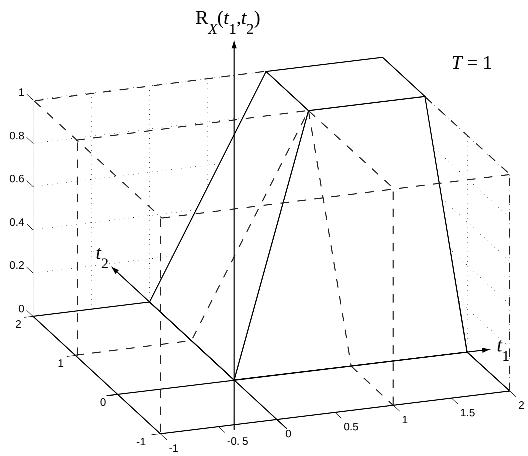
Answers to Exercises in Chapter 7 - Correlation Functions

7-1. (from Papoulis and Pillai) The random variable C is uniform in the interval $(0,T)$. Find $\mathrm{R_X}(t_1,t_2)$ if (a) $\mathrm{X}(t)=\mathrm{u}(t-C)$, (b) $\mathrm{X}(t)=\delta(t-C)$. (Use $\mathrm{R_{X'X'}}(t_1,t_2)=\dfrac{\partial^2 \mathrm{R_{XX}}(t_1,t_2)}{\partial t_1 \partial t_2}$ in (b)).

(a)

$$\mathrm{R_X}(t_1,t_2)=\begin{cases} 0 \ , \ t_1<0 \text{ or } t_2<0 \\ 1 \ , \ t_1>T \text{ and } t_2>T \\ \min(t_1,t_2)/T \ , \ \text{otherwise} \end{cases}$$

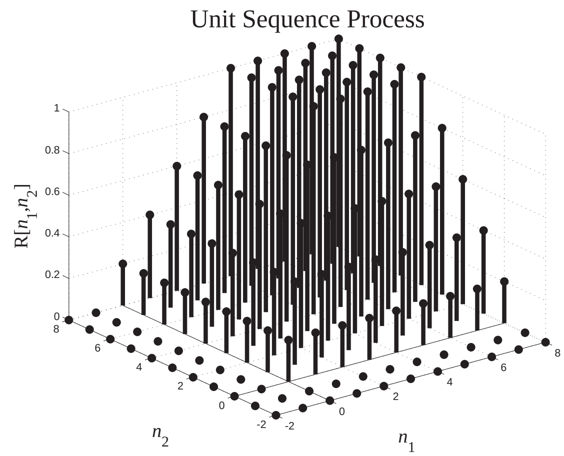


(b) $\mathrm{R_{X'X'}}(t_1,t_2)=\dfrac{1}{T}\delta(t_1-t_2)$

7-2. The random variable C is equally likely to have any value in $\{0,1,2,3,4\}$. Find $\mathrm{R_X}[n_1,n_2]$ if (a) $\mathrm{X}[n]=\mathrm{u}[n-C]$, (b) $\mathrm{X}[n]=\delta[n-C]$.

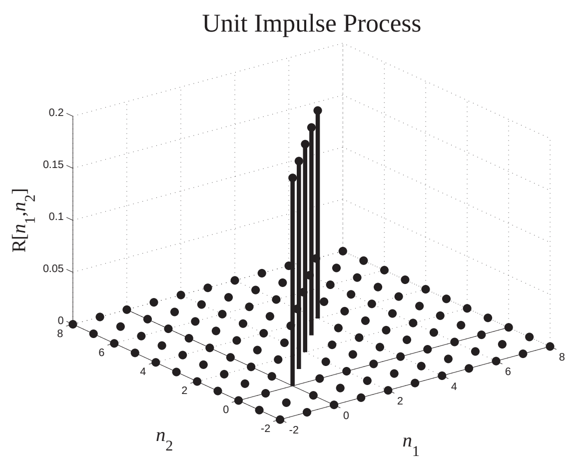
(a)

$$\mathrm{R_X}[n_1,n_2]=\begin{cases} 0 \ , \ n_1<0 \text{ or } n_2<0 \\ 1 \ , \ n_1\ge 4 \text{ and } n_2\ge 4 \\ \dfrac{\min(n_1,n_2)+1}{5} \ , \ \text{otherwise} \end{cases}$$



