The Laplace Transform

Introduction

- There are two common approaches to the developing and understanding the Laplace transform
- It can be viewed as a generalization of the CTFT to include some signals with infinite energy
- It can be seen as a natural consequence of the fact that an LTI system excited by a complex exponential responds with another complex exponential

Generalization of the CTFT

The CTFT expresses a time-domain signal as a linear combination of complex sinusoids of the form, $e^{j\omega t}$. In the generalization of the CTFT to the Laplace transform the complex sinusoids become complex *exponentials* of the form, e^{st},where s can have any complex value . Replacing the complex sinusoids with complex exponentials leads to this definition of the Laplace transform,

$$\mathcal{L}(\mathrm{x}(t)) = \mathrm{X}(s) = \int_{-\infty}^{\infty} \mathrm{x}(t) e^{-st} dt$$

A function and its Laplace transform form a transform pair which is conveniently indicated by the notation,

$$\mathrm{x}(t) \xleftrightarrow{\mathcal{L}} \mathrm{X}(s)$$

Pierre-Simon Laplace

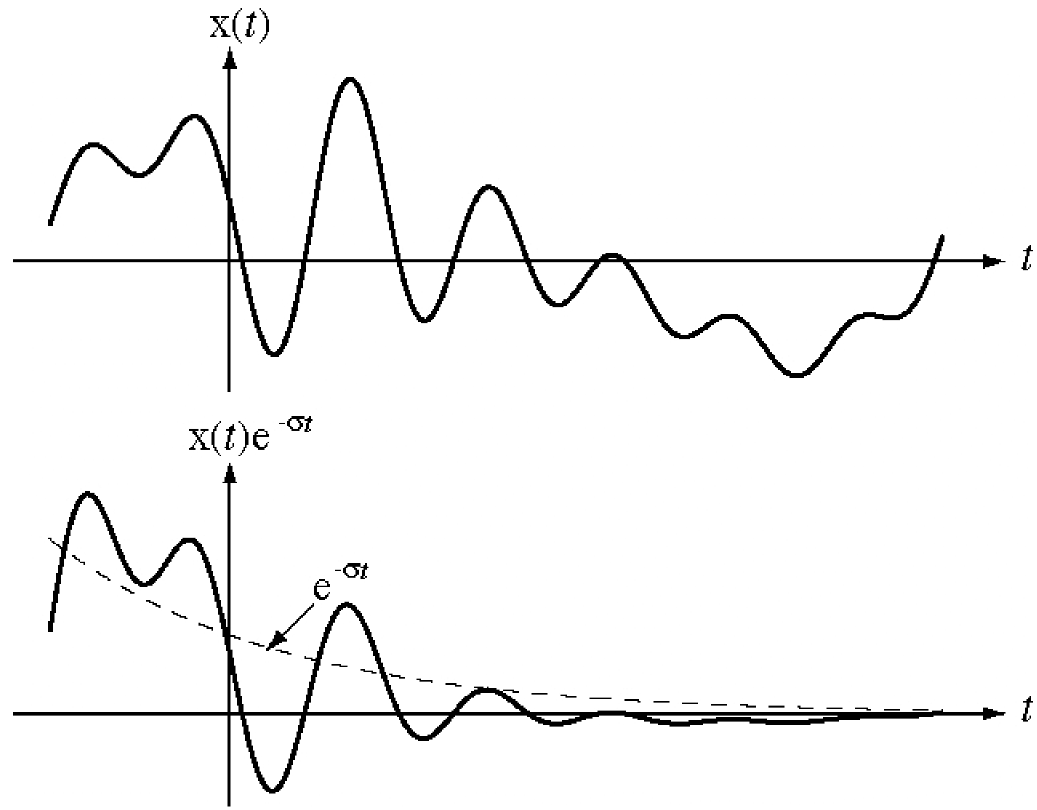
3/23/1749 - 3/2/1827

Generalization of the CTFT

The variable, s, is viewed as a generalization of the variable, ω, of the form, $s = \sigma + j\omega$. Then, when the real part of s, σ, is zero, the Laplace transform reduces to the CTFT. Using $s = \sigma + j\omega$ the Laplace transform is

$$\mathrm{X}(s) = \int_{-\infty}^{\infty} \mathrm{x}(t) e^{-(\sigma + j\omega)t} dt$$

$$= \mathcal{F}\left[\mathrm{x}(t) e^{-\sigma t}\right]$$



and can be viewed as the CTFT of, not x(t), but rather $\mathrm{x}(t) e^{-\sigma t}$.

