Ideal DT Lowpass
Filter Magnitude Frequency
Response

Discrete-Time Filters

Almost-Ideal DT Lowpass Filter Magnitude Frequency Response in dB

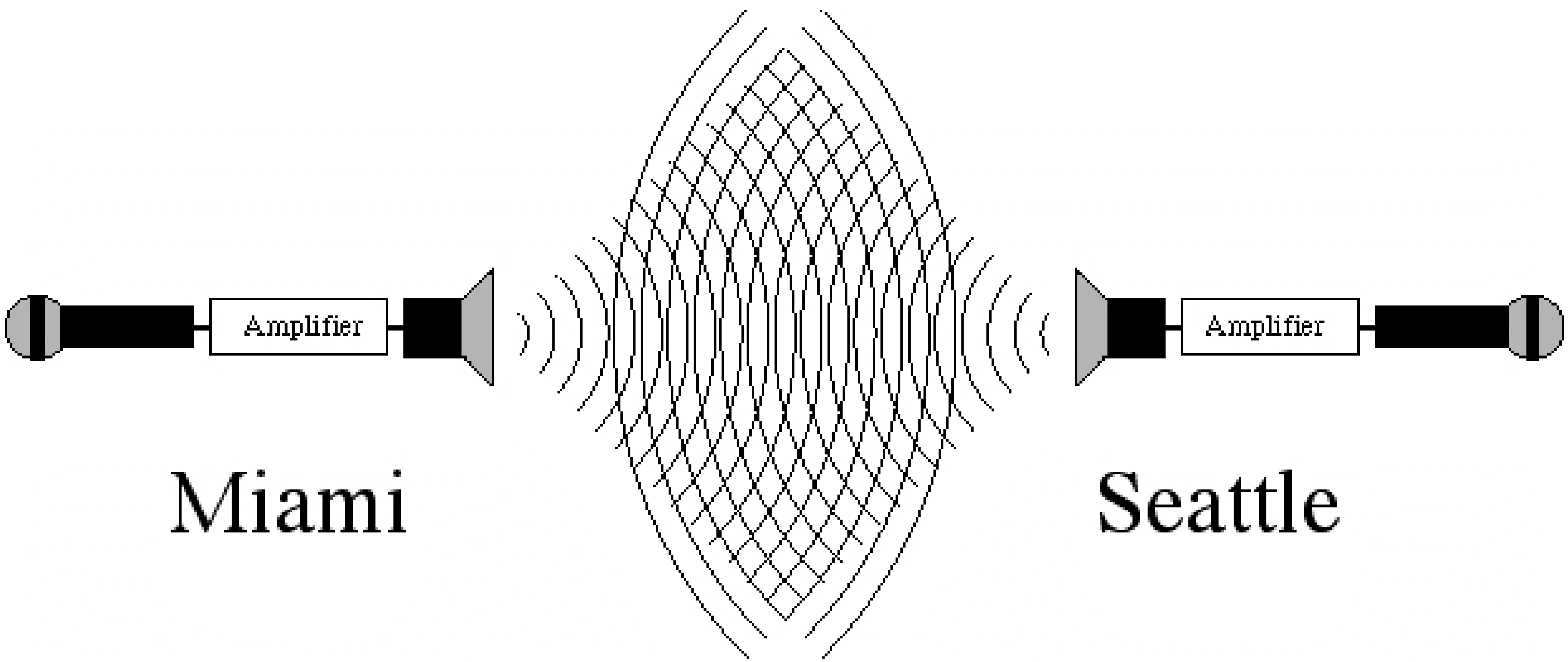


Advantages of Discrete-Time Filters

- They are almost insensitive to environmental effects
- CT filters at low frequencies may require very large components, DT filters do not
- DT filters are often programmable making them easy to modify
- DT signals can be stored indefinitely on magnetic media, stored CT signals degrade over time
- DT filters can handle multiple signals by multiplexing them

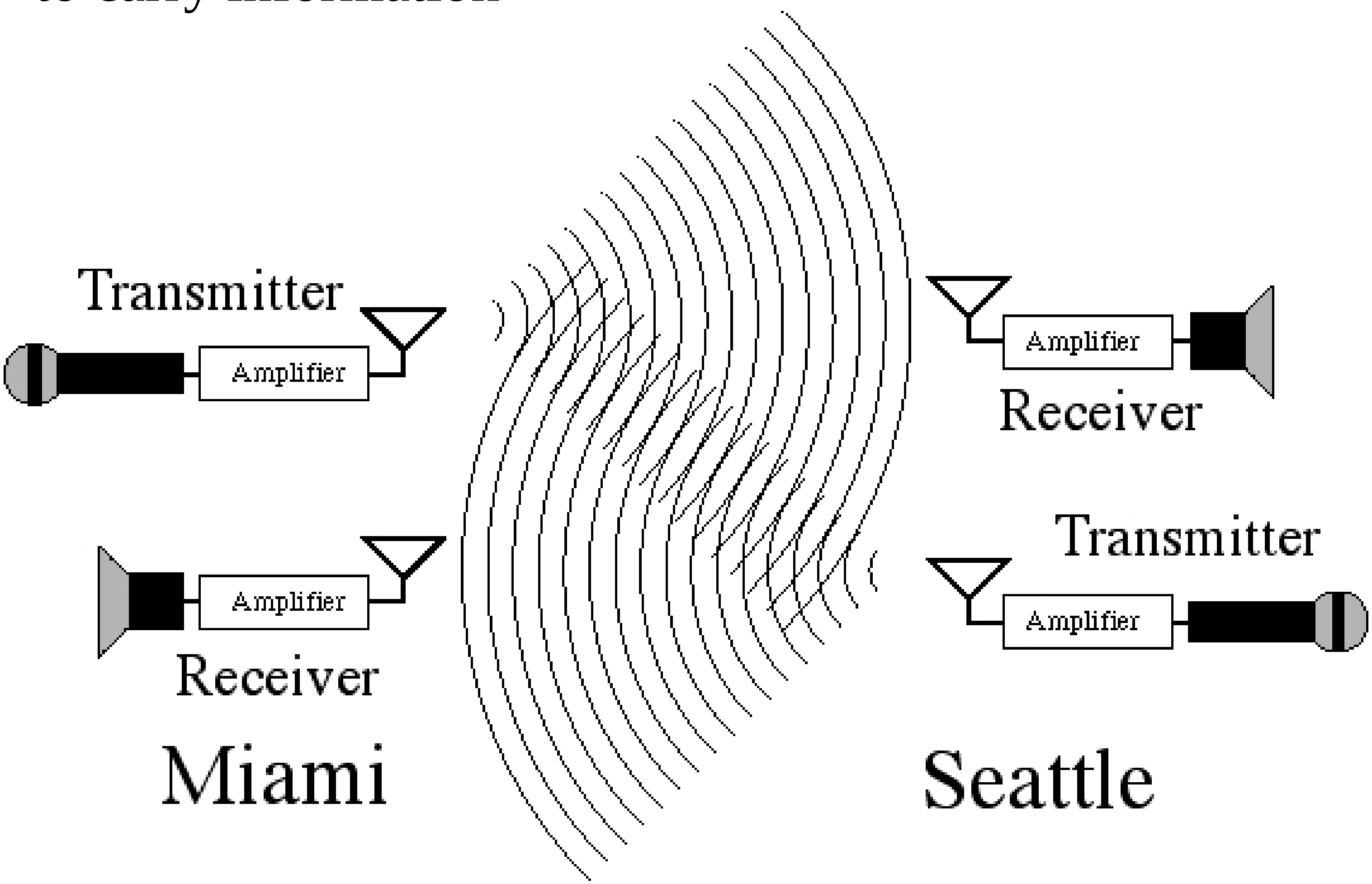
Communication Systems

A naive, absurd communication system

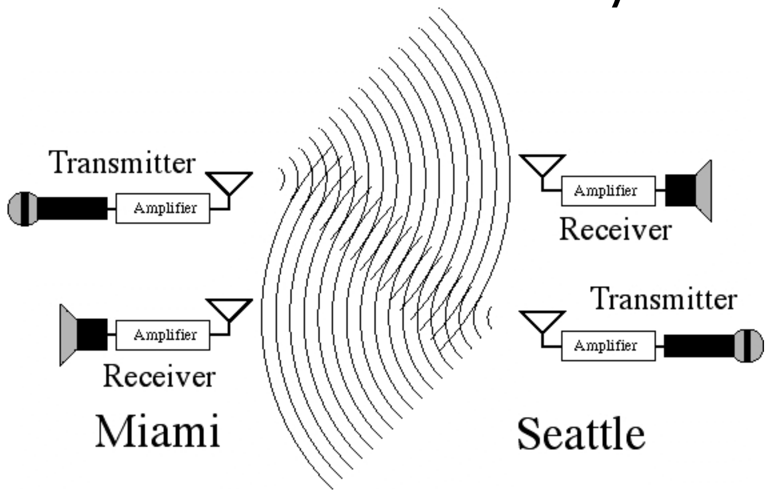


Communication Systems

A better communication system using electromagnetic waves to carry information



Communication Systems



Problems

Antenna inefficiency at audio frequencies

All transmissions from all transmitters are in the same bandwidth, thereby interfering with each other

