Linear Systems

Time-Domain Analysis

For any linear, time-invariant (LTI) system, the response y is the convolution of the excitation x with the impulse response h.

$$\mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t) \text{ or } \mathrm{y}[n] = \mathrm{x}[n] * \mathrm{h}[n]$$

In the case of non-deterministic random processes this operation cannot be done because the signals are random and cannot, therefore, be described mathematically.

If $\mathrm{X}(t)$ excites a system and $\mathrm{Y}(t)$ is the response then the convolution integral is

$$\mathrm{Y}(t) = \int_{-\infty}^{\infty} \mathrm{X}(t-\tau)\mathrm{h}(\tau)\,d\tau$$

Time-Domain Analysis

We cannot directly evaluate

$$\mathrm{Y}(t) = \int_{-\infty}^{\infty} \mathrm{X}(t-\tau)\,\mathrm{h}(\tau)\,d\tau$$

but we can find the expected value.

$$\mathrm{E}(\mathrm{Y}(t)) = \mathrm{E}\left(\int_{-\infty}^{\infty} \mathrm{X}(t-\tau)\,\mathrm{h}(\tau)\,d\tau\right)$$

If the stochastic process is bounded and the system is stable

$$\mathrm{E}(\mathrm{Y}(t)) = \int_{-\infty}^{\infty} \mathrm{E}(\mathrm{X}(t-\tau))\,\mathrm{h}(\tau)\,d\tau$$

Time-Domain Analysis

If the random process X is stationary

$$\mathrm{E}\left(\mathrm{Y}(t)\right) = \int_{-\infty}^{\infty} \mathrm{E}\left(\mathrm{X}(t-\tau)\right)\mathrm{h}(\tau)\,d\tau \Rightarrow \mathrm{E}(\mathrm{Y}) = \mathrm{E}(\mathrm{X})\int_{-\infty}^{\infty} \mathrm{h}(\tau)\,d\tau$$

Using

$$\int_{-\infty}^{\infty} \mathrm{h}(t)\,dt = \mathrm{H}(0) \Rightarrow \mathrm{E}(Y) = \mathrm{E}(X)\,\mathrm{H}(0)$$

where H is the Fourier transform of h, we see that the expected value of the response is the expected value of the excitation multiplied by the zero-frequency response of the system. If the system is DT the corresponding result is

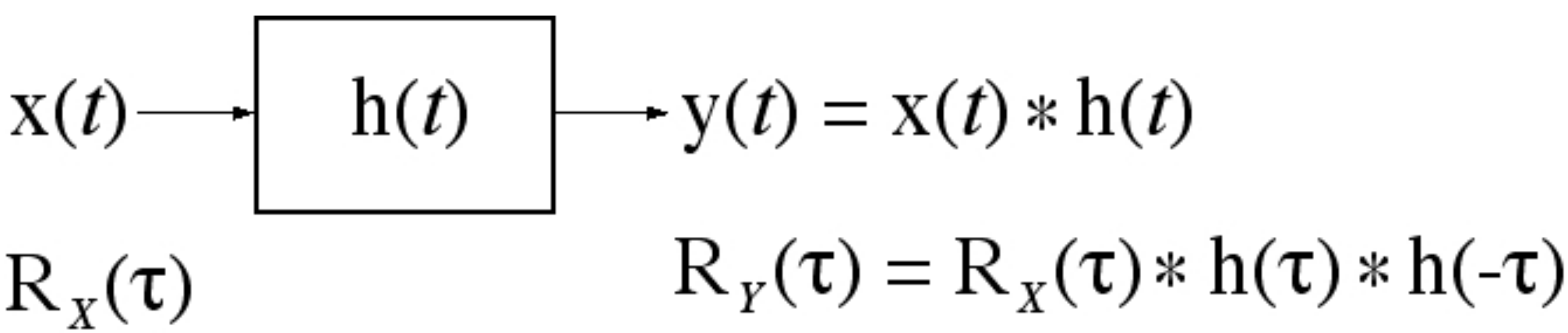
$$\mathrm{E}(Y) = \mathrm{E}(X)\sum_{n=-\infty}^{\infty} \mathrm{h}[n]$$

