Answers to Exercises in Chapter 3 - Multiple Random Variables

Mean, Standard Deviation and Expectation

3-1. Four independent random variables X_1, X_2, X_3 and X_4 each Gaussian distributed with a mean of 0 and a standard deviation of 4 are combined to form a new random variable

$$Y = X_1^2 + X_2^2 + X_3^2 + X_4^2 .$$

(a) What is the expected value of Y? 64

(b) What is the variance of Y? 2048

(c) What is the probability of the event $50 < Y < 80$? 0.24

3-2. Random variable X has a variance of 20 and Y has a variance of 5. Their correlation coefficient is 0.7.

(a) Find the variance of their sum 39

(b) Find the variance of their difference. 11

3-3. Two independent random variables X and Y have variances $\sigma_X^2 = 12$ and $\sigma_Y^2 = 18$.

If W and Z are $W = -4X + 2Y$ and $Z = 3X - 6Y$,

(a) Find the variances of W and Z. 264 and 756

(b) Find the correlation coefficient of W and Z. -0.672

3-4. A rural electric cooperative has a base load which is constant at 25% of its nominal power capability and three large industrial customers who operate independently and whose power demand and probability of demanding that power are listed below.

Industrial Customer	Power Demand	Probability of Demanding Power
1	30%	0.3
2	25%	0.4
3	35%	0.25

(a) What are the average power demanded by the system and the standard deviation of the average power demanded by the system? 52.75% 23.84%

(b) What is the probability that the power demand on the system will exceed its nominal power capability? 0.03

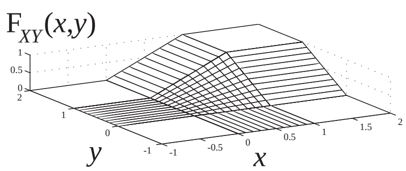
Joint Probability Density

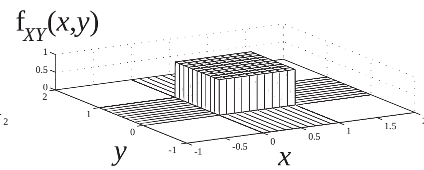
3-5. Two CV random variables X and Y have a joint distribution function given by

$$F_{XY}(x,y) = \begin{cases} 0 & , \ x < 0 \text{ or } y < 0 \\ xy & , \ 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ x & , \ y \ge 1 \text{ and } 0 \le x \le 1 \\ y & , \ x \ge 1 \text{ and } 0 \le y \le 1 \\ 1 & , \ x > 1 \text{and } y > 1 \end{cases} .$$

(a) Graph this distribution function. (See (b)).

(b) Graph the joint PDF.





(c) Find the probability of the event $\{X \le 1/2 \cap Y > 1/2\}$. 1/

(d) Find $E(XY)$. 1/4

3-6. Two CV random variables X and Y have a joint PDF given by

$$f_{XY}(x,y) = K(x+1)(y+1/2)\text{rect}(x/2)\text{rect}(y)$$

(a) Find K. 1

(b) Find the joint distribution function $F_{XY}(x,y)$.

$$F_{XY}(x,y) = \begin{cases} 0 \ , \ x < -1 \text{ or } y < -1/2 \\ \left(\dfrac{x^2}{2} + x + 1/2\right)\left(\dfrac{y^2+y}{2} + 1/8\right) \ , \ -1 < x < 1 \text{ and } -1/2 < y < 1/2 \\ \dfrac{1}{2}\left(\dfrac{x^2}{2} + x + 1/2\right) \ , \ -1 < x < 1 \text{ and } y > 1/2 \\ 2\left(\dfrac{y^2+y}{2} + 1/8\right) \ , \ -1/2 < y < 1/2 \text{ and } x > 1 \\ 1 \ , \ x > 1 \text{ and } y > 1/2 \end{cases}$$

(c) Find the probability of the event $\{X \le 1/2 \cap Y > 0\}$. 27/64

(d) Find the marginal PDF $f_X(x)$ $\qquad f_X(x) = \dfrac{1}{2}(x+1)\text{rect}(x/2)$

(e) Find $E(XY)$. 1/18

(f) Find $f_{X|Y}(x)$ and $f_{Y|X}(y)$.