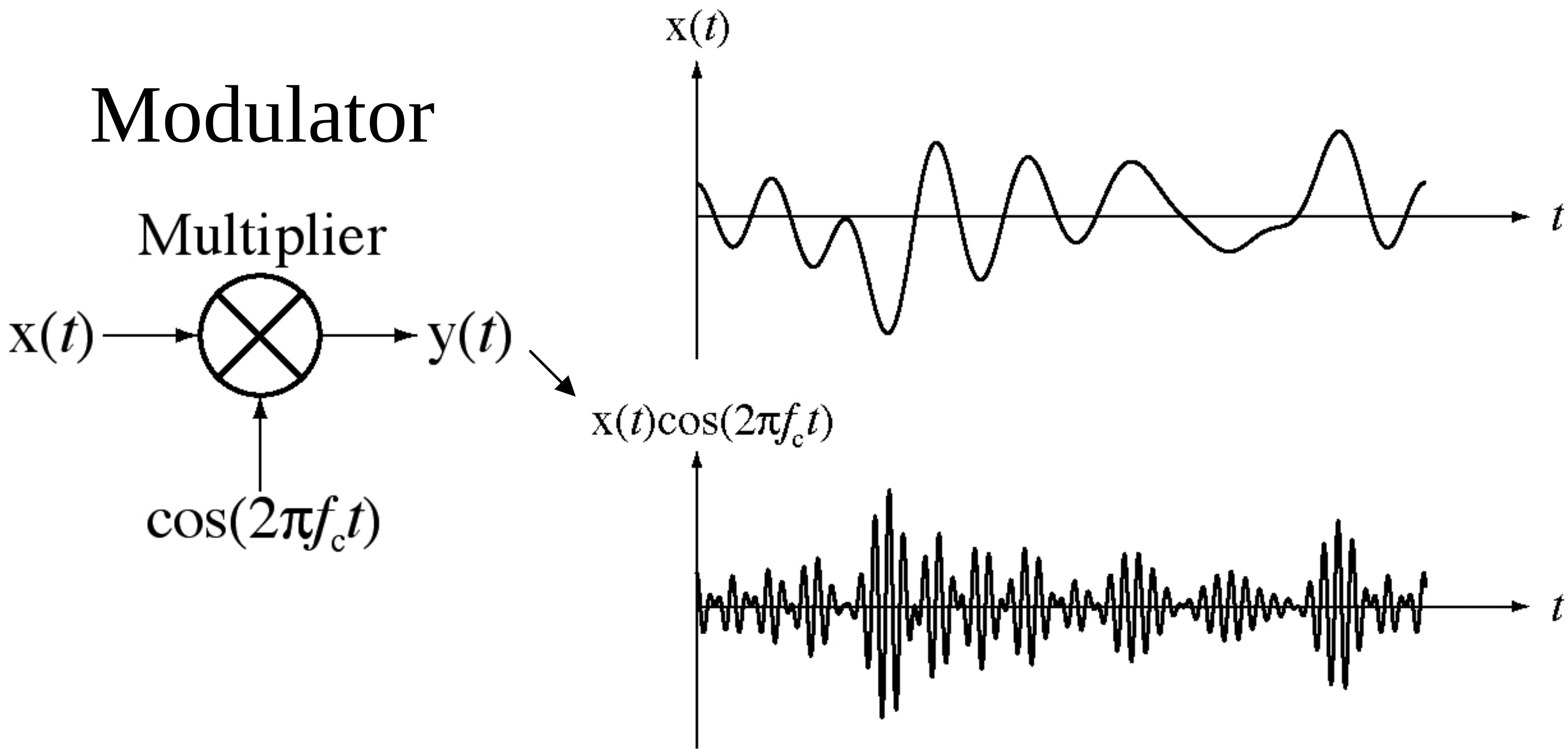
Solution *Frequency multiplexing* using modulation

Communication Systems

Double-Sideband Suppressed-Carrier (DSBSC) Modulation

$$y(t) = x(t)\cos(2\pi f_c t)$$

Modulator

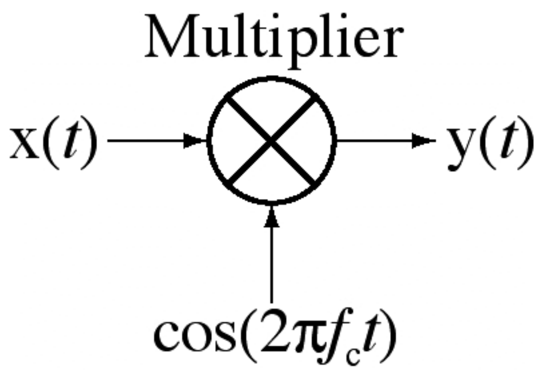


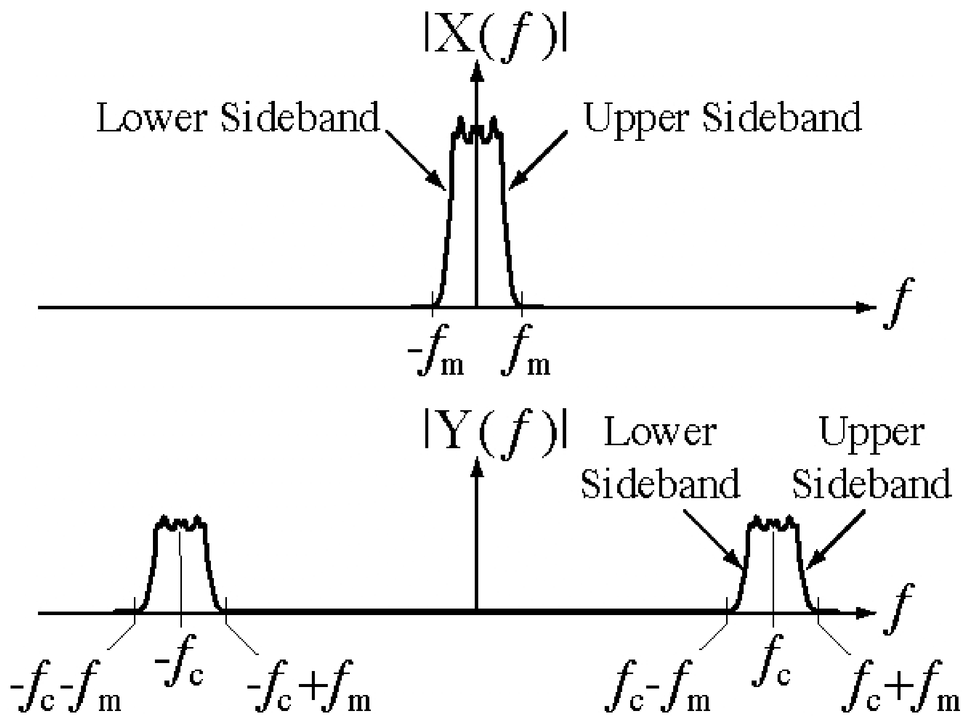
Communication Systems

Double-Sideband Suppressed-Carrier (DSBSC) Modulation

$$\mathrm{Y}(f) = \mathrm{X}(f) * \frac{1}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right]$$

Modulator



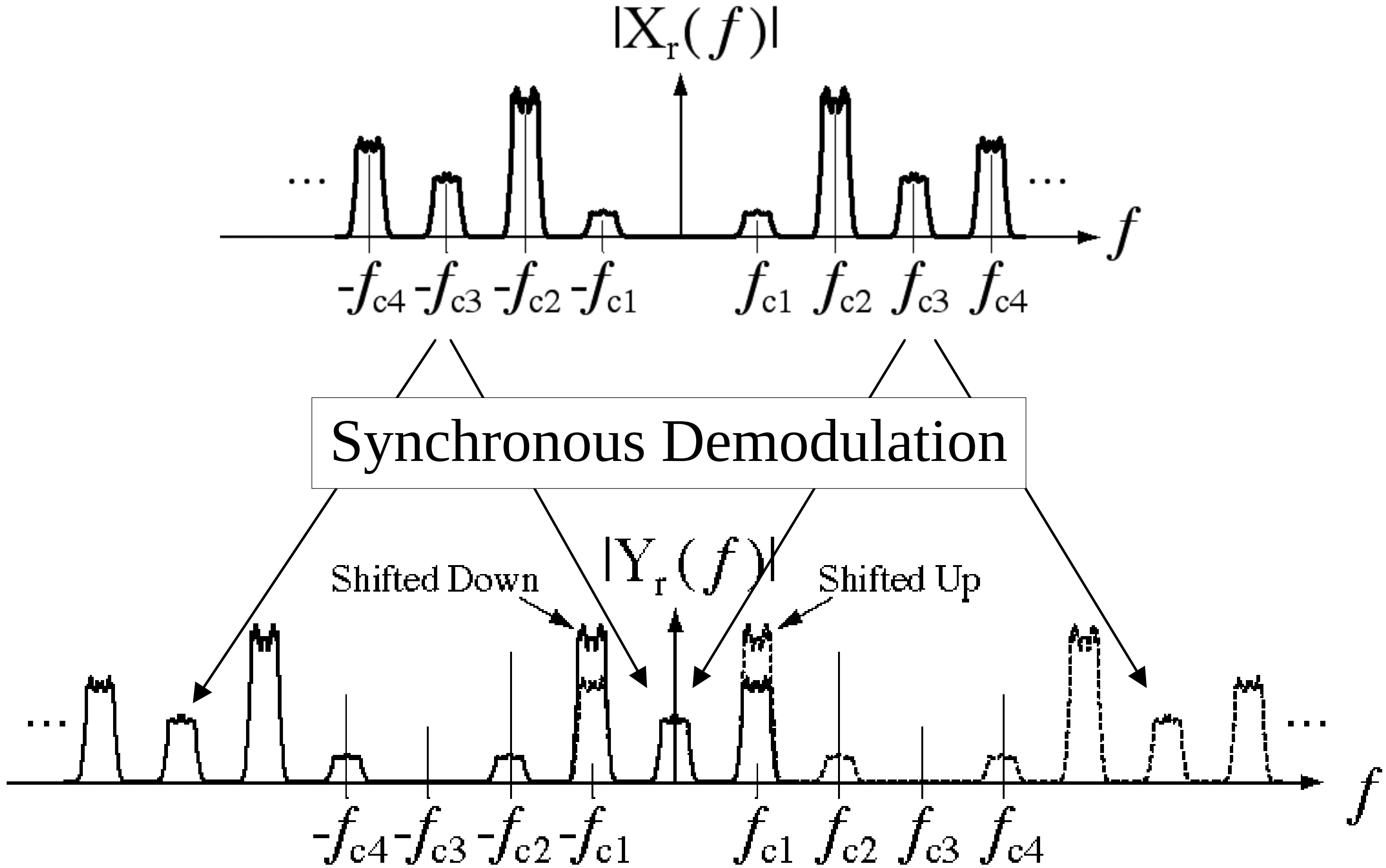


Frequency multiplexing is using a different carrier frequency, f_c , for each transmitter.

