Answers to Exercises in Chapter 9 - Linear Systems with Random Excitation

9-1. A filter has poles at $s = -6$, $s = -10$ and $s = -20$. The zero-frequency filter gain is one.

(a) Write the power transfer function $|\mathrm{H}(f)|^2$ for this filter.

$$|\mathrm{H}(f)|^2 = \frac{1200^2}{\left((2\pi f)^2 + 36\right)\left((2\pi f)^2 + 100\right)\left((2\pi f)^2 + 400\right)}$$

(b) If the excitation $\mathrm{X}(t)$ to the filter is a sample function from a stochastic CTCV process having an autocorrelation function of

$$\mathrm{R}_X(\tau) = 5\,\mathrm{sinc}(50\tau)$$

find the PSD of the response $\mathrm{Y}(t)$.

$$\mathrm{G}_\mathrm{Y}(f) = \frac{144000\,\mathrm{rect}(f/50)}{\left((2\pi f)^2 + 36\right)\left((2\pi f)^2 + 100\right)\left((2\pi f)^2 + 400\right)}$$

9-2. A linear system has an s-domain transfer function of $\mathrm{H}(s) = \dfrac{s}{s^2 + 2s + 5}$. White noise $\mathrm{X}(t)$ having a PSD of $10^{-14}\ \mathrm{V}^2/\mathrm{Hz}$ is applied as the excitation.

(a) Write the PSD of the response $\mathrm{Y}(t)$.

$$\mathrm{G}_\mathrm{Y}(f) = \frac{10^{-14}}{(2\pi)^2} \frac{f^2}{f^4 - \frac{6}{(2\pi)^2} f^2 + \frac{25}{(2\pi)^4}}$$

(b) Find the mean-squared value of the output. $\mathrm{E}(Y^2) = 6.6304 \times 10^{-16}$

9-3. Derive general expressions for the CPSD's $\mathrm{G}_{\mathrm{YZ}}(f)$ and $\mathrm{G}_{\mathrm{ZY}}(f)$ of the responses of the system below in terms of the PSD of the excitation $\mathrm{G}_\mathrm{X}(f)$ and the transfer functions of the systems $\mathrm{H}_\mathrm{Y}(s)$ and $\mathrm{H}_Z(s)$. Then if

$$\mathrm{H}_{Y,s}(s) = \frac{a}{s+a} \quad \text{and} \quad \mathrm{H}_{Z,s}(s) = \frac{s}{s+a}$$

and the excitation is white noise with PSD $\mathrm{G}_\mathrm{X}(f) = A$ find the CPSD's of the system responses.

$$\mathrm{G}_{\mathrm{YZ}}(f) = \mathrm{G}_\mathrm{X}(f)\mathrm{H}_\mathrm{Y}^*(f)\mathrm{H}_Z(f)$$

$$\mathrm{G}_{\mathrm{ZY}}(f) = \mathrm{G}_\mathrm{X}(f)\mathrm{H}_\mathrm{Y}(f)\mathrm{H}_Z^*(f)$$

$$\mathrm{G}_\mathrm{X}(f) = A \ , \ \mathrm{G}_{\mathrm{YZ}}(f) = A\frac{j2\pi fa}{(2\pi f)^2 + a^2} \ , \ \mathrm{G}_{\mathrm{ZY}}(f) = -A\frac{j2\pi f}{(2\pi f)^2 + a^2} = \mathrm{G}_{\mathrm{YZ}}(-f)$$

9-4. The impulse response of a system is $\mathrm{h}(t) = 8\,\mathrm{tri}\left(\frac{t-4}{4}\right)$.

(a) Find its equivalent noise bandwidth. $B = 0.08333$

(b) If the excitation of this system is white noise with a PSD of $10^{-6}\ \mathrm{V}^2/\mathrm{Hz}$, find the mean-squared value of the response. $\mathrm{E}\left(\mathrm{Y}^2\right) = 8.333\times10^{-8}$

9-5. White noise X with a double-sided power spectral density $\mathrm{G}_X(f) = 0.1$ and a Gaussian pdf is the excitation signal for an ideal lowpass filter with a bandwidth of 30 Hz. The filter's response signal is Y.

