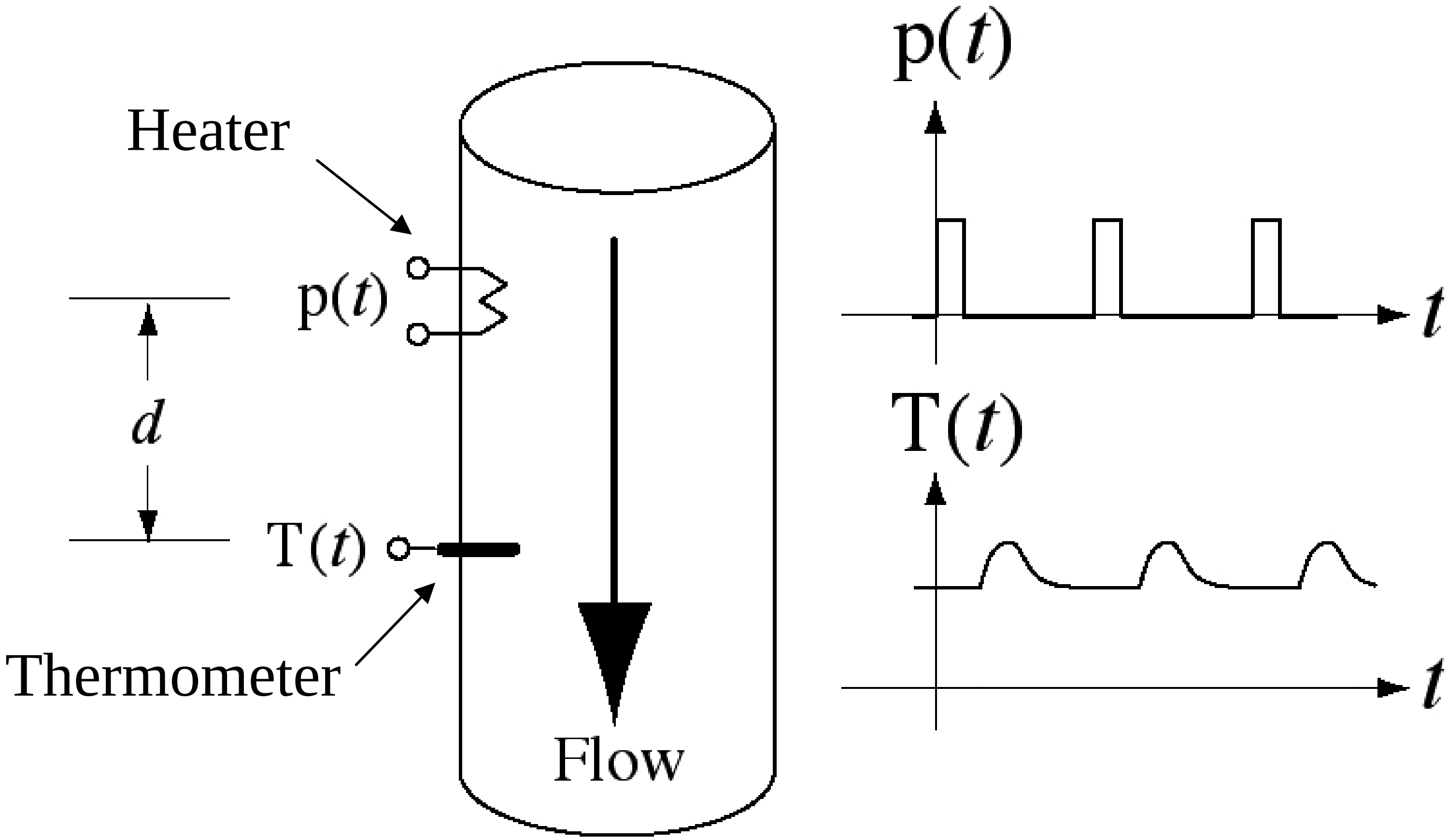
Correlation, Energy Spectral Density and Power Spectral Density

Introduction

- Relationships between signals can be just as important as characteristics of individual signals
- The relationships between excitation and/or response signals in a system can indicate the nature of the system

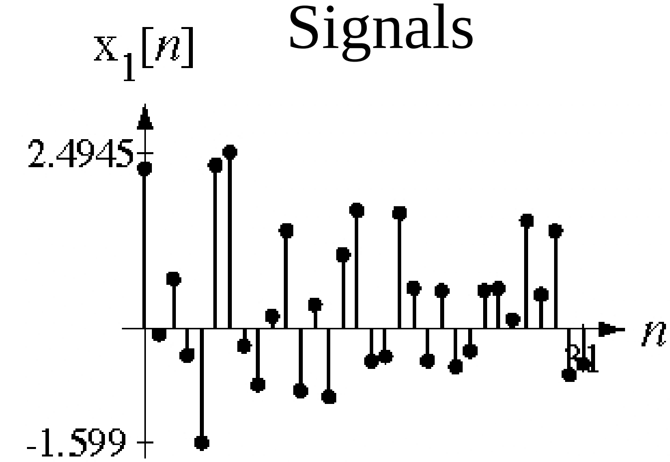
Flow Velocity Measurement

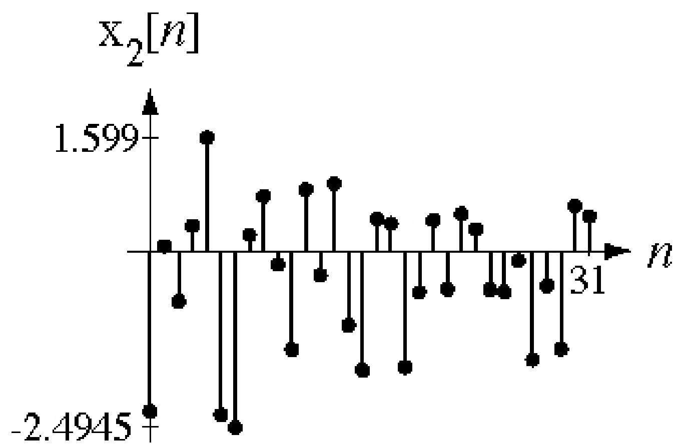


The relative timing of the two signals, p(*t*) and T(*t*), and the distance, *d*, between the heater and thermometer together determine the flow velocity.

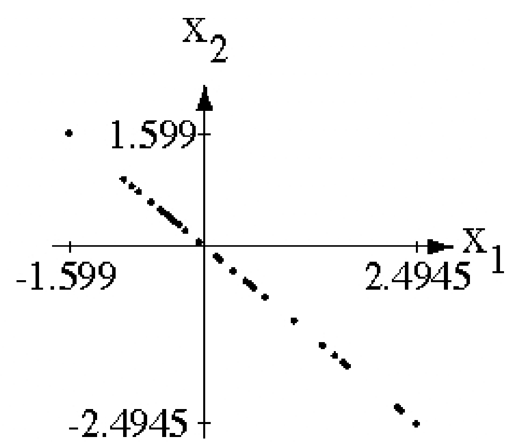
Correlograms

Signals





Correlogram

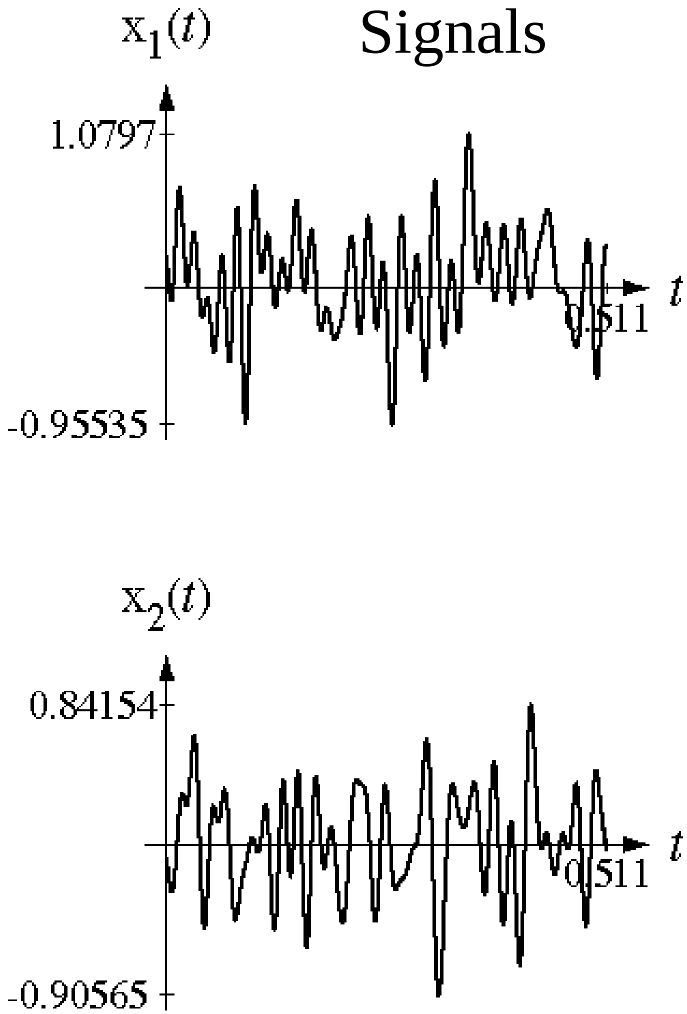


Two Completely Negatively Correlated DT Signals

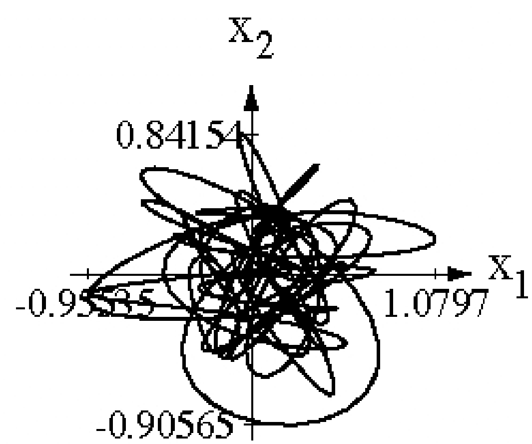
Correlograms

Two Uncorrelated CT Signals

Signals

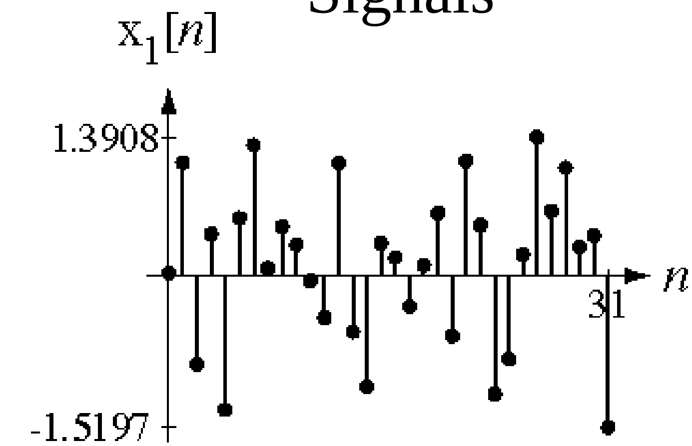


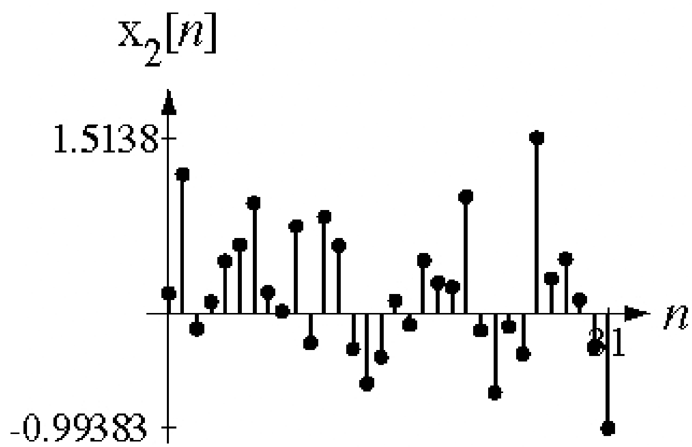
Correlogram



Correlograms

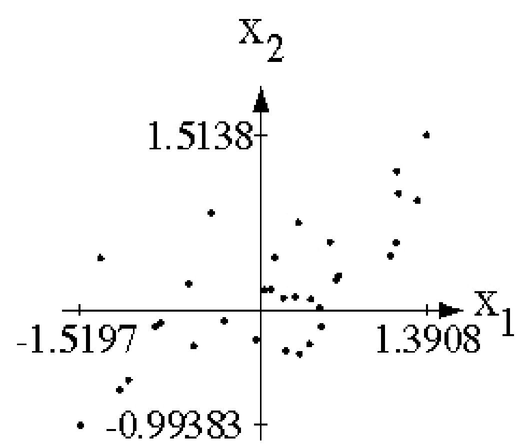
Signals





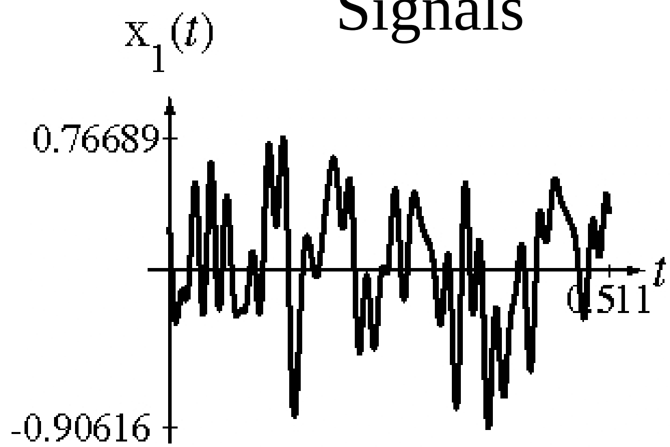
Two Partially Correlated DT Signals

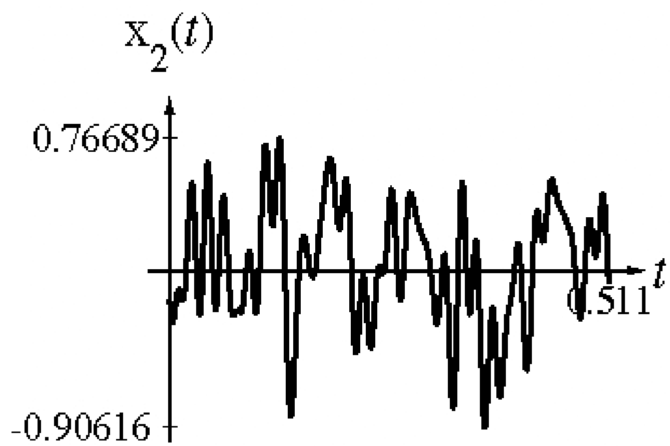
Correlogram



Correlograms

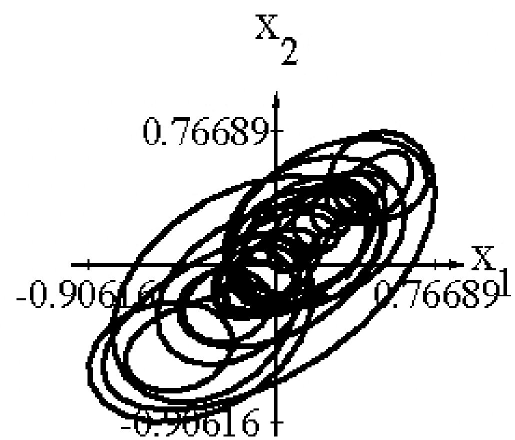
Signals





These two CT signals are not strongly correlated but would be if one were shifted in time the right amount