Communication Systems

Double-Sideband Suppressed-Carrier (DSBSC) Modulation

Typical received signal by an antenna

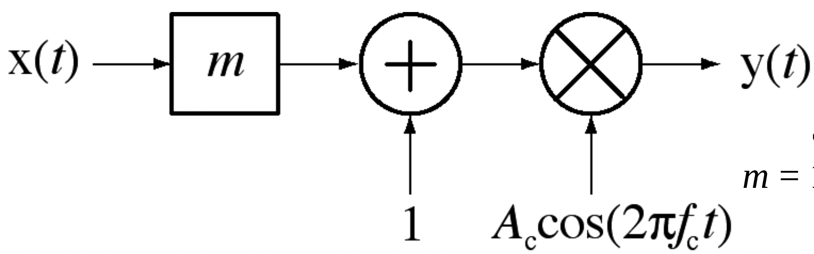


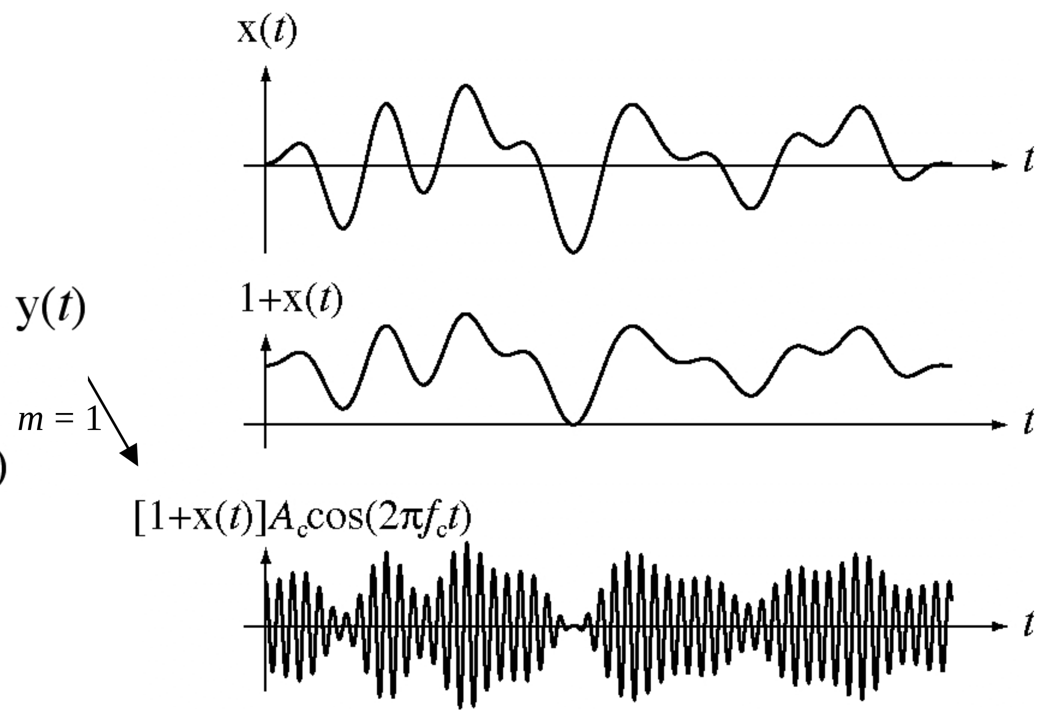
Communication Systems

Double-Sideband Transmitted-Carrier (DSBTC) Modulation

$$y(t) = [1 + m\,x(t)]A_c \cos(2\pi f_c t)$$

Modulator

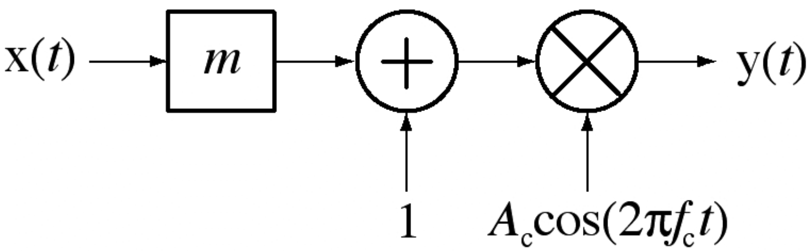


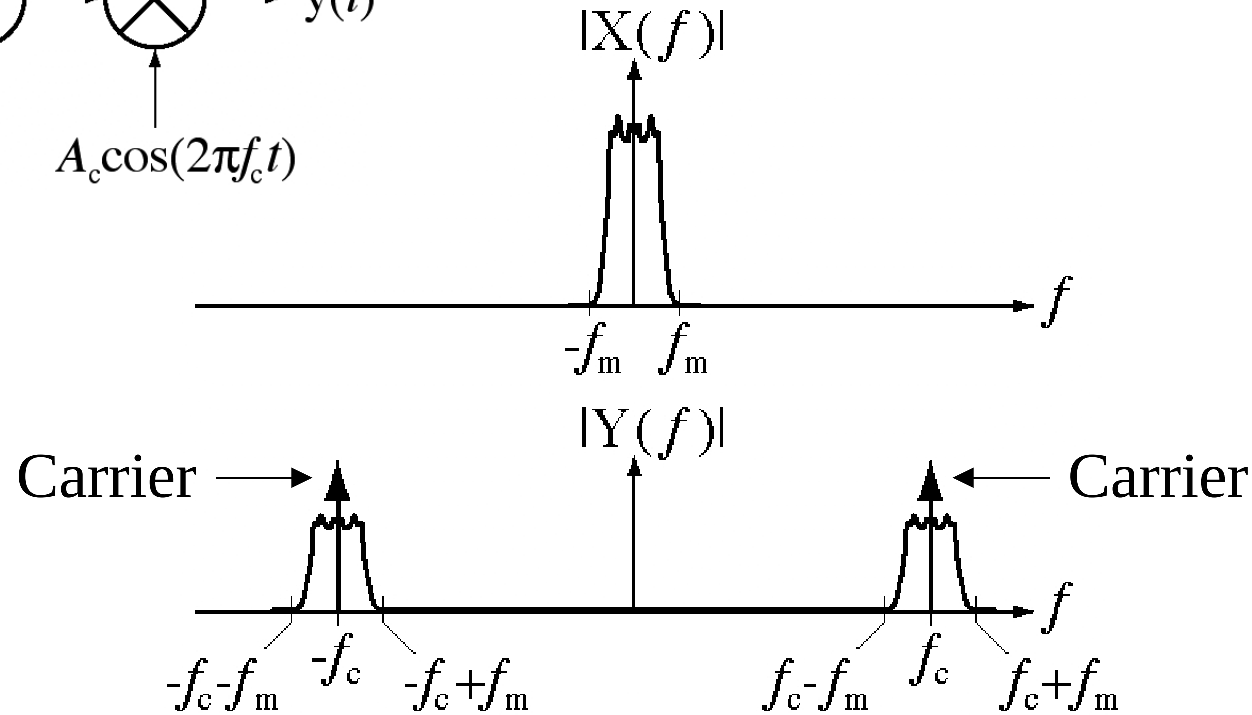


Communication Systems

Double-Sideband Transmitted-Carrier (DSBTC) Modulation

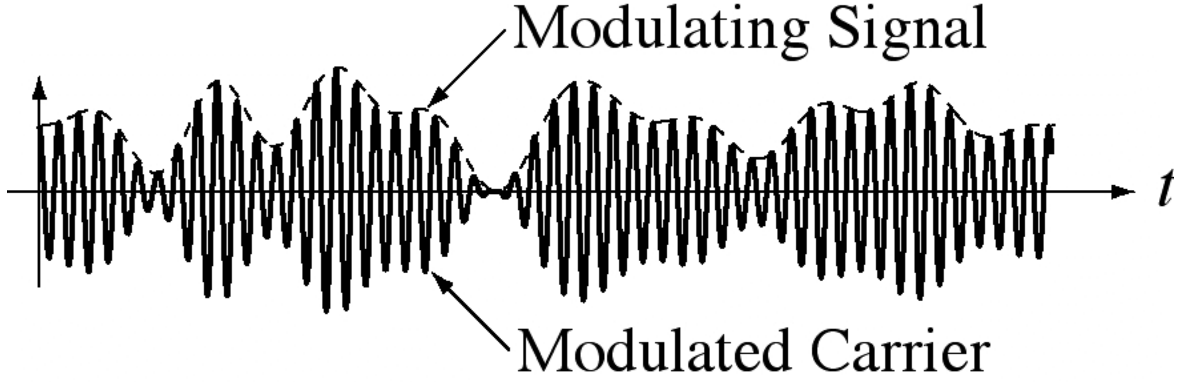
Modulator



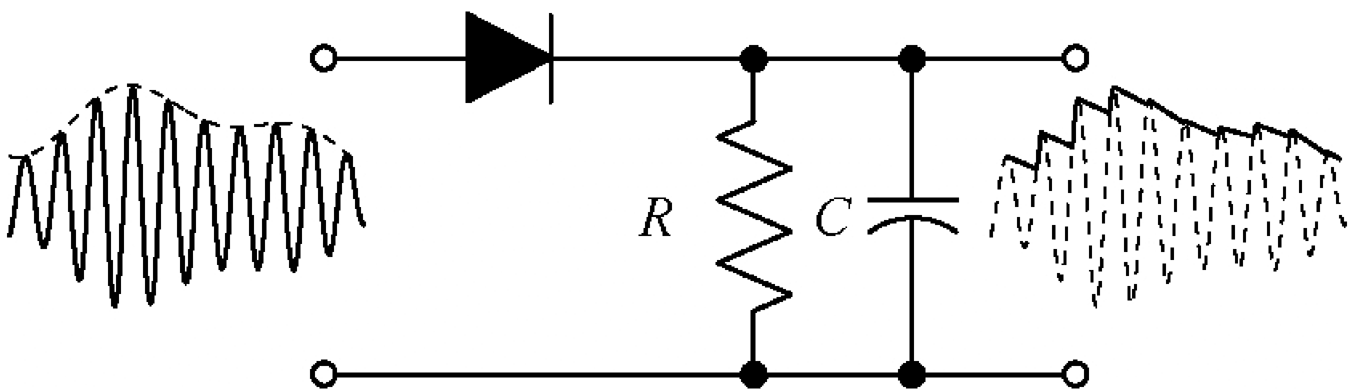


Communication Systems

Double-Sideband Transmitted-Carrier (DSBTC) Modulation

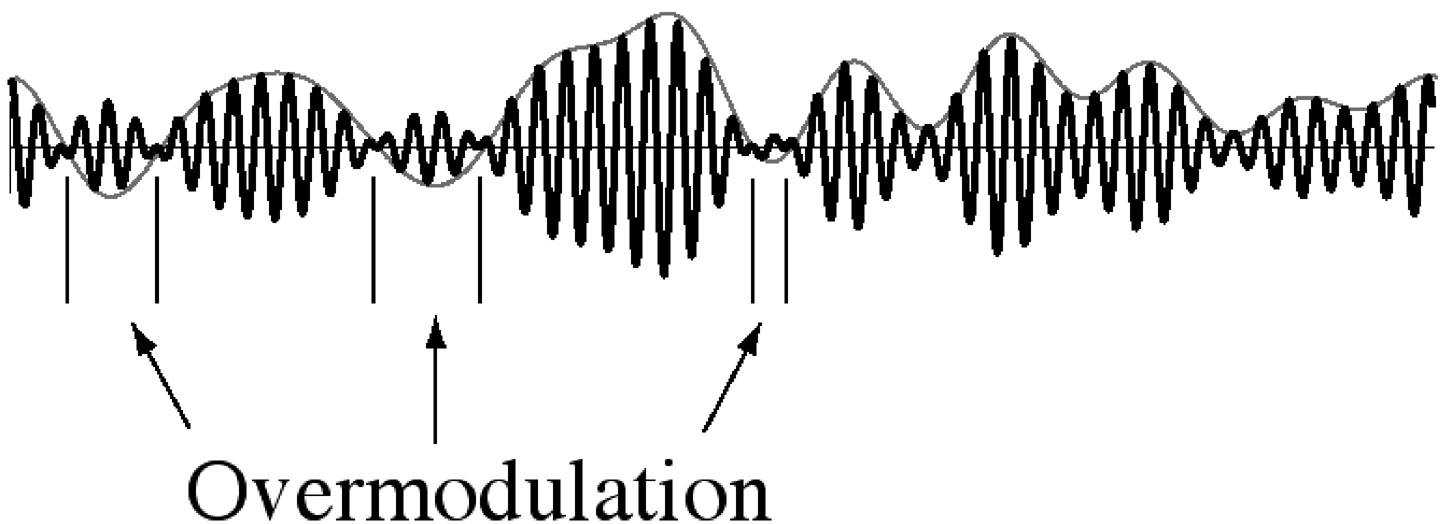


Envelope Detector



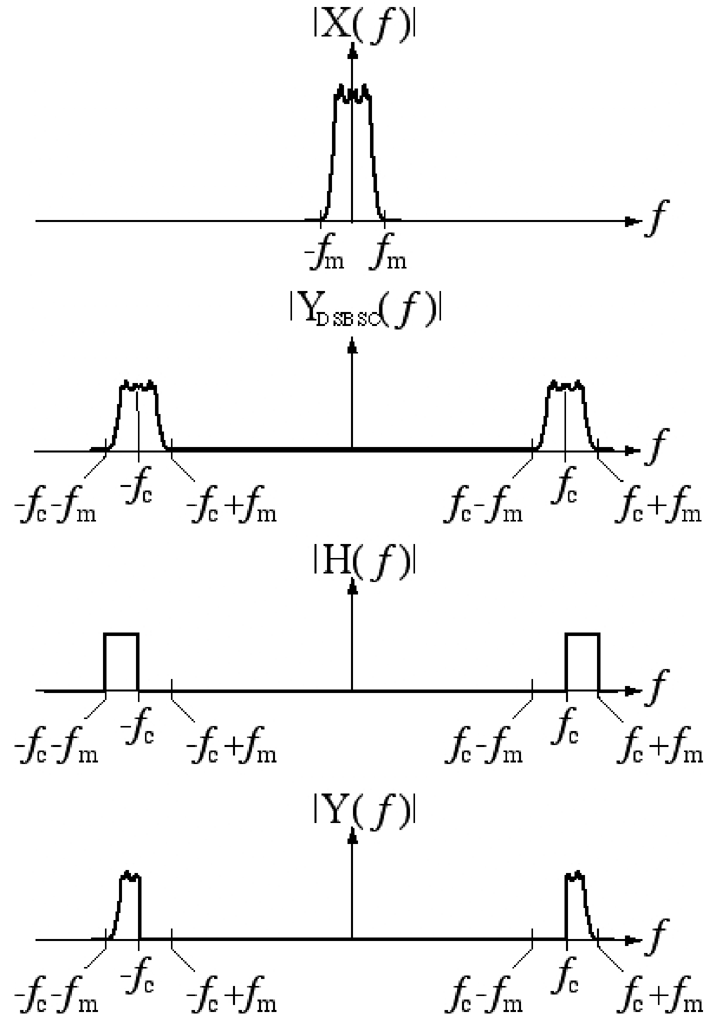
Communication Systems

Double-Sideband Transmitted-Carrier (DSBTC) Modulation

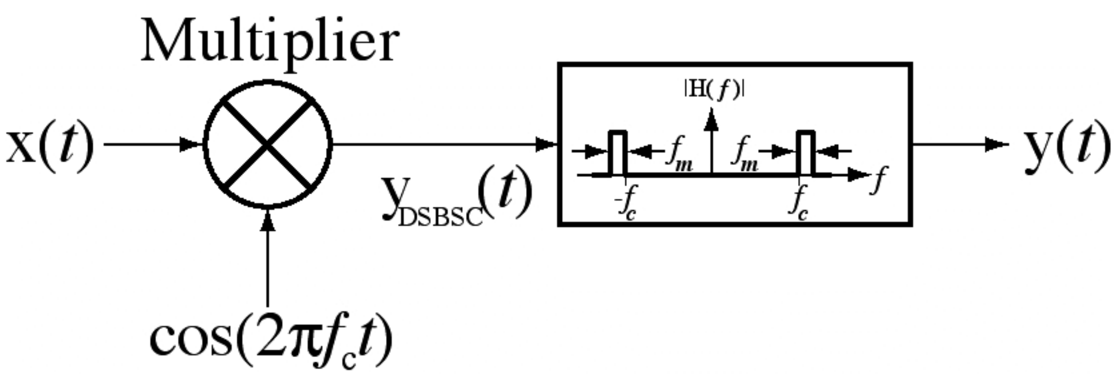


Communication Systems

Single-Sideband Suppressed-Carrier (SSBSC) Modulation

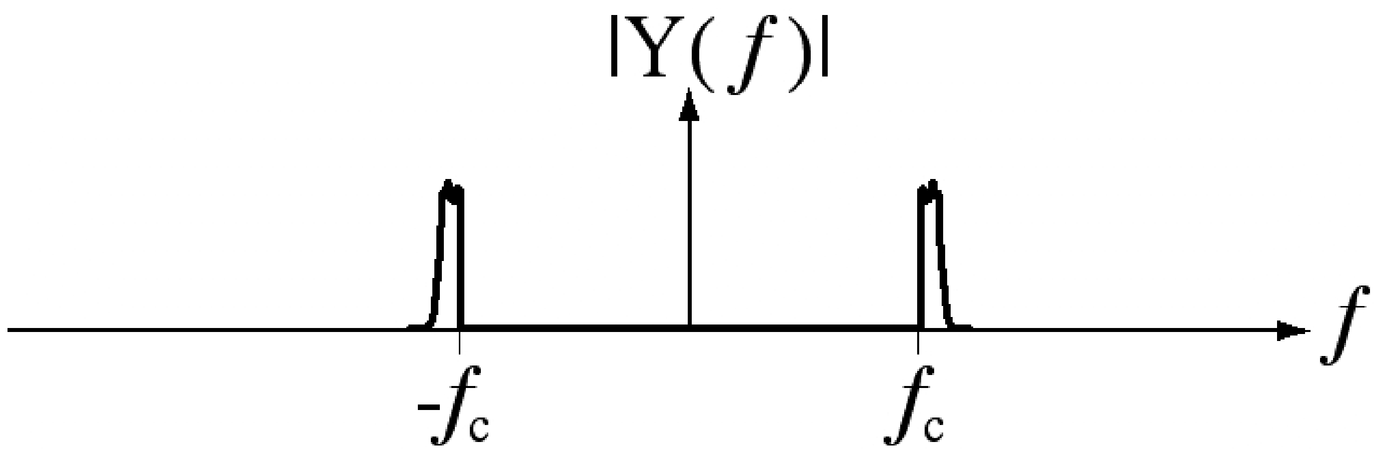


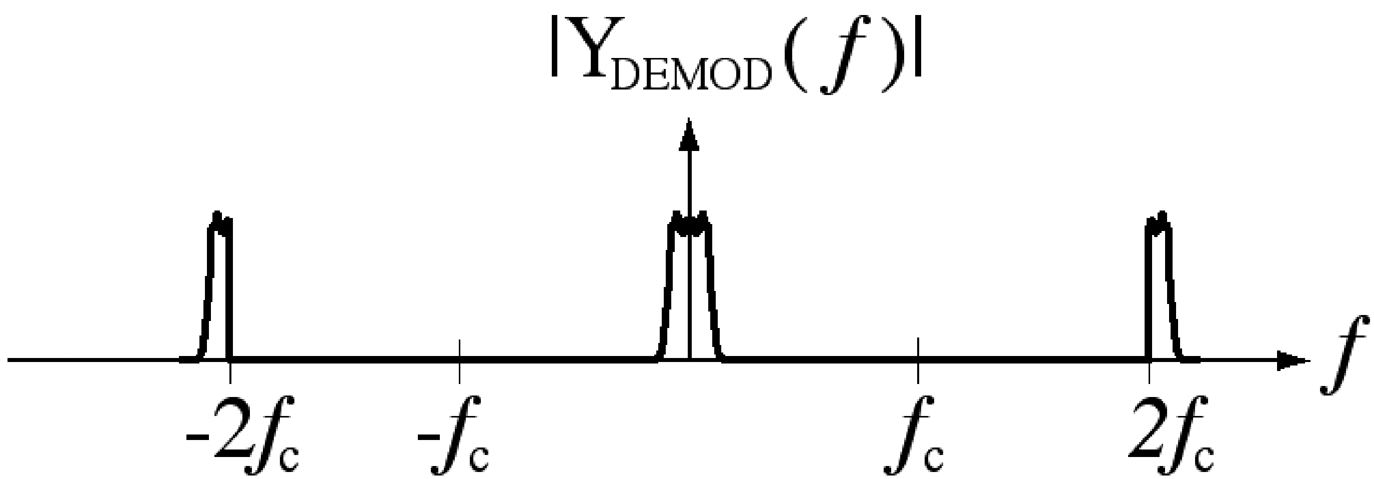
Modulator



Communication Systems

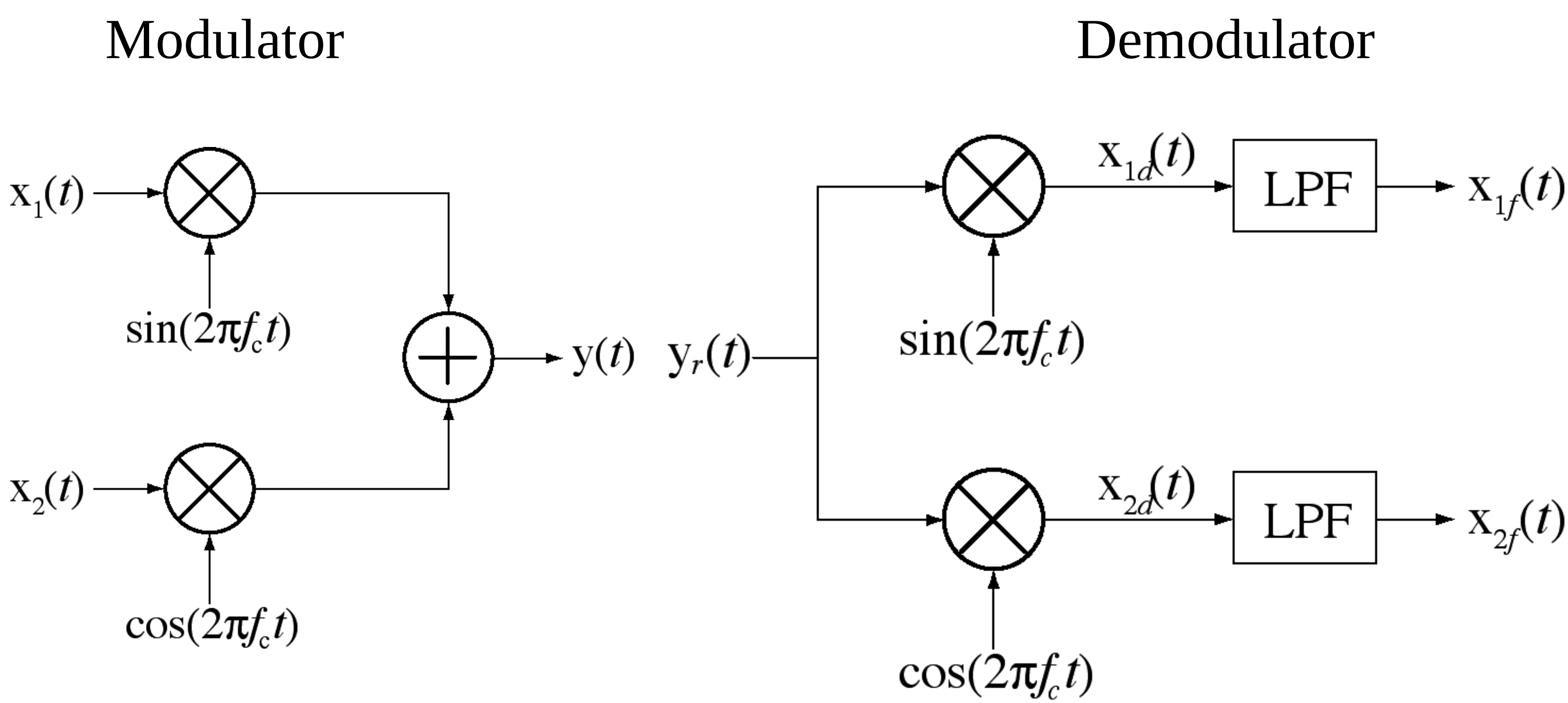
Single-Sideband Suppressed-Carrier (SSBSC) Modulation





Communication Systems

Quadrature Carrier Modulation



