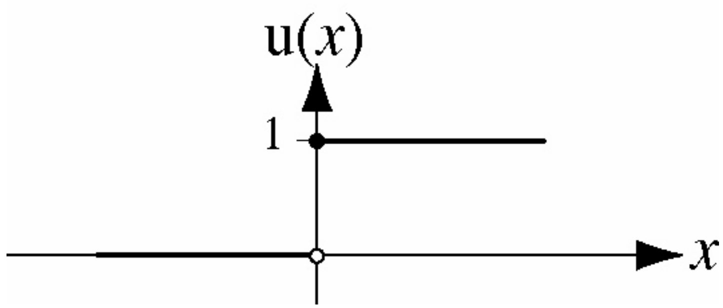
Probability

Mathematical Preliminaries

The Unit Step Function

$$\mathrm{u}(x) = \begin{cases} 1 & , \ x \ge 0 \\ 0 & , \ x < 0 \end{cases}$$



Mathematical Preliminaries

The Unit Impulse

$$\mathrm{g}(0) = \int_{-\infty}^{\infty} \delta(x)\mathrm{g}(x)\,dx$$

$$\delta(x) = 0 \;\;,\;\; x \neq 0 \quad \text{and} \quad \int_{\alpha}^{\beta} \delta(x)\,dx = \begin{cases} 1 \;\;,\;\; \alpha < 0 < \beta \\ 0 \;\;,\;\; \text{otherwise} \end{cases}$$

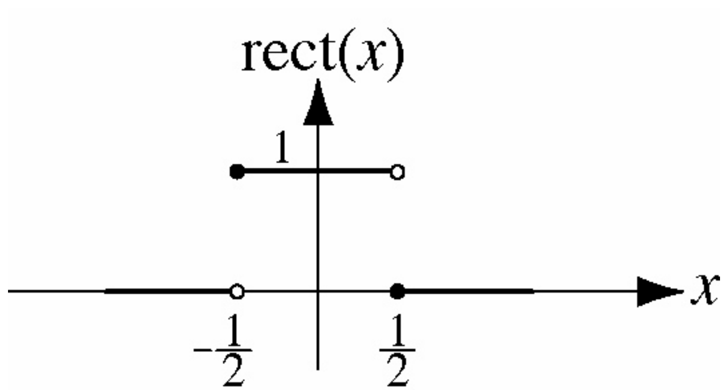
The Sampling Property $$\int_{-\infty}^{\infty} \mathrm{g}(x)\delta(x - x_0)\,dt = \mathrm{g}(x_0)$$

The Scaling Property $$\delta\big(a(x - x_0)\big) = \frac{1}{|a|}\delta(x - x_0)$$

Mathematical Preliminaries

The Unit Rectangle Function

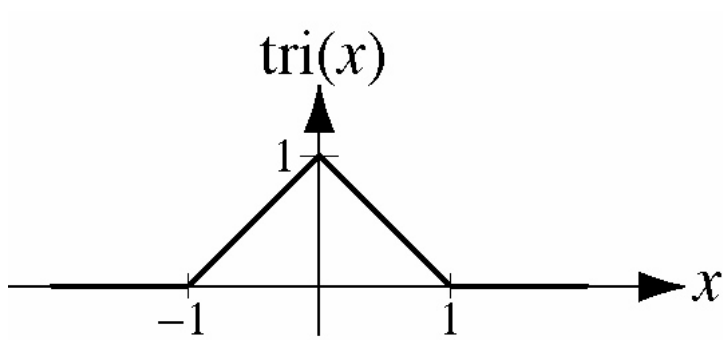
$$\mathrm{rect}(x) = \mathrm{u}(x+1/2) - \mathrm{u}(x-1/2)$$



Mathematical Preliminaries

The Unit Triangle Function

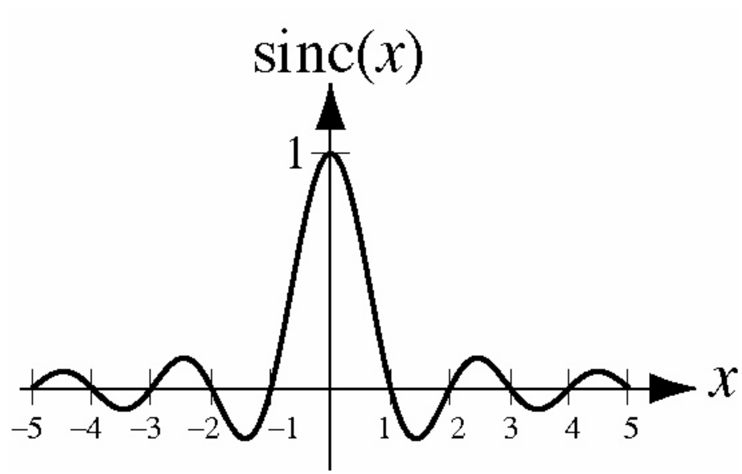
$$\text{tri}(x) = \begin{cases} 1-|x| & , \ |x|<1 \\ 0 & , \ |x| \geq 1 \end{cases}$$



Mathematical Preliminaries

The Unit Sinc Function

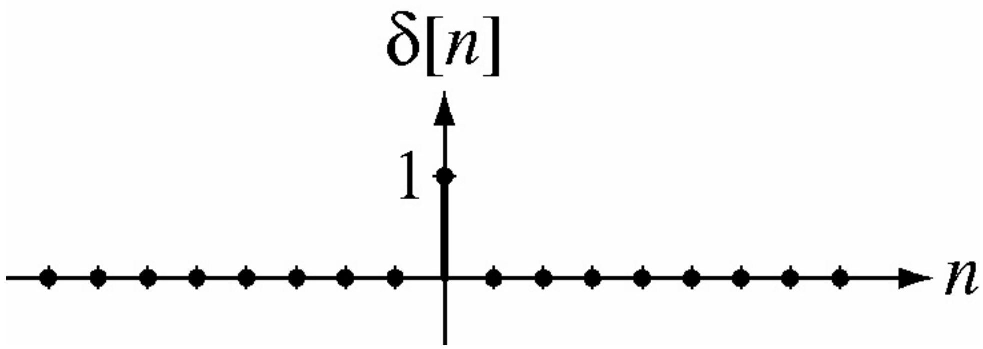
$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