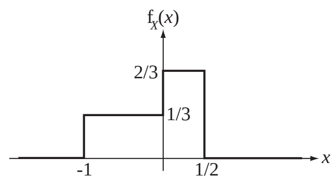
$$f_{X|Y}(x) = (1/2)(x+1)\text{rect}(x/2) \text{ and } f_{Y|X}(y) = 2(y+1/2)\text{rect}(y)$$

3-7. Two CV random variables X and Y have a joint PDF

$$f_{XY}(x,y) = \begin{cases} 2/3 \ , & -2 < x < 0 \ \text{ and } \ -1/2 < y < 0 \\ 1/3 \ , & 0 < x < 1/2 \ \text{ and } \ 0 < y < 2 \\ 0 \ , & \text{elsewhere} \end{cases}$$

(a) Find $\mathrm{E}(XY)$. 1/4

(b) Find and graph versus *x* the marginal PDF $f_X(x)$.



(c) What is the probability of the event which is the intersection of the events $X < 1/4$ and $Y > 1$? 1/12

3-8. For each joint PDF determine whether *X* and *Y* are uncorrelated and find their correlation $\mathrm{E}(XY)$.

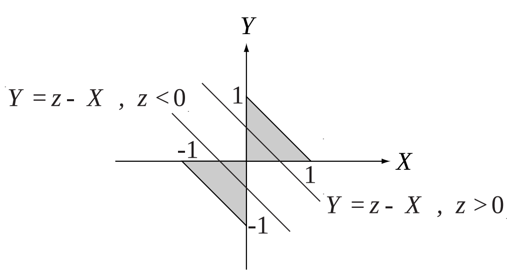
(a) $f_{XY}(x,y) = K(x^2 + y^2)\mathrm{rect}(x)\mathrm{rect}(y/2)$ $\quad \mathrm{E}(XY) = 0$

(b) $f_{XY}(x,y) = K(x + xy + y)\mathrm{rect}(x - 1/2)\mathrm{rect}(y - 1/2)$ $\quad \mathrm{E}(XY) = 0.3556$

(c) $f_{XY}(x,y) = K\dfrac{x}{y}\mathrm{rect}(x-1)\mathrm{rect}(y-1)$ $\quad \mathrm{E}(XY) = 0.986$

3-9. The joint PDF of the random variables *X* and *Y* is given by

$$f_{XY}(x,y) = \begin{cases} 1 \ , & \text{in shaded area} \\ 0 \ , & \text{otherwise} \end{cases} .$$



(a) Find the marginal PDF's of *X* and *Y*. Are *X* and *Y* independent?

$$f_X(x) = \left\{\begin{matrix} 0 & , & x < -1 \\ x+1 & , & -1 \le x < 0 \\ 1-x & , & 0 \le x < 1 \\ 0 & , & x \ge 1 \end{matrix}\right\} \quad , \quad f_Y(y) = \left\{\begin{matrix} 0 & , & y < -1 \\ y+1 & , & -1 \le y < 0 \\ 1-y & , & 0 \le y < 1 \\ 0 & , & y \ge 1 \end{matrix}\right\}$$

X and *Y* are not independent.

(b) Let $Z = X + Y$. Find $F_Z(z)$ and $f_Z(z)$.

$$F_Z(z) = \frac{1}{2}\left\{\left(1 - z^2\right)\left[u(z+1) - u(z)\right]\left(z^2 + 1\right)\left[u(z) - u(z-1)\right]\right\} + u(z-1)$$

$$f_Z(z) = |z|\left(u(z+1) - u(z-1)\right)$$

Linear Combinations of Random Variables and the Gaussian Distribution

3-10. A CV random variable X has a PDF

$$f_X(x) = (1/5)e^{-x/5}\,u(x)$$

and an independent CV random variable Y has a PDF

$$f_Y(y) = (1/3)e^{-y/3}\,u(y) \quad .$$

(a) Find the probability density function of the random variable $Z = X - Y$, graph it and verify that its area is one.