Generalization of the CTFT

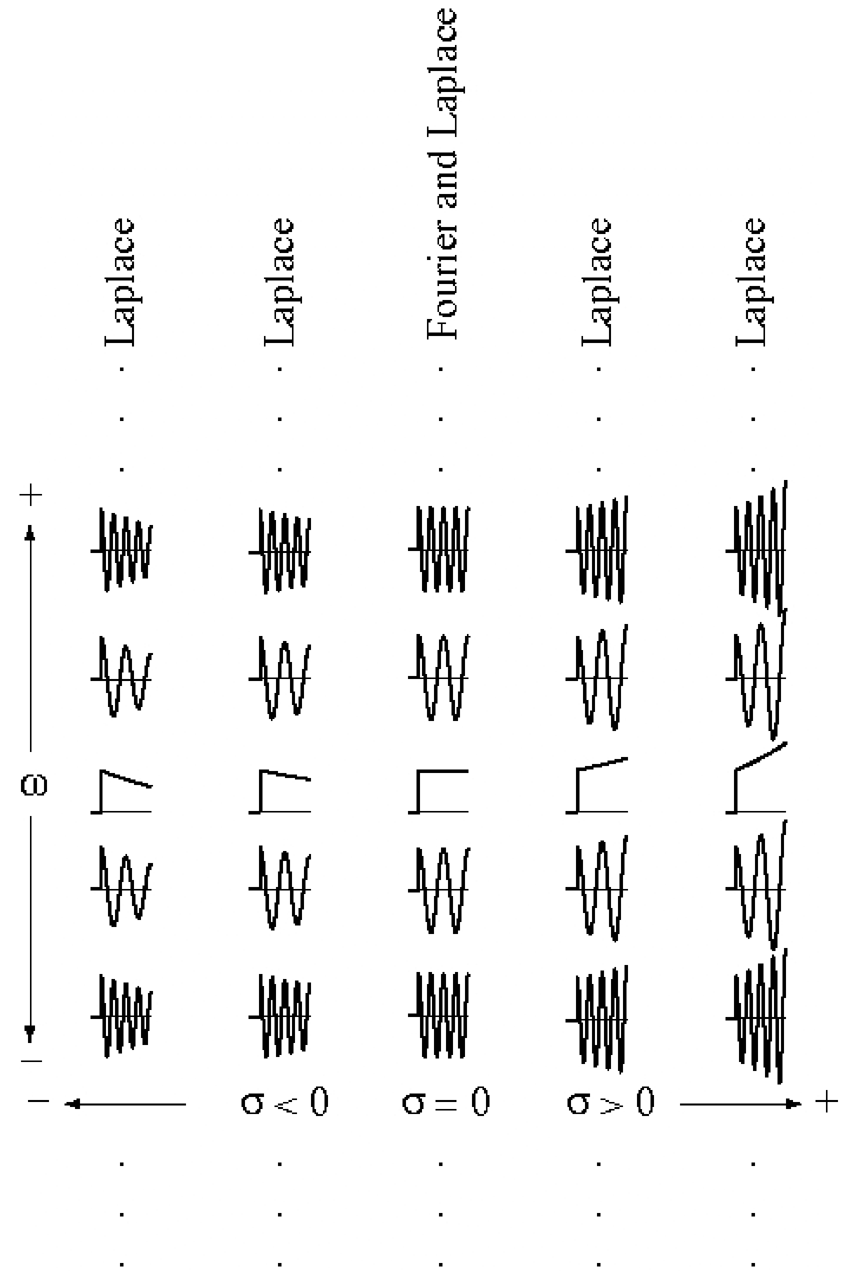
The extra factor, $e^{-\sigma t}$, is sometimes called a *convergence* factor because, when chosen properly, it makes the integral converge for some signals when it would not otherwise converge. For example, strictly speaking, the signal, $A\mathrm{u}(t)$, does not have a CTFT because the integral does not converge. But if it is multiplied by the convergence factor, and the real part of s, σ, is chosen appropriately, the CTFT integral will converge.

Generalization of the CTFT

The CTFT uses only complex sinusoids. The Laplace transform uses the more general complex exponentials.

$$e^{j\omega t} \rightarrow e^{st}$$

$$s = \sigma + j\omega$$



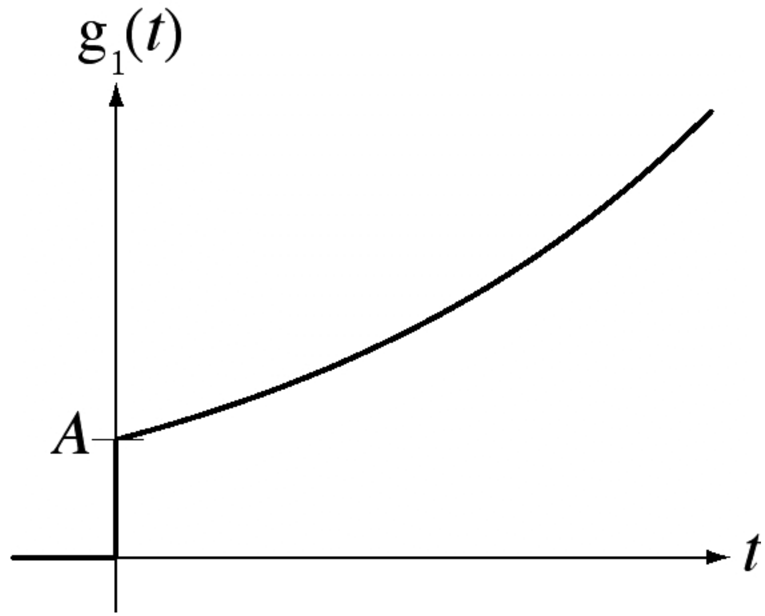
Complex Exponential Excitation

If a continuous-time LTI system is excited by a complex exponential, $\mathrm{x}(t) = Ae^{st}$, the response is also a complex exponential of the same functional form except multiplied by a complex constant. The response is the convolution of the excitation with the impulse response and, as first shown in Chapter 3, that turns out to be

$$\mathrm{y}(t) = \underbrace{Ae^{st}}_{\mathrm{x}(t)} \underbrace{\int_{-\infty}^{\infty} \mathrm{h}(\tau) e^{-s\tau} d\tau}_{\substack{\text{Laplace transform} \\ \text{of } \mathrm{h}(t)}}$$

So for any particular value of *s* the complex constant is the Laplace transform of the impulse response.

