

Power Spectral Density

Derivation

In applying frequency-domain techniques to the analysis of random signals the natural approach is to Fourier transform the signals.

Unfortunately the Fourier transform of a stochastic process does not, strictly speaking, exist because it has infinite signal energy.

But the Fourier transform of a truncated version of a stochastic process does exist.

Derivation

For a CT stochastic process let

$$\mathbf{X}_T(t) = \begin{cases} \mathbf{X}(t) & , |t| \leq T/2 \\ 0 & , |t| > T/2 \end{cases} = \mathbf{X}(t) \text{rect}(t/T)$$

The Fourier transform is

$$\mathcal{F}(\mathbf{X}_T(t)) = \int_{-\infty}^{\infty} \mathbf{X}_T(t) e^{-j2\pi ft} dt \quad , \quad T < \infty$$

Using Parseval's theorem,

$$\int_{-T/2}^{T/2} |\mathbf{X}_T(t)|^2 dt = \int_{-\infty}^{\infty} |\mathcal{F}(\mathbf{X}_T(t))|^2 df$$

Dividing through by T ,

$$\frac{1}{T} \int_{-T/2}^{T/2} \mathbf{X}_T^2(t) dt = \frac{1}{T} \int_{-\infty}^{\infty} |\mathcal{F}(\mathbf{X}_T(t))|^2 df$$

Derivation

$$\frac{1}{T} \int_{-T/2}^{T/2} X_T^2(t) dt = \frac{1}{T} \int_{-\infty}^{\infty} \left| \mathcal{F}(X_T(t)) \right|^2 df \leftarrow \begin{array}{c} \text{Average signal power} \\ \text{over time, } T \end{array}$$

If we let T approach infinity, the left side becomes the average power over all time. On the right side, the Fourier transform is not defined in that limit. But it can be shown that even though the Fourier transform does not exist, *its expected value does*. Then

$$\mathbb{E} \left(\frac{1}{T} \int_{-T/2}^{T/2} X_T^2(t) dt \right) = \mathbb{E} \left(\frac{1}{T} \int_{-\infty}^{\infty} \left| \mathcal{F}(X_T(t)) \right|^2 df \right)$$

Derivation

Taking the limit as T approaches infinity,

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} E(X^2) dt = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-\infty}^{\infty} E \left[\left| \mathcal{F}(X_T(t)) \right|^2 \right] df$$

$$\langle E(X^2) \rangle = \int_{-\infty}^{\infty} \lim_{T \rightarrow \infty} E \left[\frac{\left| \mathcal{F}(X_T(t)) \right|^2}{T} \right] df$$

The integrand on the right side is identified as **power spectral density (PSD)**.

$$G_X(f) = \lim_{T \rightarrow \infty} E \left(\frac{\left| \mathcal{F}(X_T(t)) \right|^2}{T} \right)$$