Mathematical Preliminaries

The Discrete-Time Unit Impulse Function

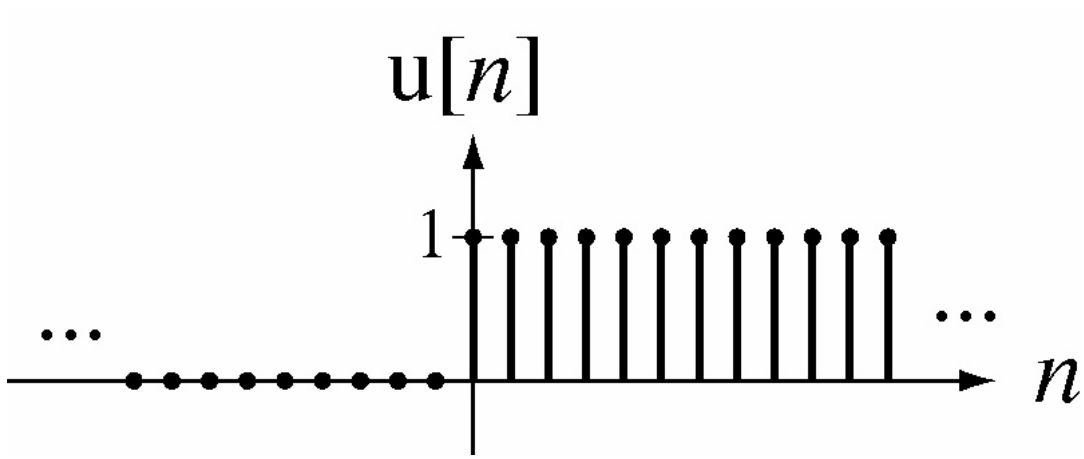
$$\delta[n] = \begin{cases} 1 \ , & n = 0 \\ 0 \ , & n \neq 0 \end{cases}$$



Mathematical Preliminaries

The Discrete-Time Unit Sequence Function

$$\mathrm{u}[n] = \begin{cases} 1 \ , & n \geq 0 \\ 0 \ , & n < 0 \end{cases}$$



Definitions

- An **experiment** is any operation performed according to a **procedure** with a specified set of **observations**.
- A **trial** is any single performance of an experiment.
- An **outcome** is the result of performing a single experiment.
- A **probability space** is the set of all possible outcomes.
- An **event** is a subset of the probability space.

Set Theory and Probability

If S is a probability space

$$\mathrm{P}[S] = 1$$

The null set $\varnothing$ is associated with the impossible event.

$$\mathrm{P}[\varnothing] = 0$$

A subset A of S is any set whose elements are also elements of S. If S has n elements there are 2^n distinct subsets of S, each of which is an event.

For example if S is the set $\{H, T\}$ the four events are

$$\{\varnothing\}, \{H\}, \{T\}, \{H, T\}$$

Set Theory and Probability

If a die is tossed

there are 6 outcomes

{1,2,3,4,5,6}.

There are 64 events.

{∅}	1 subset
{1},{2},{3},{4},{5},{6}	6 subsets
{1,2},{1,3},{1,4},{1,5},{1,6} {2,3},{2,4},{2,5},{2,6} {3,4},{3,5},{3,6} {4,5},{4,6} {5,6}	15 subsets
{1,2,3},{1,2,4},{1,2,5},{1,2,6},{1,3,4},{1,3,5}, {1,3,6},{1,4,5},{1,4,6},{1,5,6},{2,3,4}, {2,3,5},{2,3,6},{2,4,5},{2,4,6},{2,5,6} {3,4,5},{3,4,6},{3,5,6},{4,5,6}	20 subsets
{1,2,3,4},{1,2,3,5},{1,2,3,6},{1,2,4,5},{1,2,4,6}, {1,2,5,6},{1,3,4,5},{1,3,4,6},{1,3,5,6}, {1,4,5,6},{2,3,4,5},{2,3,4,6},{2,3,5,6}, {2,4,5,6},{3,4,5,6}	15 subsets
{1,2,3,4,5},{1,2,3,4,6},{1,2,3,5,6},{1,2,4,5,6}, {1,3,4,5,6},{2,3,4,5,6}	6 subsets
{1,2,3,4,5,6}	1 subset

Set Theory and Probability

The notation $A = \left\{ e^x \mid x = -2, 0, 3, 4 \right\}$ means the set of powers of e

$$e^{-2}\ ,\ e^{0}\ ,\ e^{3}\ ,\ e^{4}$$

Some sets have infinitely many elements. They are normally specified by their properties rather than by enumeration (since enumeration is impossible). The set of real numbers between 0 and 1 is described by $\left\{ 0 < x < 1 \right\}$.

Set Theory and Probability

If A is the subset

$$A = \left\{ A, 2, 3, 4, 5, 6, 7, 8, 9, 10, \text{J}, \text{Q}, \text{K} \right\}$$

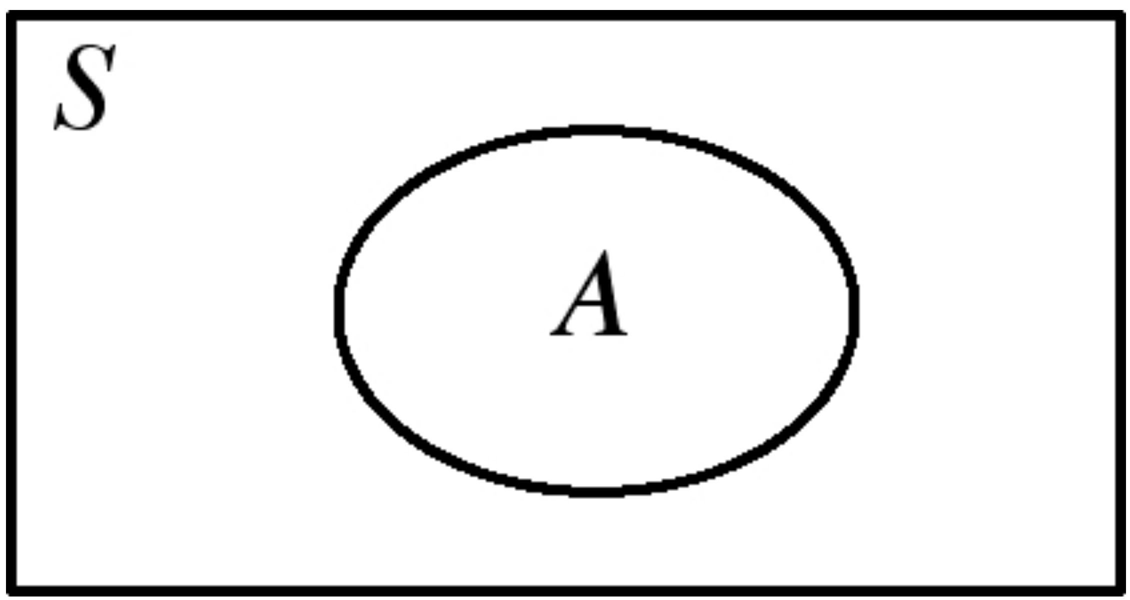
of spades in the set of playing cards S $(A \subset S)$ we can say that the probability of choosing a spade from S is

$$\text{P}\left[\text{a spade is chosen} \right] = \text{P}\left[A \right]$$

Set Theory and Probability

Venn Diagrams

The statement "*A* is a subset of *S*" can be represented geometrically by this Venn diagram



Set Theory and Probability

The union of two sets $A \cup B$ or $A + B$ is the set consisting of all the members which are in A or B or both.

The intersection of two sets $A \cap B$ or AB is the set consisting of all the members which are in both A and B

The complement of a set $\overline{A}$ is the set consisting of all the members that are not in A.