Phase and Group Delay

- Through the time- shifting property of the Fourier transform, a linear phase shift as a function of frequency corresponds to simple delay
- Most real system transfer functions have a *non-linear* phase shift as a function of frequency
- Non-linear phase shift delays some frequency components more than others
- This leads to the concepts of *phase delay* and *group delay*

Phase and Group Delay

To illustrate phase and group delay let a system be excited by

$$\mathrm{x}(t) = A\underbrace{\cos(\omega_m t)}_{\text{Modulation}}\underbrace{\cos(\omega_c t)}_{\text{Carrier}}$$

$$\mathrm{X}(j\omega) = \frac{A\pi}{2}\begin{bmatrix}\delta(\omega-\omega_c-\omega_m)+\delta(\omega-\omega_c+\omega_m) \\ +\delta(\omega+\omega_c-\omega_m)+\delta(\omega+\omega_c+\omega_m)\end{bmatrix}$$

an amplitude-modulated carrier. To keep the analysis simple suppose that the system has a transfer function whose magnitude is the constant, 1, over the frequency range,

$$\omega_c - \omega_m < |\omega| < \omega_c + \omega_m$$

and whose phase is

$$\phi(\omega)$$

Phase and Group Delay

The system response is

$$\mathrm{Y}(j\omega)=\frac{A\pi}{2}\begin{bmatrix}\delta(\omega-\omega_c-\omega_m)+\delta(\omega-\omega_c+\omega_m)\\+\delta(\omega+\omega_c-\omega_m)+\delta(\omega+\omega_c+\omega_m)\end{bmatrix}e^{j\phi(\omega)}$$