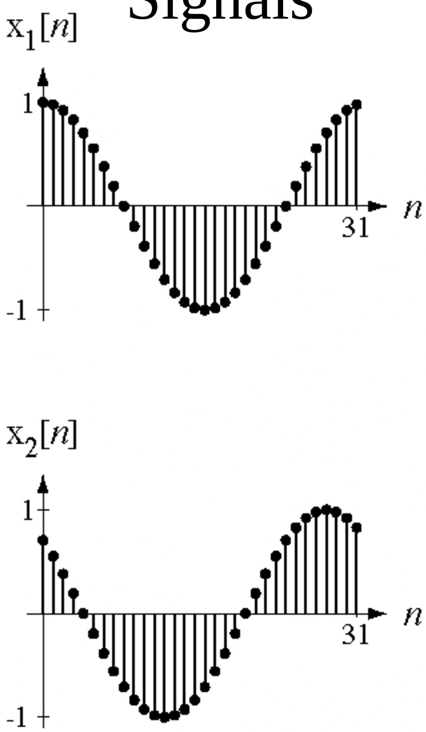
Correlogram



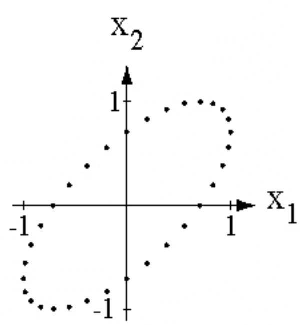
Correlograms

DT Sinusoids With a Time Delay

Signals

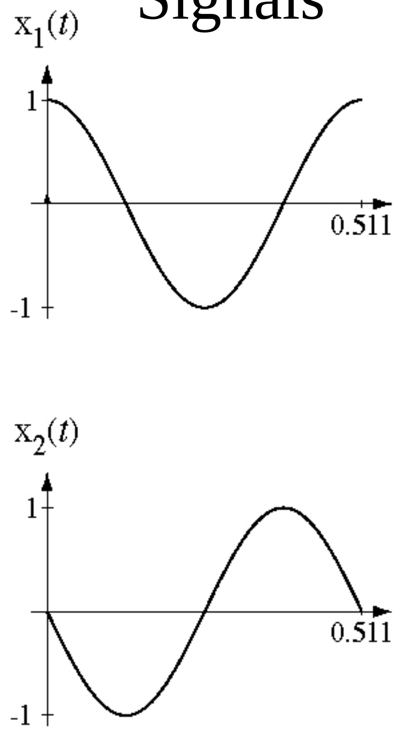


Correlogram

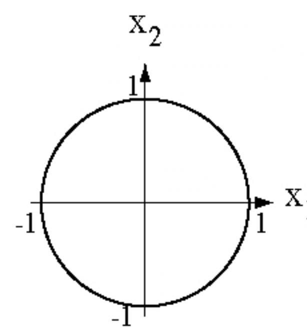


CT Sinusoids With a Time Delay

Signals



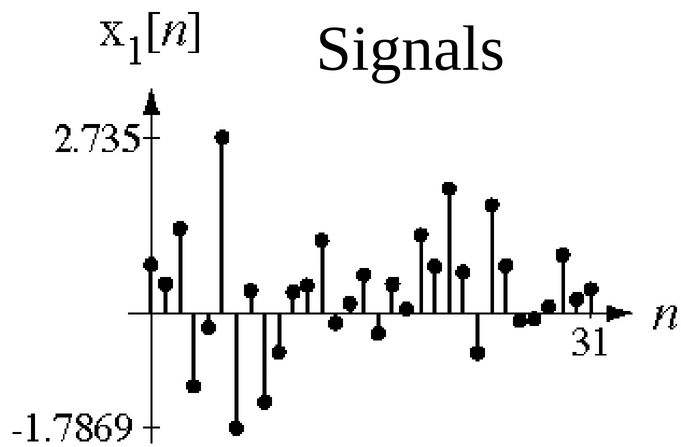
Correlogram

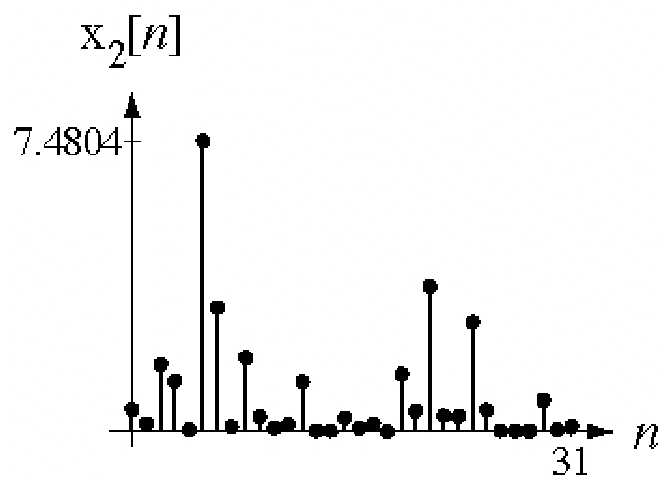


Correlograms

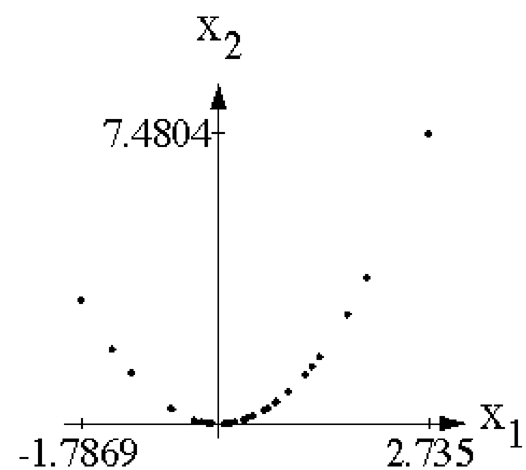
Two Non-Linearly Related DT Signals

Signals





Correlogram

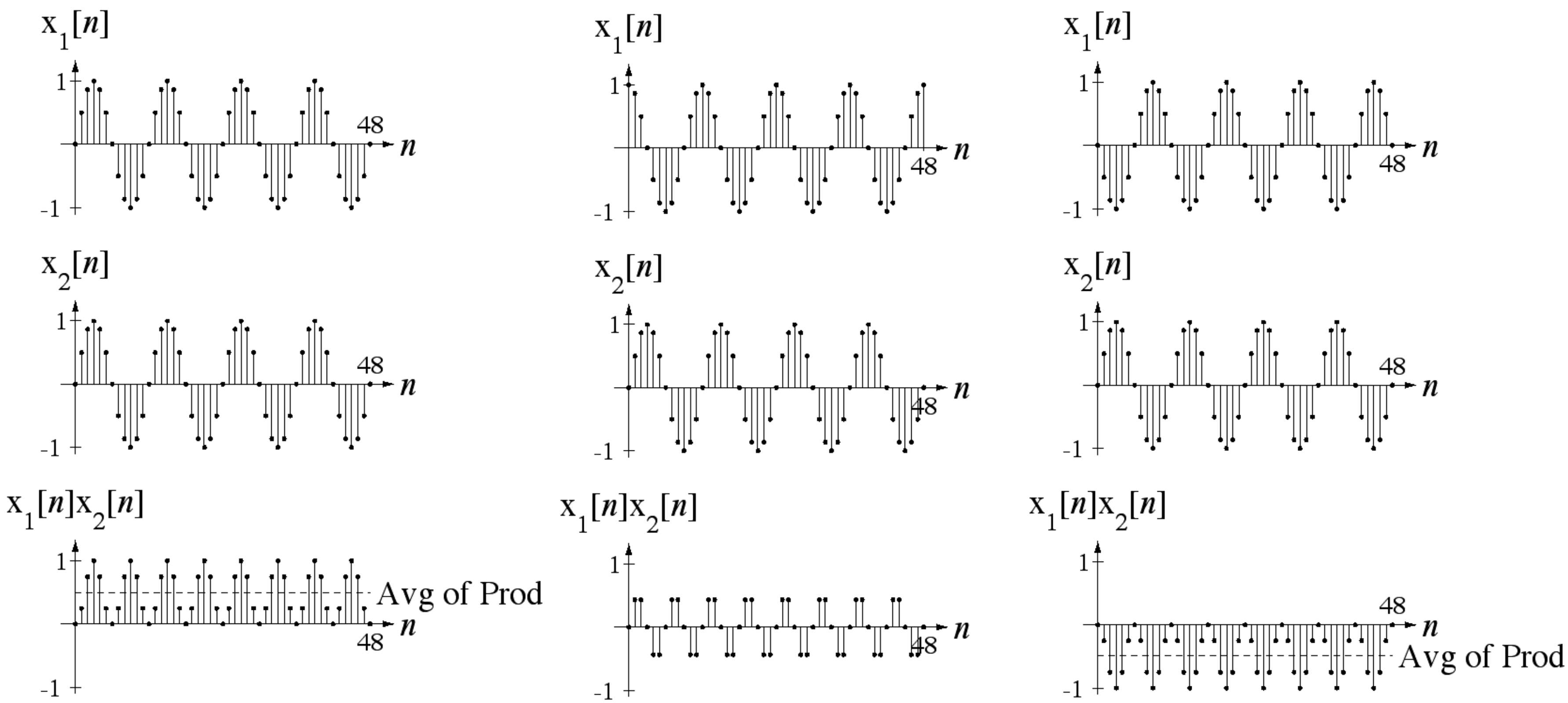


The Correlation Function

Positively Correlated DT Sinusoids with Zero Mean

Uncorrelated DT Sinusoids with Zero Mean

Negatively Correlated DT Sinusoids with Zero Mean

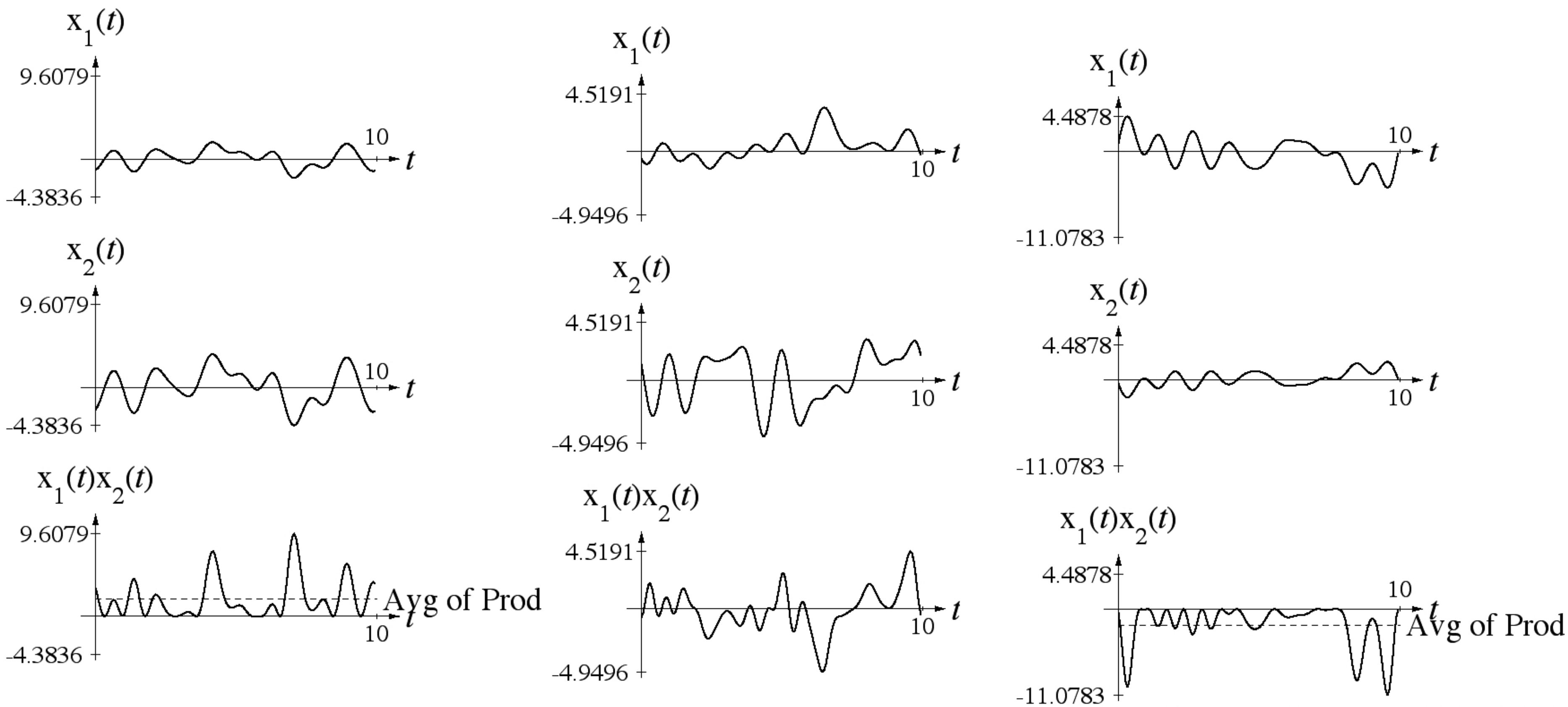


The Correlation Function

Positively Correlated Random CT Signals with Zero Mean

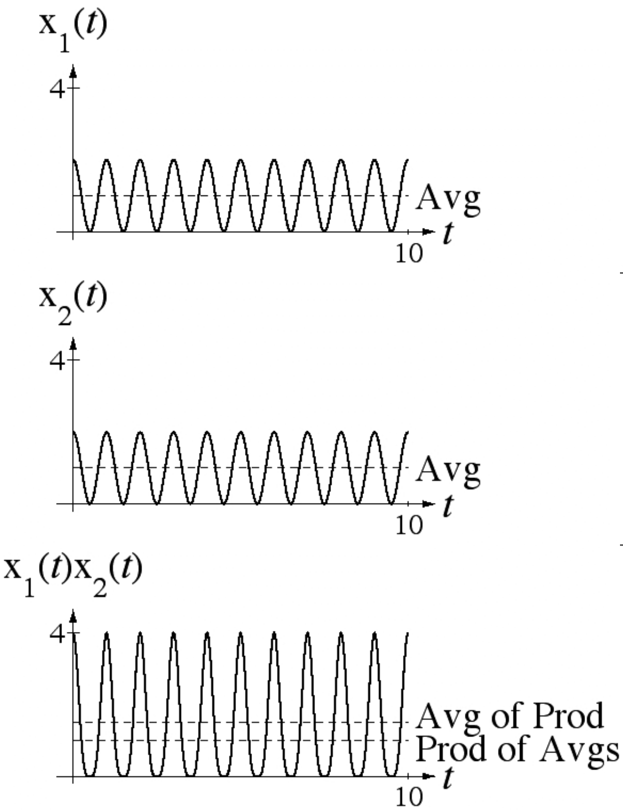
Uncorrelated Random CT Signals with Zero Mean