(a) What is the expected value of Y $\mathrm{E}(Y)$? 0

(b) What is the average signal power of Y $\mathrm{E}(Y^2)$? 6

(c) What is the probability that, at some randomly-chosen time, Y is greater than 2 $\mathrm{P}[Y > 2]$? 0.208

9-6. A deterministic CTCV stochastic process $\{\mathrm{X}(t)\}$ has sample functions of the form

$$\mathrm{X}(t) = A + B\sin(300\pi t - \theta) \ .$$

A is a Rayleigh-distributed random variable with a mean-squared value of 8,

B is a Gaussian-distributed random variable with a mean of 3 and a variance of 4

and θ is uniformly distributed from $-\pi$ to π .

All three random variables are mutually independent. This sample function is the excitation of a system having an impulse response of

$$\mathrm{h}(t) = 200e^{-200t}\,\mathrm{u}(t)$$

(a) Find the response $\mathrm{Y}(t)$ of the system. $\mathrm{Y}(t) = A + B\left[\frac{60000\cos(300\pi t - \theta) + 40000\sin(300\pi t - \theta)}{(300\pi)^2 + (200)^2}\right]$

(b) Find the mean value of the response $\mathrm{E}(\mathrm{Y}(t))$. $\mathrm{E}[\mathrm{Y}(t)] = 2.507$

(c) Find the mean-squared value of the response $\mathrm{E}(\mathrm{Y}^2(t))$. $\mathrm{E}\left(\mathrm{Y}^2\right) = 8.0392$

9-7. Let $\mathrm{X}(t)$ be stationary white noise with autocorrelation $\mathrm{R}_X(\tau) = 10^{-16}\delta(\tau)$. Find the autocorrelation $\mathrm{R}_Y(\tau)$ and mean-squared value $\mathrm{E}(\mathrm{Y}^2(t))$ of the response $\mathrm{Y}(t)$ of a linear system with impulse response $\mathrm{h}(t) = 10e^{-10t}\,\mathrm{u}(t)$. $\mathrm{R}_Y(\tau) = \frac{10^{-14}}{20}e^{-10|\tau|}$ $\mathrm{R}_Y(0) = 5\times10^{-16}$

9-8. Let $\mathrm{X}(t)$ and $\mathrm{Y}(t)$ be independent stationary stochastic CTCV processes each with autocorrelation $\mathrm{R}_X(\tau) = \mathrm{R}_Y(\tau) = 2 + e^{-\pi\tau^2}$. Let $\mathrm{Z}(t) = \mathrm{X}(t) + \mathrm{Y}(t)$ and let $\mathrm{W}(t)$ be the response of an ideal bandpass filter whose transfer function magnitude is

$$\left|H(f)\right| = \begin{cases} 4 & , \ 0.3 < |f| < 0.5 \\ 0 & , \ \text{otherwise} \end{cases}$$

to the excitation $Z(t)$ ($Z(t)$ going into the filter and $W(t)$ coming out).

(a) What is the square of the expected value of $X(t)$, $E^2(X(t))$? 2

(b) What is the mean-squared value of $X(t)$, $E(X^2(t))$? 3

(c) What are the three possible values of the expected value of $Z(t)$, $E(Z(t))$? $\pm 2\sqrt{2}$ and 0.

(d) What are the two possible mean-squared values of $Z(t)$, $E(Z^2(t))$? $E(Z^2(t)) = 6 \pm 4$

(e) What is the expected value of $W(t)$, $E(W(t))$? 0

(f) What is the mean-squared value of $W(t)$, $E(W^2(t))$? $E(W^2) = 3.872$

9-9. A very useful device for rejecting periodic signals and their harmonics is the finite-time integrator. Its effective impulse response is

$$h(t) = u(t) - u(t-T) = \text{rect}((t - T/2)/T) \ .$$

(a) Show that any periodic signal with period T is completely rejected by this system. Proof.

(b) Let the excitation of this system $X(t)$ be white noise with a PSD of A. Find the mean value and mean-squared value of the response, the autocorrelation of the response and the crosscorrelation between the excitation and the response.

$$E(Y) = 0$$

$$E(Y^2) = AT$$

$$R_Y(\tau) = AT\,\text{tri}(\tau/T)$$

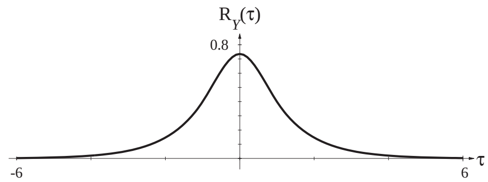
$$R_{XY}(\tau) = A\,\text{rect}((\tau - T/2)/T)$$

9-10. Repeat Exercise 9-9 with the autocorrelation of X changed from $R_X(\tau) = A\delta(\tau)$ to $R_X(\tau) = Ae^{-a|\tau|}$.