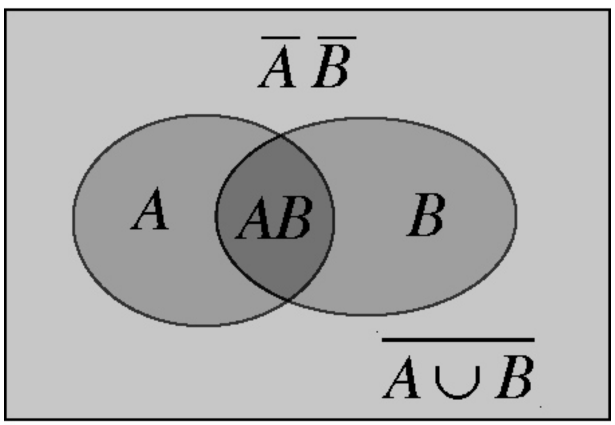
The difference $A - B$ is the set consisting of all the members that are in A and not in B.

Set Theory and Probability

DeMorgan's Laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \quad , \quad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

$$\overline{A + B} = \overline{A}\,\overline{B} \quad , \quad \overline{AB} = \overline{A} + \overline{B}$$



Set Theory and Probability

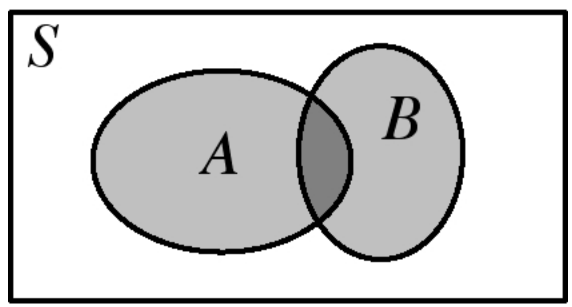
The Three Postulates of Probability Theory

If A and B are subsets of S, $\mathrm{P}[A] \geq 0$, $\mathrm{P}[S] = 1$

If $AB = \varnothing$, then $\mathrm{P}[A+B] = \mathrm{P}[A] + \mathrm{P}[B]$

Using these postulates it can be shown that

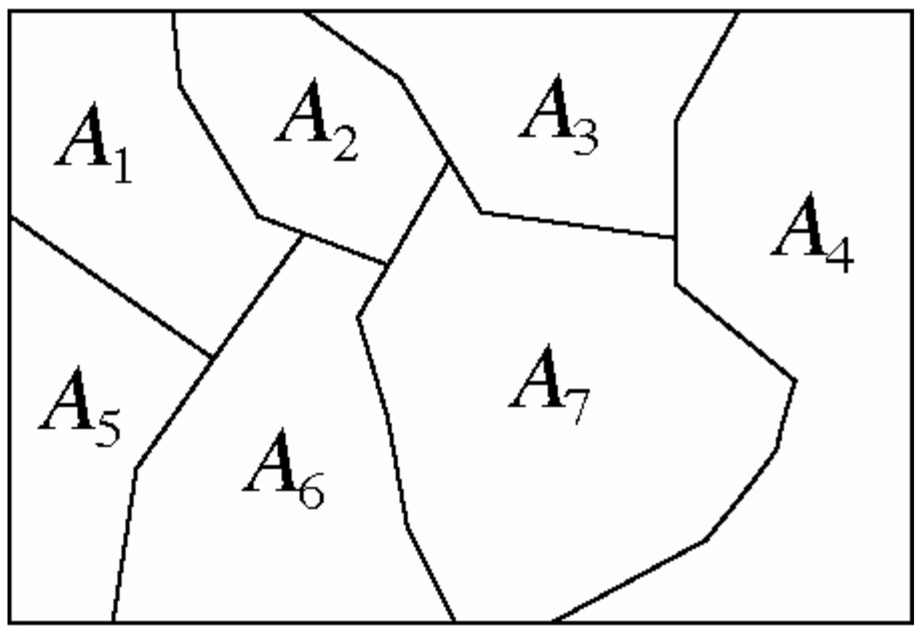
$$\mathrm{P}[A+B] = \mathrm{P}[A] + \mathrm{P}[B] - \mathrm{P}[AB]$$



If $AB = \varnothing$ then A and B are **mutually exclusive** or **disjoint**.

Set Theory and Probability

A **partition** of a set *S* is a collection of mutually exclusive subsets of *S* whose union is the set *S*.



Set Theory and Probability

If the outcomes of an experiment are $\{\zeta_1, \zeta_2, \cdots, \zeta_N\}$ and the probability of the *i*th outcome is $P[\zeta_i] = p_i$ then $\sum_{i=1}^{N} p_i = 1$. The sum of the probabilities of all the possible outcomes of an experiment must be one.

Relative Frequency and Probability

The relative frequency of occurrence of an event A is defined as $\mathrm{r}(A)=\frac{N_A}{N}$ where N_A is the number of times the event A occurred in N trials of an experiment. The probability of the event A is the limit of the relative frequency of occurrence as the number of trials N approaches infinity.

$$\mathrm{P}[A]=\lim_{N\to\infty}\mathrm{r}(A)=\lim_{N\to\infty}\frac{N_A}{N}$$

Conditional Probability

Conditional probability is the probability that an event occurs, given that another event also occurs. The conditional probability of event *A* given that event *B* also occurs is defined by

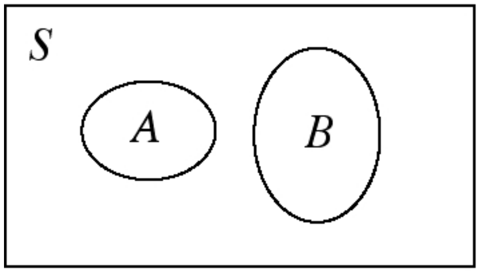
$$\mathrm{P}\left[A \mid B\right] \equiv \frac{\mathrm{P}\left[A \cap B\right]}{\mathrm{P}\left[B\right]} \quad , \quad \mathrm{P}\left[B\right] > 0$$

If *A* and *B* are mutually exclusive (disjoint) their intersection is the null set and

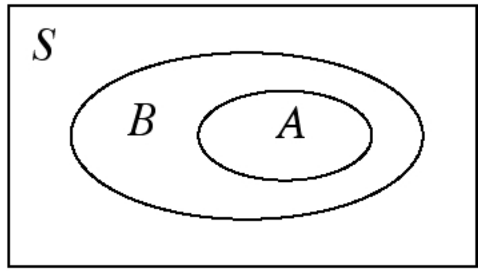
$$\mathrm{P}\left[A \mid B\right] = \frac{\mathrm{P}\left[\varnothing\right]}{\mathrm{P}\left[B\right]} = 0$$

Conditional Probability

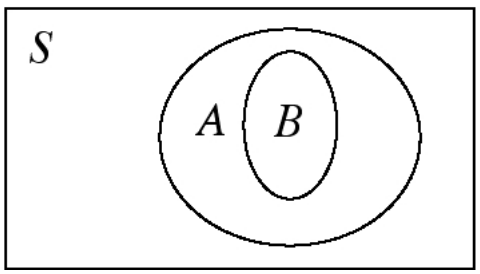
A and *B* are disjoint



A is a subset of *B*



B is a subset of *A*



If A is a subset of B,

$$\mathrm{P}\left[A \mid B\right] = \frac{\mathrm{P}\left[AB\right]}{\mathrm{P}\left[B\right]}$$

$$= \frac{\mathrm{P}\left[A\right]}{\mathrm{P}\left[B\right]} \geq \mathrm{P}\left[A\right]$$