Negatively Correlated Random CT Signals with Zero Mean

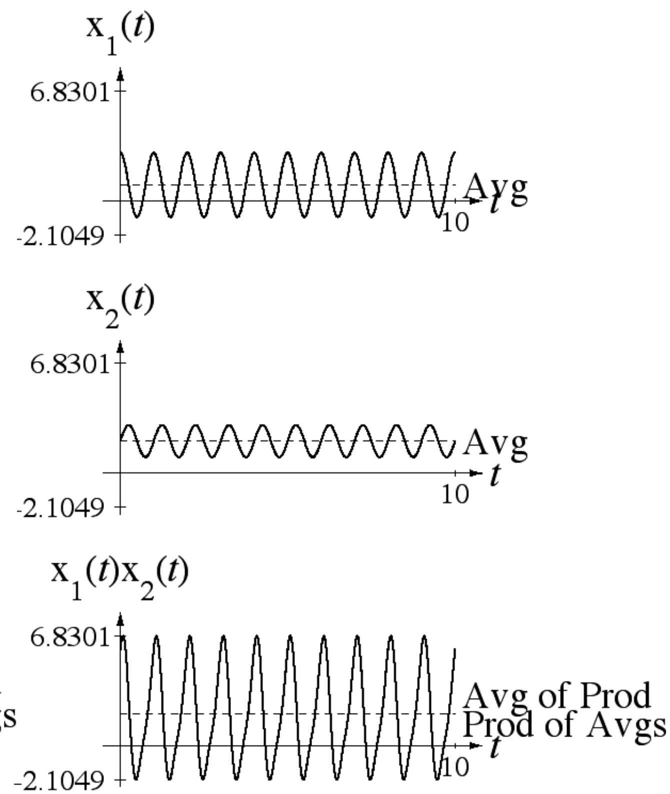


The Correlation Function

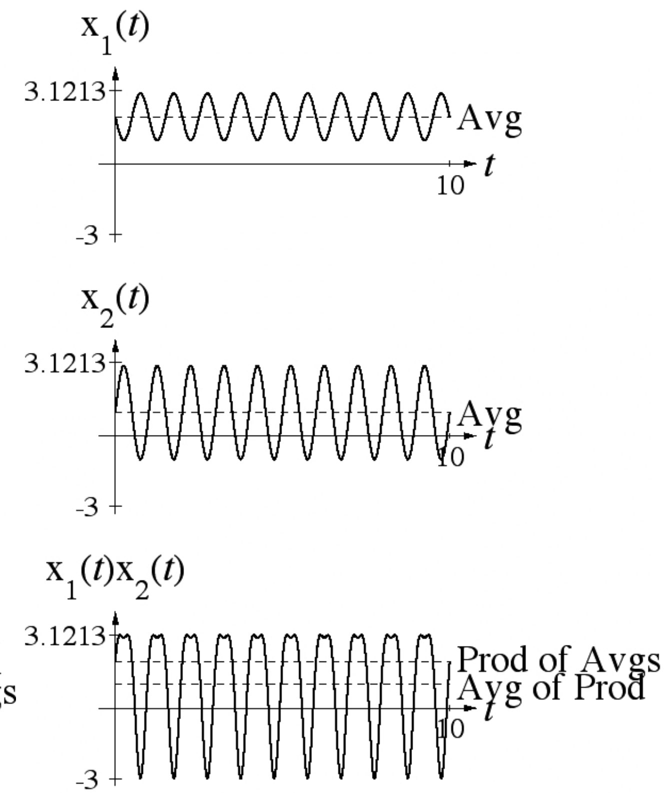
Positively Correlated CT Sinusoids with Non-zero Mean



Uncorrelated CT Sinusoids with Non-zero Mean



Negatively Correlated CT Sinusoids with Non-zero Mean

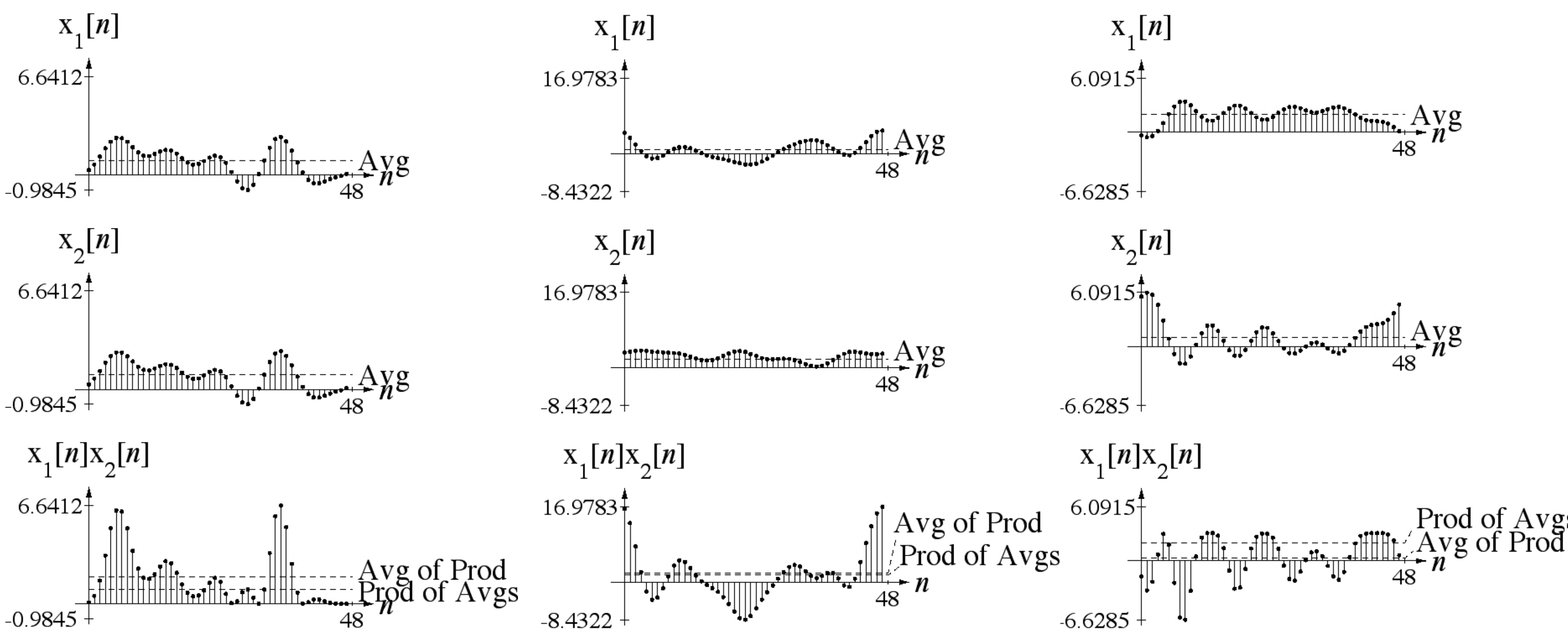


The Correlation Function

Positively Correlated Random DT Signals with Non-zero Mean

Uncorrelated Random DT Signals with Non-zero Mean

Negatively Correlated Random DT Signals with Non-zero Mean



Correlation of Energy Signals

The correlation between two energy signals, x and y, is the area under (for CT signals) or the sum of (for DT signals) the product of x and y*.

$$\int_{-\infty}^{\infty} \mathrm{x}(t)\mathrm{y}^*(t)\,dt \qquad \text{or} \qquad \sum_{n=-\infty}^{\infty} \mathrm{x}[n]\mathrm{y}^*[n]$$

The correlation *function* between two energy signals, x and y, is the area under (CT) or the sum of (DT) that product *as a function of how much* y *is shifted* relative to x.

$$\mathrm{R}_{\mathrm{xy}}(\tau) = \int_{-\infty}^{\infty} \mathrm{x}(t)\mathrm{y}^*(t+\tau)\,dt \qquad \text{or} \qquad \mathrm{R}_{\mathrm{xy}}[m] = \sum_{n=-\infty}^{\infty} \mathrm{x}[n]\mathrm{y}^*[n+m]$$

In the very common case in which x and y are both real,

$$\mathrm{R}_{\mathrm{xy}}(\tau) = \int_{-\infty}^{\infty} \mathrm{x}(t)\mathrm{y}(t+\tau)\,dt \qquad \text{or} \qquad \mathrm{R}_{\mathrm{xy}}[m] = \sum_{n=-\infty}^{\infty} \mathrm{x}[n]\mathrm{y}[n+m]$$

Correlation of Energy Signals

The correlation function for two real energy signals is very similar to the convolution of two real energy signals.

$$\mathrm{x}(t) * \mathrm{y}(t) = \int_{-\infty}^{\infty} \mathrm{x}(t-\tau)\mathrm{y}(\tau)d\tau \quad \text{or} \quad \mathrm{x}[n] * \mathrm{y}[n] = \sum_{m=-\infty}^{\infty} \mathrm{x}[n-m]\mathrm{y}[m]$$

Therefore it is possible to use convolution to find the correlation function.

$$\mathrm{R}_{\mathrm{xy}}(\tau) = \mathrm{x}(-\tau) * \mathrm{y}(\tau) \quad \text{or} \quad \mathrm{R}_{\mathrm{xy}}[m] = \mathrm{x}[-m] * \mathrm{y}[m]$$

It also follows that

$$\mathrm{R}_{\mathrm{xy}}(\tau) \xleftrightarrow{\mathcal{F}} \mathrm{X}^*(f)\mathrm{Y}(f) \quad \text{or} \quad \mathrm{R}_{\mathrm{xy}}[m] \xleftrightarrow{\mathcal{F}} \mathrm{X}^*(F)\mathrm{Y}(F)$$

Correlation of Power Signals

The correlation function between two power signals, x and y, is the average value of the product of x and y* as a function of how much y* is shifted relative to x.

$$\mathrm{R_{xy}}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_T \mathrm{x}(t)\mathrm{y}^*(t+\tau)\,dt \quad \text{or} \quad \mathrm{R_{xy}}[m] = \lim_{N \to \infty} \frac{1}{N} \sum_{n=\langle N \rangle} \mathrm{x}[n]\mathrm{y}^*[n+m]$$

If the two signals are both periodic and their fundamental periods have a finite least common period,

$$\mathrm{R_{xy}}(\tau) = \frac{1}{T} \int_T \mathrm{x}(t)\mathrm{y}^*(t+\tau)\,dt \quad \text{or} \quad \mathrm{R_{xy}}[m] = \frac{1}{N} \sum_{n=\langle N \rangle} \mathrm{x}[n]\mathrm{y}^*[n+m]$$

where *T* or *N* is any integer multiple of that least common period. For real periodic signals these become

$$\mathrm{R_{xy}}(\tau) = \frac{1}{T} \int_T \mathrm{x}(t)\mathrm{y}(t+\tau)\,dt \quad \text{or} \quad \mathrm{R_{xy}}[m] = \frac{1}{N} \sum_{n=\langle N \rangle} \mathrm{x}[n]\mathrm{y}[n+m]$$

