Answers to Exercises in Chapter 8 - Power Spectral Density

8-1. A stationary random process has a PSD of $G_X(f) = 5\,\text{tri}(f/2)$. Find $E(X^2)$. 10

8-2. A stationary DTCV process has a PSD of

$$G_X(\Omega) = \sum_{k=-\infty}^{\infty} 18\,\text{rect}(10(\Omega - 2\pi k)) \;.$$

Find $E(X^2)$. 0.2865

8-3. A stochastic CTDV process with a PSD of $G_X(f)$ has a mean-squared value of 64. Find the mean-squared values of Y for each of the following PSD's.

(a) $G_Y(f) = 10\,G_X(f)$ 640

(b) $G_Y(f) = G_X(10f)$ 6.4

(c) $G_Y(f) = G_X(f/10)$ 640

(d) $G_Y(f) = 10\,G_X(10f)$ 64

8-4. Find the mean-squared value of a stationary stochastic CTCV process whose PSD is

$$G_X(f) = 25\delta(f) + \frac{f^2}{f^4 + 17f^2 + 16} \;.$$

25.628

8-5. A stochastic CTDV process has an autocorrelation function of $R_X(\tau) = 60\,\text{tri}(\tau/6)$.

(a) Find the variance of this random process. 60

(b) Find the PSD of this random process. $G_X(f) = 360\,\text{sinc}^2(6f)$

(c) What is the relation between the null frequencies of the PSD and the first null of the autocorrelation function? (A null is where a function goes to zero.)

The null frequencies of the PSD are at integer multiples of the reciprocal of the value of τ at the first null of the autocorrelation.

8-6. A stochastic DTCV process has an autocorrelation function of

$$R_X[m] = 5\,\text{sinc}(n/80) + 4 .$$

(a) Find the variance of this stochastic process. 5

(b) Find the PSD of this stochastic process.

$$G_X(F) = 400 \sum_{k=-\infty}^{\infty} \text{rect}(80(F - k))$$

or

$$G_X(\Omega) = 400 \sum_{k=-\infty}^{\infty} \text{rect}(40(\Omega - 2\pi k)/\pi)$$

8-7. A stationary stochastic CTCV process has a PSD of

$$G_X(f) = 20\left[\text{rect}\left(\frac{f-50}{30}\right) + \text{rect}\left(\frac{f+50}{30}\right)\right]$$

(a) Find the mean-squared value of *X*. 1200

(b) Find the autocorrelation function of *X*. $R_X(\tau) = 1200\,\text{sinc}(30t)\cos(100\pi\tau)$

8-8. A stationary stochastic CTCV process has a PSD of $G_X(f) = \dfrac{500}{f^2+9}$.

(a) Find the PSD of bandlimited white noise $Y(t)$ that has the same value at zero frequency and the same mean-squared value as the PSD of *X*. $G_Y(f) = \dfrac{500}{9}\text{rect}\left(\dfrac{f}{3\pi}\right)$.

(b) Find the autocorrelation function of $\{X(t)\}$. $R_X(\tau) = \dfrac{500\pi}{3}e^{-6\pi|\tau|}$

(c) Find the autocorrelation function of $\{Y(t)\}$. $R_Y(\tau) = \dfrac{500\pi}{3}\text{sinc}(3\pi\tau)$

(d) Verify that their values at $\tau = 0$ and their total areas are the same. Proof.

8-9. A stationary stochastic CTCV process has a PSD of $G_X(f) = 5\,\text{rect}(f/250)$

(a) Find the autocorrelation function of $\{X(t)\}$. $R_X(\tau) = 1250\,\text{sinc}(250\tau)$

(b) Find τ at the first null in the autocorrelation function. τ_{min} = 4 ms

(c) Find the correlation between samples if this process is sampled at a rate of 250 Hz. Repeat if the sampling rate is 400 Hz. Repeat if the sampling rate is 200 Hz.

$R_X(1/250) = 0$ $R_X(1/400) = 588.16$ $R_X(1/200) = -216.08$

8-10. An ergodic stochastic CTCV process $\{X(t)\}$ has a power spectral density

$$G_X(f) = 20\,\text{sinc}^2(f/10) \quad .$$

(a) What is the mean-squared value of this random process? 200

(b) What is the highest sampling rate at which samples from this process are statistically independent?
10 Hz

8-11. A stochastic CTCV process $\{X(t)\}$ has a power spectral density which is a constant *K* from $f = -f_m$ to $f = f_m$ and zero elsewhere. A 50% duty-cycle square wave oscillating between *A* and *B* is added to a sample function of this random process to form $Z(t)$. What is the signal power (mean-squared value) of $Z(t)$?

$$2Kf_m + \frac{(A^2+B^2)}{2}$$

8-12. A stationary stochastic CTCV process $\{X(t)\}$ has a PSD of

$$\mathrm{G}_X(f) = \frac{100}{f^2 + 100}$$

and an independent stationary stochastic process $\{\mathrm{Y}(t)\}$ has a PSD of

$$\mathrm{G}_Y(f) = \frac{f^2}{f^2 + 100} \quad .$$

Let $\mathrm{Z}(t) = \mathrm{X}(t) + \mathrm{Y}(t)$.