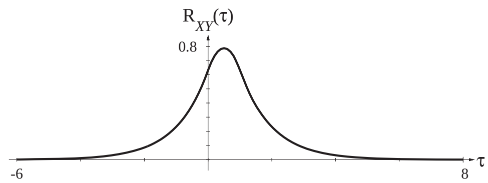
$$E(Y) = 0$$

$$E(Y^2) = \frac{2A}{a^2}\left(aT + e^{-aT} - 1\right)$$

$$R_Y(\tau) = \frac{A}{a^2}\begin{cases} 2a(T - |\tau|) + (e^{-aT} - 2)e^{-a|\tau|} + e^{-a(T-|\tau|)} & , \ 0 < |\tau| < T \\ e^{-a|\tau|}\left(e^{-aT} - 2 + e^{aT}\right) & , \ |\tau| > T \end{cases}$$



$$\mathrm{R}_{XY}(\tau)=\frac{A}{a}\begin{cases}e^{a\tau}\left(1-e^{-aT}\right) & , \ \tau<0 \\ 2-\left[e^{-a\tau}+e^{a(\tau-T)}\right] & , \ 0<\tau<T \\ e^{-a\tau}\left(e^{aT}-1\right) & , \ \tau>T\end{cases}$$



9-11. A stationary random CTCV process X is the excitation of a linear time-invariant system and the response is Y. The power spectral density of X is

$$\mathrm{G}_{X1}(f)=100\begin{cases}1-|f|/1000 & , \ |f|<1000 \\ 0 & , \ \text{otherwise}\end{cases}=100\,\mathrm{tri}(f/1000) \ .$$

The transfer function of the system is

$$\mathrm{H}(f)=\begin{cases}5 & , \ 400<|f|<600 \\ 0 & , \ \text{otherwise}\end{cases}=5\left[\mathrm{rect}\left(\frac{f-500}{200}\right)+\mathrm{rect}\left(\frac{f+500}{200}\right)\right] .$$

(a) What is the expected value of X $\mathrm{E}(\mathrm{X})$? 0

(b) What is the mean-squared value of X $\mathrm{E}(\mathrm{X}^2)$? 100,000

(c) What is the expected value of Y $\mathrm{E}(Y)$? 0

(d) What is the mean-squared value of Y $\mathrm{E}(Y^2)$? 500,000

(e) Let the power spectral density of X be changed to $\mathrm{G}_{X2}(f)=\mathrm{G}_{X1}(f)+4000\delta(f)$ and redo parts (a) through (d) for this new situation.

$$\mathrm{E}(X)=\pm 63.24 \qquad \mathrm{E}(X^2)=104,000$$
$$\mathrm{E}(Y)=0 \qquad \mathrm{E}(Y^2)=500,000$$

9-12. In the system below, the signals X and N are both bandlimited white noise with a bandwidth of 20 kHz (and mean values of zero). That is,

$$\mathrm{G}_X(f)=\mathrm{G}_N(f)=0 \ , \ |f|>20 \text{ kHz} .$$

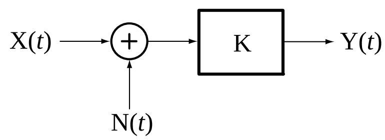
The signals are statistically independent. The variance of N is 10. The variance of the response signal $\mathrm{Y}(t)$ is estimated twice by sampling $\mathrm{Y}(t)$ at the Nyquist rate of 40 kHz (making the samples uncorrelated) for 1 second and estimating its variance from those samples. The first time the signal $\mathrm{X}(t)$ is not present and the second time it is present. The results are

Measurement #1 with $\mathrm{X}(t)$ not present (set to zero). $\sigma_{\mathrm{Y}}^2 = 200$

Measurement #2 with $\mathrm{X}(t)$ present. $\sigma_{\mathrm{Y}}^2 = 300$

(a) What is the best estimate of the gain constant, K? 4.47

(b) What is the best estimate of the variance of $\mathrm{X}(t)$, σ_{X}^2? 5



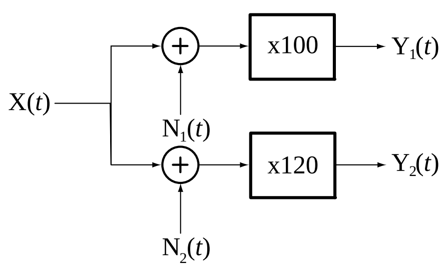
9-13. In the system below, all three inputs X, N_1 and N_2 are bandlimited white noise with a bandwidth of 100 kHz. That is,