Correlation of Power Signals

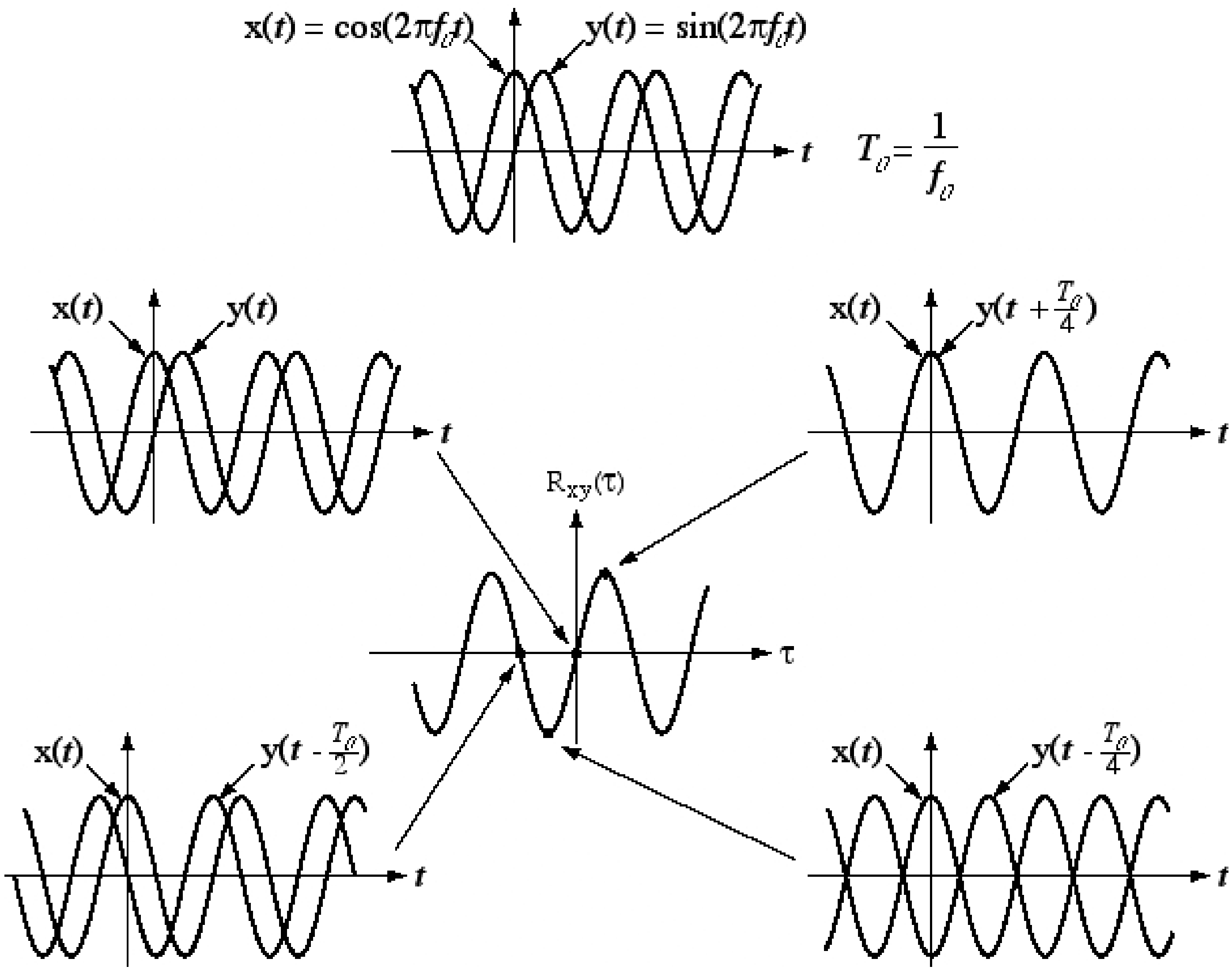
Correlation of real periodic signals is very similar to periodic convolution

$$\mathrm{R}_{\mathrm{xy}}(\tau) = \frac{\mathrm{x}(-\tau) \circledast \mathrm{y}(\tau)}{T} \quad \text{or} \quad \mathrm{R}_{\mathrm{xy}}[m] = \frac{\mathrm{x}[-m] \circledast \mathrm{y}[m]}{N}$$

$$\mathrm{R}_{\mathrm{xy}}(\tau) \xleftrightarrow{\mathcal{FS}} \mathrm{X}^*[k]\mathrm{Y}[k] \quad \text{or} \quad \mathrm{R}_{\mathrm{xy}}[m] \xleftrightarrow{\mathcal{FS}} \mathrm{X}^*[k]\mathrm{Y}[k]$$

where it is understood that the period of the periodic convolution is any integer multiple of the least common period of the two fundamental periods of x and y.

Correlation of Power Signals



Correlation of Sinusoids

- The correlation function for two sinusoids of different frequencies is always zero. (pp. 588-589)

Autocorrelation

A very important special case of correlation is *autocorrelation*. Autocorrelation is the correlation of a function with a shifted version of *itself*. For energy signals,

$$\mathrm{R_{xx}}(\tau) = \int_{-\infty}^{\infty} \mathrm{x}(t)\mathrm{x}^*(t+\tau)\,dt \quad \text{or} \quad \mathrm{R_{xx}}[m] = \sum_{n=-\infty}^{\infty} \mathrm{x}[n]\mathrm{x}^*[n+m]$$

At a shift, τ or *m*, of zero,

$$\mathrm{R_{xx}}(0) = \int_{-\infty}^{\infty} |\mathrm{x}(t)|^2\,dt \quad \text{or} \quad \mathrm{R_{xx}}[0] = \sum_{n=-\infty}^{\infty} |\mathrm{x}[n]|^2$$

which is the signal energy of the signal. For power signals,

$$\mathrm{R_{xx}}(0) = \lim_{T\to\infty}\frac{1}{T}\int_T |\mathrm{x}(t)|^2\,dt \quad \text{or} \quad \mathrm{R_{xx}}[0] = \lim_{N\to\infty}\frac{1}{N}\sum_{n=\langle N\rangle} |\mathrm{x}[n]|^2$$

which is the average signal power of the signal.

Properties of Autocorrelation

For real signals, autocorrelation is an even function.

$$\mathrm{R}_{\mathrm{xx}}(\tau) = \mathrm{R}_{\mathrm{xx}}(-\tau) \qquad \text{or} \qquad \mathrm{R}_{\mathrm{xx}}[m] = \mathrm{R}_{\mathrm{xx}}[-m]$$

Autocorrelation magnitude can never be larger than it is at zero shift.

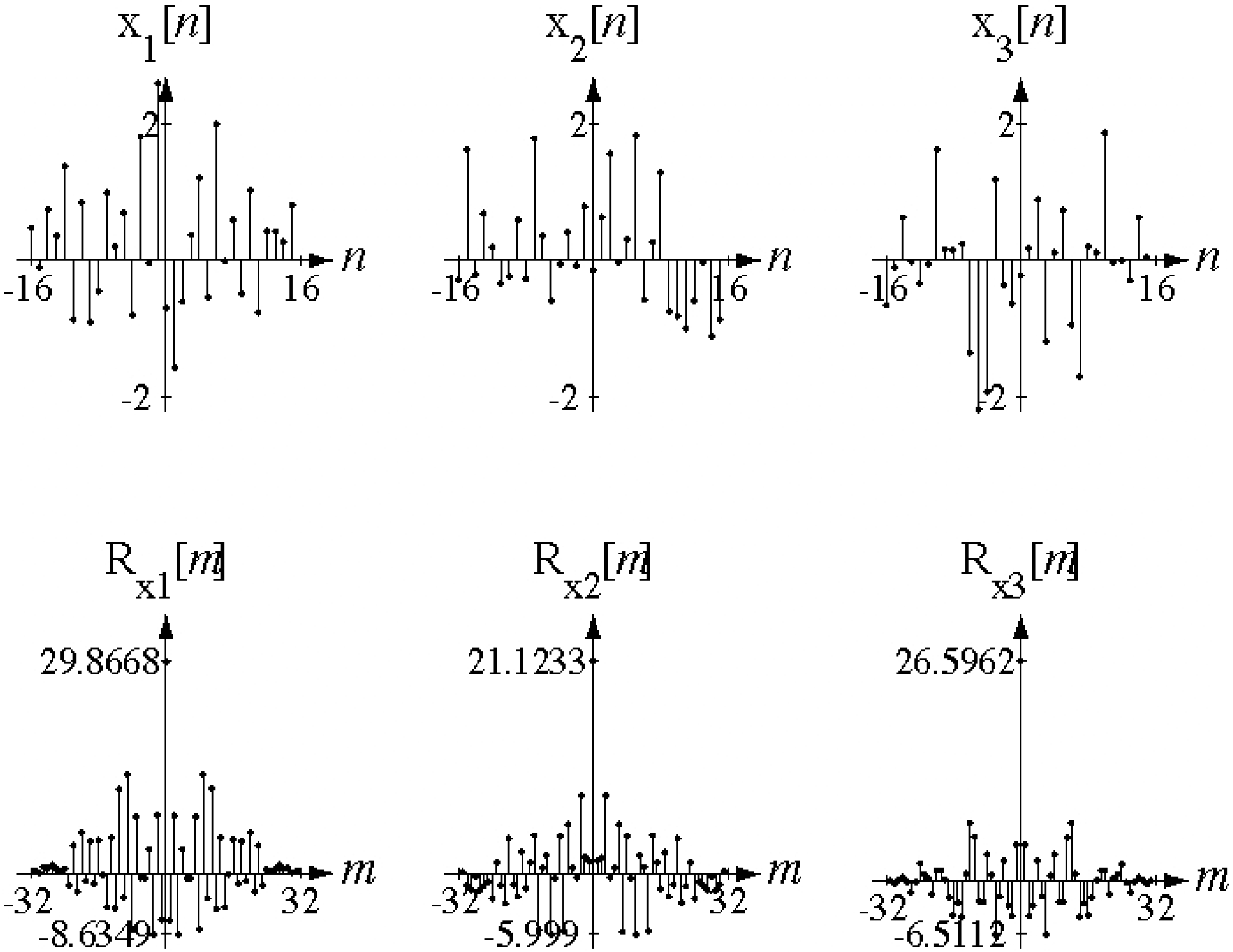
$$\mathrm{R}_{\mathrm{xx}}(0) \geq \left|\mathrm{R}_{\mathrm{xx}}(\tau)\right| \qquad \text{or} \qquad \mathrm{R}_{\mathrm{xx}}[0] \geq \left|\mathrm{R}_{\mathrm{xx}}[m]\right|$$

If a signal is time shifted its autocorrelation does not change.

The autocorrelation of a sum of sinusoids of different frequencies is the sum of the autocorrelations of the individual sinusoids.

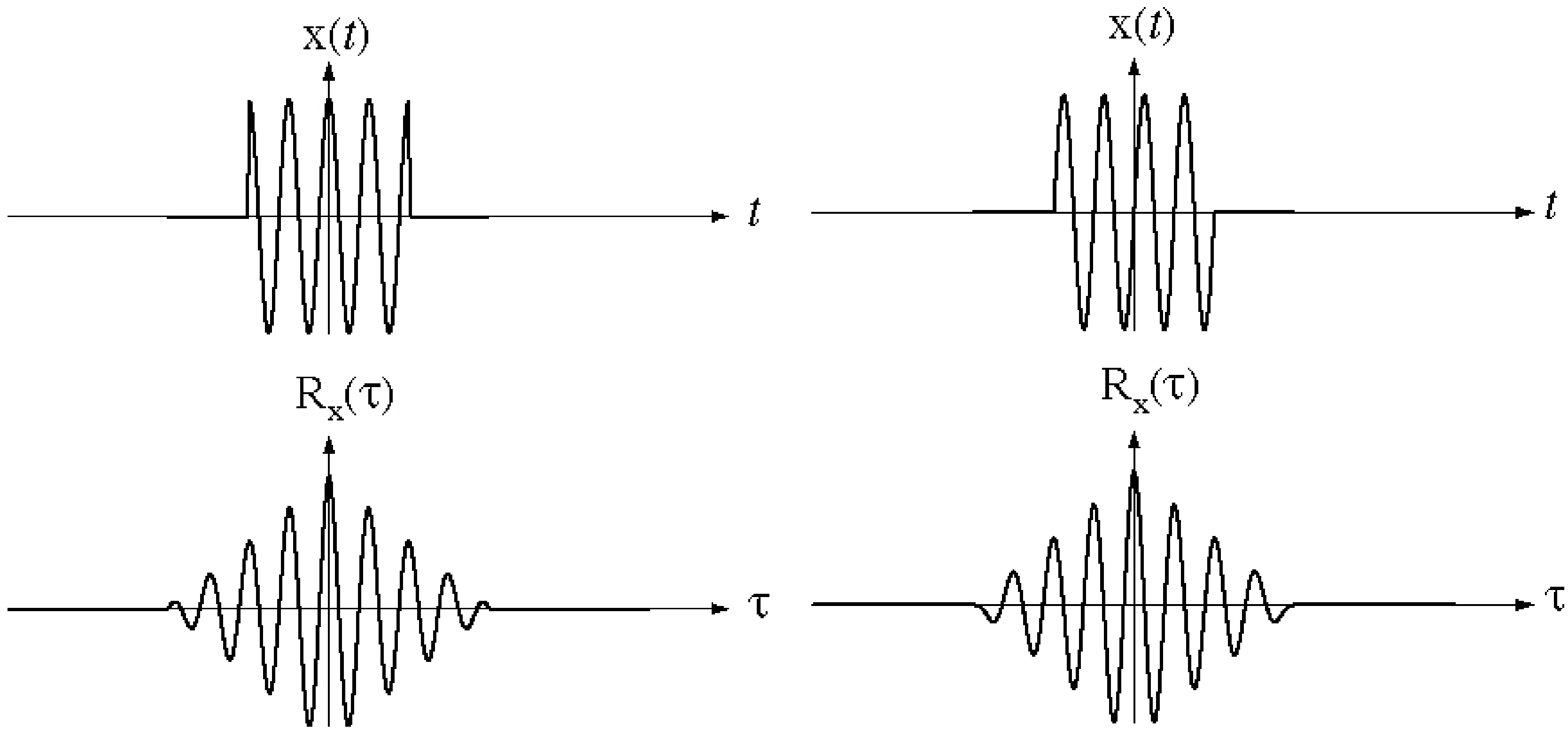
Autocorrelation Examples

Three different random DT signals and their autocorrelations. Notice that, even though the signals are different, their autocorrelations are quite similar, all peaking sharply at a shift of zero.