Time-Domain Analysis

It can be shown (and is in the text) that the autocorrelation of the excitation and the autocorrelation of the response are related by

$\mathrm{R}_Y(\tau) = \mathrm{R}_X(\tau) * \mathrm{h}(\tau) * \mathrm{h}(-\tau)$ or $\mathrm{R}_Y[m] = \mathrm{R}_X[m] * \mathrm{h}[m] * \mathrm{h}[-m]$

This result leads to a way of thinking about the analysis of LTI systems with random excitation.



Time-Domain Analysis

It can be shown (and is in the text) that the cross correlation between the excitation and the response is

$$\mathrm{R}_{XY}\left(\tau\right)=\mathrm{R}_{X}\left(\tau\right)*\mathrm{h}\left(\tau\right) \text{ or } \mathrm{R}_{XY}\left[m\right]=\mathrm{R}_{X}\left[m\right]*\mathrm{h}\left[m\right]$$

and

$$\mathrm{R}_{YX}\left(\tau\right)=\mathrm{R}_{X}\left(\tau\right)*\mathrm{h}\left(-\tau\right) \text{ or } \mathrm{R}_{YX}\left[m\right]=\mathrm{R}_{X}\left[m\right]*\mathrm{h}\left[-m\right]$$

Frequency-Domain Analysis

The frequency-domain relationship between excitation and response of an LTI system is the Fourier transform of the time-domain relationship.

$$\mathrm{R}_Y(\tau) = \mathrm{R}_X(\tau) * \mathrm{h}(\tau) * \mathrm{h}(-\tau) \xleftrightarrow{\mathcal{F}} \mathrm{G}_Y(f) = \mathrm{G}_X(f)\mathrm{H}(f)\mathrm{H}^*(f) = \mathrm{G}_X(f)\left|\mathrm{H}(f)\right|^2$$

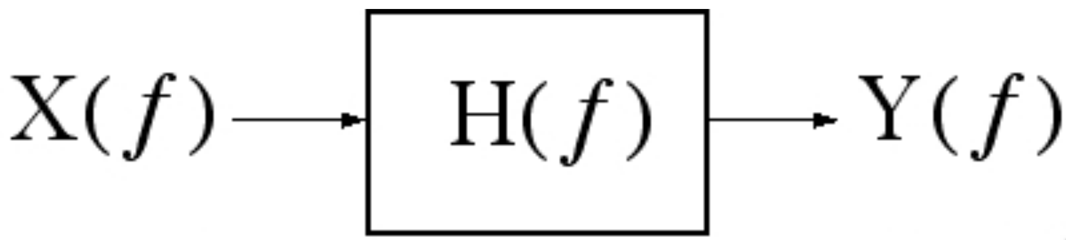
$$\mathrm{R}_Y[m] = \mathrm{R}_X[m] * \mathrm{h}[m] * \mathrm{h}[-m] \xleftrightarrow{\mathcal{F}} \mathrm{G}_Y(F) = \mathrm{G}_X(F)\mathrm{H}(F)\mathrm{H}^*(F) = \mathrm{G}_X(F)\left|\mathrm{H}(F)\right|^2$$

The mean-squared value of the response is

$$\mathrm{E}(Y^2) = \int_{-\infty}^{\infty} \mathrm{G}_Y(f)\,df = \int_{-\infty}^{\infty} \mathrm{G}_X(f)\left|\mathrm{H}(f)\right|^2 df$$

$$\mathrm{E}(Y^2) = \int_{-\infty}^{\infty} \mathrm{G}_Y(F)\,dF = \int_{-\infty}^{\infty} \mathrm{G}_X(F)\left|\mathrm{H}(F)\right|^2 dF$$

Frequency-Domain Analysis



$G_X(f)$

$$G_Y(f) = G_X(f)|H(f)|^2$$

Frequency-Domain Analysis

Equivalent Noise Bandwidth