(a) Find the PSD of $\mathrm{Z}(t)$. $\mathrm{G}_Z(f) = 1$

(b) Find $\mathrm{G}_{\mathrm{XY}}(f)$. $\mathrm{G}_{\mathrm{XY}}(f) = 0$.

(c) Find $\mathrm{G}_{\mathrm{XZ}}(f)$. $\mathrm{G}_{\mathrm{XZ}}(f) = \mathrm{G}_{\mathrm{X}}(f) = \dfrac{100}{f^2 + 100}$.

8-13. (from Papoulis and Pillai) Find $\mathrm{G}(\omega)$ if (a) $\mathrm{R}(\tau) = e^{-\alpha\tau^2}$ (b) $\mathrm{R}(\tau) = e^{-\alpha\tau^2}\cos(\omega_0\tau)$.

(a) $\mathrm{G}(\omega) = \sqrt{\dfrac{\pi}{\alpha}} e^{-\frac{\omega^2}{4\alpha}}$

(b) $\mathrm{G}(\omega) = \dfrac{1}{2}\sqrt{\dfrac{\pi}{\alpha}}\left[e^{-\frac{(\omega-\omega_0)^2}{4\alpha}} + e^{-\frac{(\omega+\omega_0)^2}{4\alpha}}\right]$

8-14. Show that if $\mathrm{Y}(t) = \mathrm{X}(t+a) - \mathrm{X}(t-a)$, then

$\mathrm{R}_{\mathrm{Y}}(\tau) = 2\mathrm{R}_{\mathrm{X}}(\tau) - \mathrm{R}_{\mathrm{X}}(\tau + 2a) - \mathrm{R}_{\mathrm{X}}(\tau - 2a)$ and $\mathrm{G}(\omega) = 4\mathrm{G}_X(\omega)\sin^2(a\omega)$

Proof.

8-15. (from Papoulis and Pillai) Find $\mathrm{R}(\tau)$ if (a) $\mathrm{G}(\omega) = \dfrac{1}{1+\omega^4}$, (b) $\mathrm{G}(\omega) = \dfrac{1}{(4+\omega^2)^2}$.

(a) $\mathrm{R}(\tau) = \dfrac{1}{2\sqrt{2}}\left(e^{-|\tau|/\sqrt{2}}\cos\left(\tau/\sqrt{2}\right) + e^{-|\tau|/\sqrt{2}}\sin\left(|\tau|/\sqrt{2}\right)\right)$

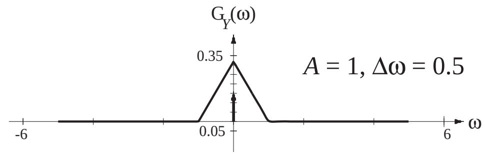
(b) $\mathrm{R}(\tau) = \dfrac{e^{2|\tau|}}{32}\left(1 + 2|\tau|\right)$

8-16. (from Papoulis and Pillai) The process $\mathrm{X}(t)$ is normal with zero mean. Show that if $\mathrm{Y}(t) = \mathrm{X}^2(t)$, then

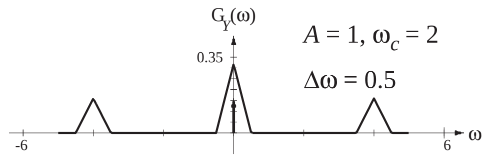
$$\mathrm{G}_Y(\omega) = 2\pi \mathrm{R}_X^2(0)\delta(\omega) + (1/\pi)\mathrm{S}_X(\omega) * \mathrm{S}_X(\omega) \quad .$$

Plot $\mathrm{G}_Y(\omega)$ if $\mathrm{G}_X(\omega)$ is (a) ideal LP ; (b) ideal BP.

(a)


$G_Y(\omega)$
0.35
0.05
-6
6
ω
$A = 1$, $\Delta\omega = 0.5$

(b)


$G_Y(\omega)$
0.35
-6
6
ω
$A = 1$, $\omega_c = 2$
$\Delta\omega = 0.5$