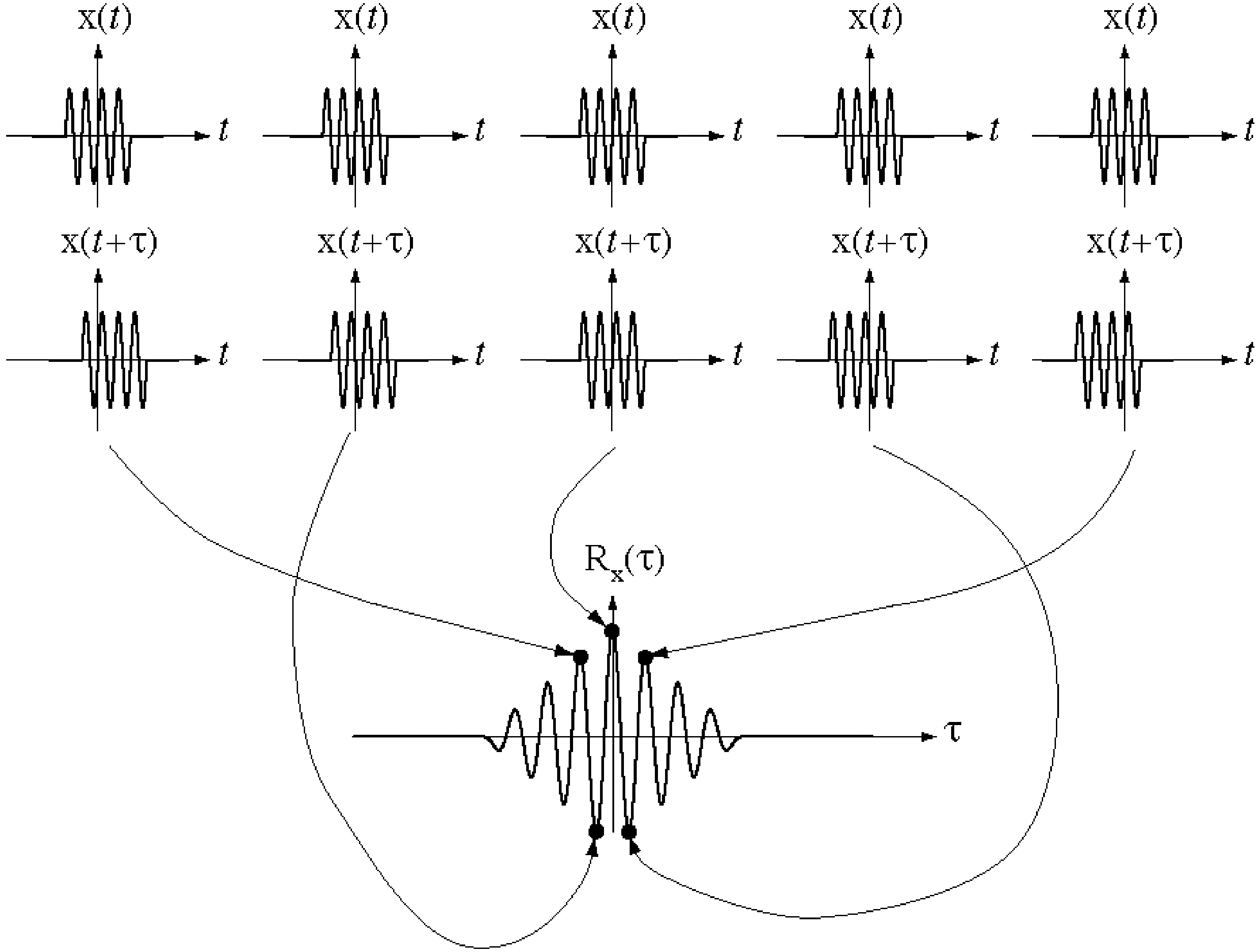


Autocorrelation Examples

Autocorrelations for a cosine "burst" and a sine "burst". Notice that they are almost (but not quite) identical.



Autocorrelation Examples

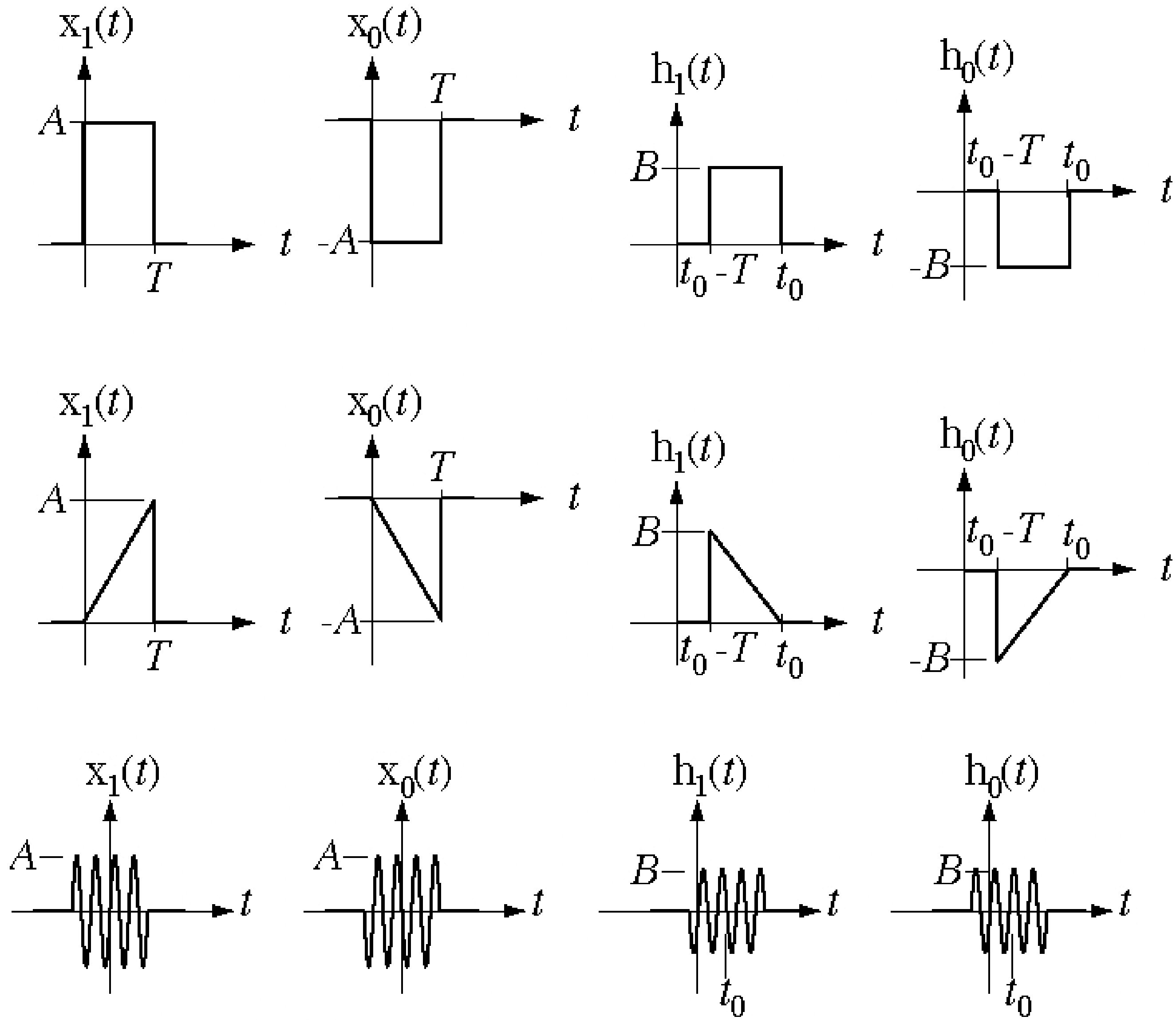


Matched Filters

- A very useful technique for detecting the presence of a signal of a certain shape in the presence of noise is the *matched filter*.
- The matched filter uses correlation to detect the signal so this filter is sometimes called a *correlation filter*
- It is often used to detect 1's and 0's in a binary data stream

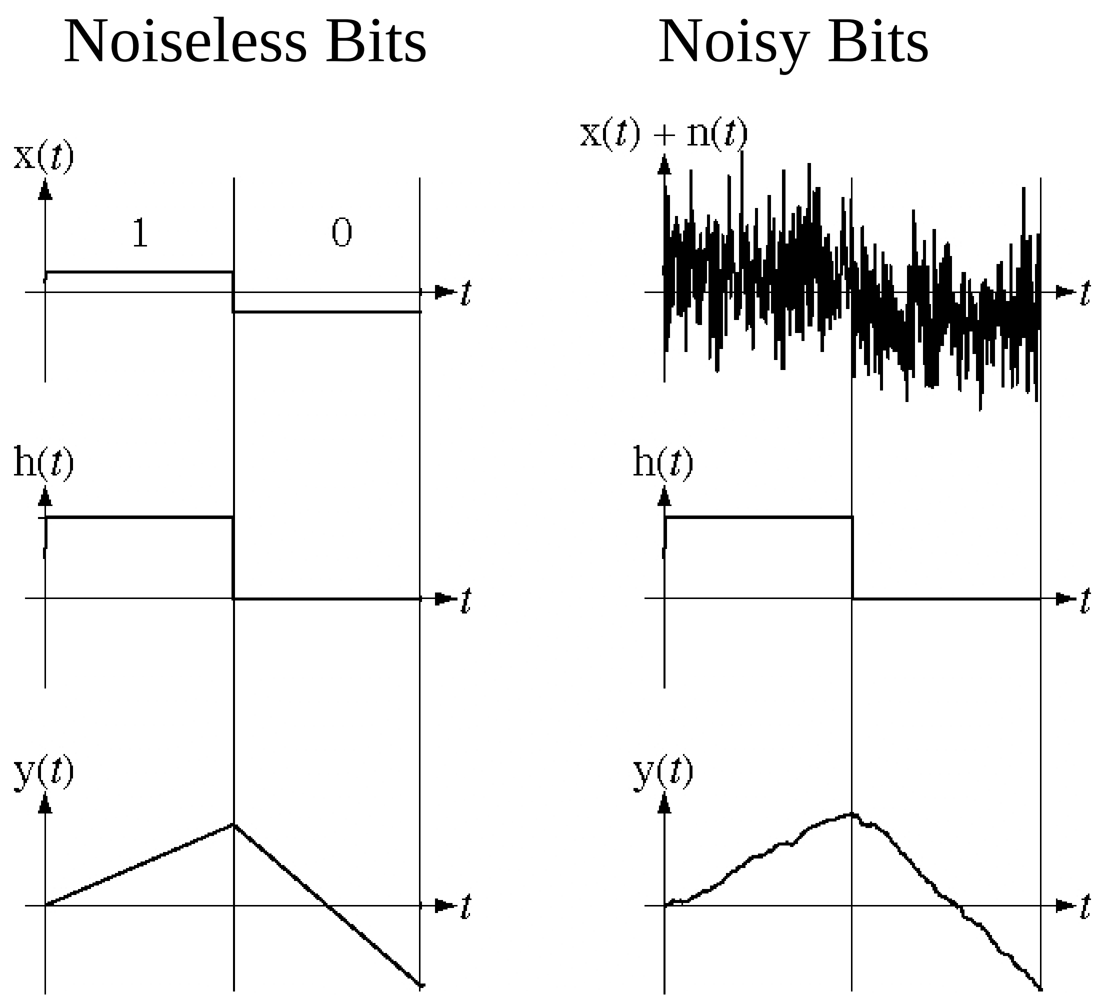
Matched Filters

It has been shown that the optimal filter to detect a noisy signal is one whose impulse response is proportional to the time inverse of the signal. Here are some examples of waveshapes encoding 1's and 0's and the impulse responses of matched filters.



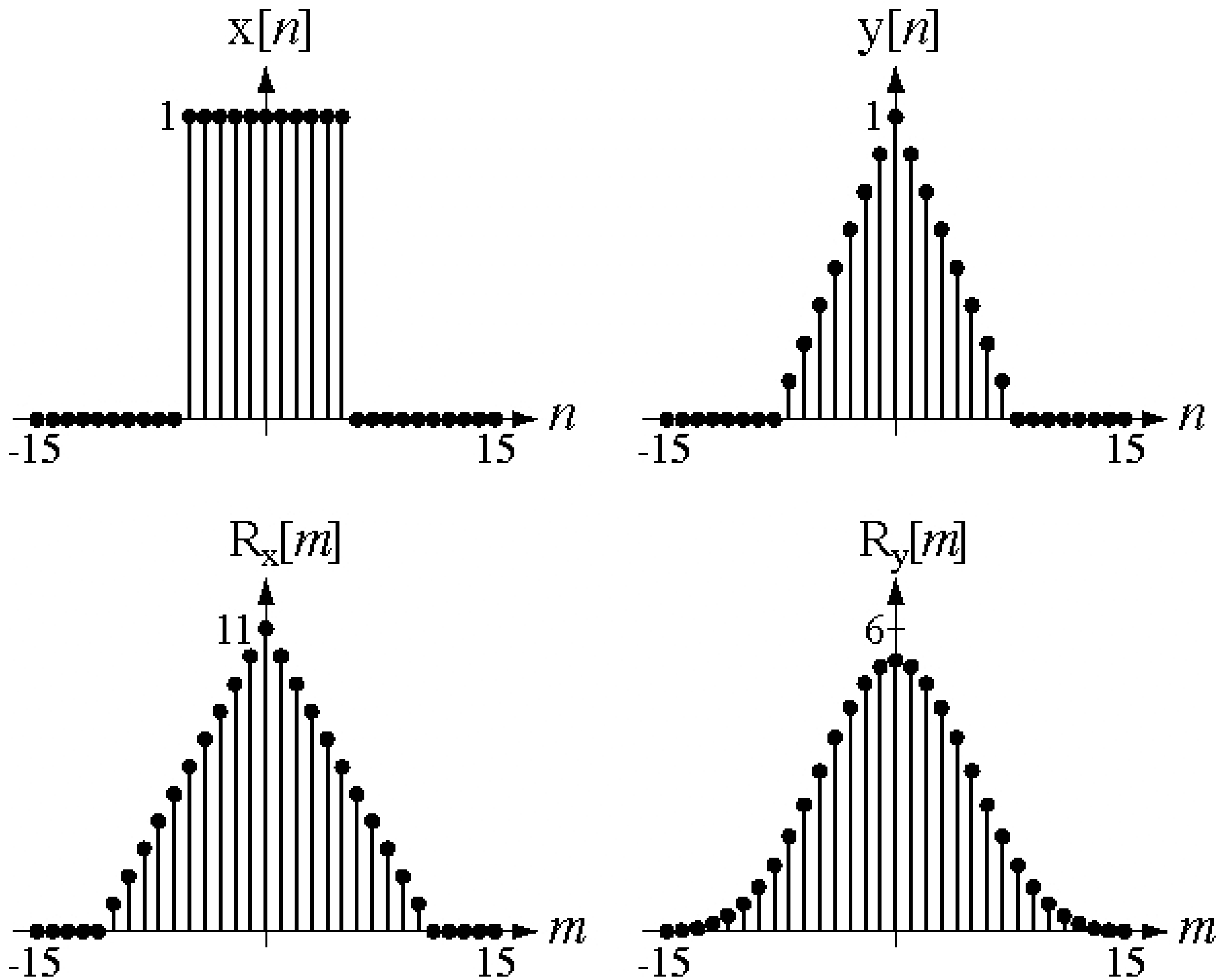
Matched Filters

Even in the presence of a large additive noise signal this matched filter indicates with a high response level the presence of a 1 and with a low response level the presence of a 0. Since the 1 and 0 are encoded as the negatives of each other, one matched filter optimally detects both.