Region of Convergence



The causal function,

$$g_1(t) = Ae^{\alpha t}\,u(t)\;,\;\alpha > 0$$

does not have a CTFT, even in the generalized sense. But it does have a Laplace transform which is

$$G_1(s) = \int_{-\infty}^{\infty} Ae^{\alpha t}\,u(t)e^{-st}dt = A\int_{0}^{\infty} e^{-(s-\alpha)t}dt = A\int_{0}^{\infty} e^{(\alpha-\sigma)t}e^{-j\omega t}dt$$

This integral converges *if* $\sigma > \alpha$ and this inequality defines what is known as the *region of convergence (ROC)* of this Laplace transform.

Region of Convergence

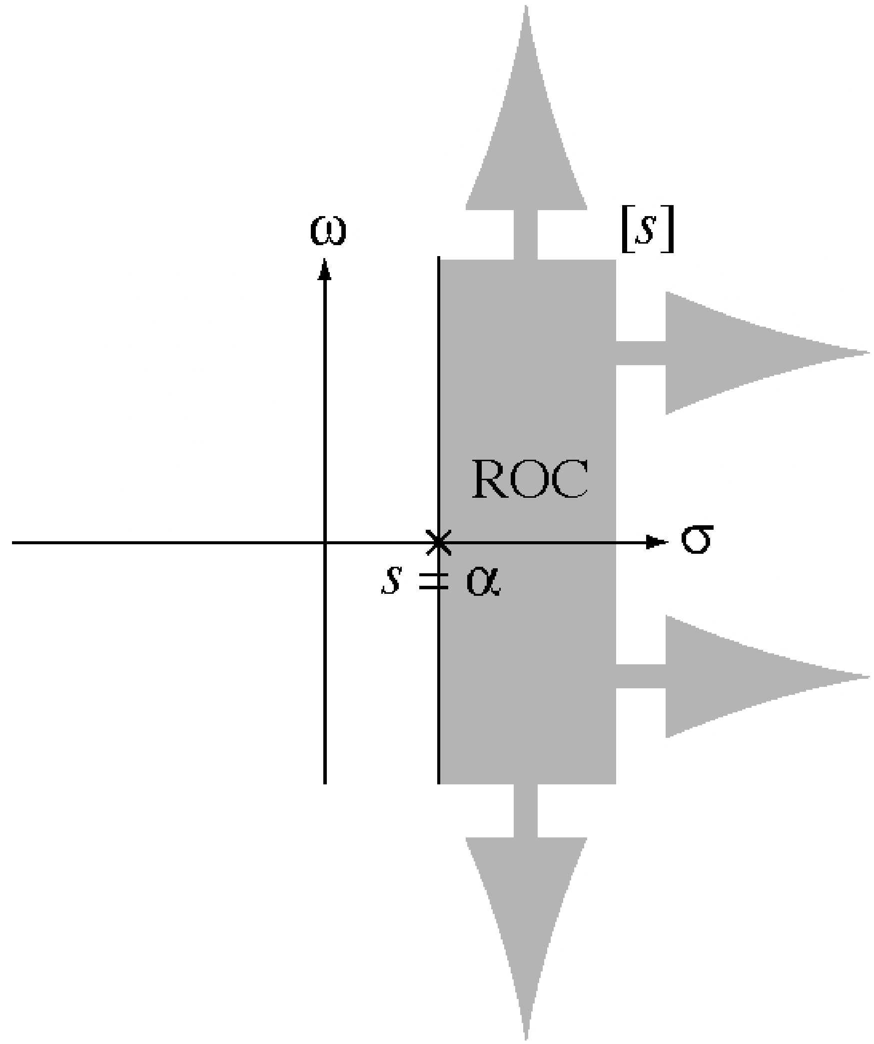
The ROC of the integral

$$G_1(s) = A\int_0^{\infty} e^{(\alpha-\sigma)t} e^{-j\omega t} dt$$

is the region of the *s* plane for which $\sigma > \alpha$ and the integral is

$$G_1(s) = \frac{A}{s-\alpha}$$

This function has a pole at $s = \alpha$ and the ROC is the region to the right of that point.