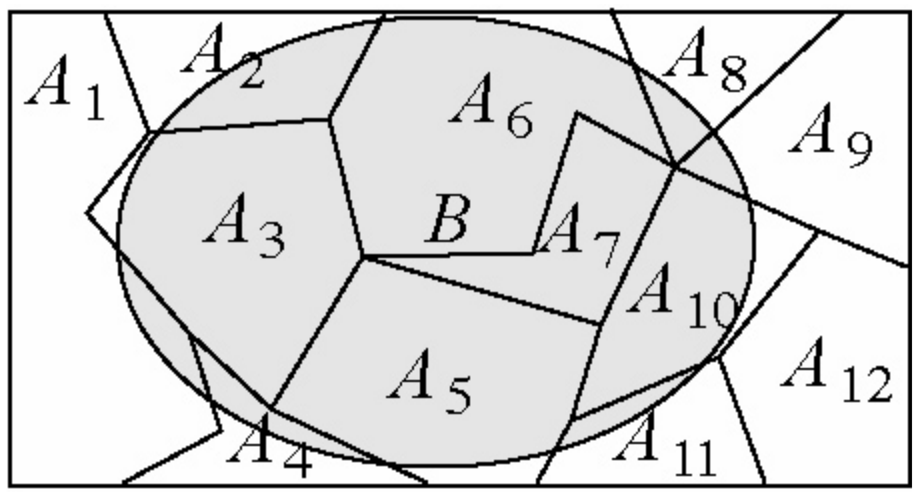
If B is a subset of A,

$$\mathrm{P}\left[A \mid B\right] = \frac{\mathrm{P}\left[AB\right]}{\mathrm{P}\left[B\right]} = \frac{\mathrm{P}\left[B\right]}{\mathrm{P}\left[B\right]} = 1$$

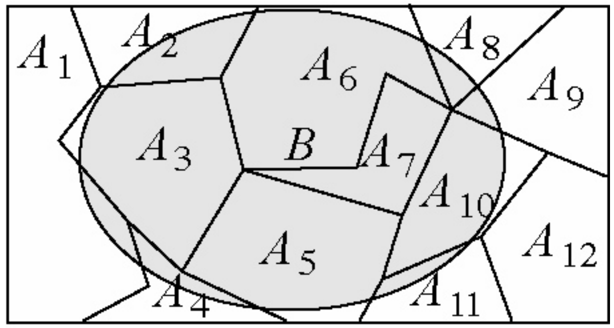
Conditional Probability

Suppose a probability space is divided into N mutually exclusive events $\left\{A_1, A_2, \cdots, A_N\right\}$ and that the event B intersects some or all of those events. Then

$$A_1 + A_2 + \cdots + A_N = S$$



Conditional Probability



The probability of B is

$$B = B\left(A_1 + A_2 + \cdots + A_N\right) = BA_1 + BA_2 + \cdots + BA_N$$

$$\mathrm{P}\left[B\right] = \mathrm{P}\left[BA_1 + BA_2 + \cdots + BA_N\right]$$

$$\mathrm{P}\left[B\right] = \mathrm{P}\left[BA_1\right] + \mathrm{P}\left[BA_2\right] + \cdots + \mathrm{P}\left[BA_N\right]$$

Then using

$$\mathrm{P}\left[A \mid B\right] \equiv \frac{\mathrm{P}\left[AB\right]}{\mathrm{P}\left[B\right]} \ , \ \mathrm{P}\left[B\right] > 0$$

$$\mathrm{P}\left[B\right] = \mathrm{P}\left[B \mid A_1\right]\mathrm{P}\left[A_1\right] + \mathrm{P}\left[B \mid A_2\right]\mathrm{P}\left[A_2\right] + \cdots + \mathrm{P}\left[B \mid A_N\right]\mathrm{P}\left[A_N\right]$$

Conditional Probability

Bayes' Theorem

From the definition of conditional probability

$$\mathrm{P}\left[A \mid B\right] \equiv \frac{\mathrm{P}\left[AB\right]}{\mathrm{P}\left[B\right]} \quad , \quad \mathrm{P}\left[B\right] > 0$$

Exchanging the roles of A and B,

$$\mathrm{P}\left[B \mid A\right] = \frac{\mathrm{P}\left[BA\right]}{\mathrm{P}\left[A\right]} = \frac{\mathrm{P}\left[AB\right]}{\mathrm{P}\left[A\right]} \quad , \quad \mathrm{P}\left[A\right] > 0$$

Then $\mathrm{P}\left[AB\right] = \mathrm{P}\left[A \mid B\right]\mathrm{P}\left[B\right] = \mathrm{P}\left[B \mid A\right]\mathrm{P}\left[A\right]$

and

$$\mathrm{P}\left[A \mid B\right] = \frac{\mathrm{P}\left[B \mid A\right]\mathrm{P}\left[A\right]}{\mathrm{P}\left[B\right]} \quad , \quad \mathrm{P}\left[B\right] > 0 \quad \leftarrow \text{Bayes' Theorem}$$

Conditional Probability

Example: Suppose in a thrift store there are 3 boxes labeled "Small", "Medium" and "Large"containing shoes.

Shoe Size	Small	Medium	Large	Totals
6	15	5	1	21
7	10	8	3	21
8	9	10	4	23
9	6	12	3	21
10	6	9	7	22
11	3	6	10	19
12	2	4	14	20
Totals	51	54	42	

If a box is first chosen at random and then a shoe is chosen at random from that box, what is the probability of choosing a size-9 shoe?

Conditional Probability

Example

The probability of choosing the ith box is A_i and $\mathrm{P}\left[A_i\right]=1/3$, for any i. Let the event "a size-9 shoe is chosen" be B.

$$\mathrm{P}\left[B \mid A_1\right]=\frac{6}{51} \;,\; \mathrm{P}\left[B \mid A_2\right]=\frac{12}{54}=\frac{2}{9} \;,\; \mathrm{P}\left[B \mid A_3\right]=\frac{3}{42}=\frac{1}{14}$$

The probability of choosing a size-9 shoe is then

$$\mathrm{P}\left[B\right]=\frac{1}{3}\times\frac{6}{51}+\frac{1}{3}\times\frac{2}{9}+\frac{1}{3}\times\frac{1}{14}=0.1371$$

Conditional Probability

Example

What is the probability that, if a size-9 shoe is chosen, that it came from box 3? Using Bayes' Theorem

$$\mathrm{P}\left[A_3 \mid B\right] = \frac{\mathrm{P}\left[B \mid A_3\right]\mathrm{P}\left[A_3\right]}{\mathrm{P}\left[B\right]} = \frac{1/14 \times 1/3}{0.1371} = 0.1737$$

Conditional Probability

Example

The army has an image analysis system that recognizes the two types of enemy tanks and classifies them as type A or type B. Its identification is right 90% of the time but wrong 10% of the time. That means that, on average, 10% of type A tanks are classified as type B and 10% of type B tanks are classified as type A. In a battle the enemy's tanks are 80% type A and 20% type B. At random an enemy tank appears and is identified as type B. What is the probability that the tank really is type B?