(b)

$$R_X[n_1,n_2] = \begin{cases} 0 \ , \ n_1 < 0 \text{ or } n_2 < 0 \text{ or } n_1 > 4 \text{ or } n_2 > 4 \text{ or } n_1 \neq n_2 \\ 1/5 \ , \ n_1 = n_2 \text{ and } 0 \leq n_1 \leq 4 \text{ and } 0 \leq n_2 \leq 4 \end{cases}$$



7-3. (from Papoulis and Pillai) The random variables A and B are independent and Gaussian with zero mean and standard deviation σ and p is the probability that the stochastic CTCV process $X(t) = A - Bt$ crosses the t axis in the interval $(0,T)$. Show that $\pi p = \tan^{-1}(T)$. (*Hint:* $p = P(0 \leq A/B \leq T)$.)

Proof.

7-4. (from Papoulis and Pillai) The stochastic CTCV process $X(t)$ is WSS and Gaussian with $R_X(\tau) = 4e^{-2|\tau|}$. (a) Find $P[X(t) \leq 3]$. (b) Find $E([X(t+1) - X(t-1)]^2)$.

(a) 0.9332
(b) 7.8535

7-5. The stochastic DTDV process $X[n]$ is WSS and geometrically distributed with $R_X[m] = 10(0.8)^{|m|}$. (a) Find $P[X[n] \leq 5]$. (b) Find $E([X[n+1] - X[n-1]]^2)$.

(a) 0.9222

(b) 7.2

7-6. (from Papoulis and Pillai) Given a Gaussian stochastic CTCV process $X(t)$ with $R_X(\tau) = 4e^{-2|\tau|}$, we form the random variables $Z = X(t+1)$, $W = X(t-1)$ (a) Find $E(ZW)$ and $E((Z+W)^2)$, (b) find $f_Z(z)$, $P[Z<1]$ and $f_{ZW}(z,w)$.

(a) 0.0733, 8.1465

(b) $P[Z<1] = 0.6915$

$$f_{ZW}(z,w) = \frac{\exp\left[-\frac{(z/2)^2 - 0.00915zw + (w/2)^2}{1.99933}\right]}{25.13}$$

$$f_Z(z) = \frac{1}{2\sqrt{2\pi}} e^{z^2/8} \ .$$

7-7. (from Cooper and McGillem) A WSS stochastic CTCV process having sample functions $X(t)$ has an autocorrelation function $R_X(\tau) = 5e^{-5|\tau|}$. Another WSS stochastic process has sample functions

$$Y(t) = X(t) + bX(t-0.1) \ .$$

(a) Find the value of b that minimizes the mean-squared value of Y. -0.6065

(b) Find the value of the minimum mean-squared value of Y. 3.1608

(c) If $|b| \le 1$, find the maximum mean-squared value of Y. 16.065.

7-8. Consider a WSS stochastic CTDV process having sample functions illustrated in Figure E7-8.

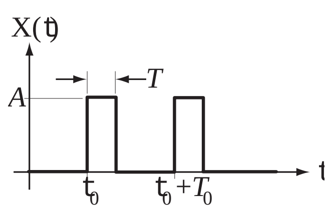
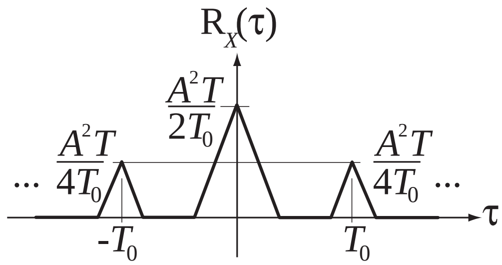


Figure E7-8 A sample function from a WSS stochastic DTDV process

In this process a pulse of height A and width T is equally likely to occur or not occur at times $t_0 \pm nT_0$. The probability of occurrence of the pulse is independent of the probabilities of occurrence of all other pulses. The time t_0 is random and uniformly distributed over $t = 0$ to $t = T_0$ and the pulse width T is less than half the pulse spacing T_0. Find the autocorrelation function for this process.

$$R_X(\tau) = \frac{A^2T}{4T_0}\text{tri}\left(\frac{\tau}{T}\right) + \frac{A^2T}{4T_0}\sum_{n=-\infty}^{\infty}\text{tri}\left(\frac{\tau - nT_0}{T}\right) \ .$$



7-9. Consider a WSS stochastic DTDV process having the sample functions illustrated in Figure E7-9.