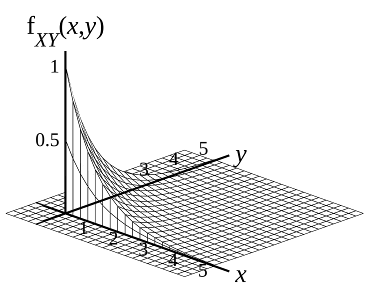
$$f_Z(z) = \frac{1}{8}\begin{cases} e^{-z/5} & , \ z \ge 0 \\ e^{z/3} & , \ z < 0 \end{cases}$$

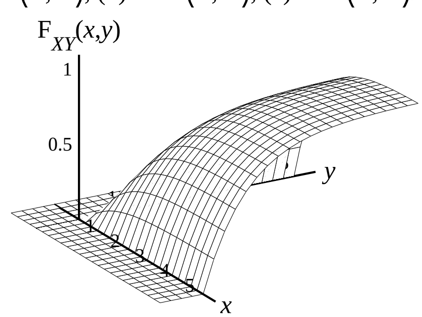
(b) Find the probability that $-1 < Z \le 1$. 0.2195

3-11. X and Y are independent, identically distributed (i.i.d.) random variables with common PDF

$$f_X(x) = e^{-x}\,u(x) \qquad f_Y(x) = e^{-y}\,u(y)$$

Find the PDF of the following random variables (a) $\min(X,Y)$, (b) $\max(X,Y)$, (c) $\min(X,Y)/\max(X,Y)$.





(a) $F_Z(z) = P\left[\min(X,Y) \le z\right]$

$$f_Z(z) = 2e^{-2z}u(z)$$

(b) $F_Z(z) = P\left[\max(X,Y) \le z\right]$

$$f_Z(z) = 2e^{-z}\left(1 - e^{-z}\right)u(z)$$

(c) $Z = \min(X,Y)/\max(X,Y)$ and X and Y are never negative.

$$f_Z(z) = \frac{2\,\mathrm{rect}(z - 1/2)}{(1+z)^2}$$

3-12. X and Y are independent and uniform in the interval $(0,a)$. Find the PDF of (a) X/Y, (b) $Y/(X+Y)$, (c) $|X - Y|$.

(a) $\mathrm{f}_Z(z) = \frac{1}{2}[\mathrm{u}(z) - \mathrm{u}(z-1)] + \frac{1}{2z^2}\mathrm{u}(z-1)$

(b) $\mathrm{f}_Z(z) = \frac{1}{2(1-z)^2}\left[\mathrm{u}(z) - \mathrm{u}\left(z - \frac{1}{2}\right)\right] + \frac{1}{2z^2}\left[\mathrm{u}\left(z - \frac{1}{2}\right) - \mathrm{u}(z-1)\right]$

(c) $\mathrm{f}_Z(z) = \frac{2}{a}\left(1 - \frac{z}{a}\right)[\mathrm{u}(z) - \mathrm{u}(z-a)]$

3-13. The joint PDF of X and Y is given by

$$\mathrm{f}_{XY}(x,y) = \begin{cases} 2(1-x) & , \ 0 < x \le 1, 0 < y \le 1 \\ 0 & , \ \text{otherwise} \end{cases} .$$

Determine the PDF of $Z = XY$.

$$\mathrm{f}_Z(z) = 2[z - 1 - \ln(z)]$$

3-14. X and Y are independent uniformly distributed random variables on $(0,1)$. Find the joint PDF of $X + Y$ and $X - Y$.