Conditional Probability

Example

$$\mathrm{P}\left[B \mid B\ \mathrm{ID}\right] = \frac{\mathrm{P}\left[B\ \mathrm{ID} \middle| B\right]\mathrm{P}\left[B\right]}{\mathrm{P}\left[B\ \mathrm{ID}\right]}$$ Bayes' Theorem

Conditional probability of identifying a type B tank as type B

$$\mathrm{P}\left[B\ \mathrm{ID} \middle| B\right] = 0.9$$

Probability of a type B tank appearing $\mathrm{P}\left[B\right] = 0.2$

Total probability of identifying a tank as type B

$$\mathrm{P}\left[B\ \mathrm{ID}\right] = \mathrm{P}\left[B\ \mathrm{ID} \middle| B\right]\mathrm{P}\left[B\right] + \mathrm{P}\left[B\ \mathrm{ID} \middle| A\right]\mathrm{P}\left[A\right]$$

$$\mathrm{P}\left[B\ \mathrm{ID}\right] = 0.9 \times 0.2 + 0.1 \times 0.8 = 0.26$$

Probability that the identification as type B is correct

$$\mathrm{P}\left[B \mid B\ \mathrm{ID}\right] = \frac{0.9 \times 0.2}{0.26} = 0.692$$

Independent Events

If A and B are independent

$$\mathrm{P}\left[A \mid B\right] = \mathrm{P}\left[A\right] = \frac{\mathrm{P}\left[AB\right]}{\mathrm{P}\left[B\right]} \quad \text{and} \quad \text{therefore} \quad \mathrm{P}\left[AB\right] = \mathrm{P}\left[A\right]\mathrm{P}\left[B\right]$$

If two events are mutually exclusive they cannot be independent (unless at least one of them has zero probability). For example, when tossing a coin once, if a head occurs a tail cannot. Therefore

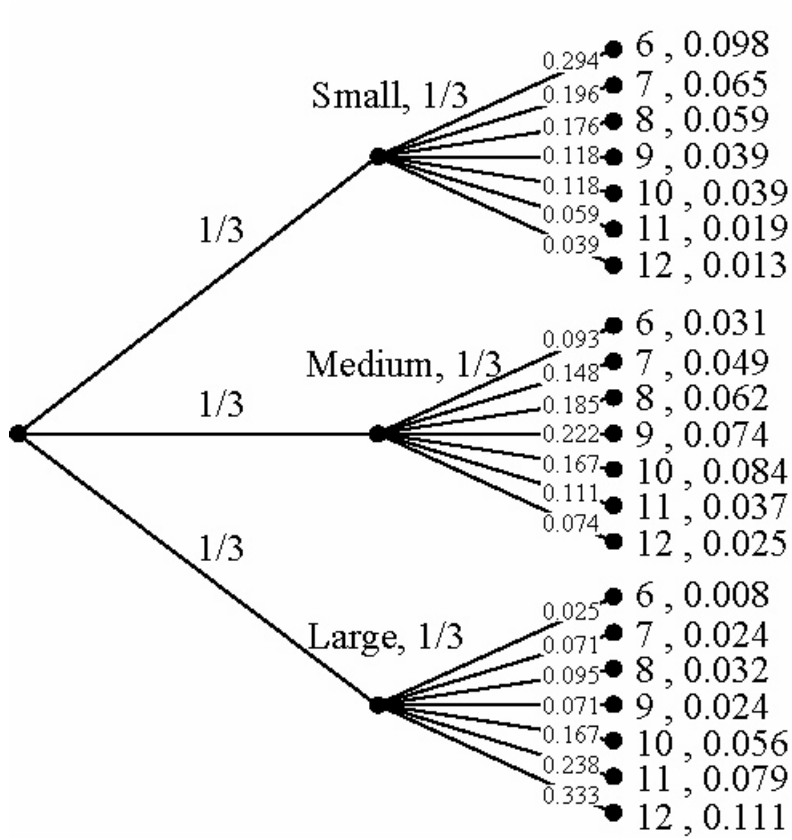
$$\mathrm{P}\left[T \mid H\right] = 0$$

If a head and a tail were independent then we should have

$$\mathrm{P}\left[T \mid H\right] = \mathrm{P}\left[T\right] = 1/2$$

Tree Diagrams

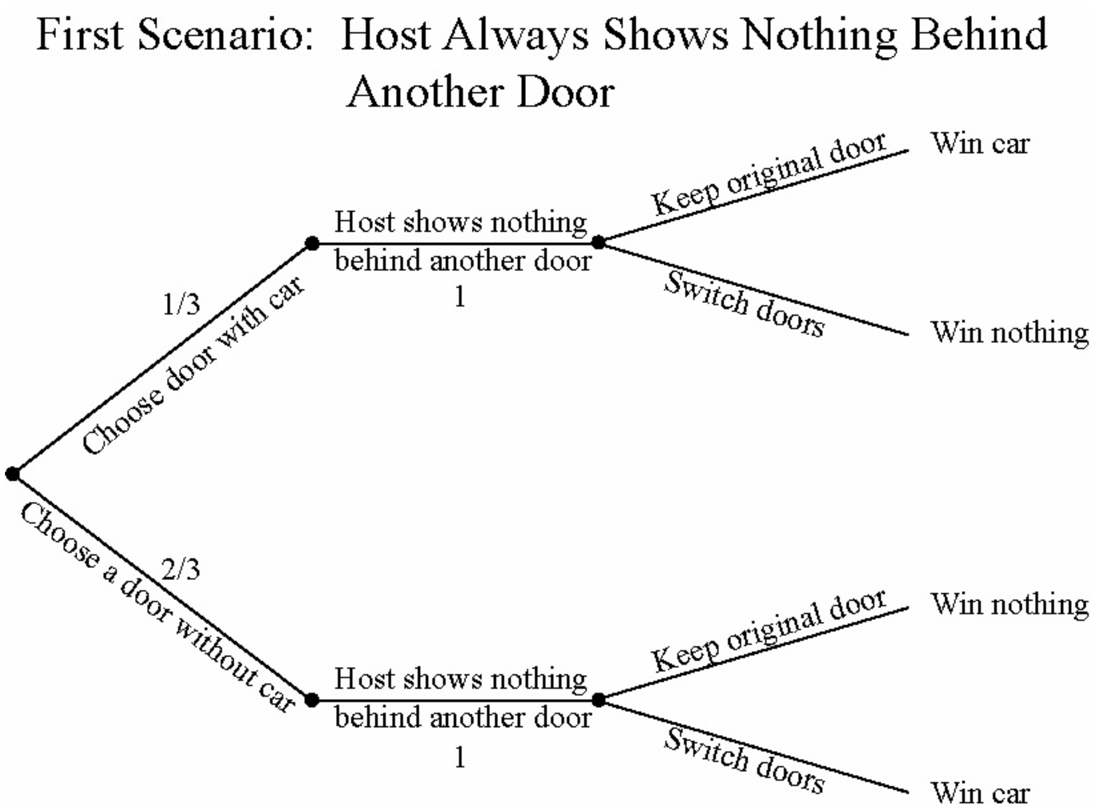
- A graphical technique known as a tree diagram can sometimes be very helpful in complicated problems in conditional probability