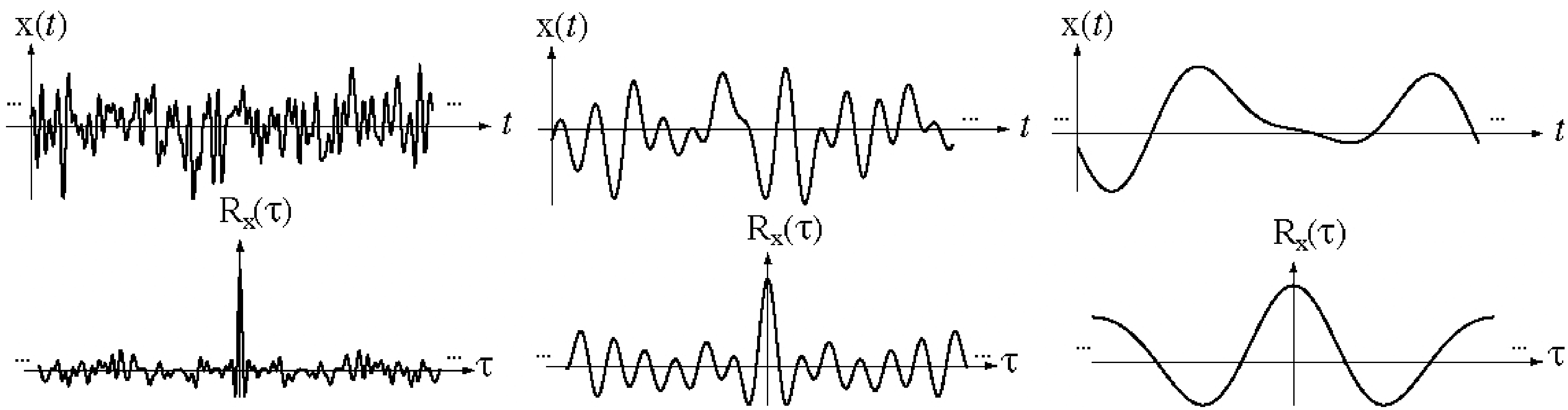
Autocorrelation Examples

Two familiar DT signal shapes and their autocorrelations.



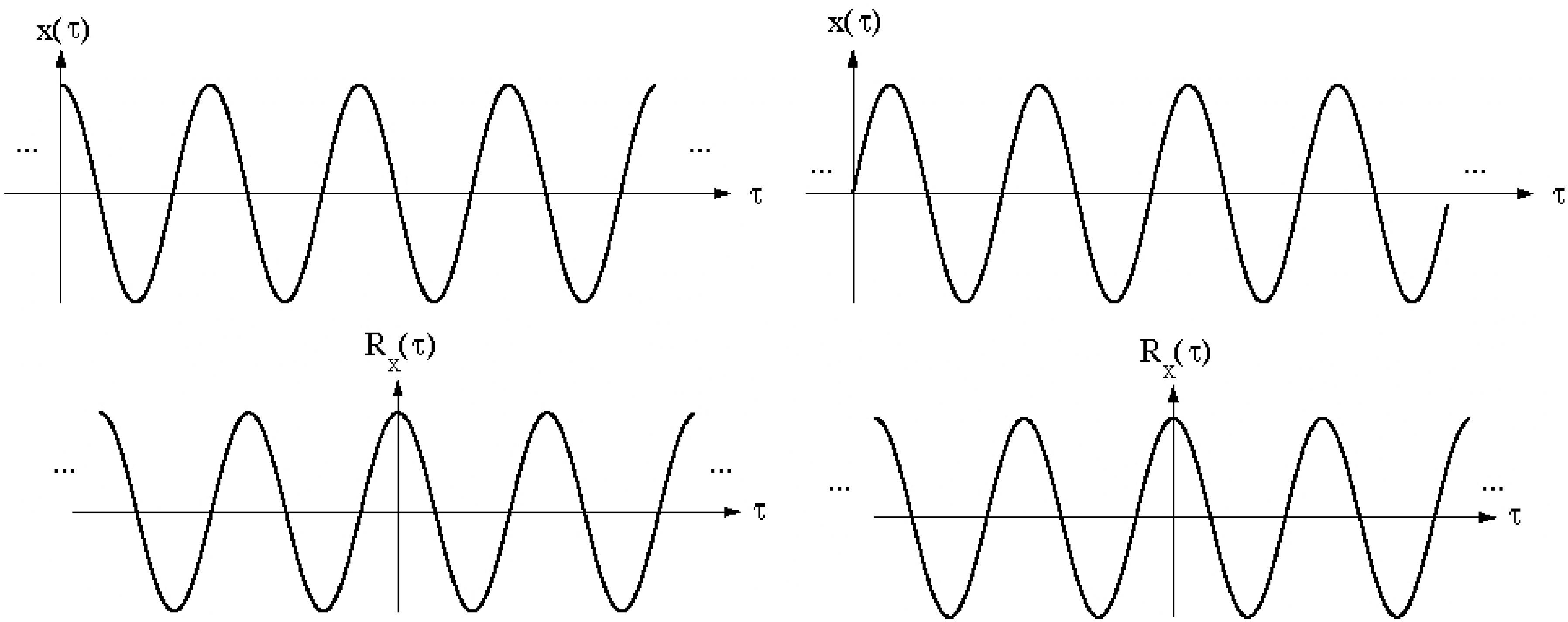
Autocorrelation Examples

Three random power signals with different frequency content and their autocorrelations.



Autocorrelation Examples

Autocorrelation functions for a cosine and a sine. Notice that the autocorrelation functions are identical even though the signals are different.



Autocorrelation Examples

- One way to simulate a random signal is with a summation of sinusoids of different frequencies and random phases
- Since all the sinusoids have different frequencies the autocorrelation of the sum is simply the sum of the autocorrelations
- Also, since a time shift (phase shift) does not affect the autocorrelation, when the phases are randomized the signals change, but not their autocorrelations

Autocorrelation Examples

Let a random signal be described by

$$\mathrm{x}(t) = \sum_{k=1}^{N} A_k \cos(2\pi f_{0k} t + \theta_k)$$

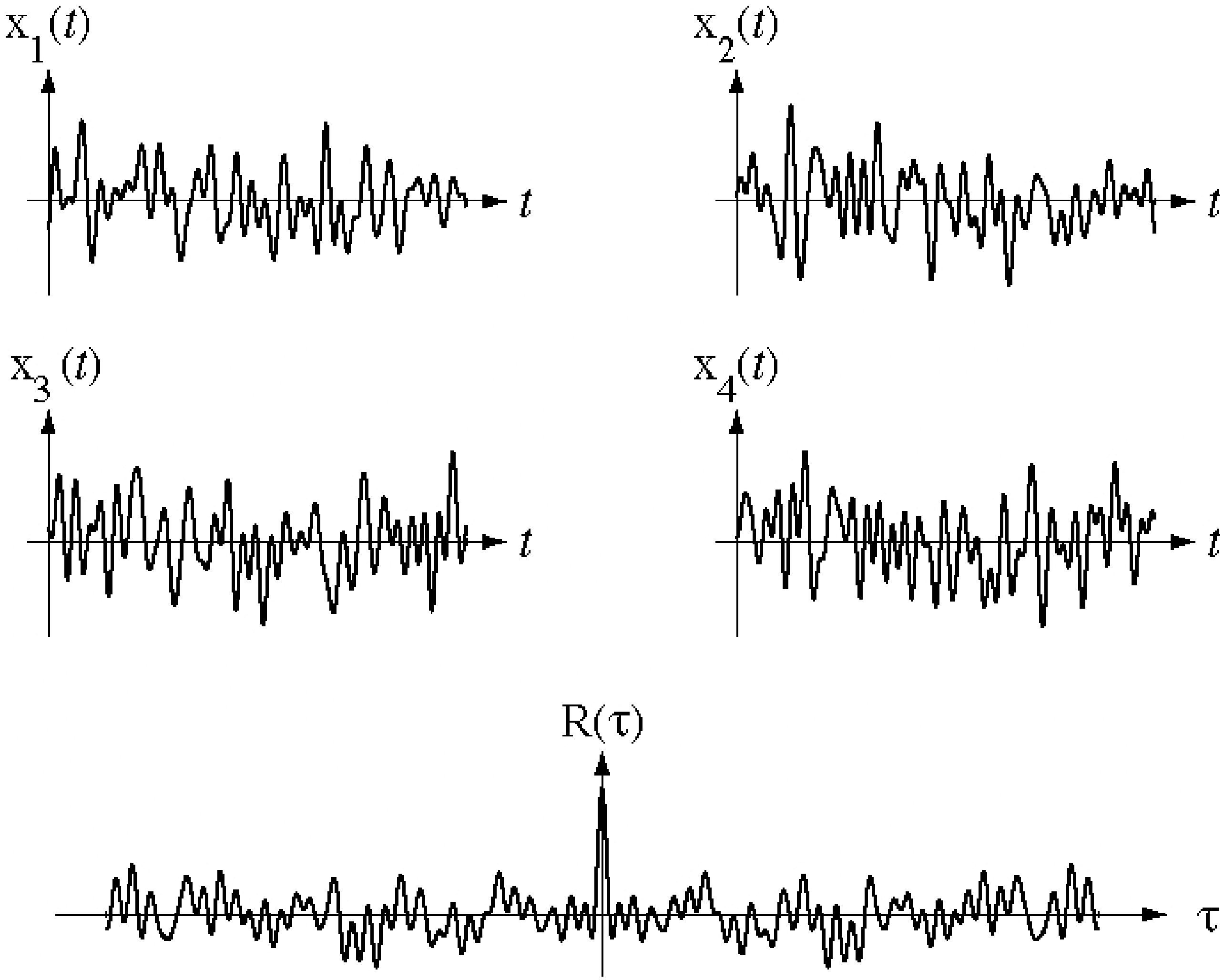
Since all the sinusoids are at different frequencies,

$$\mathrm{R}_{\mathrm{x}}(\tau) = \sum_{k=1}^{N} \mathrm{R}_k(\tau)$$

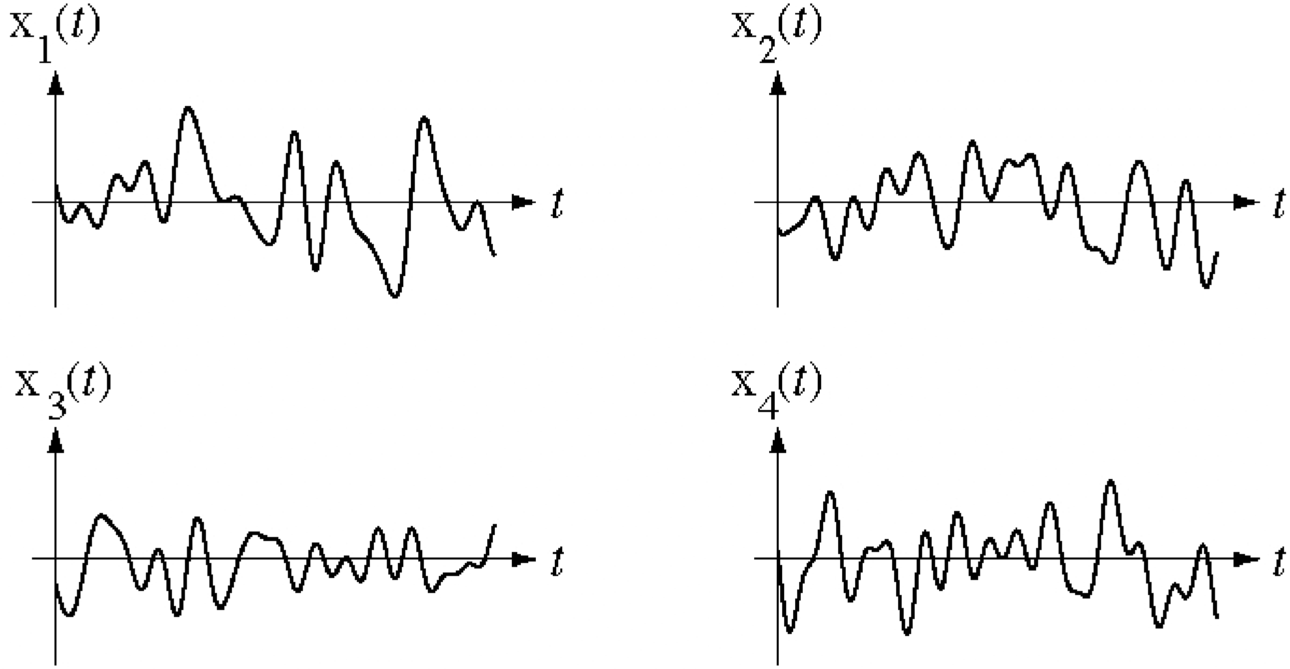
where $\mathrm{R}_k(\tau)$ is the autocorrelation of $A_k \cos(2\pi f_{0k} t + \theta_k)$.