Derivation

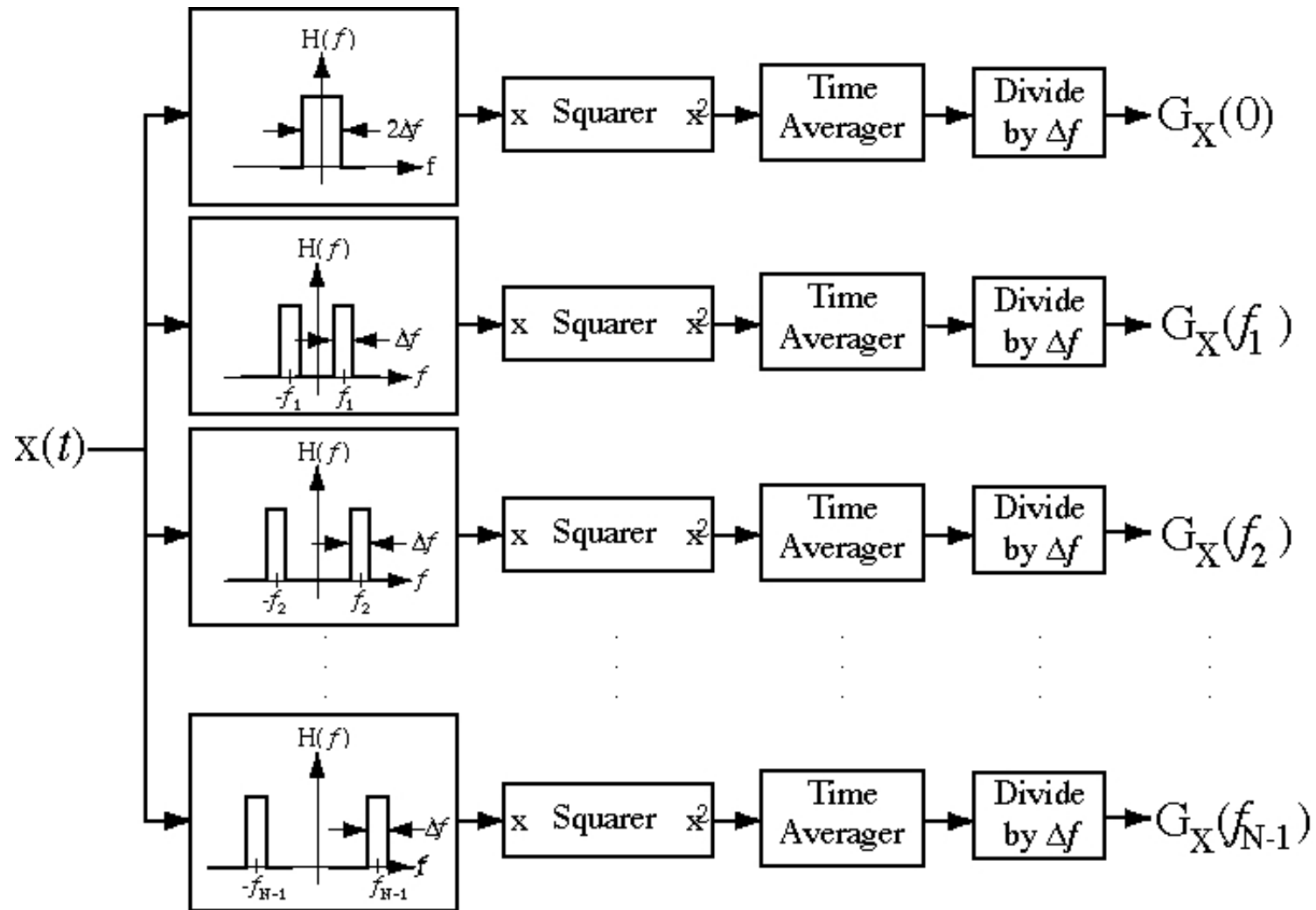
$$\int_{-\infty}^{\infty} G_X(f) df = \text{mean - squared value of } \{X(t)\}$$

$$\int_{-\infty}^{\infty} G_X(f) df = \text{average power of } \{X(t)\}$$

PSD is a description of the variation of a signal's power versus frequency.

PSD can be (and often is) conceived as single-sided, in which all the power is accounted for in positive frequency space.

PSD Concept



PSD of DT Stochastic Processes

$$G_X(F) = \lim_{N \rightarrow \infty} E \left(\frac{|\mathcal{F}(X_T[n])|^2}{N} \right)$$

$$\int_1 G_X(F) dF = \text{mean - squared value of } \{X[n]\}$$

$$\frac{1}{2\pi} \int_{2\pi} G_X(\Omega) d\Omega = \text{mean - squared value of } \{X[n]\}$$

where the Fourier transform is the discrete-time Fourier transform (DTFT) defined by

$$x[n] = \mathcal{F}^{-1}(X(F)) = \int_1 X(F) e^{j2\pi Fn} dF \xleftrightarrow{\mathcal{F}} X(F) = \mathcal{F}(x[n]) = \sum_{n=-\infty}^{\infty} x[n] e^{-j2\pi Fn}$$

$$x[n] = \mathcal{F}^{-1}(X(\Omega)) = \frac{1}{2\pi} \int_{2\pi} X(\Omega) e^{j\Omega n} d\Omega \xleftrightarrow{\mathcal{F}} X(\Omega) = \mathcal{F}(x[n]) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\Omega n}$$