Region of Convergence

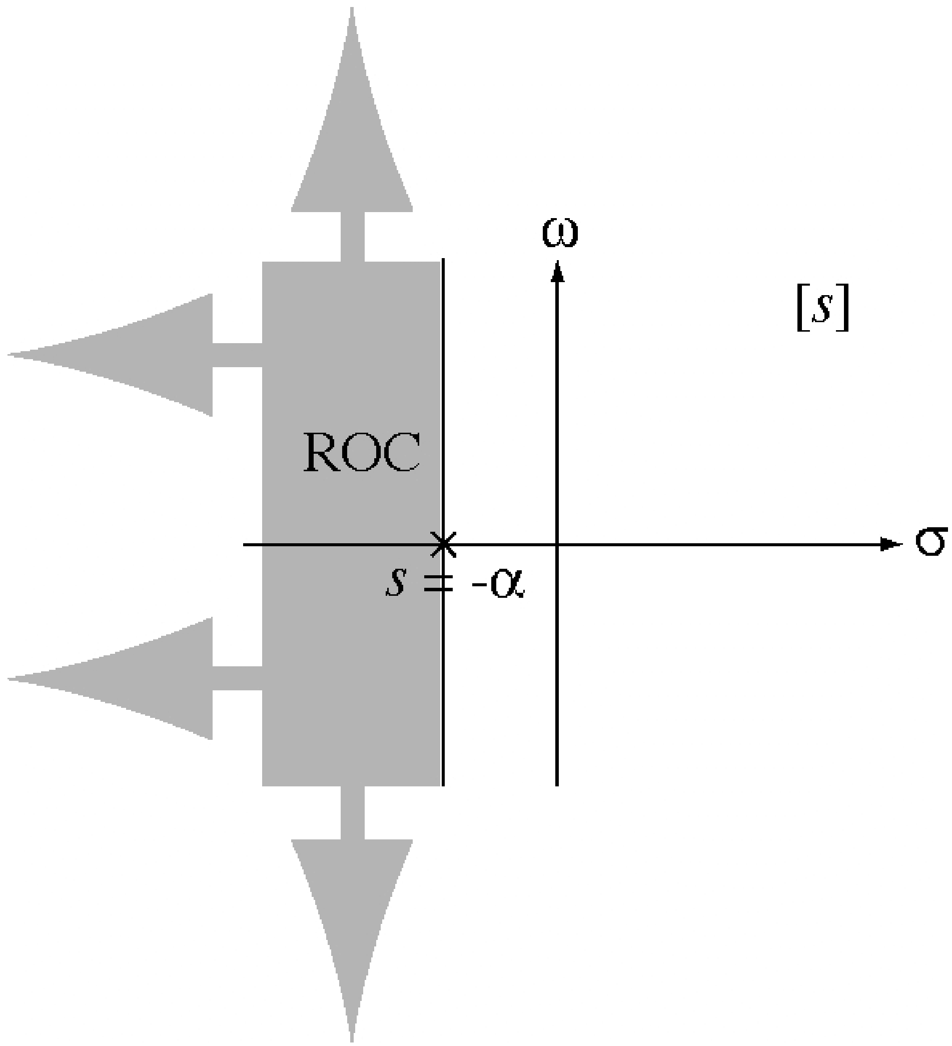
By similar reasoning, the ROC of Laplace transform of the anti-causal function,

$$g_2(t) = Ae^{-\alpha t}\,u(-t)$$
$$= g_1(-t)\,,\ \alpha > 0$$

is the region, $\sigma < -\alpha$, in the s plane and the integral is

$$G_2(s) = -\frac{A}{s+\alpha}$$

This function has a pole at $s = -\alpha$ and the ROC is the region to the left of that point.



Region of Convergence

The following two Laplace transform pairs illustrate the importance of the region of convergence.

$$e^{-\alpha t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+\alpha}\ ,\ \ \sigma > -\alpha$$

$$-e^{-\alpha t}\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+\alpha}\ ,\ \ \sigma < -\alpha$$

The two time-domain functions are different but the algebraic expressions for their Laplace transforms are the same. Only the ROC's are different.

Unilateral Laplace Transform

Definition $\longrightarrow$ $$\mathrm{G}(s) = \int_{0^-}^{\infty} \mathrm{g}(t) e^{-st} dt$$

Only the lower limit of the integral has changed

A function and its Laplace transform are uniquely related only if the function is causal.

The ROC is always the region of the *s* plane to the right of the pole in the transform with the most positive real part.

Any function which grows no faster than an exponential in positive time has a unilateral Laplace transform.