Autocorrelation Examples

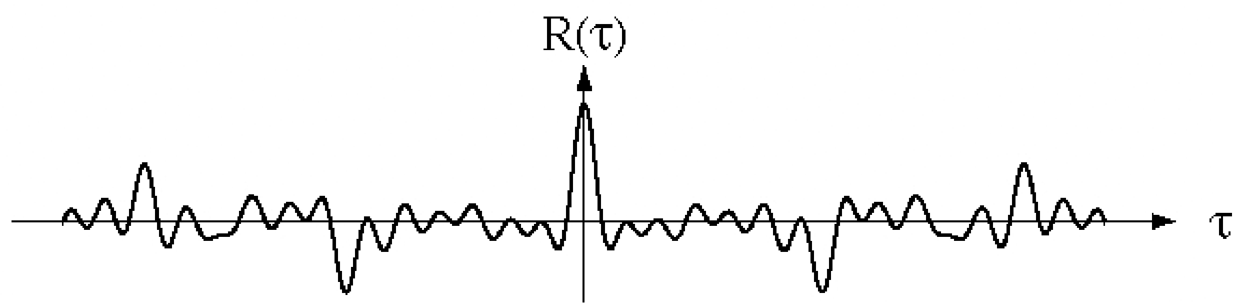
Four Different Random Signals with Identical Autocorrelations



Autocorrelation Examples

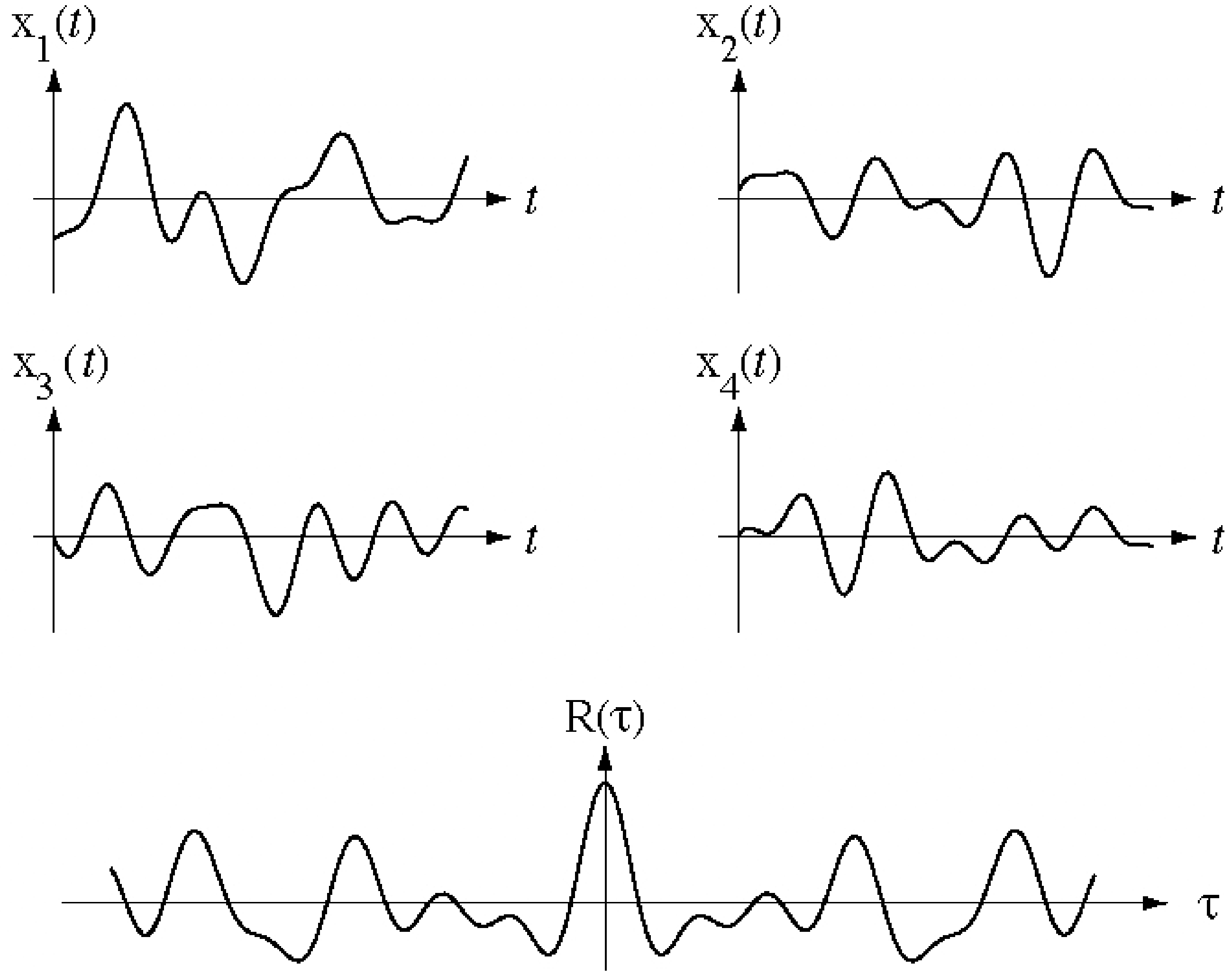


Four Different Random Signals with Identical Autocorrelations



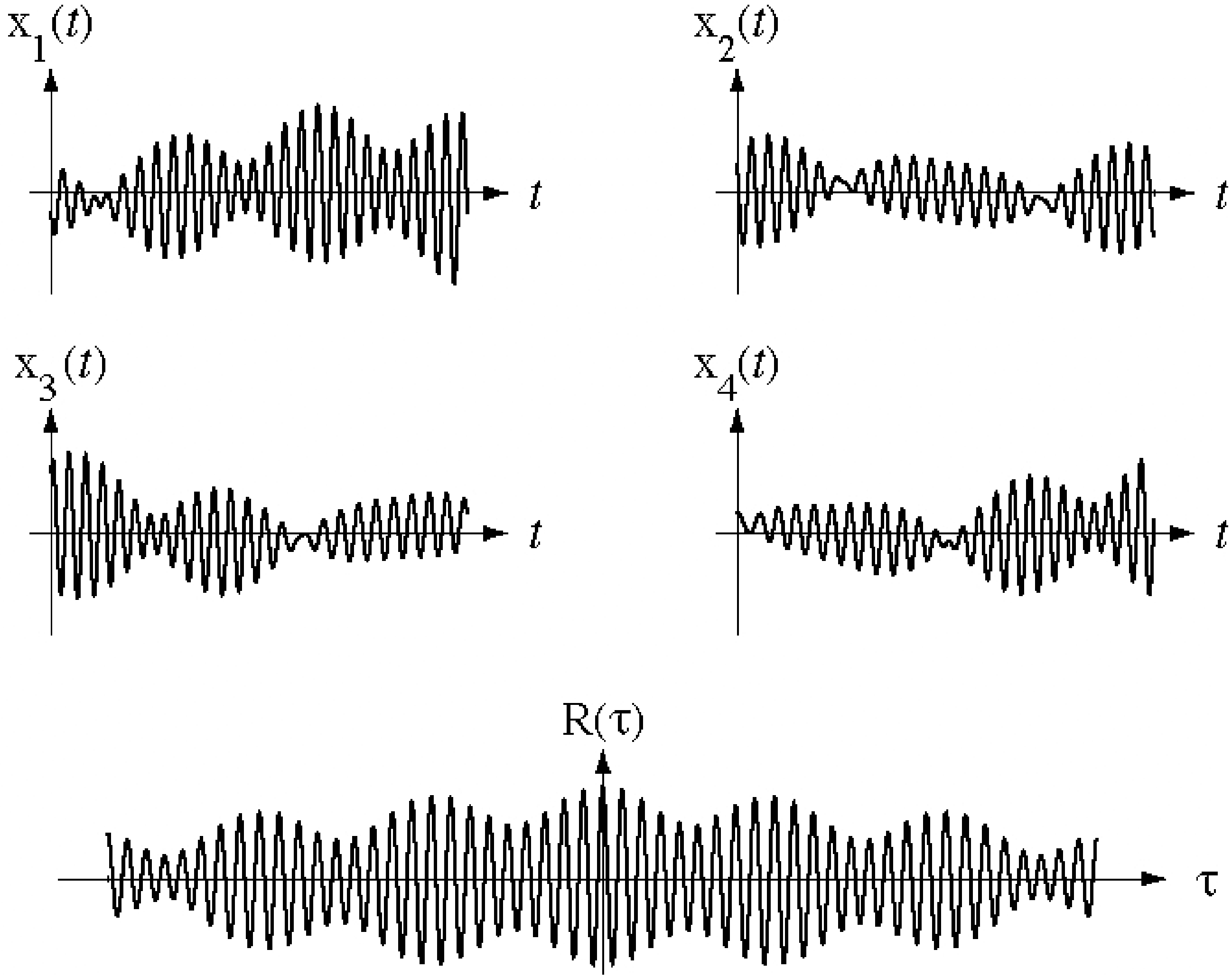
Autocorrelation Examples

Four Different Random Signals with Identical Autocorrelations



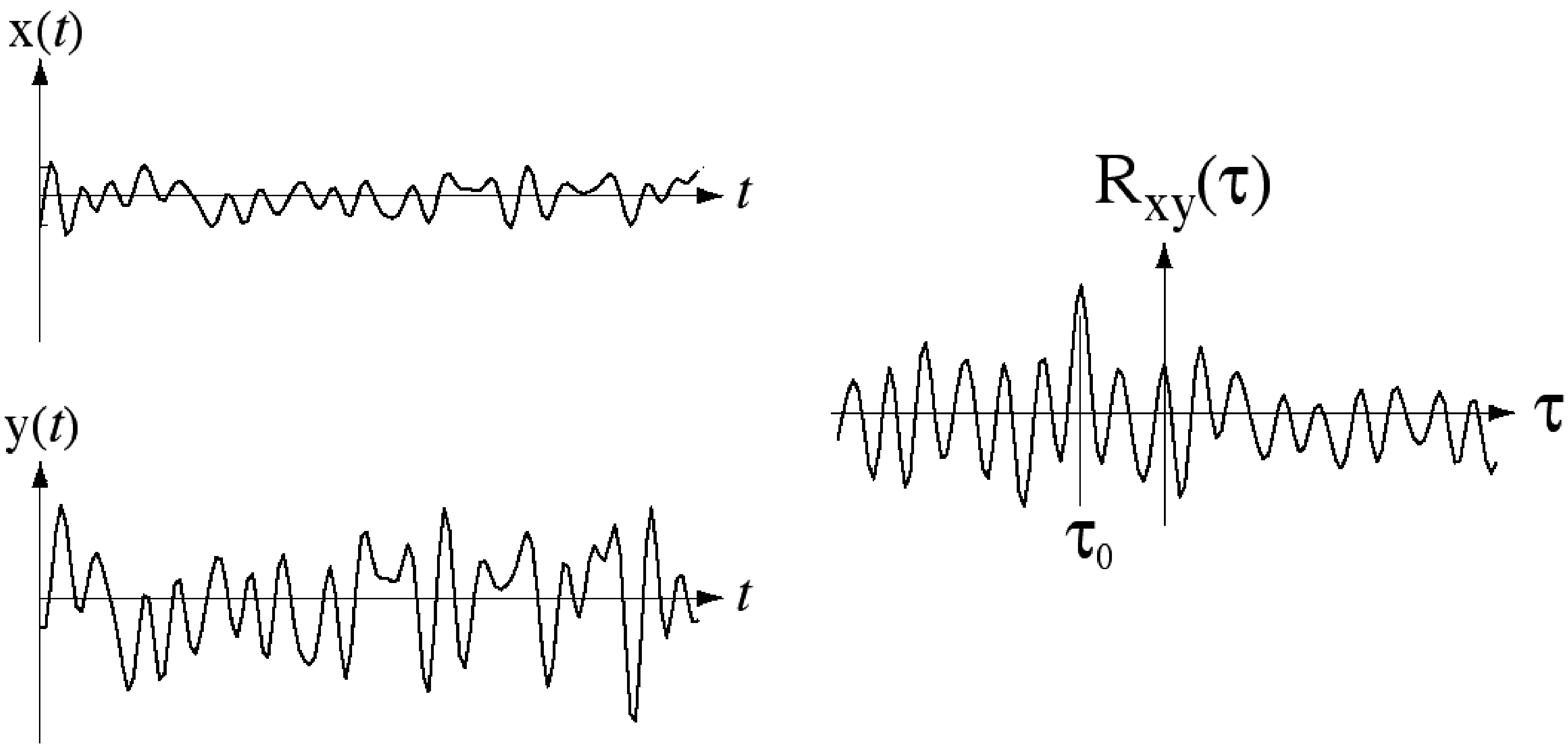
Autocorrelation Examples

Four Different Random Signals with Identical Autocorrelations



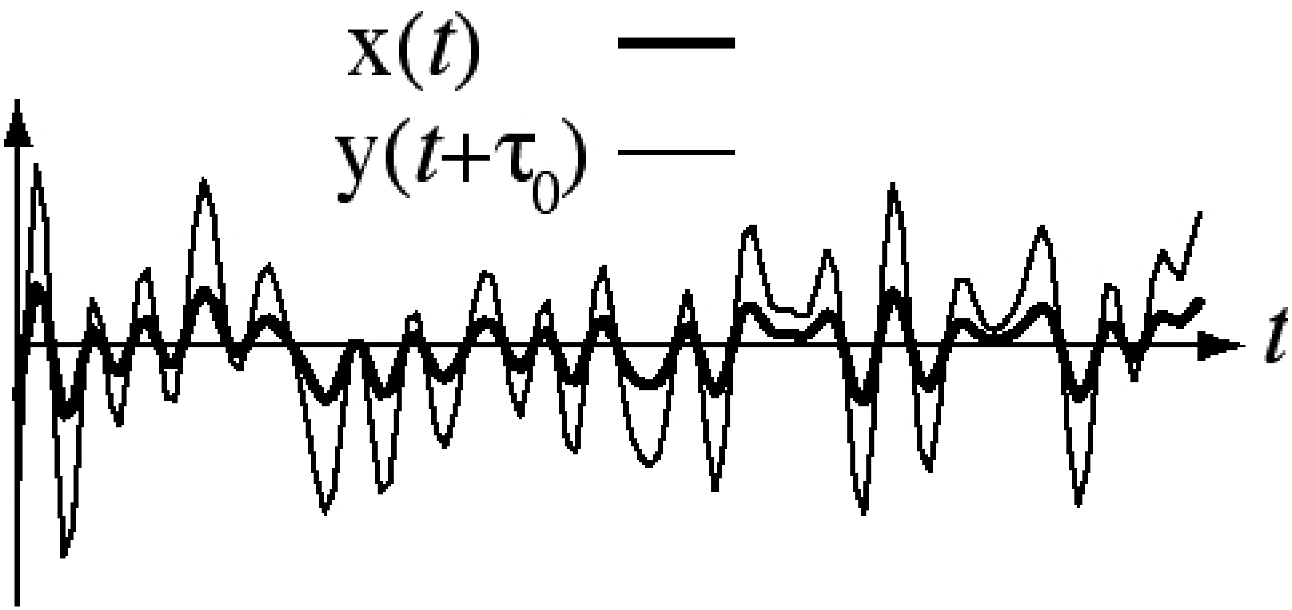
Cross Correlation

Cross correlation is really just “correlation” in the cases in which the two signals being compared are different. The name is commonly used to distinguish it from autocorrelation.