After some considerable algebra, the time-domain response can be written as

$$\mathrm{y}(t)=\underbrace{A\cos\left(\omega_c\left(t+\frac{\phi(\omega_c+\omega_m)+\phi(\omega_c-\omega_m)}{2\omega_c}\right)\right)}_{\text{Carrier}}\underbrace{\cos\left(\omega_m\left(t+\frac{\phi(\omega_c+\omega_m)-\phi(\omega_c-\omega_m)}{2\omega_m}\right)\right)}_{\text{Modulation}}$$

Phase and Group Delay

$$y(t) = A\cos\left(\omega_c\left(t + \frac{\phi(\omega_c+\omega_m)+\phi(\omega_c-\omega_m)}{2\omega_c}\right)\right)\cos\left(\omega_m\left(t + \frac{\phi(\omega_c+\omega_m)-\phi(\omega_c-\omega_m)}{2\omega_m}\right)\right)$$

Carrier — Modulation

In this expression it is apparent that the carrier is shifted in time by

$$\frac{\phi(\omega_c+\omega_m)+\phi(\omega_c-\omega_m)}{2\omega_c}$$

and the modulation is shifted in time by

$$\frac{\phi(\omega_c+\omega_m)-\phi(\omega_c-\omega_m)}{2\omega_m}$$

Phase and Group Delay

If the phase function is a linear function of frequency,

$$\phi(\omega) = -K\omega$$

the two delays are the same, -*K*. If the phase function is the non-linear function,

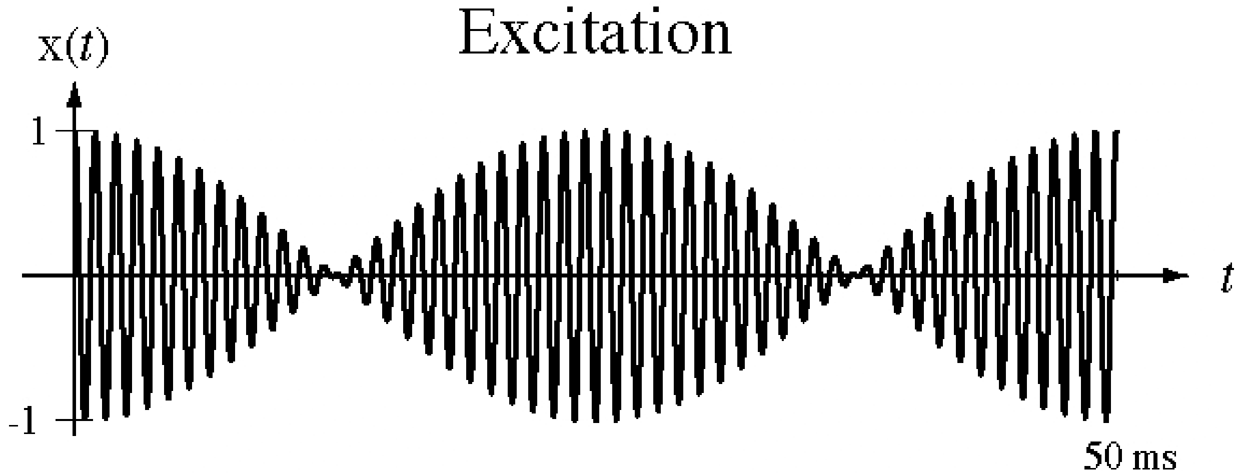
$$\phi(\omega) = -\tan^{-1}\left(2\frac{\omega}{\omega_c}\right)$$