PSD and Autocorrelation

It can be shown (and is in the text) that PSD and autocorrelation form a Fourier transform pair.

$$G_x(f) = \mathcal{F}(R_x(\tau)) \text{ or } G_x(F) = \mathcal{F}(R_x[m])$$

White Noise

White noise is a CT stochastic process whose PSD is constant.

$$G_X(f) = A$$

Signal power is the integral of PSD over all frequency space.

Therefore the power of white noise is infinite.

$$E(X^2) = \int_{-\infty}^{\infty} A df \rightarrow \infty$$

No real physical process may have infinite signal power.

Therefore white noise cannot exist. However, many real and important stochastic processes have a PSD that is almost constant over a very wide frequency range.

White Noise

In many kinds of noise analysis, a type of random variable known as **bandlimited white noise** is used. Bandlimited white noise is simply the response of an ideal lowpass filter that is excited by white noise.

The PSD of bandlimited white noise is constant over a finite frequency range and zero outside that range.

Bandlimited white noise has finite signal power.

Cross Power Spectral Density

PSD is the Fourier transform of autocorrelation.

Cross power spectral density is the Fourier transform of cross correlation.

Properties:

$$R_{XY}(t) \xleftrightarrow{\mathcal{F}} G_{XY}(f) \text{ or } R_{XY}[n] \xleftrightarrow{\mathcal{F}} G_{XY}(F)$$

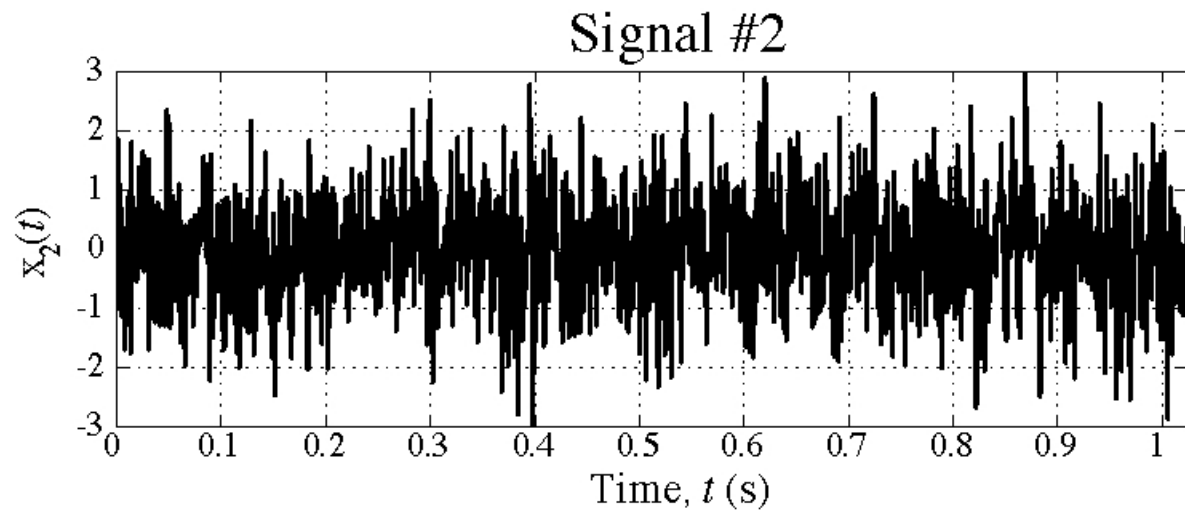
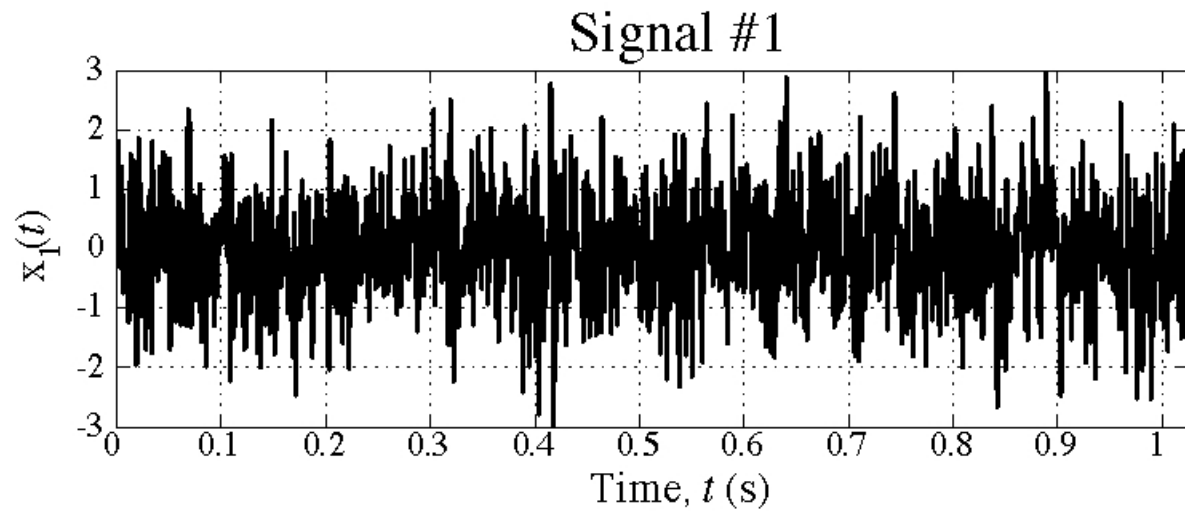
$$G_{XY}(f) = G_{YX}^*(f) \text{ or } G_{XY}(F) = G_{YX}^*(F)$$