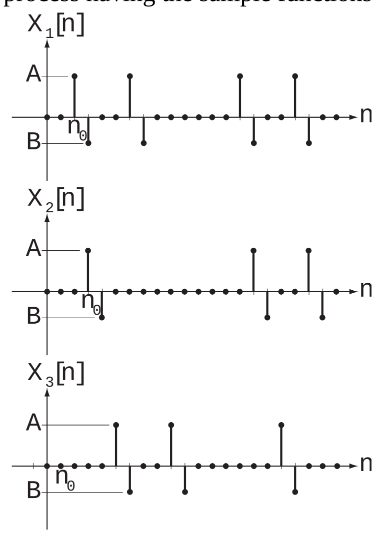


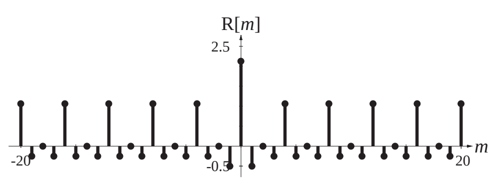
Figure E7-9 A DTCV bipolar-pulse stochastic process

In this process bipolar pulse pairs occur every integer multiple of 4 in discrete time with probability p. The probability of occurrence of each pulse pair is independent of the occurrence of any other pulse pair. The positions n_0 of the first pulse pair after $n = 0$ (whether or not the pulse pair actually occurs) are uniformly distributed on $\{1,2,3,4\}$.

(a) Find the autocorrelation function for this process.

$$\mathrm{R}[m] = \frac{p}{4}\left(\begin{array}{l} (1-p)\left[\left(A^2+B^2\right)\delta[m] + AB\left(\delta[m+1]+\delta[m-1]\right)\right] \\ +p\sum_{k=-\infty}^{\infty}\left\{\left(A^2+B^2\right)\delta[m-4k] + AB\left(\delta[m-4k-1]+\delta[m-4k-3]\right)\right\} \end{array} \right)$$

(b) If $p = 1/2$, $A = 4$ and $B = -1$ graph the autocorrelation function.



7-10. Which of the following functions could or could not be an autocorrelation function of a real WSS stochastic process and why?

(a) $3\sin(200\pi\tau)$ Cannot be

(b) $3\cos(500\pi\tau)$ Can be

(c) $10\,\text{sinc}(\tau/4)$ Can be

(d) $4\,\text{rect}(10\tau)$ Cannot be

(e) $12\,\text{tri}(\tau/24)$ Can be

(f) $12\left[\text{tri}\left(\frac{\tau-18}{24}\right)+\text{tri}\left(\frac{\tau+18}{24}\right)\right]$ Cannot be

(g) $12\left[\text{tri}\left(\frac{\tau-6}{24}\right)+\text{tri}\left(\frac{\tau+6}{24}\right)\right]$ Cannot be

(h) $10\cos(20\pi\tau)\text{tri}(\tau/3)$ Can be

(i) $10\sin^2(20\pi\tau)$ Cannot be

(j) $\delta(\tau-5)+\delta(\tau+5)$ Cannot be

(k) $\delta(\tau-5)+4\delta(\tau)+\delta(\tau+5)$ Can be

(l) $\delta(\tau-5)+1.5\delta(\tau)+\delta(\tau+5)$ Cannot be

7-11. A WSS stochastic CTCV process has an autocorrelation function

$$\text{R}_X(\tau)=10\cos(4\pi\tau)\text{tri}(\tau/3)+4\cos(6\pi\tau)+12\,.$$

(a) Find its mean value, mean-squared value and standard deviation. 26, 3.742

(b) Does the autocorrelation have a sinusoidal component and, if so, what is its frequency? Yes, at 3 Hz.