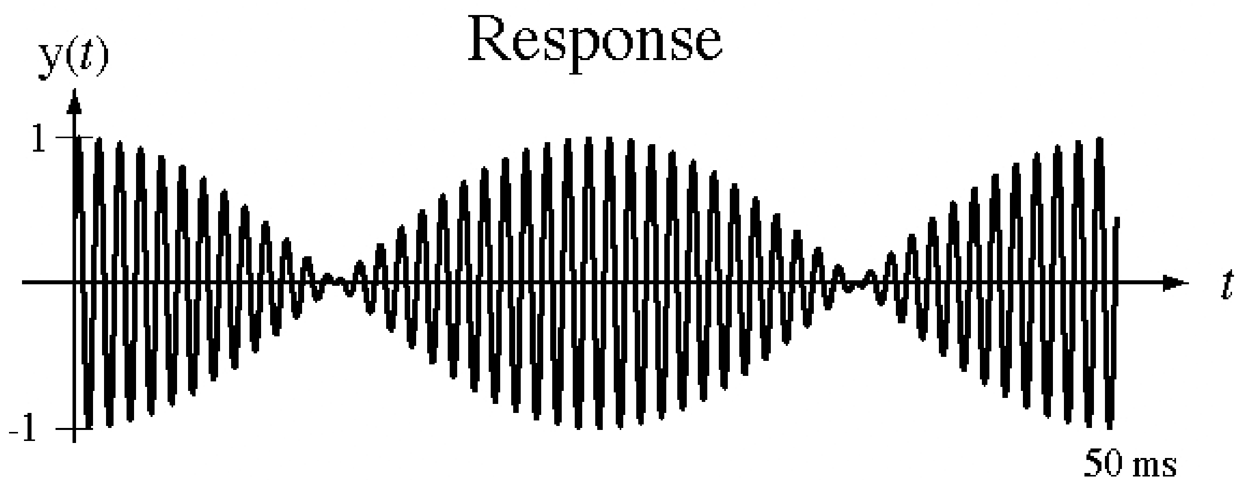
which is typical of a single-pole lowpass filter, with

$$\omega_c = 10\omega_m$$

the carrier delay is $\frac{1.107}{\omega_c}$ and the modulation delay is $\frac{0.4}{\omega_c}$

Phase and Group Delay

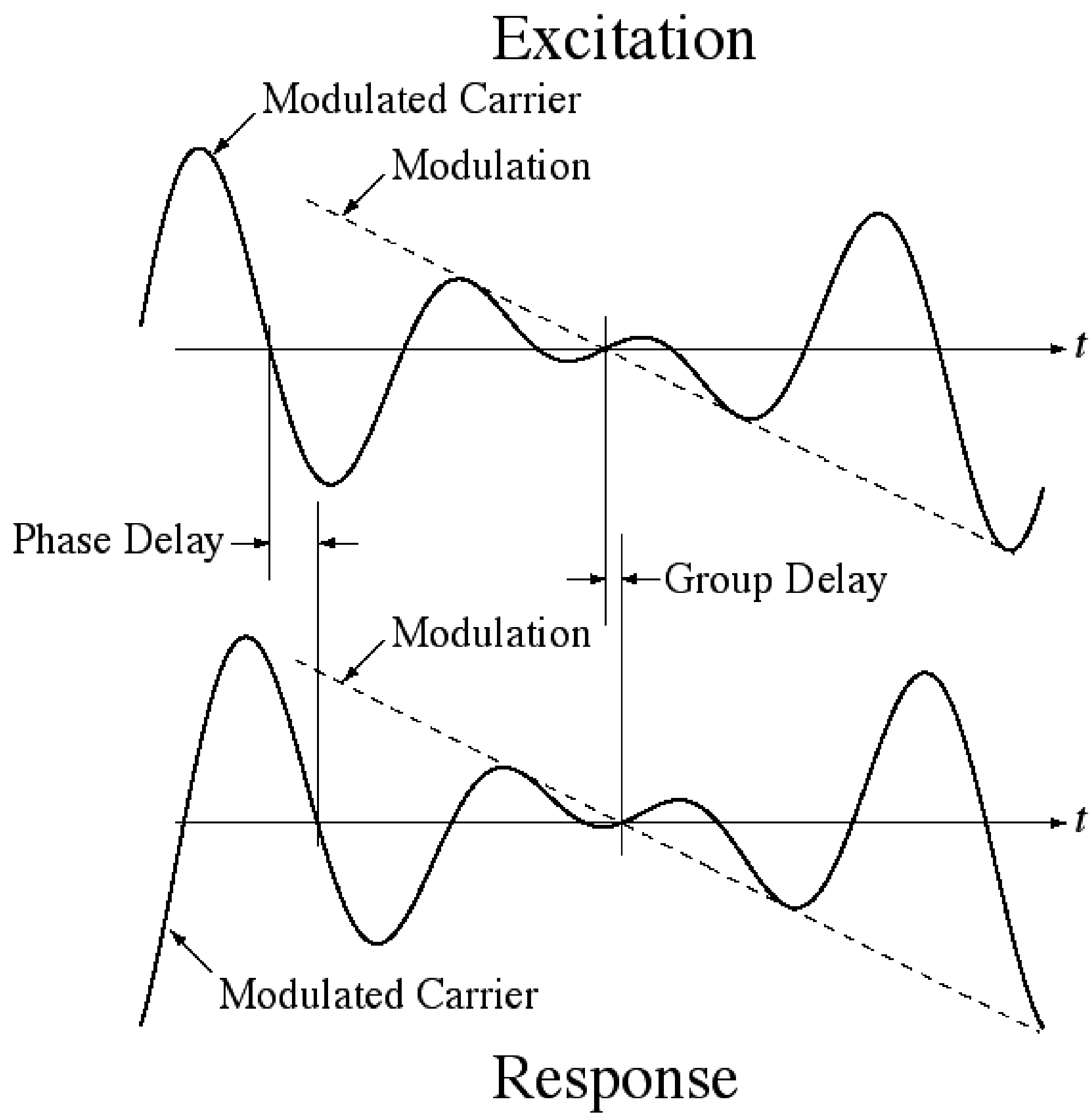




On this scale the delays are difficult to see.

Phase and Group Delay

In this magnified view the difference between carrier delay and modulation delay is visible. The delay of the carrier is phase delay and the delay of the modulation is group delay.



Phase and Group Delay

The expression for modulation delay,

$$\frac{\phi(\omega_c+\omega_m)-\phi(\omega_c-\omega_m)}{2\omega_m}$$

approaches

$$\left[\frac{d}{df}(\phi(\omega))\right]_{\omega=\omega_c}$$

as the modulation frequency approaches zero. In that same limit the expression for carrier delay,

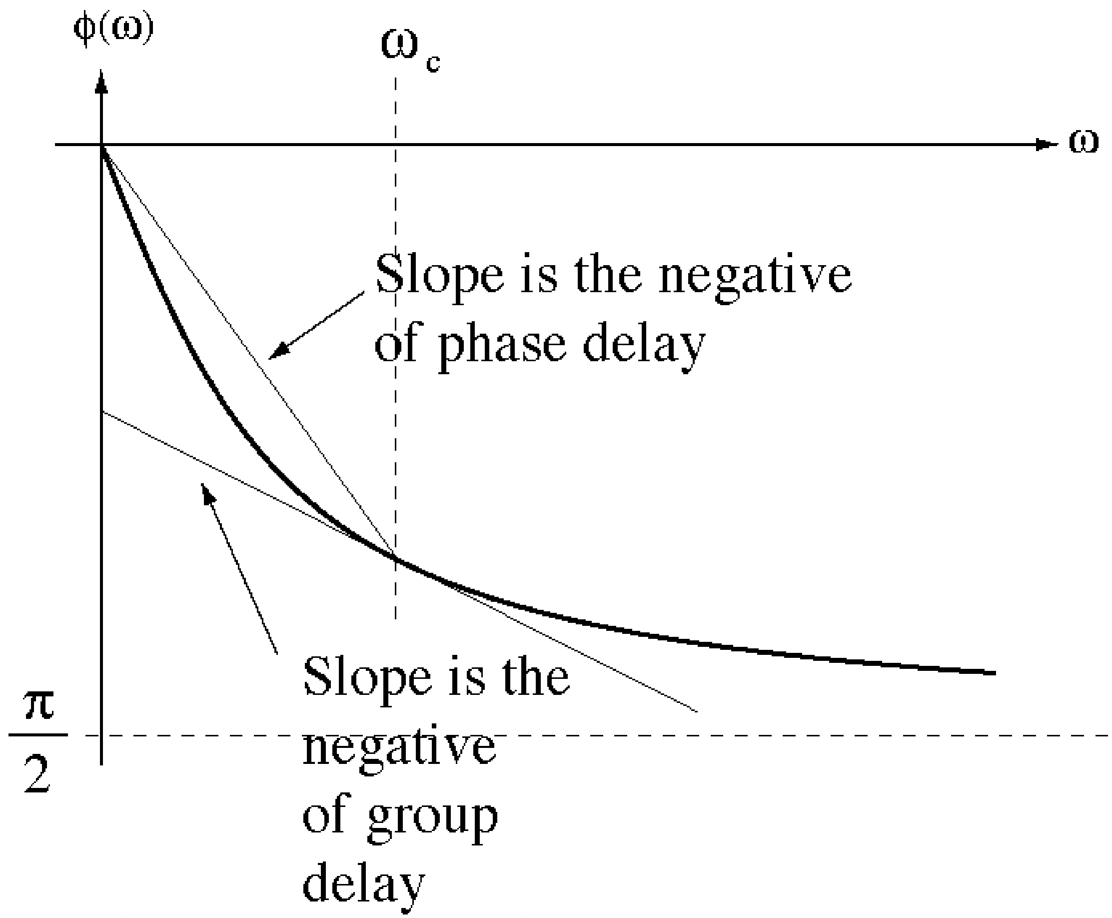
$$\frac{\phi(\omega_c+\omega_m)+\phi(\omega_c-\omega_m)}{2\omega_c}$$

approaches

$$\frac{\phi(\omega_c)}{\omega_c}$$

Phase and Group Delay

Carrier time shift is proportional to phase shift at any frequency and modulation time shift is proportional to *the derivative with respect to frequency of the phase shift.*