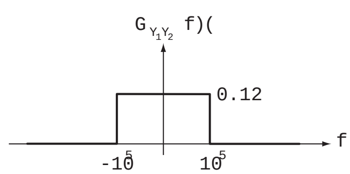
$$\mathrm{G}_{\mathrm{X}}(f) = \mathrm{G}_{N_1}(f) = \mathrm{G}_{N_2}(f) = 0 \ \ , \ |f| > 100 \text{ kHz} .$$

All three signals are mutually statistically independent. Their mean-squared values are

$$\mathrm{E}(\mathrm{X}) = 2 \ \ , \ \ \mathrm{E}(N_1) = 1 \ \ , \ \ \mathrm{E}(N_2) = 1 \ \ .$$

X is added to N_1 and N_2 and then amplified by the two amplifiers with gains of 100 and 120, as indicated in the diagram. Sketch the cross power spectral density $\mathrm{G}_{\mathrm{Y}_1\mathrm{Y}_2}(f)$ of the outputs Y_1 and Y_2.





9-14. A composite excitation $\mathrm{x}_i(t)$ consists of a signal $\mathrm{s}_i(t)$ plus noise $\mathrm{n}_i(t)$. That is $\mathrm{x}_i(t) = \mathrm{s}_i(t) + \mathrm{n}_i(t)$. The signal $\mathrm{s}_i(t)$ is given by $\mathrm{s}_i(t) = 100\sin(10{,}000\pi t)$. The noise is white with a power spectral density of 0.1. $\mathrm{x}_i(t)$ is the excitation of a linear system with a transfer function of

$$\mathrm{H}(f) = \frac{50}{1 + jf / 5000} \ \ .$$

The response $\mathrm{x}_o(t)$ consists of the sum of the response $\mathrm{s}_o(t)$ to the signal $\mathrm{s}_i(t)$ and the response $\mathrm{n}_o(t)$ to the noise $\mathrm{n}_i(t)$. That is, $\mathrm{x}_o(t) = \mathrm{s}_o(t) + \mathrm{n}_o(t)$. Find the ratio of the power in $\mathrm{s}_o(t)$ to the power in $\mathrm{n}_o(t)$, the signal-to-noise ratio.

$$\text{SNR} \cong 1.5915$$

9-15. A circuit consists of a resistor R at an absolute temperature T and a capacitor C in parallel. (The double-sided PSD of the short-circuit Johnson noise current of a resistor is $2kT/R$ and the double-sided PSD of the open-circuit Johnson noise voltage is $2kTR$ where k is Boltzmann's constant, $1.38\times10^{-23}\,\text{J/K}$.)

(a) Find a general formula for the mean-squared voltage across the capacitor (and the resistor) in terms of *k*, *R*, *T* and *C*. This formula should contain no integral signs, Fourier transforms or convolution operators. You may use the integration formula

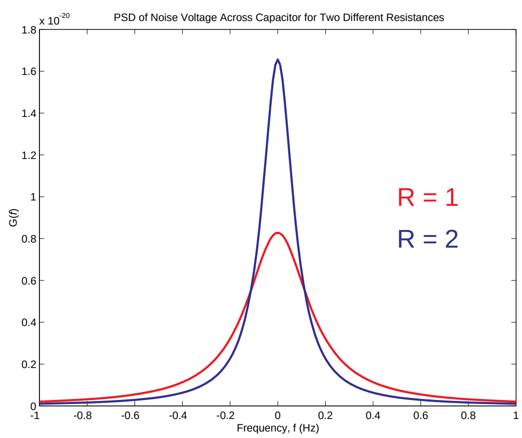
$$\int \frac{dx}{a^2+(bx)^2} = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right)$$

if needed during the process of finding the formula.