(c) What is the smallest positive value of τ at which $\text{X}(t)$ and $\text{X}(t+\tau)$ are uncorrelated? $\tau = 0.109$

7-12. An ergodic, stochastic CTCV process $\{\text{X}(t)\}$ has the autocorrelation function $\text{R}_\text{X}(\tau)$ graphed below. A sample function $\text{X}(t)$ of that random process is sampled at a rate $f_s = 1/T_s$. A DTCV random variable is formed according to the following formula.

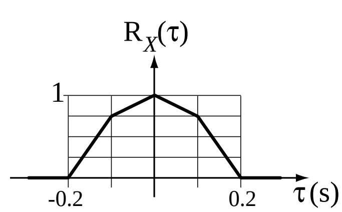
$$\text{Y}[0]=\frac{\text{X}(0)+2\,\text{X}(T_s)+\text{X}(2T_s)}{4}$$

$$\text{Y}[1]=\frac{\text{X}(3T_s)+2\,\text{X}(4T_s)+\text{X}(5T_s)}{4}$$

$$\vdots \qquad \vdots$$

$$\text{Y}[k]=\frac{\text{X}(3kT_s)+2\,\text{X}((3k+1)T_s)+\text{X}((3k+2)T_s)}{4}$$

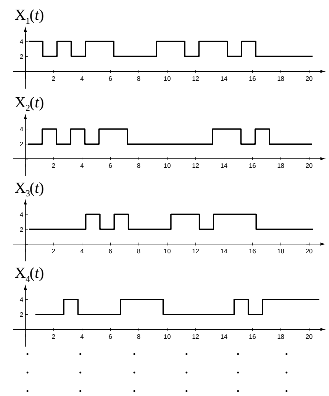
$$\vdots \qquad \vdots$$

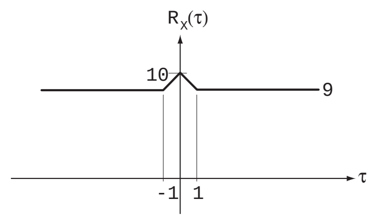


(a) If the sampling rate f_s is 5 Hz, what is the variance of the random variable Y? 3/8

(b) If the sampling rate f_s is 10 Hz, what is the variance of the random variable Y? 3/4

7-13. An ergodic, deterministic stochastic CTDV process has sample functions each of which is a sequence of contiguous rectangular pulses of width 1 second. The pulses can have only two possible amplitudes 2 and 4 with equal probability. The pulse amplitudes occur randomly and are all statistically independent of each other. Sketch the autocorrelation function for this random process.





7-14. Let the function

$$g(\tau) = \begin{cases} 1 - |\tau|/a & , \ |\tau| < 1 \\ 0 & , \ \text{otherwise} \end{cases} .$$