Group delay is defined as

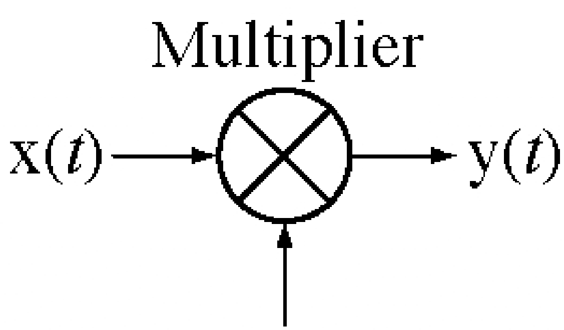
$$\tau(\omega) = -\frac{d}{d\omega}(\phi(\omega))$$

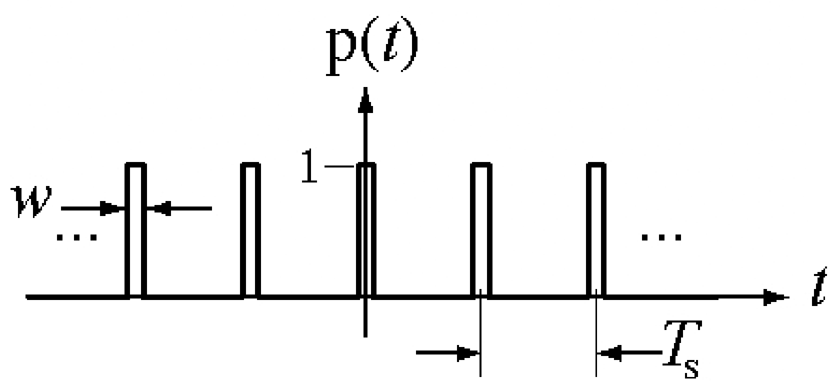
When the modulation time shift is negative, the group delay is positive.

Pulse Amplitude Modulation

Pulse amplitude modulation is like DSBSC modulation except that the "carrier" is a rectangular pulse train,

Modulator





$$\mathrm{p}(t) = \mathrm{rect}\left(\frac{t}{w}\right) * \frac{1}{T_s}\mathrm{comb}\left(\frac{t}{T_s}\right)$$

Pulse Amplitude Modulation

The response of the pulse modulator is

$$\mathrm{y}(t) = \mathrm{x}(t)\mathrm{p}(t) = \mathrm{x}(t)\left[\mathrm{rect}\left(\frac{t}{w}\right) * \frac{1}{T_s}\mathrm{comb}\left(\frac{t}{T_s}\right)\right]$$

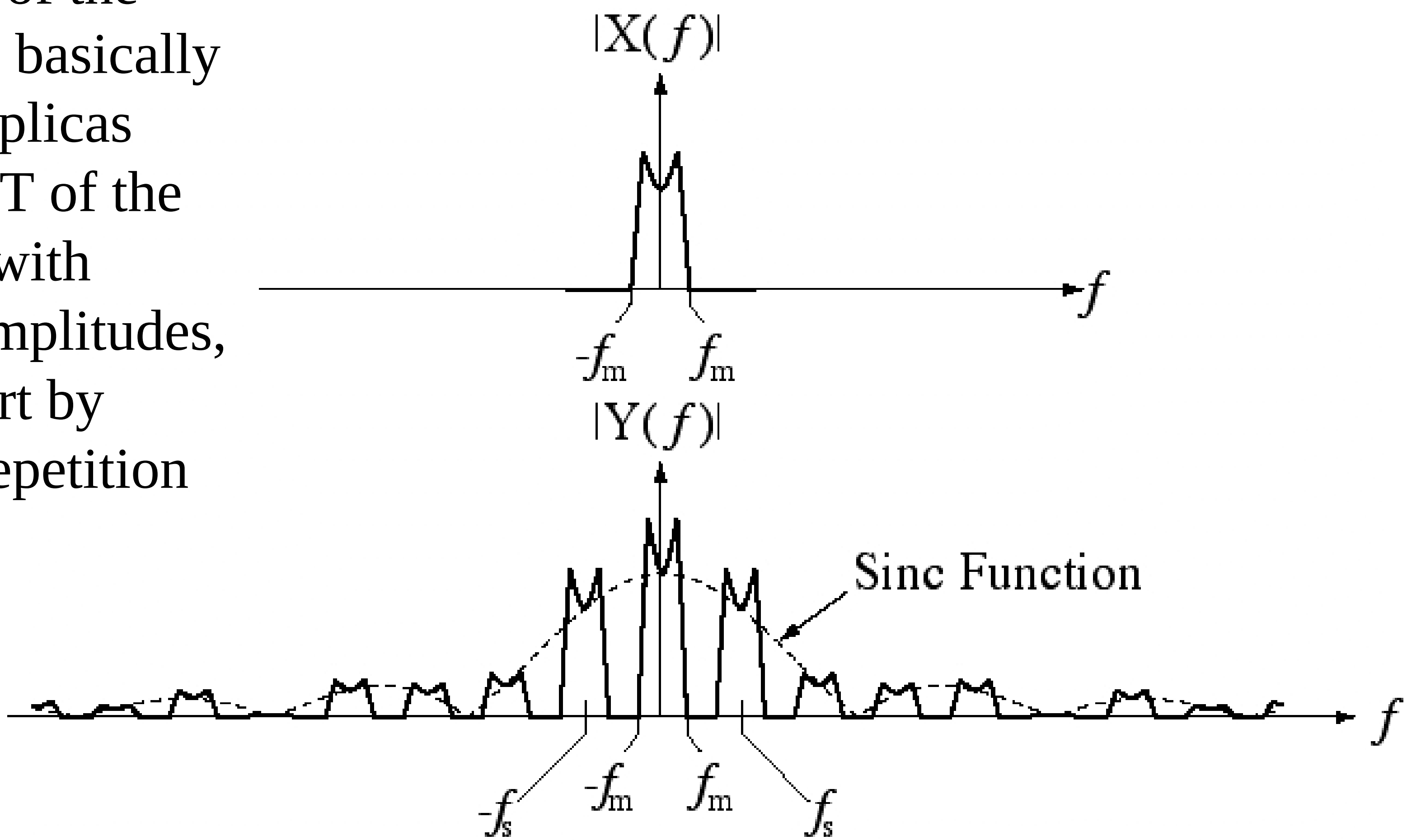
and its CTFT is

$$\mathrm{Y}(f) = wf_s \sum_{k=-\infty}^{\infty} \mathrm{sinc}(wkf_s)\,\mathrm{X}(f - kf_s)$$

where $f_s = \dfrac{1}{T_s}$

Pulse Amplitude Modulation

The CTFT of the response is basically multiple replicas of the CTFT of the excitation with different amplitudes, spaced apart by the pulse repetition rate.