Using a current source model for the Johnson noise of the resistor, the transfer function from the noise current to the capacitor voltage is the impedance of the parallel R-C circuit. The mean-squared noise voltage is

$$\text{E}\left(V^2\right) = \frac{kT}{C}$$

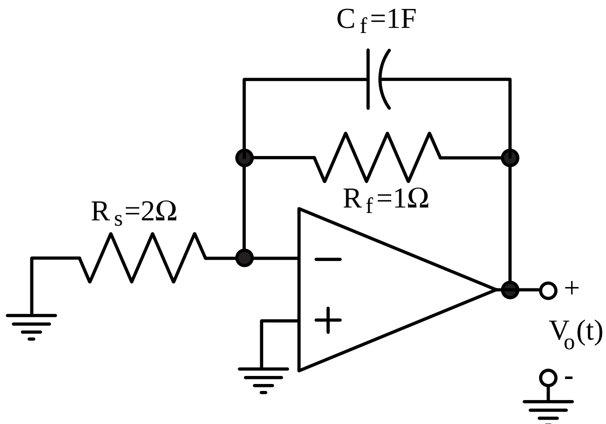
(b) You should find that the mean-squared voltage is the independent of resistance *R*. By sketching the PSD of the noise voltage across the capacitor for the two cases $R=1$, $C=1$, $T=300$ and $R=2$, $C=1$, $T=300$ explain how that can happen even though the PSD of the resistor Johnson noise current (or voltage) noise is a function of *R*.



9-16. In the circuit below the operational amplifier is ideal. It has infinite gain, infinite input impedance, zero output impedance, zero noise and infinite bandwidth. This holds the voltage at the inverting input at zero volts. The two resistors are real and generate Johnson noise. The double-sided power spectral density of the equivalent noise current for the Johnson noise of a resistor is $2kT/R$ and the double-sided power spectral density of the equivalent noise voltage for the Johnson noise of a resistor is $2kT\text{R}$. The amplifier and resistors are all in thermal equilibrium at 300 K. The transfer function from the noise current of either resistor injected into the operational amplifier's inverting input to the output voltage is given by

$$\text{H}(f) = \frac{\text{V}_o(f)}{\text{I}(f)} = -\text{Z}_f(f)$$

where Z_f is the impedance of the feedback network of R_f in parallel with C_f. The noise voltages of the two resistors are independent. What is the mean-squared noise voltage $\text{E}\left(V_o^2\right)$ of the response?



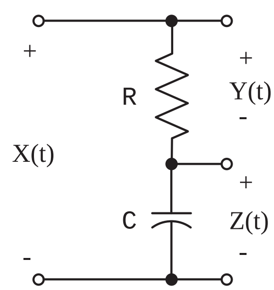
$$\mathrm{E}\left(V_o^2\right) = 6.21\times 10^{-21}$$

9-17. White noise is the excitation of a filter whose impulse response is

$$\mathrm{h}(t) = \mathrm{u}(t) - \mathrm{u}(t-4) = \mathrm{rect}\left(\frac{t-2}{4}\right) .$$

What is the maximum rate at which the response of the filter can be sampled and still have no correlation between samples? 0.25 Hz

9-18. For the circuit below, if the excitation $\mathrm{X}(t)$ is white noise with a PSD of $A\ \mathrm{V}^2/\mathrm{Hz}$, find the cross-correlation function between the two responses $\mathrm{Y}(t)$ and $\mathrm{Z}(t)$, $\mathrm{R}_{\mathrm{YZ}}(\tau)$.



$$\mathrm{R}_{\mathrm{YZ}}(\tau) = \frac{A}{2RC} e^{-|\tau|/RC}\,\mathrm{sgn}(\tau)$$

9-19. The mean value of a stationary stochastic CTCV process can be estimated by averaging a finite number of samples. Let the number of samples be N. Then the estimate is

$$\bar{X} = \frac{1}{N}\sum_{n=1}^{N} \mathrm{X}(nT_s) = \frac{1}{N}\sum_{n=1}^{N} \mathrm{X}[n]$$

where T_s is the time between samples.

(a) Find the variance of the estimate if the samples are uncorrelated.

$$\sigma_{\bar{X}}^2 = \frac{\sigma_{\mathrm{X}}^2}{N}$$

(b) Find the variance of the estimate if the stochastic process has an autocorrelation function $\mathrm{R_X}(\tau)$.

$$\sigma_{\bar{X}}^2 = \frac{\sigma_{\mathrm{X}}^2}{N} + \frac{\displaystyle\sum_{\substack{n=1\\n\neq m}}^{N}\sum_{m=1}^{N}\mathrm{C_X}\left((m-n)T_s\right)}{N^2}$$