$\text{Re}(G_{XY}(f))$ and $\text{Re}(G_{YX}(f))$ are both even

$\text{Im}(G_{XY}(f))$ and $\text{Im}(G_{YX}(f))$ are both odd

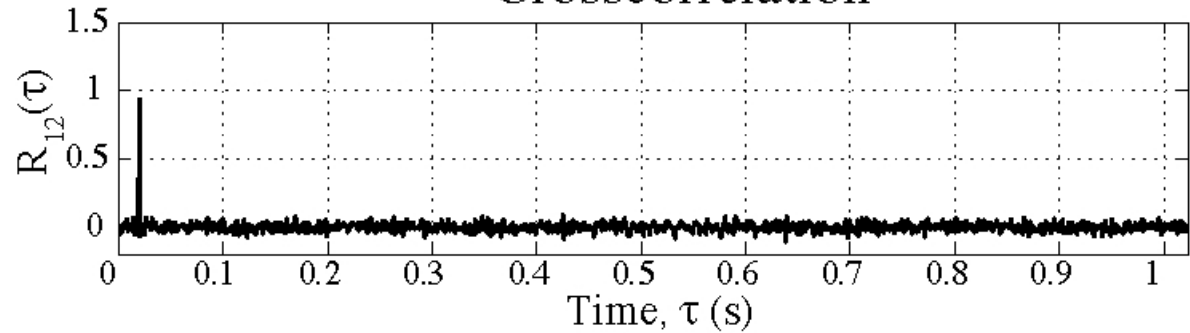
Cross Power Spectral Density

Two Random
Signals



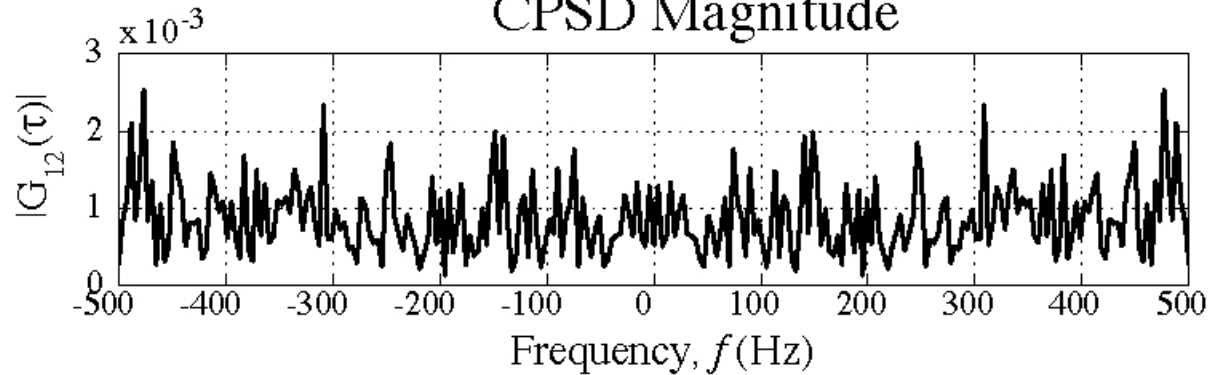
Cross Power Spectral Density

Crosscorrelation

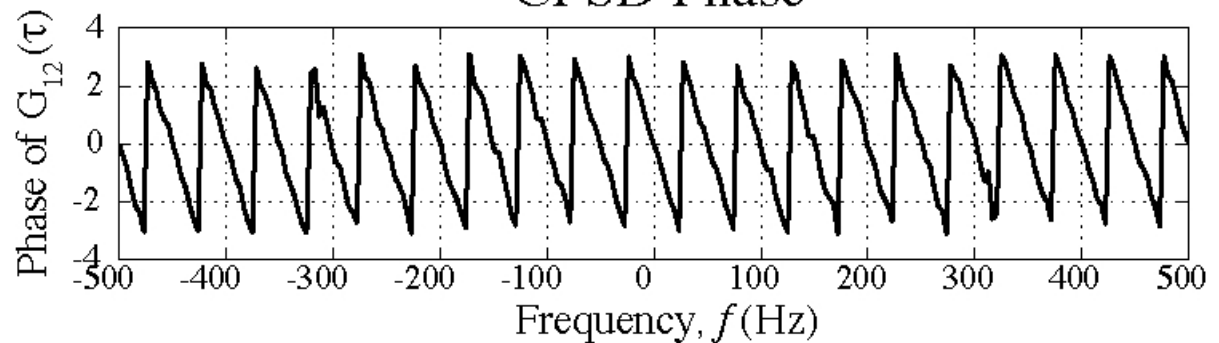


Crosscorrelation
and CPSD of
Two Random
Signals

CPSD Magnitude



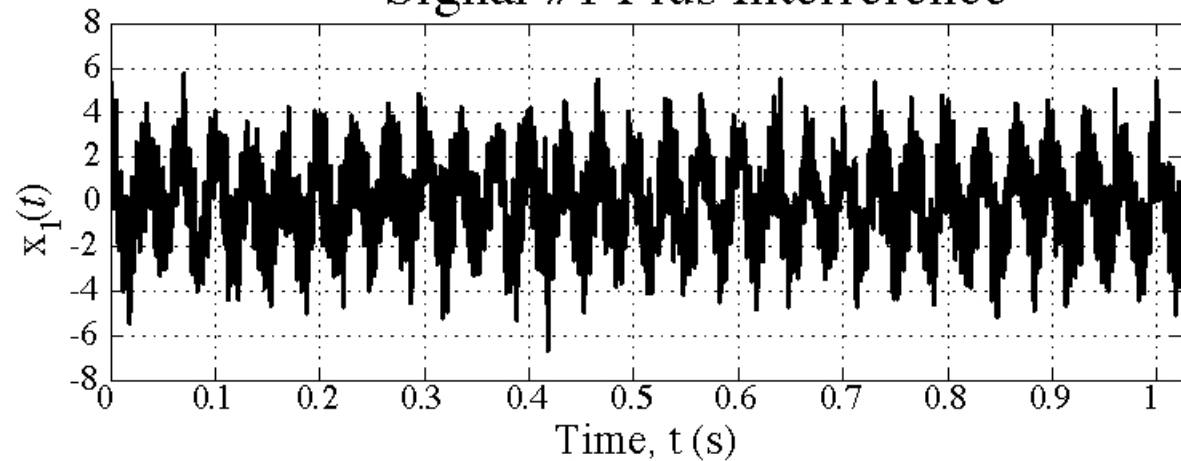