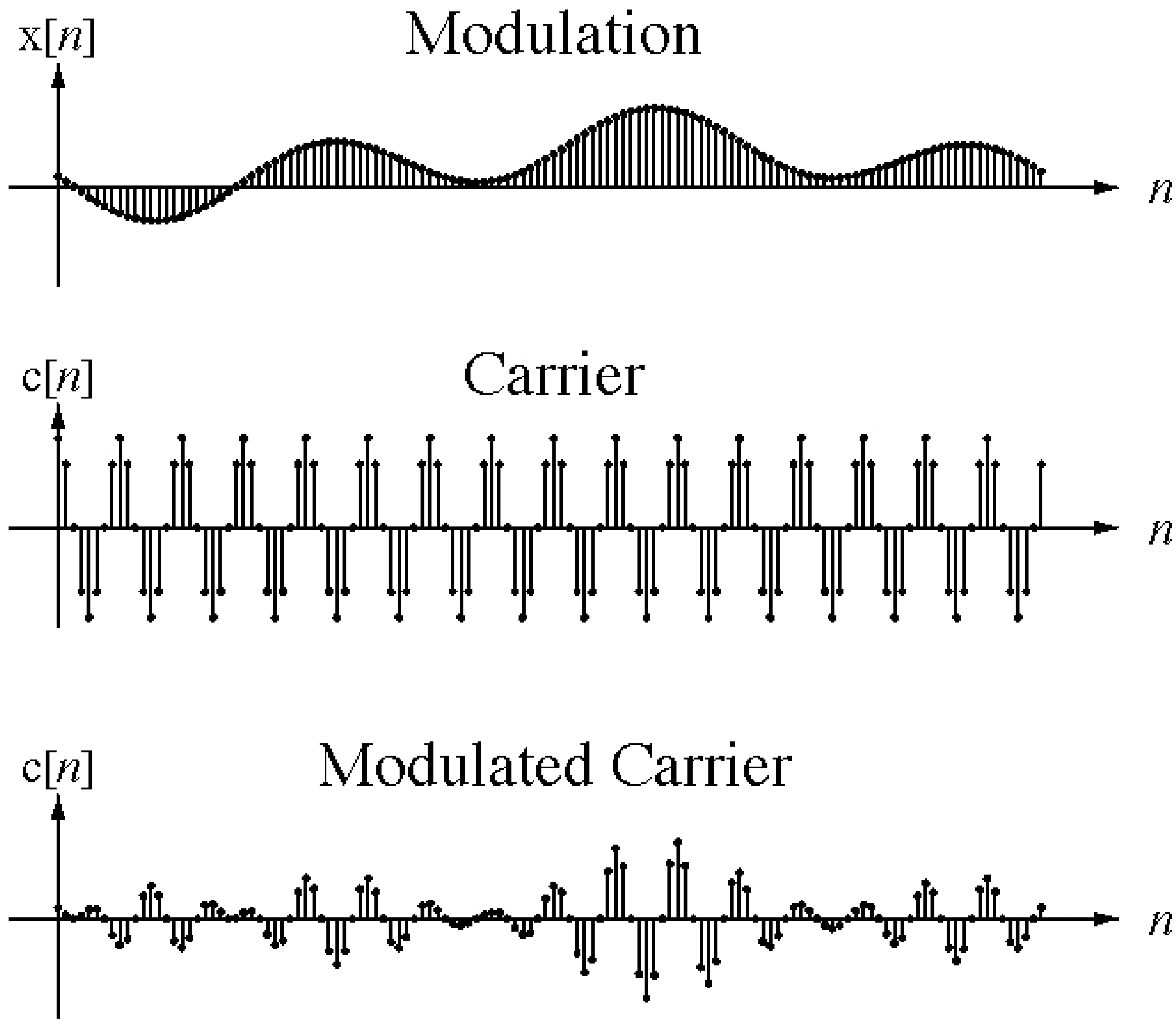
Discrete-Time Modulation

Discrete-time modulation is analogous to continuous-time modulation. A modulating signal multiplies a carrier. Let the carrier be

$$c[n] = \cos(2\pi F_0 n)$$

If the modulation is x[n], the response is

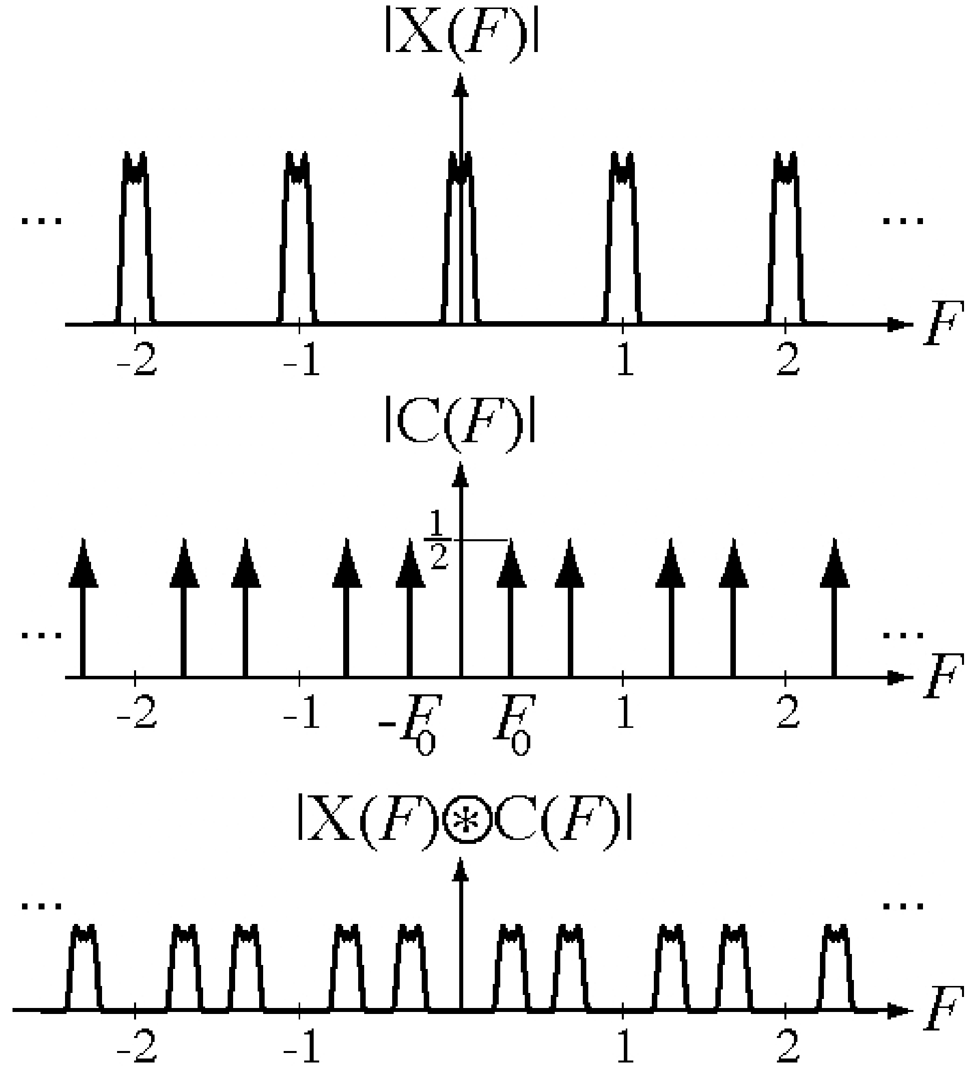
$$y[n] = x[n]\cos(2\pi F_0 n)$$



Discrete-Time Modulation

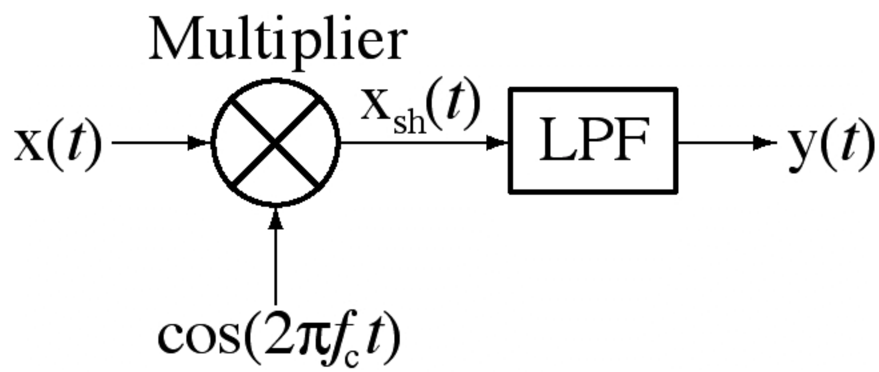
$$\mathrm{Y}(F) = \mathrm{X}(F) \circledast \mathrm{C}(F)$$

$$= \frac{1}{2}\left[\mathrm{X}(F - F_0) + \mathrm{X}(F + F_0)\right]$$



Spectral Analysis

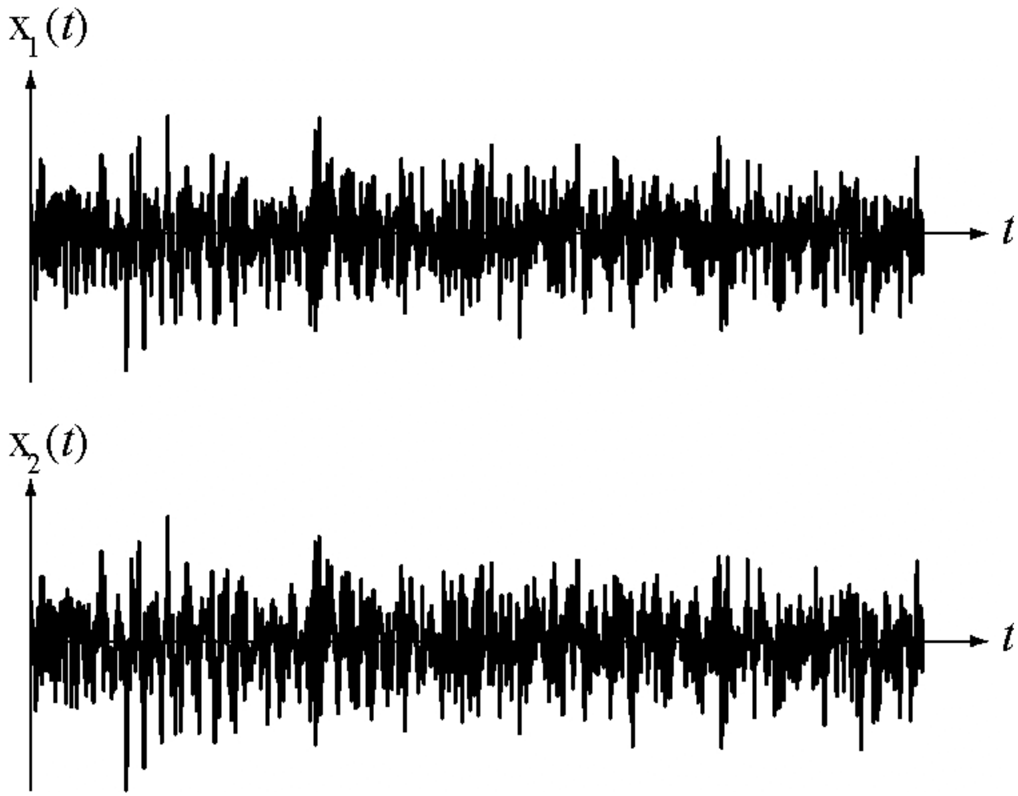
The heart of a "swept-frequency" type spectrum analyzer is a multiplier, like the one introduced in DSBSC modulation, plus a lowpass filter.



Multiplying by the cosine shifts the spectrum of x(t) by f_c and the signal power shifted into the passband of the lowpass filter is measured. Then, as the frequency, f_c , is slowly "swept" over a range of frequencies, the spectrum analyzer measures its signal power versus frequency.

Spectral Analysis

One benefit of spectral analysis is illustrated below.



These two signals are different but exactly how they are different is difficult to see by just looking at them.

Spectral Analysis

The magnitude spectra of the two signals reveal immediately what the difference is. The second signal contains a sinusoid, or something close to a sinusoid, that causes the two large "spikes".