Unilateral Laplace Transform

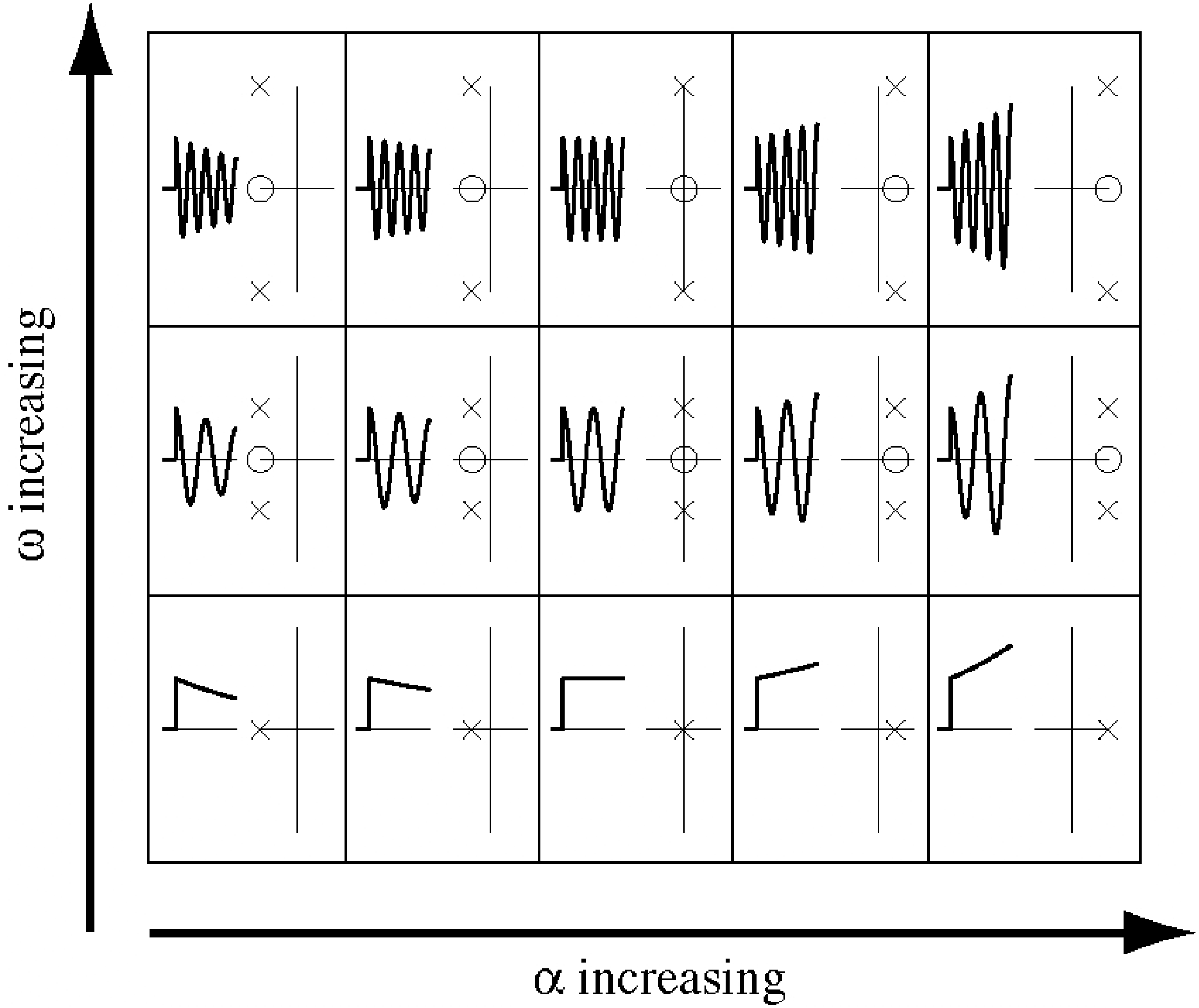
The unilateral Laplace transform will be referred to simply as *the* Laplace transform and the bilateral Laplace transform will be identified specifically. There is an inversion integral which is the same for both forms,

$$\mathcal{L}^{-1}\left(\mathrm{G}(s)\right) = \mathrm{g}(t) = \frac{1}{j2\pi}\int_{\sigma - j\infty}^{\sigma + j\infty} \mathrm{G}(s) e^{+st} ds$$

The inversion integral is rarely used in practice. Instead tables and properties are used.

Poles and Zeros

The poles and zeros of any Laplace-transform expression characterize the time-domain function completely, except for a scale factor.



Properties

Let g(t) and h(t) both be causal functions and let them form the following transform pairs,

$$\mathrm{g}(t) \xleftrightarrow{\mathcal{L}} \mathrm{G}(s) \qquad \mathrm{h}(t) \xleftrightarrow{\mathcal{L}} \mathrm{H}(s)$$

Linearity

$$\alpha\,\mathrm{g}(t) + \beta\,\mathrm{h}(t) \xleftrightarrow{\mathcal{L}} \alpha\,\mathrm{G}(s) + \beta\,\mathrm{H}(s)$$

Time Shifting

$$\mathrm{g}(t - t_0) \xleftrightarrow{\mathcal{L}} \mathrm{G}(s)e^{-st_0} \quad , \quad t_0 > 0$$

Complex-Frequency Shifting

$$e^{s_0 t}\,\mathrm{g}(t) \xleftrightarrow{\mathcal{L}} \mathrm{G}(s - s_0)$$

Properties

Time Scaling

$$g(at) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{a} G\left(\frac{s}{a}\right) \ , \ \ a > 0$$

Frequency Scaling

$$\frac{1}{a} g\left(\frac{t}{a}\right) \stackrel{\mathcal{L}}{\longleftrightarrow} G(as) \ , \ \ a > 0$$

Time Differentiation Once

$$\frac{d}{dt}(g(t)) \stackrel{\mathcal{L}}{\longleftrightarrow} s\,G(s) - g(0^-)$$