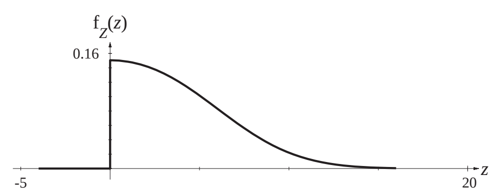
$$\mathrm{f}_{UV}(u,v) = \frac{1}{2}\mathrm{f}_{XY}\left(\frac{u+v}{2}, \frac{u-v}{2}\right) = \frac{1}{2}\begin{cases} 1 & , \ 0 \le u+v \le 2 \ , \ 0 \le u-v \le 2 \\ 0 & , \ \text{otherwise} \end{cases}$$

Central Limit Theorem and Gaussian Distributions

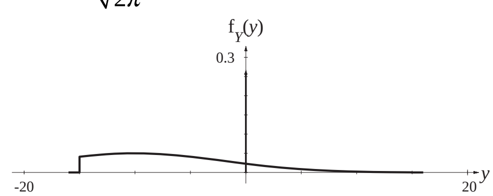
3-15. A resistor in a circuit has a Gaussian noise voltage across it with zero mean and a mean-squared value of $10^{-12}\ \mathrm{V}^2$. What percentage of the time is the voltage across the resistor greater than $1\ \mu\mathrm{V}$? About 16%

3-16. A random variable X is Gaussian distributed and $\mathrm{P}[X > 2] = 0.3$ and $\mathrm{P}[X > 5] = 0.1$. What are the expected value $\mathrm{E}(X)$ and the variance σ_X^2 of X? $\sigma_X^2 = 15.6$, $\mathrm{E}(X) = -0.1$.

3-17. A Gaussian random variable X has a mean of 3 and a variance of 16. If $Y = |X|$ find the mean and variance of Y and graph its PDF. $\mathrm{E}(Y) = 4.0531$ $\sigma_Y^2 = 8.572$



3-18. A Gaussian random variable X has a mean of -10 and a variance of 64. If $Y = X\,\mathrm{rect}(X/30)$, find the expected value of Y and graph its PDF. $\mathrm{E}(Y) = \frac{1}{\sqrt{2\pi}}[-8 \times (-0.815)] - 7.331 = 2.6 - 7.331 = -4.731$



3-19. X and Y are independent identically-distributed Gaussian random variables with zero mean and common variance σ^2. Find the PDF of (a) $Z = \sqrt{X^2 + Y^2}$, (b) $W = X^2 + Y^2$ and (c) $U = X - Y$.

(a) $$f_Z(z) = \frac{z}{\sigma^2} e^{-z^2/2\sigma^2} u(z)$$

(b) $$f_W(w) = \frac{e^{-w/2\sigma^2}}{2\sigma^2} u(w)$$

(c) $$f_U(u) = \frac{1}{\left(\sqrt{2}\sigma\right)\sqrt{2\pi}} e^{-u^2/2\left(\sqrt{2}\sigma\right)^2}$$

3-20. A system consists of two cascaded subsystems 1 and 2 cascaded with another subsystem which consists of two parallel-connected systems 3 and 4. The subsystems have constant failure rates with MTTF's of

$$\tau_1 = 5000 \text{ hours} \ , \ \tau_2 = 3000 \text{ hours} \ , \ \tau_3 = 10000 \text{ hours} \ , \ \tau_4 = 1000 \text{ hours} \ .$$

Assuming that the system only fails if both parallel-connected subsystems fail what is the MTTF of the overall system?

The overall system MTTF is 1620 hours.

General

3-21. A CV random signal X has a Rayleigh PDF and a mean value of 10 and is added to noise N that is uniformly distributed with a mean value of zero and a variance of 12. X and N are statistically independent and can be observed only as $Y = X + N$.

(a) Find, sketch and label the conditional PDF $f_{X|Y}(x)$ as a function of x for Y = 0, 6 and 12.

(b) If an observation yields a value of Y = 12, what is the best estimate of the true value of X? 7.98