Thrift Store Tree Diagram

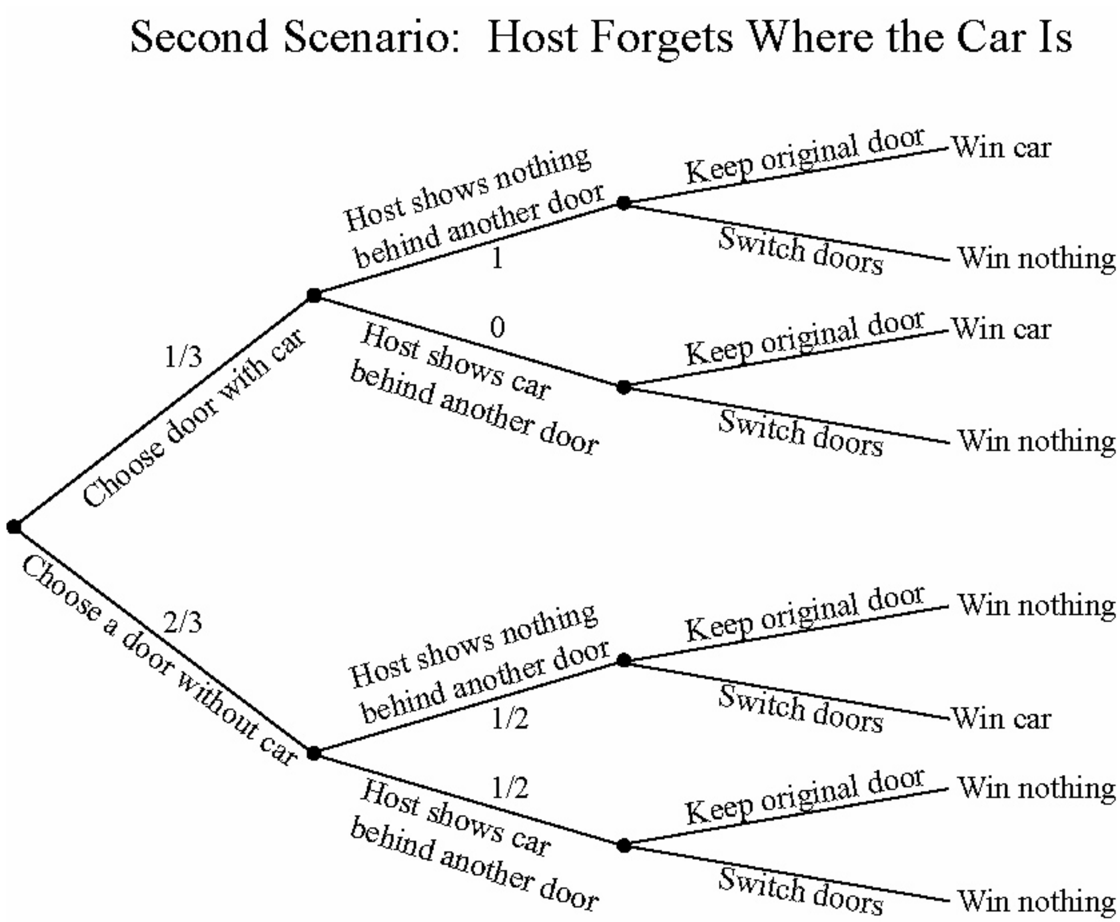
Tree Diagrams

On a game show a contestant is allowed to choose any of three doors and will win what is behind the door. Behind one of the doors is a car. Behind each of the other two doors are nothing. The contestant chooses a door. But before the door is opened the host (knowing where the car is) opens one of the other doors to reveal that there is nothing behind it. Then the contestant is given the opportunity to keep that door or to switch to the other unopened door. Should he switch?



Tree Diagrams

Now imagine that the scenario is the same except that the host forgets which door the car is behind.



Combined Experiments

Let one experiment have a probability space S_1 and let another experiment have a probability space S_2. $S_1 = \{\alpha_1, \alpha_2, \cdots, \alpha_N\}$ and $S_2 = \{\beta_1, \beta_2, \cdots, \beta_M\}$. A combined experiment is one which consists of performing both of these experiments. The probability space of the combined experiment is the cartesian product of the two individual probability spaces.

$$S = S_1 \times S_2 = \begin{Bmatrix} (\alpha_1,\beta_1),(\alpha_1\beta_2),\cdots,(\alpha_1,\beta_M), \\ (\alpha_2,\beta_1),(\alpha_2,\beta_2),\cdots,(\alpha_2,\beta_M), \\ \vdots \quad \vdots \quad \vdots \quad \vdots \\ (\alpha_N,\beta_1),(\alpha_N,\beta_2),\cdots,(\alpha_N,\beta_M) \end{Bmatrix}$$

If A_1 is an event in S_1 and A_2 is an event in S_2 and $A = A_1 \times A_2$ then A is an event in S.

Permutations and Combinations

Most of the really interesting problems in probability involve selecting objects from one group to form a second group.

As objects are taken from the first group they may be replaced, leaving the first group unchanged, or they may not be replaced, thereby changing the first group.

The sequence in which the second group of objects is chosen may be significant or insignificant.

Permutations and Combinations

How many distinct sequences of k objects can be formed by taking them from n distinct objects with replacement?

There are n ways of choosing the first object.

There are n ways of choosing the second object and all succeeding objects because the object is always replaced.

So there are $\underbrace{n \times n \times \cdots \times n}_{k \text{ of these}} = n^k$ ways of choosing k objects from n distinct objects with replacement.

Permutations and Combinations

There are $4^2 = 16$ ways of choosing two objects from the set $\{a, b, c, d\}$ with replacement.

$$\begin{array}{cccc} aa & ab & ac & ad \\ ba & bb & bc & bd \\ ca & cb & cc & cd \\ da & db & dc & dd \end{array}$$

Permutations and Combinations

How many distinct sequences of k objects can be formed by taking them from n distinct objects without replacement?

There are n ways of choosing the first object.

There are $n-1$ ways of choosing the second object, $n-2$ ways of choosing the third object, etc...

So there are

$$\underbrace{n\times(n-1)\times(n-2)\cdots\times(n-k+1)}_{k\text{ of these}}=\frac{n!}{(n-k)!}$$

distinct sequences of k objects taken from n distinct objects without replacement.

Permutations and Combinations

The $\frac{5!}{(5-2)!} = \frac{120}{6} = 20$ distinct sequences of 2 objects chosen from the set $\{a,b,c,d,e\}$ without replacement are

$$\begin{array}{cccc} ab & ac & ad & ae \\ ba & bc & bd & be \\ ca & cb & cd & ce \\ da & db & dc & de \\ ea & eb & ec & ed \end{array}$$

Permutations and Combinations

Given a sequence of k distinct objects, if they are rearranged into a different sequence of the same k objects, that new sequence is called a **permutation** of the original sequence. The process of permuting a sequence of k distinct objects is the same as choosing k objects from a group of k distinct objects without replacement. The number of distinct permutations of k distinct objects is

$\frac{k!}{(k-k)!}k! = \frac{k!}{0!} = k!$ (where it is understood that $0! = 1$) .