Time Differentiation Twice

$$\frac{d^2}{dt^2}(g(t)) \stackrel{\mathcal{L}}{\longleftrightarrow} s^2\,G(s) - s\,g(0^-) - \frac{d}{dt}(g(t))_{t=0^-}$$

Properties

Complex-Frequency Differentiation

$$-t\,g(t) \xleftrightarrow{\mathcal{L}} \frac{d}{ds}(G(s))$$

Multiplication-Convolution Duality

$$g(t) * h(t) \xleftrightarrow{\mathcal{L}} G(s)H(s) \qquad g(t)h(t) \xleftrightarrow{\mathcal{L}} \frac{1}{j2\pi}\int_{\sigma-j\infty}^{\sigma+j\infty} G(w)H(s-w)\,dw$$

Integration

$$\int_{0^-}^{t} g(\tau)\,d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s}G(s)$$

Initial Value Theorem

$$g(0^+) = \lim_{s\to\infty} s\,G(s)$$

Properties

Final Value Theorem

$$\lim_{t \to \infty} g(t) = \lim_{s \to 0} s\,G(s)$$

This theorem only applies if the limit actually exists. It is possible for the limit, $\lim_{s \to 0} s\,G(s)$, to exist even though the limit, $\lim_{t \to \infty} g(t)$ does not exist. For example,

$$x(t) = \cos(\omega_0 t) \xleftrightarrow{\mathcal{L}} X(s) = \frac{s}{s^2 + \omega_0^2}$$

$$\lim_{s \to 0} s\,X(s) = \lim_{s \to 0} \frac{s^2}{s^2 + \omega_0^2} = 0$$

but $\lim_{t \to \infty} \cos(\omega_0 t)$ does not exist.

Partial-Fraction Expansion

The inverse Laplace transform can always be found (in principle at least) by using the inversion integral. But that is rare in engineering practice. The most common type of Laplace-transform expression is a ratio of polynomials in *s*,

$$\mathrm{G}(s) = \frac{b_N s^N + b_{N-1} s^{N-1} + \cdots + b_1 s + b_0}{s^D + a_{D-1} s^{D-1} + \cdots a_1 s + a_0}$$

The denominator can be factored, putting it into the form,

$$\mathrm{G}(s) = \frac{b_N s^N + b_{N-1} s^{N-1} + \cdots + b_1 s + b_0}{(s - p_1)(s - p_2)\cdots(s - p_D)}$$

Partial-Fraction Expansion

For now, assume that there are no repeated poles and that $D > N$, making the fraction *proper* in s. Then it is possible to write the expression in the form,

$$\mathrm{G}(s) = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \cdots + \frac{K_D}{s - p_D}$$

where

$$\frac{b_N s^N + b_{N-1} s^{N-1} + \cdots b_1 s + b_0}{(s - p_1)(s - p_2)\cdots(s - p_D)} = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \cdots + \frac{K_D}{s - p_D}$$

The K's can be found be any convenient method.

Partial-Fraction Expansion

Multiply both sides of the previous expression by $s - p_1$.

$$(s-p_1)\frac{b_N s^N + b_{N-1}s^{N-1} + \cdots + b_1 s + b_0}{(s-p_1)(s-p_2)\cdots(s-p_D)} = \left[\begin{array}{l} (s-p_1)\dfrac{K_1}{s-p_1} + (s-p_1)\dfrac{K_2}{s-p_2} + \cdots \\ \quad + (s-p_1)\dfrac{K_D}{s-p_D} \end{array}\right]$$

Then, since the expression must be valid for any value of s, let $s = p_1$. All the terms on the right except one are then zero and

$$K_1 = \frac{b_N p_1^N + b_N p_1^{N-1} + \cdots + b_1 p_1 + b_0}{(p_1 - p_2)\cdots(p_1 - p_D)}$$

All the K's can be found by the same method and the inverse Laplace transform is then found by table look-up.

Partial-Fraction Expansion

If the expression has a repeated pole of the form,

$$\mathrm{G}(s) = \frac{b_N s^N + b_{N-1} s^{N-1} + \cdots + b_1 s + b_0}{(s - p_1)^2 (s - p_3) \cdots (s - p_D)}$$

the partial fraction expansion is of the form,

$$\mathrm{G}(s) = \frac{K_{12}}{(s - p_1)^2} + \frac{K_{11}}{s - p_1} + \frac{K_3}{s - p_3} + \cdots + \frac{K_D}{s - p_D}$$

and K_{12} can be found using the same method as before.
But K_{11} cannot be found using the same method.

Partial-Fraction Expansion

Instead K_{11} can be found by using the more general formula

$$K_{qk} = \frac{1}{(m-k)!} \frac{d^{m-k}}{ds^{m-k}} \left[\left(s - p_q\right)^m \mathrm{H}(s) \right]_{s \to p_q} \quad , \quad k = 1, 2, \quad , m$$

where m is the order of the qth pole, which applies to repeated poles of any order (pp. 647-649).

If the expression is not a proper fraction in s the partial-fraction method will not work. But it is always possible to synthetically divide the numerator by the denominator until the *remainder* is a proper fraction and then apply partial-fraction expansion (p. 649).