For what values of a could this function be an autocorrelation function? $a = 1$

7-15. Generate samples from a sample function of an ergodic stochastic CTCV process with the instruction sequence

```
N = 10000 ;                    %   Number of time samples
X = randn(N,1) ;                %   Gaussian distributed random values
[b,a] = butter(3,0.05) ;      %   Third-order Butterworth lowpass filter
X = filtfilt(b,a,X) ;         %   Compute filtered signal
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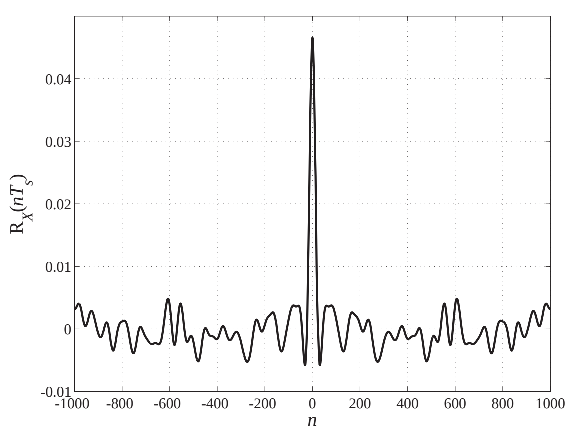
Then compute and graph an estimate of the autocorrelation function using the approximation given in the text.



7-16. Find the mean and variance of stochastic CTCV processes having the following autocorrelation functions.

(a) $R_X(\tau) = 18e^{-|\tau|}$ 0,18

(b) $R_X(\tau) = 6\dfrac{|\tau|+3}{|\tau|+2}$ ±2.449 ,3

(c) $R_X(\tau) = 4e^{-\tau}\cos(10\pi\tau)$ 0,4

7-17. Two independent WSS stochastic CTCV processes with sample functions $X(t)$ and $Y(t)$ have autocorrelation functions

$$R_X(\tau) = 3\operatorname{sinc}(8\tau) \ , \ R_Y(\tau) = 12e^{-5|\tau|}$$

(a) Find the autocorrelation functions of $Z_s(t) = X(t) + Y(t)$ and $Z_d(t) = X(t) - Y(t)$.

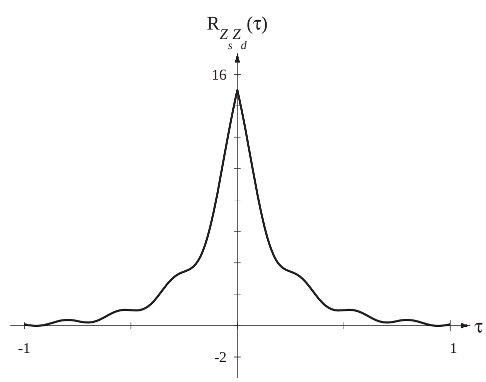
$$R_{Z_s}(\tau) = 3\operatorname{sinc}(8\tau) + 12e^{-5|\tau|}$$

$$R_{Z_d}(\tau) = 3\operatorname{sinc}(8\tau) + 12e^{-5|\tau|}$$

(c) Find the cross-correlations $R_{Z_sZ_d}(\tau)$ and $R_{Z_dZ_s}(\tau)$ and compare their maximum values to the upper limit stated above in the text.

$$R_{Z_sZ_d}(\tau) = 3\operatorname{sinc}(8\tau) + 12e^{-5|\tau|} = R_{Z_sZ_d}(-\tau)$$

$$\left|R_{Z_sZ_d}(\tau)\right| \le \sqrt{R_{Z_s}(0)R_{Z_s}(0)} \ , \ R_{Z_s}(0) = 15 \ \text{and} \ R_{Z_d}(0) = 15 \ , \ \left|R_{Z_sZ_d}(\tau)\right| \le 15$$



7-18. A WSS stochastic CTCV process has an autocorrelation function

$$R_X(\tau) = 15\operatorname{tri}(\tau/4)$$

(a) Find $R_{X'X}(\tau)$

$$R_{X'X}(\tau) = \frac{15}{4}\begin{cases} 0 & , \ \tau \le -4 \\ -1 & , \ -4 < \tau \le 0 \\ 1 & , \ 0 < \tau \le 4 \\ 0 & , \ \tau > 4 \end{cases}$$

(b) Find $R_{X'}(\tau)$

$$R_{X'}(\tau) = \frac{15}{4}\left[-\delta(\tau+4) + 2\delta(\tau) - \delta(\tau-4)\right]$$

7-19. A common technique for optimally detecting a sinusoidal signal in the presence of broadband noise is to multiply the signal plus noise by a sinusoid of the same frequency and let the product excite a lowpass filter. The multiplying sinusoid is generated by what is called the *local oscillator*. The expected value of the filter response indicates the amplitude of the sinusoid. Let the sinusoidal signal be $X(t) = 0.01\sin(50\pi t + \theta)$ and let the autocorrelation of the noise be $R_N(\tau) = e^{-8|\tau|}$. Let the local-oscillator sinusoid be of the form, $y(t) = 10\cos(50\pi t + \phi)$. Then the product of the incoming signal and the locally-generated sinusoid is $Z(t) = y(t)X(t) + y(t)N(t)$.

(a) Find the expected value of $Z(t)$, with ϕ a constant.

$$E[Z(t)] = 0.05[\sin(100\pi t + \phi + \theta) + \sin(\phi - \theta)]$$

(b) What value of ϕ maximizes $E(\langle Z(t)\rangle)$? $\phi - \theta = \pi / 2 \pm n\pi$, $n = 0,1,2,3,\cdots$.