The 24 distinct permutations of the letters a, b, c and d are

abcd	*abdc*	*acbd*	*acdb*	*adbc*	*adcb*
bacd	*badc*	*bcad*	*bcda*	*bdac*	*bdca*
cabd	*cadb*	*cbad*	*cbda*	*cdab*	*cdba*
dabc	*dacb*	*dbac*	*dbca*	*dcab*	*dcba*

Permutations and Combinations

A **combination** of objects is a grouping without regard to order. Two permutations containing the same objects in different sequences are a single combination of those objects. The number of distinct sequences of n distinct objects taken k at a time without replacement is $\frac{n!}{(n-k)!}$. For each distinct sequence of k objects there are $k!$ distinct permutations of those k objects. So the number of combinations of k objects taken from n distinct objects is the number of distinct sequences of k objects divided by the number of distinct permutations of those k objects or

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \leftarrow \text{Binomial Coefficient}$$

Permutations and Combinations

The $\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \frac{5!}{2!3!} = \frac{120}{2 \times 6} = 10$ distinct combinations of 2 objects taken from the set $\{a, b, c, d, e\}$ without replacement are

$$\begin{matrix} ab & ac & ad & ae & bc \\ bd & be & cd & ce & de \end{matrix}$$

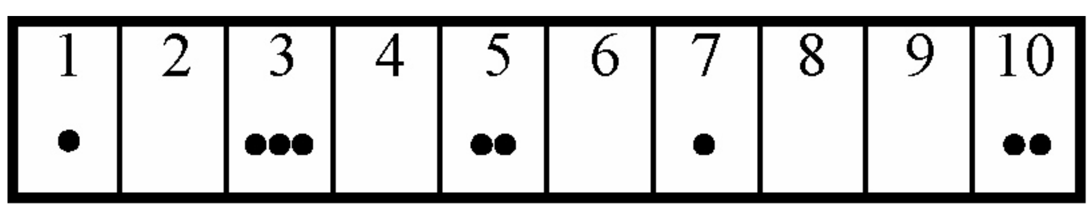
Permutations and Combinations

There is one more case to analyze, the number of combinations of k objects selected from n objects with replacement.

Imagine n slots in a linear array divided by inner walls, which can be moved, and with two fixed walls, one at each end.

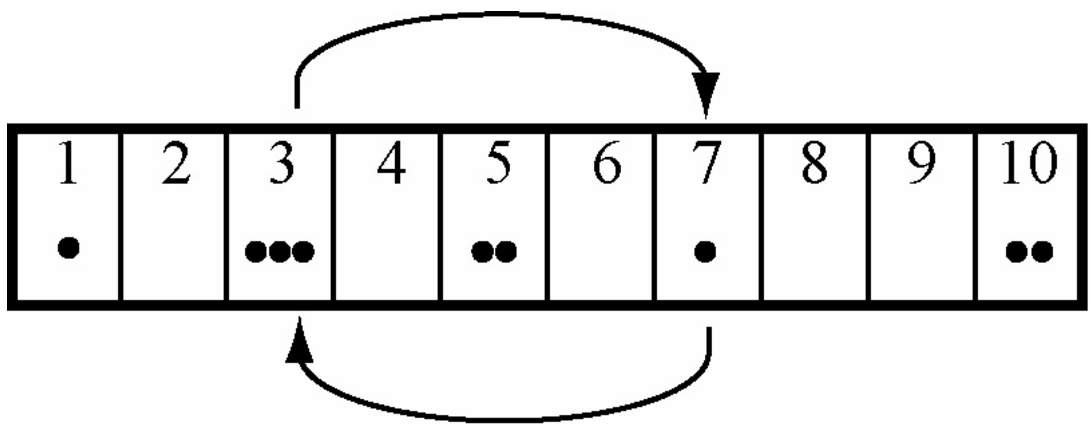
Let the slots be labeled 1 through n left-to-right.

Now suppose we randomly place a ball in a slot and the slot happens to be slot m. Then we randomly place another ball into a slot which could be any slot, including slot m, and continue until we have put k balls into the n-slot array.



Permutations and Combinations

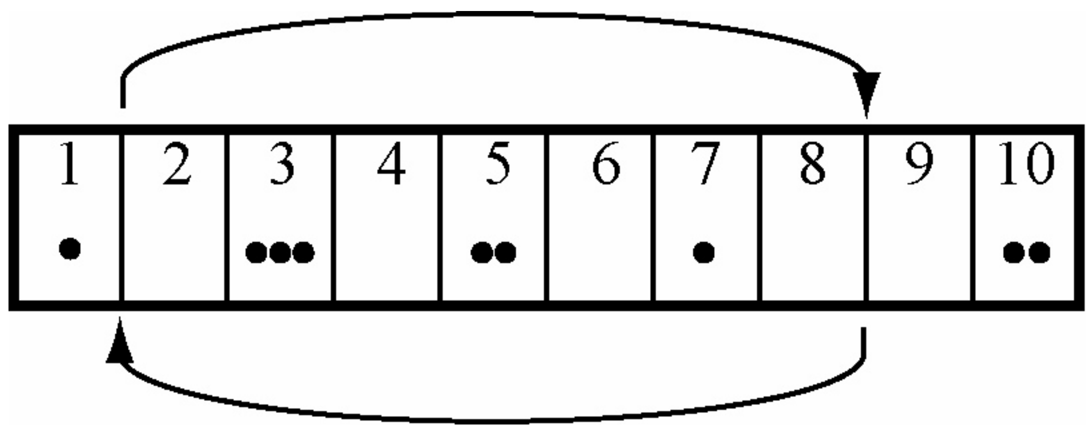
Exchange any two balls and let a ball's identification still be determined by its slot location.



This is not a new arrangement. We have the same number of balls in the same slots and the slot location determines the identity of a ball.

Permutations and Combinations

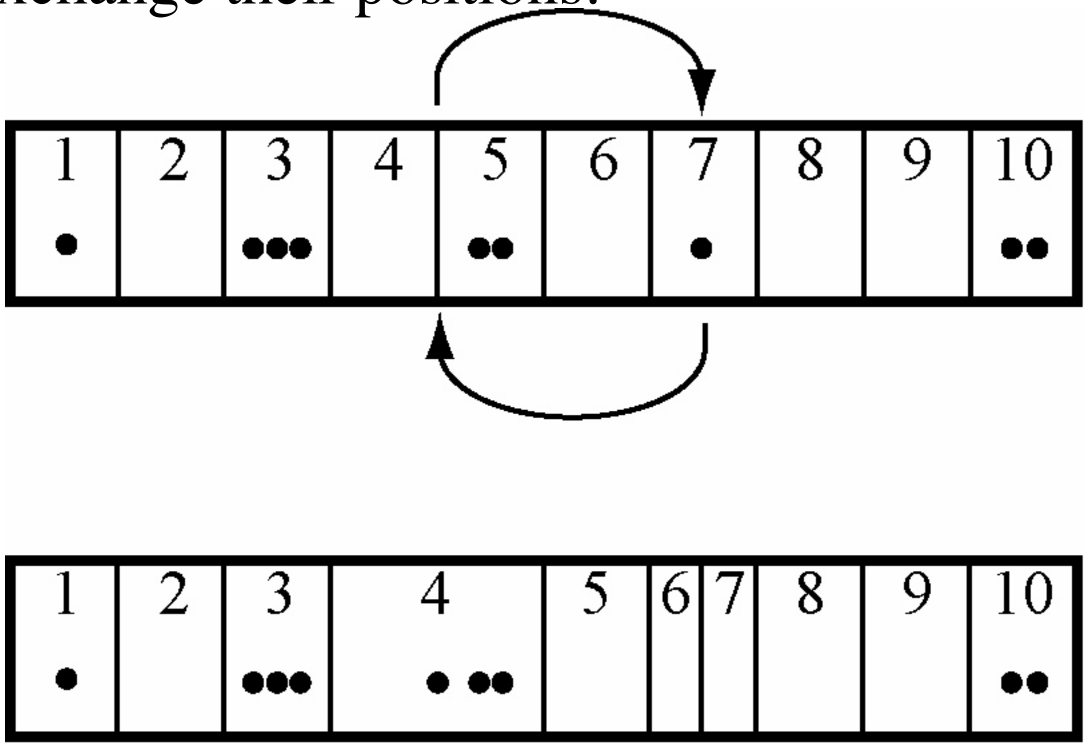
Exchange any two inner walls.