Partial-Fraction Expansion

$$\mathrm{H}(s) = \frac{10s}{(s+4)(s+9)} = \frac{K_1}{s+4} + \frac{K_2}{s+9}$$

$$K_1 = \left[\cancel{(s+4)}\frac{10s}{\cancel{(s+4)}(s+9)}\right]_{s=-4} = \left[\frac{10s}{s+9}\right]_{s=-4} = \frac{-40}{5} = -8$$

$$K_2 = \left[\cancel{(s+9)}\frac{10s}{(s+4)\cancel{(s+9)}}\right]_{s=-9} = \left[\frac{10s}{s+4}\right]_{s=-9} = \frac{-90}{-5} = 18$$

$$\mathrm{H}(s) = \frac{-8}{s+4} + \frac{18}{s+9} = \frac{-8s - 72 + 18s + 72}{(s+4)(s+9)} = \frac{10s}{(s+4)(s+9)}$$ Check.

$$\mathrm{h}(t) = \left(-8e^{-4t} + 18e^{-9t}\right)\mathrm{u}(t)$$

Partial-Fraction Expansion

$$\mathrm{H}(s) = \frac{10s^2}{(s+4)(s+9)} \leftarrow \text{ Improper in } s$$

$$\mathrm{H}(s) = \frac{10s^2}{s^2 + 13s + 36}$$

Synthetic Division

$$\begin{array}{r|l} & \qquad\qquad\quad 10 \\ s^2 + 13s + 36 & 10s^2 \\ & \underline{10s^2 + 130s + 360} \\ & \quad -130s - 360 \end{array}$$

$$\mathrm{H}(s) = 10 - \frac{130s + 360}{(s+4)(s+9)} = 10 - \left[\frac{-32}{s+4} + \frac{162}{s+9}\right]$$

$$\Downarrow$$

$$\mathrm{h}(t) = 10\delta(t) - \left[162e^{-9t} - 32e^{-4t}\right]\mathrm{u}(t)$$

Partial-Fraction Expansion

$$\mathrm{H}(s) = \frac{10s}{(s+4)^2(s+9)} = \frac{K_{12}}{(s+4)^2} + \frac{K_{11}}{s+4} + \frac{K_2}{s+9}$$

Repeated Pole

$$K_{12} = \left[\cancel{(s+4)^2} \frac{10s}{\cancel{(s+4)^2}(s+9)} \right]_{s=-4} = \frac{-40}{5} = -8$$

Using

$$K_{D,k} = \frac{1}{(m-k)!} \frac{d^{m-k}}{ds^{m-k}} \left[(s-p_D)^m \, \mathrm{H}(s) \right]_{s \to p_D} \quad , \quad k = 1, 2, \cdots, m$$

$$K_{11} = \frac{1}{(2-1)!} \frac{d^{2-1}}{ds^{2-1}} \left[(s+4)^2 \, \mathrm{H}(s) \right]_{s \to -4} = \frac{d}{ds} \left[\frac{10s}{s+9} \right]_{s \to -4}$$

Partial-Fraction Expansion

$$K_{11} = \left[\frac{(s+9)10 - 10s}{(s+9)^2} \right]_{s=-4} = \frac{18}{5}$$

$$K_2 = -\frac{18}{5} \quad \Longrightarrow \quad \mathrm{H}(s) = \frac{-8}{(s+4)^2} + \frac{\frac{18}{5}}{s+4} + \frac{-\frac{18}{5}}{s+9}$$

$$\mathrm{H}(s) = \frac{-8s - 72 + \frac{18}{5}\left(s^2 + 13s + 36\right) - \frac{18}{5}\left(s^2 + 8s + 16\right)}{(s+4)^2 (s+9)}$$

$$\mathrm{H}(s) = \frac{10s}{(s+4)^2 (s+9)} \quad \text{Check}$$

$$\Downarrow$$

$$\mathrm{h}(t) = \left(-8te^{-4t} + \frac{18}{5} e^{-4t} - \frac{18}{5} e^{-9t} \right) \mathrm{u}(t)$$

Laplace-Transform Fourier-Transform Equivalence

- If the region of convergence of the Laplace transform contains the ω axis (where σ is zero) then the Laplace transform, along that axis, is the same as the Fourier transform
- In such a case the Fourier transform can be found from the Laplace transform by substituting $j\omega$ for s

Solution of Differential Equations

The unilateral Laplace transform is particularly well suited for the solution of differential equations with initial conditions because the time differentiation properties of the unilateral Laplace transform call for the initial conditions in a systematic way. For example, the Laplace transform of the differential equation,

$$\frac{d^2}{dt^2}[\mathrm{x}(t)] + 7\frac{d}{dt}[\mathrm{x}(t)] + 12\,\mathrm{x}(t) = 0$$

is

$$s^2\,\mathrm{X}(s) - s\,\mathrm{x}(0^-) - \frac{d}{dt}(\mathrm{x}(t))_{t=0^-} + 7\left[s\,\mathrm{X}(s) - \mathrm{x}(0^-)\right] + 12\,\mathrm{X}(s) = 0$$

an algebraic equation in s which can be solved by algebraic methods. Then, when X(s) is found, its inverse Laplace transform, x(t), is the solution of the original differential equation.