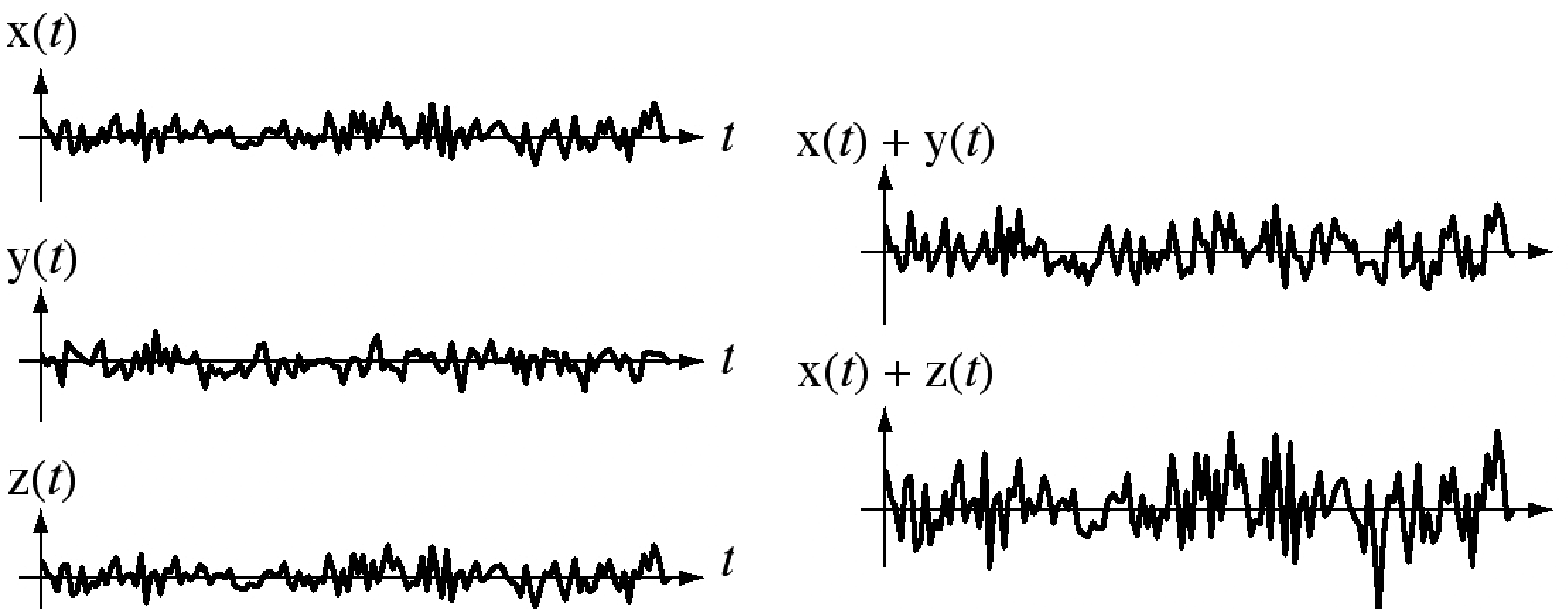
Cross Correlation

A comparison of x and y with y shifted for maximum correlation.



Cross Correlation

Below, x and z are highly positively correlated and x and y are uncorrelated. All three signals have the same average signal power. The signal power of x+z is greater than the signal power of x+y.



Correlation and the Fourier Series

Calculating Fourier series harmonic functions can be thought of as a process of correlation. Let

$$\mathrm{c}(t) = \cos\left(2\pi\left(kf_0\right)t\right) \quad \text{and} \quad \mathrm{s}(t) = \sin\left(2\pi\left(kf_0\right)t\right)$$

Then the trigonometric CTFS harmonic functions are

$$\mathrm{X}_c[k] = 2\,\mathrm{R}_{\mathrm{xc}}(0) \ , \ \mathrm{X}_s[k] = 2\,\mathrm{R}_{\mathrm{xs}}(0)$$

Also, let

$$\mathrm{z}(t) = e^{+j2\pi\left(kf_0\right)t}$$

then the complex CTFS harmonic function is

$$\mathrm{X}[k] = \mathrm{R}_{\mathrm{xz}}(0)$$

Energy Spectral Density

The total signal energy in an energy signal is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df \quad \text{or} \quad E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \int_1 |X(F)|^2 \, dF$$

The quantity, $|X(f)|^2$, or $|X(F)|^2$, is called the *energy spectral density (ESD)* of the signal, x, and is conventionally given the symbol, Ψ. That is,

$$\Psi_x(f) = |X(f)|^2 \qquad \text{or} \qquad \Psi_x(F) = |X(F)|^2$$

It can be shown that if x is a real-valued signal that the ESD is even, non-negative and real.

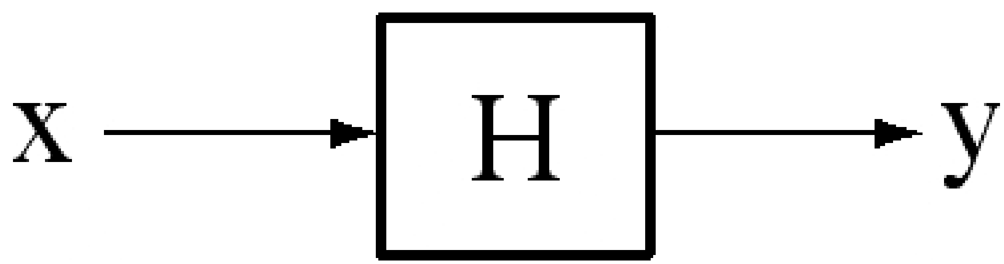
Energy Spectral Density

Probably the most important fact about ESD is the relationship between the ESD of the excitation of an LTI system and the ESD of the response of the system. It can be shown (pp. 606-607) that they are related by

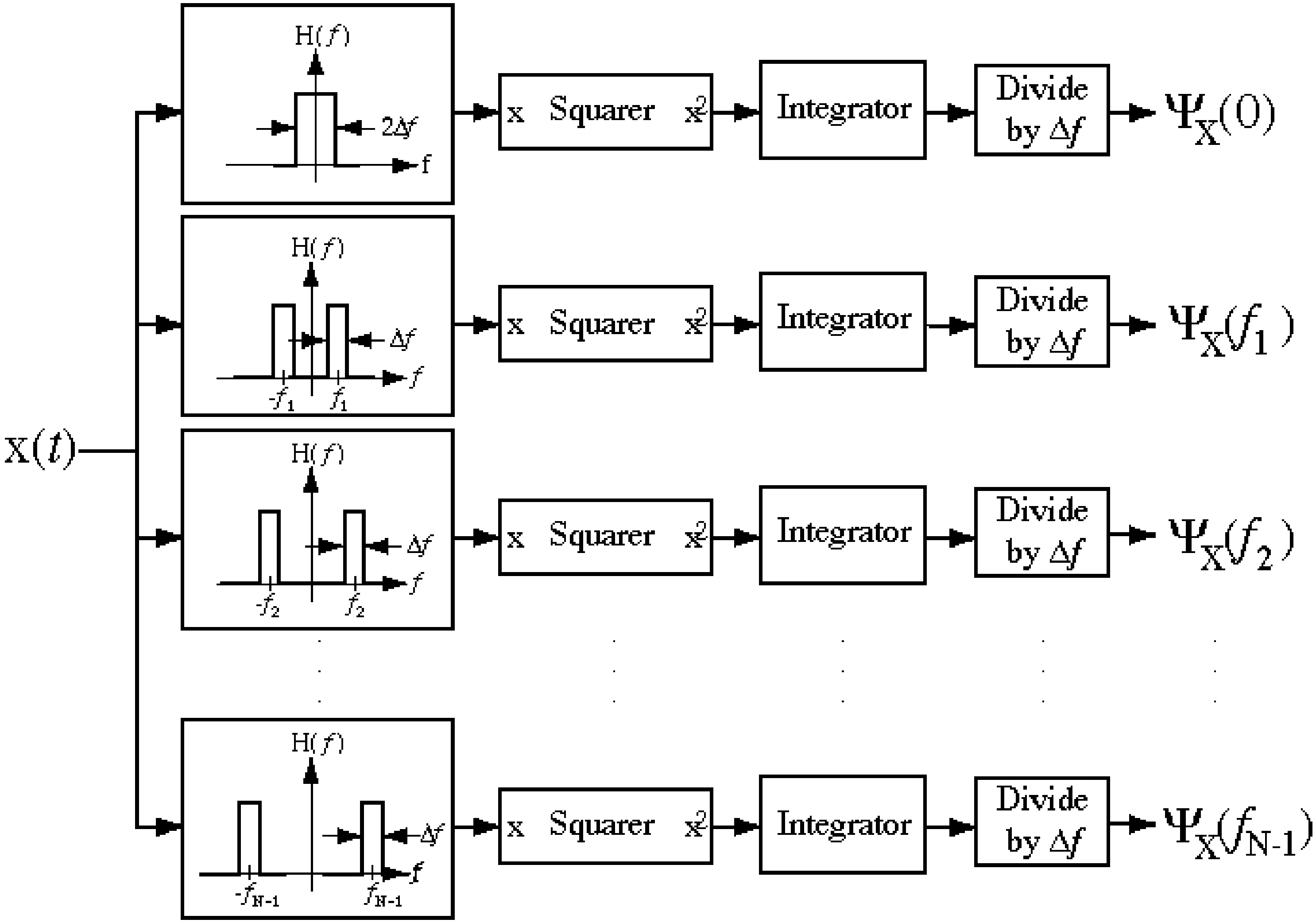
$$\Psi_y(f) = |H(f)|^2 \Psi_x(f) = H(f) H^*(f) \Psi_x(f)$$

or

$$\Psi_y(F) = |H(F)|^2 \Psi_x(F) = H(F) H^*(F) \Psi_x(F)$$



Energy Spectral Density



Energy Spectral Density

It can be shown (pp. 607-608) that, for an energy signal, ESD and autocorrelation form a Fourier transform pair.

$$\mathrm{R}_{\mathrm{x}}(t) \xleftrightarrow{\mathcal{F}} \Psi_{\mathrm{x}}(f) \quad \text{or} \quad \mathrm{R}_{\mathrm{x}}[n] \xleftrightarrow{\mathcal{F}} \Psi_{\mathrm{x}}(F)$$

Power Spectral Density

Power spectral density (PSD) applies to power signals in the same way that energy spectral density applies to energy signals. The PSD of a signal x is conventionally indicated by the notation, $\mathrm{G_x}(f)$ or $\mathrm{G_x}(F)$. In an LTI system,

$$\mathrm{G_y}(f) = |\mathrm{H}(f)|^2 \, \mathrm{G_x}(f) = \mathrm{H}(f)\mathrm{H}^*(f)\mathrm{G_x}(f)$$

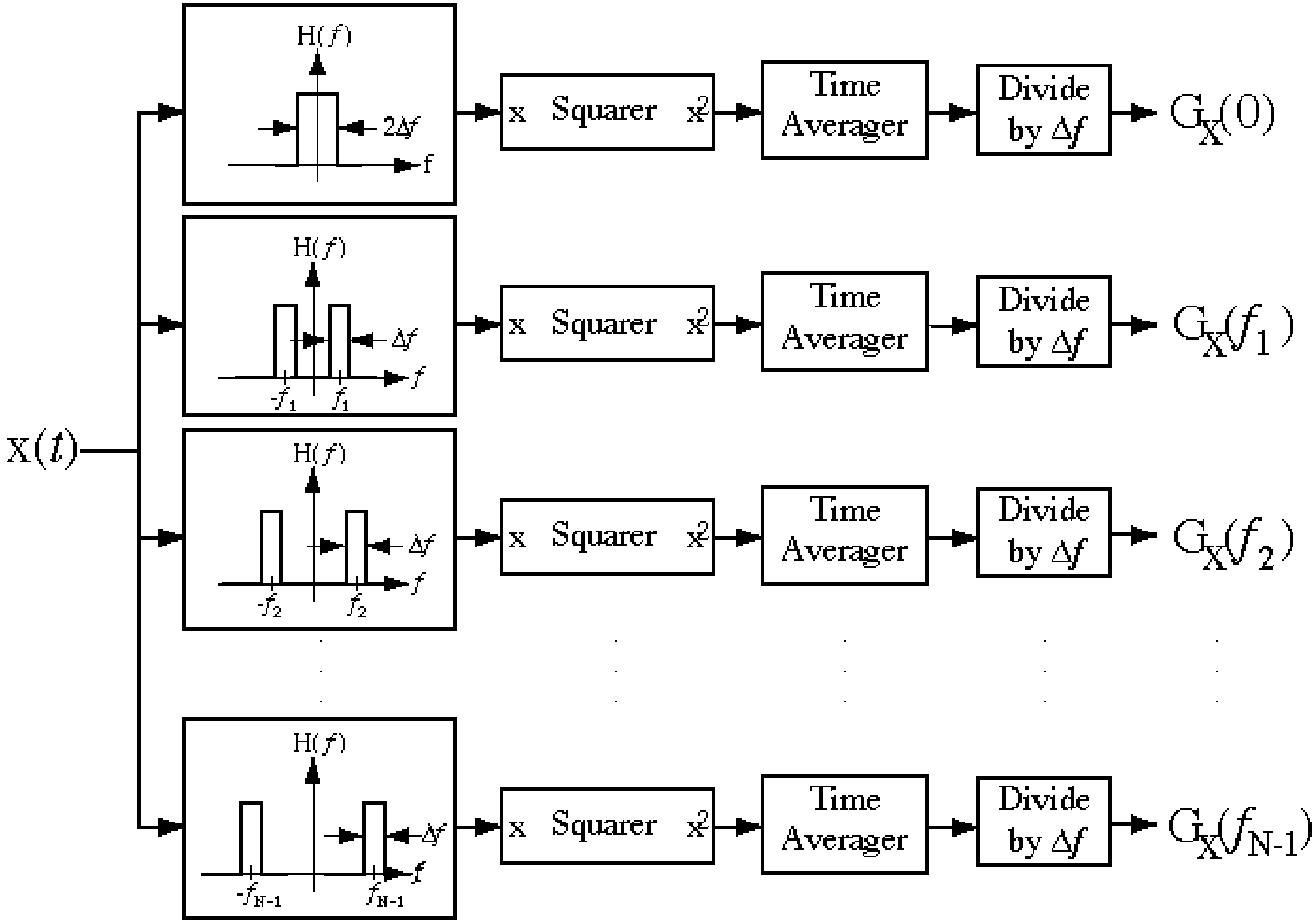
or

$$\mathrm{G_y}(F) = |\mathrm{H}(F)|^2 \, \mathrm{G_x}(F) = \mathrm{H}(F)\mathrm{H}^*(F)\mathrm{G_x}(F)$$

Also, for a power signal, PSD and autocorrelation form a Fourier transform pair.

$$\mathrm{R}(t) \xleftrightarrow{\mathcal{F}} [\mathrm{G}(f)] \quad \text{or} \quad \mathrm{R}[n] \xleftrightarrow{\mathcal{F}} \mathrm{G}(F)$$

PSD Concept


x(t)
H(f)
2Δf
f
x Squarer x²
Time Averager
Divide by Δf
G_X(0)
Δf
-f₁ f₁
G_X(f₁)
-f₂ f₂
G_X(f₂)
-f_N-1 f_N-1
G_X(f_N-1)

Typical Signals in PSD Concept