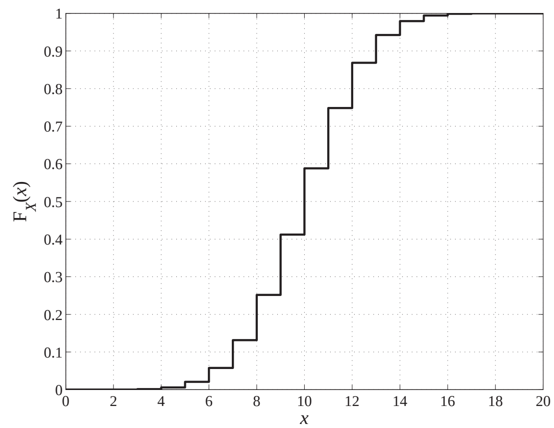
# Answers to Exercises in Chapter 2 - Random Variables

**Distribution Functions**

2-1. Let an experiment be flipping a fair coin 20 times. The outcome is the number of heads. Let this number be a random variable $X$.

(a)




(b) What is the probability that $X$ is less than 14 ? 0.942

2-2. The distribution function for a CV random variable $X$ is given by

$$\mathrm{F}_X(x) = K\left(2 - e^{-x}\right)\mathrm{u}(x)$$

(a) Find $K$. $K = 1/2$

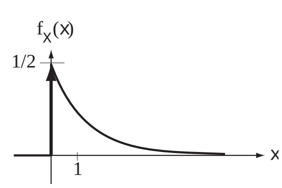
(b) What is the probability that $X$ lies in the interval $1 < X \leq \infty$ ? 0.184

(c) What is the probability that $X$ lies in the interval $1 < X \leq 2$ ? 0.1165

(d) What is the probability that $X = 0$ ? 1/2

**Probability Density Functions**

2-3. Find and sketch the probability density function corresponding to the distribution function of Exercise 2-2.




Then, using this , find

(a) The probability that $X$ lies in the interval $1 < X \leq 3$. 0.1591

(b) The probability that $X$ is less than 2. 0.9325

2-4. A CV random variable $X$ has a PDF given by $\mathrm{f}_X(x) = (1/4)\mathrm{tri}(x/4)$ .

(a) $Y = X^2$ . Find the PDF for $Y$ and find the probability that $Y$ is less than 3.

$$\mathrm{f}_Y(y) = \frac{\mathrm{tri}\left(\sqrt{y}/4\right)}{4\sqrt{y}}\mathrm{u}(y)$$

$$\mathrm{P}\left[Y < 3\right] = 0.678$$

(b) $Y = 2X + 1$ Find the PDF for $Y$ and find $\mathrm{P}\left[Y < X\right]$.

$$\mathrm{f}_Y(y) = \frac{1}{8}\mathrm{tri}\left(\frac{y-1}{8}\right)$$

$$\mathrm{P}\left[Y < X\right] = 0.28125$$

**Mean, Standard Deviation and Expectation**

2-5. A die is tossed multiple times. What is the probability that a number less than 3 will appear within the first 3 tosses?

19/27

2-6. A fair coin is flipped 8 times. What is the probability that the 4th head will occur during those 8 flips? 0.6367

2-7. Beginning with the definition of variance $\sigma_X^2 = \mathrm{E}\left(\left[X - \mathrm{E}(X)\right]^2\right)$ and using the properties of the expectation operator show that

$$\sigma_X^2 = \mathrm{E}\left(X^2\right) - \left[\mathrm{E}(X)\right]^2 \ .$$

Proof.

2-8. A coin is flipped until a head appears or $N_0$ flips have occurred, whichever occurs first. The number of flips is $N$. Find $\mathrm{E}(N)$.

$$\mathrm{E}(N) = \frac{N_0}{2^{N_0-1}} + \sum_{n=1}^{N_0-1}\frac{n}{2^n}$$

2-9. An experiment consists of tossing a fair die until the number 3 occurs or the die is tossed 4 times, whichever comes first. The number of tosses is a random variable $N$. Find $\mathrm{E}(N)$. 3.1065

2-10. A man can go to the office by two routes. On one route there is a bridge which is out with probability 0.2. Travel time to the bridge is 12 minutes and additional travel time to the office is 8 minutes. On the other route the travel time is 25 minutes. Every day when he leaves home he does not know whether the bridge is out or not. When the bridge is out he returns home and then takes the longer route. Let the travel time be the random variable $T$. What is his expected travel time $\mathrm{E}(T)$? 25.8 minutes

2-11. For the random variables of Exercises 2-3 and 2-4 (for $X$ only) find the expected values, the mean-squared values and the variances.