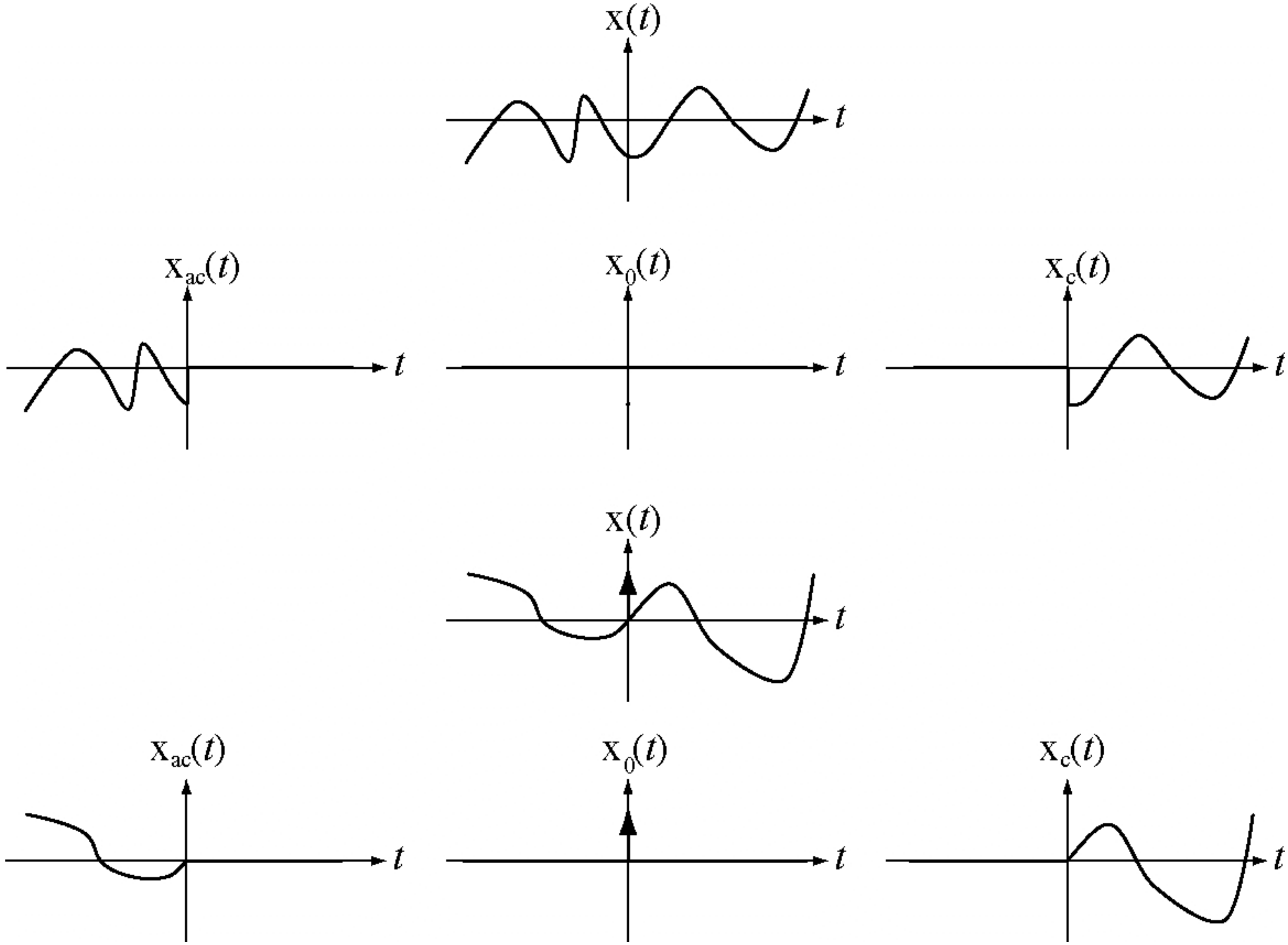
The Bilateral Laplace Transform

- Applies to non-causal signals and systems
- Can be found using tables of the unilateral Laplace transform

Any signal can be expressed as the sum of three parts, the part before time, $t = 0$ (the anticausal part), the part *at* time, $t = 0$ and the part after time, $t = 0$ (the causal part).

$$\mathrm{x}(t) = \mathrm{x}_{ac}(t) + \mathrm{x}_0(t) + \mathrm{x}_c(t)$$

The Bilateral Laplace Transform



The Bilateral Laplace Transform

- To find a bilateral Laplace transform (pp. 657-660)
- 1. Find the unilateral Laplace transform of the causal part along with its ROC
- 2. Find the unilateral Laplace transform of the time inverse of the anti-causal part along with its ROC
- 3. Change s to $-s$ in the result of step 2 and in its ROC
- 4. If there is an impulse at time, $t = 0$, find its Laplace transform and its ROC, the entire s plane
- 5. Add the results of steps 1, 3 and 4 and form the ROC from the region in the s plane common to all the ROC's. If such a region does not exist, the bilateral transform does not exist either