CPSD Phase



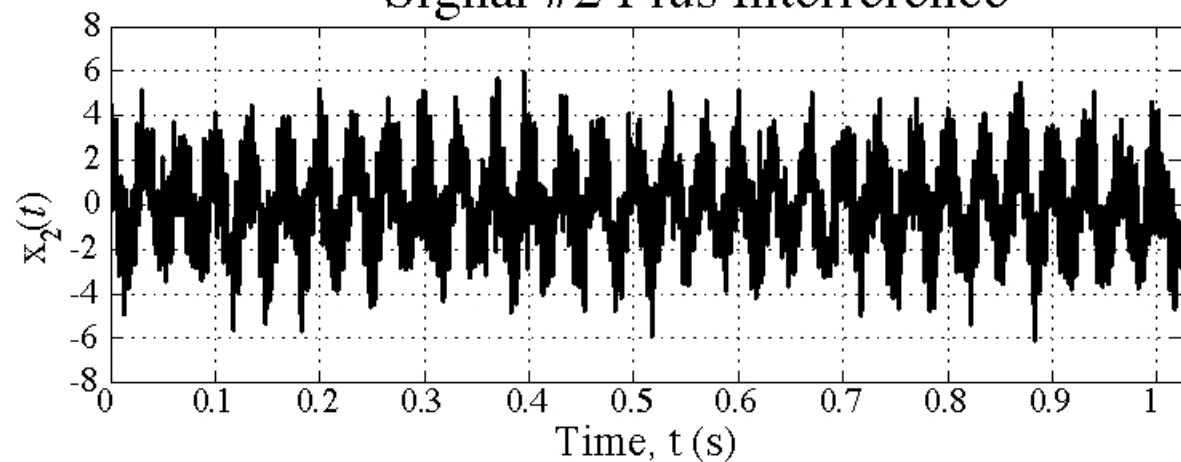
Cross Power Spectral Density

Two Random
Signals Plus
Narrowband
Interference

Signal #1 Plus Interference



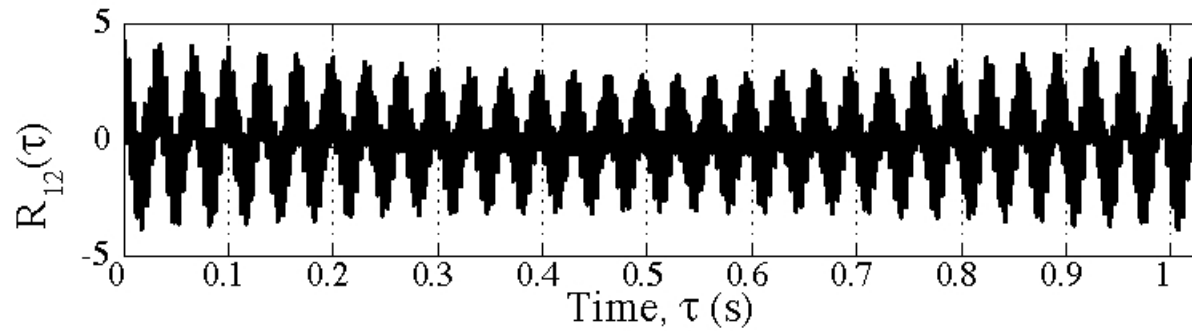
Signal #2 Plus Interference



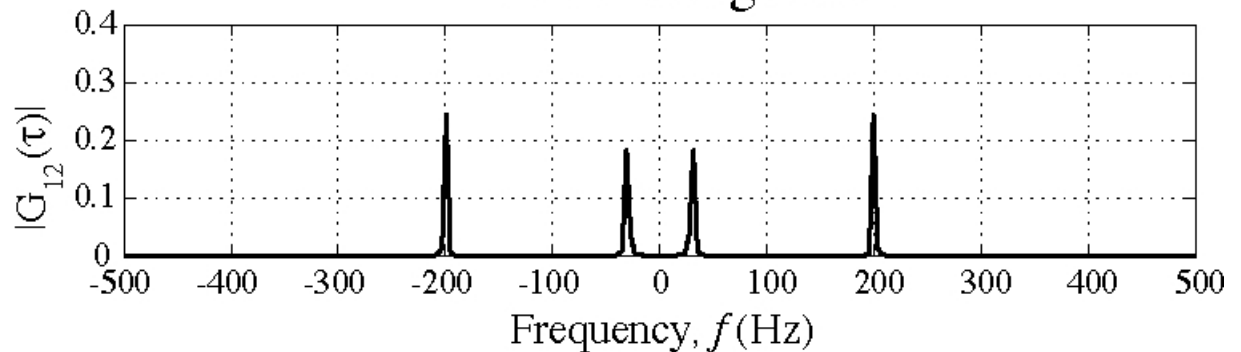
Cross Power Spectral Density

Crosscorrelation
and CPSD of
Two Random
Signals Plus
Narrowband
Interference

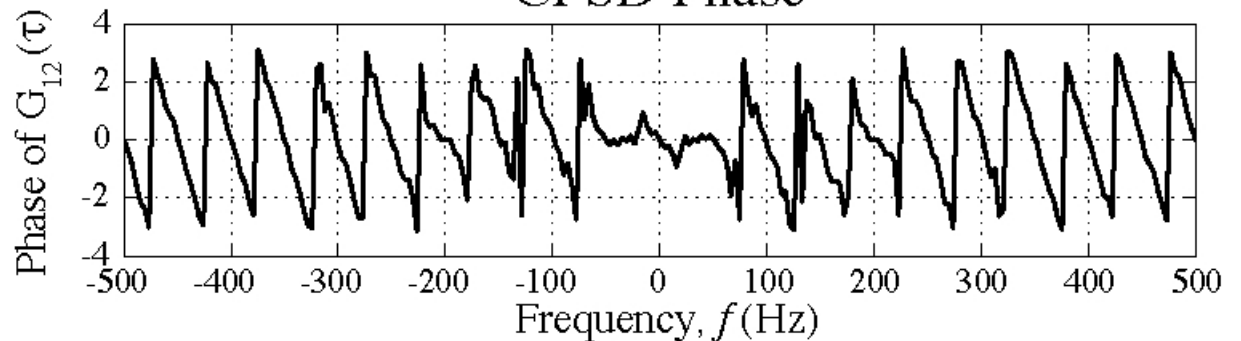
Crosscorrelation



CPSD Magnitude



CPSD Phase



Measurement of Power Spectral Density

A natural idea for estimating the PSD of an ergodic stochastic CT process is to start with the definition,

$$G_X(f) = \lim_{T \rightarrow \infty} \mathbf{E} \left(\frac{|\mathcal{F}(X_T(t))|^2}{T} \right)$$

and just not take the limit.

$$\hat{G}_X(f) = \frac{|\mathcal{F}(X_T(t))|^2}{T}$$

Unfortunately this approach yields an estimate whose variance does not decrease with increasing time T .

Measurement of Power Spectral Density

Another approach to estimating PSD is to first estimate autocorrelation and then Fourier transform that estimate.

We can estimate autocorrelation from

$$\hat{R}_X(\tau) = \frac{1}{T - \tau} \int_0^{T-\tau} X(t)X(t + \tau) dt \quad , \quad 0 \leq \tau \ll T$$

This estimate does improve with increasing time T .

Later we will see another, even better, technique for estimating PSD and CPSD.