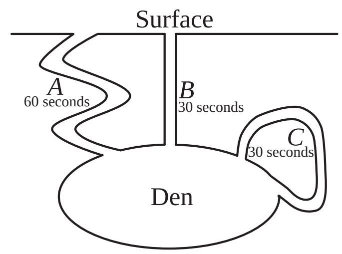
1/2, 1, 3/4
4/3, 8/3, 8/9

2-12. A DV random variable $X$ has a PDF $\mathrm{f}_X(x) = 0.2\delta(x) + 0.5\delta(x-4) + 0.3\delta(x-6)$.

(a) What is its expected value $\mathrm{E}(X)$? 3.8

(b) What is its variance $\sigma_X^2$ ? 4.36

2-13. A prairie dog is in his underground den and wants to go to the surface. There are three tunnels $A$, $B$ and $C$ he can choose. Tunnel $A$ leads to the surface and the time required to get to the surface is 60 seconds. Tunnel $B$ leads to the surface and the time required to get to the surface is 30 seconds. Tunnel $C$ does not go to the surface but instead loops back into the den and time required to get back to the den is 30 seconds. (So there are two entrances to Tunnel $C$, making a total of four tunnel entrances in the den.) The probability that the prairie dog will choose any particular tunnel entrance is the same as choosing any other tunnel entrance. This prairie dog is incapable of learning from his mistakes so when he chooses Tunnel $C$ and returns to his den the probability that he will choose any tunnel entrance on his next try is the same as it was before. Let the time to get to the surface be the random variable $T$. What is the expected time $\mathrm{E}(T)$ for the prairie dog to reach the surface? 75 seconds




**Poisson Processes and Exponential Probability Density**

2-14. Assume that lightning strikes the earth an average of 100 times every second and that it is equally likely to strike at any point on the entire surface. The surface area of the earth is approximately 500 million square kilometers. A man is born, lives and dies over a time span of 80 years.

(a) What is the probability, during his entire life, that he will be within one meter of a lightning strike exactly one time? (The man is at the center of a circular area of $\pi$ square meters and lightning strikes somewhere within that circular area.) 0.0016

(b) What is the probability, during his entire life, that he will be within 50 meters of a lightning strike three or more times? 0.775

2-15. An air conditioner compressor is designed to have a mean time to failure of 15 years. Let the time to failure be a random variable $T$ that is exponentially distributed. Find

(a) The probability that the compressor will survive more than 10 years. 0.513

(b) The probability that the compressor will fail sooner than 15 years. 0.632

(c) The probability that the compressor will fail during the 12th year. 0.031

2-16. The compressor of Exercise 2-15 is replaced when it fails with an identical model. Let $T$ be the overall lifetime of 3 such compressors.

(a) Find the expected lifetime of three compressors $\mathrm{E}(T)$. 45 years

(b) Find the probability that 3 compressors will last longer than 60 years 0.09

This can be found using the Erlang distribution with $k = 3$.

2-17. The time $T$ between emissions of beta particles by a radioactive source is random and exponentially distributed with a mean time $\mathrm{E}(T)$ between emissions of five seconds and a probability density function of the form

$$\mathrm{f}_T(t) = Ke^{-t/\mathrm{E}(T)}\,\mathrm{u}(t).$$

(a) What is value of the constant $K$? 0.2

(b) If an emission has just occurred what is the expected time until the next emission occurs? 5 seconds

(c) If $A = \{T > 1\}$, find the value of $\mathrm{E}(T \mid A)$, the expected time between emissions given that we know it is greater than one second. 6 seconds

2-18. For the compressor in Exercise 2-15, find the probability that it will survive for 30 years given that it has already survived for 15 years. Also find the conditional mean lifetime, given that it has survived 9 years.
0.368, 24 years

**Functions of a Random Variable**

2-19. A CV signal has a uniform probability density between -10 V and +10 V. It is sampled and quantized by an $n$-bit ADC into $2^n$ equally-spaced levels, two of which are -10 V and 0 V.

(a) If $n = 4$, write the probability density function for the DV random variable representing one sample, find the mean and standard deviation for the random variable and compare them with the mean and standard deviation of a CV uniform random variable from -10 V to 10 V.

$$\mathrm{f}_V(v) = \frac{1}{16}\begin{bmatrix} \delta(v+10)+\delta(v+8.75)+\delta(v+7.5)+\delta(v+6.25) \\ +\delta(v+5)+\delta(v+3.75)+\delta(v+2.5)+\delta(v+1.25) \\ +\delta(v)+\delta(v-1.25)+\delta(v-2.5)+\delta(v-3.75) \\ +\delta(v-5)+\delta(v-6.25)+\delta(v-7.5)+\delta(v-8.75) \end{bmatrix}$$

DV: $\mathrm{E}(V) = -0.625$ $\sigma_V = 5.7622$

CV: $\mathrm{E}(V) = 0$ $\sigma_V = 5.7735$

(b) If $n = 8$, write the probability density function for the DV random variable representing one sample, find the mean and standard deviation for the random variable and compare them with the mean and standard deviation of a CV uniform random variable from -10 V to 10 V.

$$\mathrm{f}_V(v) = \frac{1}{256}\left[\delta(v+10)+\delta(v+9.921875)+\cdots+\delta(v-9.921875)\right]$$

DV: $\mathrm{E}(V) = -0.03906$ $\sigma_V = 5.7735$

CV: $\mathrm{E}(V) = 0$ $\sigma_V = 5.7735$

2-20. A CV random variable $\Theta$ is uniformly distributed over the range $-\pi$ to $\pi$. Another random variable $X$ is related to $\Theta$ by $X = \sin(\Theta)$ .

(a) Find the PDF of $X$. $\mathrm{f}_X\left(x\right) = \dfrac{1}{\pi\sqrt{1-x^2}}\operatorname{rect}\left(x/2\right)$

(b) Find the mean value of $X$. 0

(c) Find the variance of $X$. 1/2

(d) Find the probability that X is greater than 0.5. 1/3

2-21. A CV random variable $X$ has a uniform probability density function between -3 and +6. Another random variable $Y$ is created through the function in Figure E2-21 relating $Y$ to $X$.

(a) What is the probability of the event $X = 2$ ?

Zero. The PDF of $X$ is continuous.

(b) What is the probability of the event $Y = 2$? 4/9

(c) Sketch the PDF of $Y$.

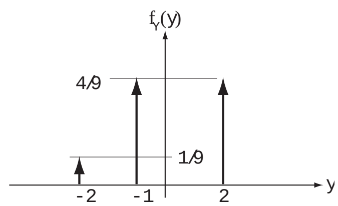



Figure E2-21

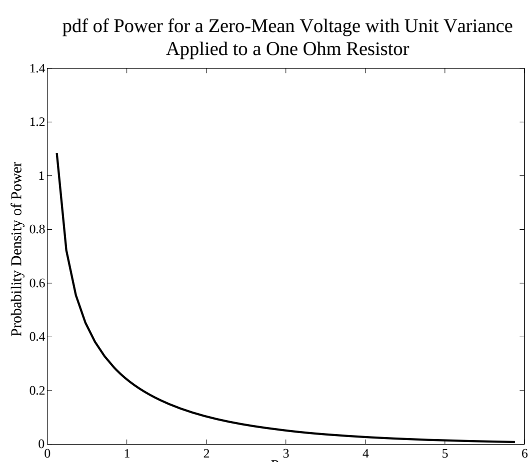
2-22. Let the PDF of the voltage $V$ across a resistor be Gaussian with mean $\mu_V$ and standard deviation $\sigma_V$

$$\mathrm{f}_V\left(v\right) = \frac{1}{\sigma_V\sqrt{2\pi}}e^{-\left(v-\mu_V\right)^2/2\sigma_V^2} .$$

(a) Find a general expression for the  of the power $P$ delivered to the resistor.

$$P = \left(\mu_V^2 + \sigma_V^2\right)/R$$

(b) If the voltage has zero mean and a variance of one and the resistance is one ohm graph the PDF of the power.




2-23. A system consists of two cascaded subsystems 1 and 2 cascaded with another subsystem which consists of two parallel-connected systems 3 and 4. The subsystems have constant failure rates with MTTF's of

$$\tau_1 = 5000 \text{ hours} \ , \ \tau_2 = 3000 \text{ hours} \ , \ \tau_3 = 10000 \text{ hours} \ , \ \tau_4 = 1000 \text{ hours} \ .$$

Assuming that the system only fails if both parallel-connected subsystems fail what is the MTTF of the overall system? 1620 hours