Again, nothing changes because the balls are still in slots with the same identification.

Permutations and Combinations

Now imagine that we select any ball and any inner wall and exchange their positions.



Now we have made a change because the ball is now in a different slot and is therefore identified differently.

Permutations and Combinations

If we permute all balls and inner walls as a group through all their possible permutations we will make $(k+n-1)!$ permutations (k balls and $n-1$ inner walls). But permuting an inner wall with another inner wall or permuting a ball with another ball does not make a new combination. So if we divide the $(k+n-1)!$ permutations of all entities by the $k!$ permutations of the balls and the $(n-1)!$ permutations of the inner walls we have the number of distinct combinations of k objects with n possible identities with replacement which is

$$\frac{(k+n-1)!}{k!(n-1)!} = \binom{k+n-1}{k} = \binom{k+n-1}{n-1}$$

Permutations and Combinations

There are $\left(\begin{array}{c} 3+3-1 \\ 3 \end{array} \right) = \frac{5!}{3!2!} = \frac{120}{6 \times 2} = 10$ distinct combinations of 3 objects taken from the set $\{a, b, c\}$ with replacement.

$$\begin{array}{ccccc} aaa & bbb & ccc & abb & acc \\ aab & bcc & aac & bbc & abc \end{array}$$

Permutations and Combinations

If n objects are divided into c classes and members of a class are indistinguishable from each other the number of distinct permutations of the n objects is $\frac{n!}{n_1!n_2!\cdots n_c!}$ where n_i is the number of members in the ith class. The $\frac{4!}{2!1!1!}=\frac{24}{2\times 1\times 1}=12$ distinct permutations of the set $\{a,a,b,c\}$ are

aabc	*aacb*	*abac*	*abca*	*acab*	*acba*
baac	*baca*	*bcaa*	*caab*	*caba*	*cbaa*

Bernoulli Trials

In performing an experiment multiple times let there be an event A with a probability of p and the complementary event $\overline{A}$ with a probability of $1-p$. If the experiment is repeated n times, what is the probability that the event A occurs exactly k times? The probability of the event A occurring k times in the first k trials is

$$\underbrace{\mathrm{P}\left[A\right]\mathrm{P}\left[A\right]\cdots\mathrm{P}\left[A\right]}_{k\text{ occurrences}}\underbrace{\mathrm{P}\left[\overline{A}\right]\mathrm{P}\left[\overline{A}\right]\cdots\mathrm{P}\left[\overline{A}\right]}_{n\text{-}k\text{ occurrences}}=p^k\left(1-p\right)^{n-k}$$

But this is only one way that the event could occur k times in n trials. This is an example of permutations with two classes of indistinguishable members A and $\overline{A}$ and the total number of distinguishable permutations is $\binom{n}{k}=\dfrac{n!}{k!\left(n-k\right)!}$.

Bernoulli Trials

The probability of exactly k occurrences of the event A in n trials is

$$\mathrm{P}\left[k\ A\text{'s in any order in } n \text{ trials}\right]=\binom{n}{k}p^k\left(1-p\right)^{n-k}$$

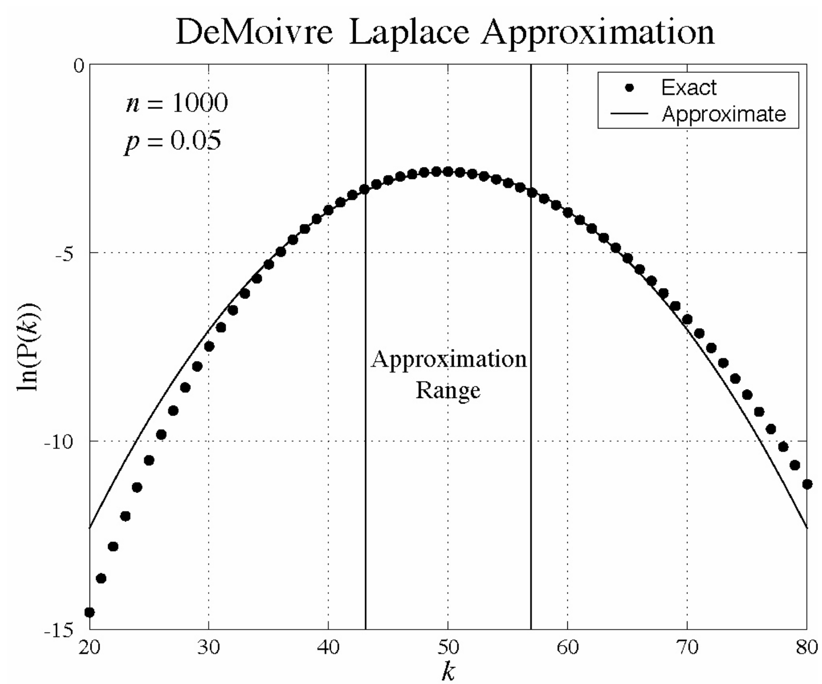
Let the probability of the A event be 0.2 and let k be 3 and let n be 5. Then

$$\mathrm{P}\left[3\ A\text{'s in any order in 5 trials}\right]=\binom{5}{3}p^2\left(1-p\right)^{4-2}=0.0512$$

The DeMoivre-Laplace Approximation

If $np(1-p) >> 1$ and $|k-np| < \sqrt{np(1-p)}$ then

$$\binom{n}{k} p^k (1-p)^{n-k} \cong \frac{1}{\sqrt{2\pi np(1-p)}} e^{-\frac{(k-np)^2}{2np(1-p)}}$$



Generalization of Bernoulli Trials

We can extend Bernoulli-trial theory to the probability of a certain number of occurrences of each of more than two events in n trials for experiments in which there are more than two events.

Suppose there are three events. The number of ways event 1 can occur k_1 times is $\binom{n}{k_1} = \frac{n!}{k_1!(n-k_1)!}$.

Generalization of Bernoulli Trials

For each permutation there are $n-k_1$ events of the other two types. We can permute each of those by letting events 2 and 3 take on all possible positions where event 1 does not occur. For each permutation of event 1 there are $\frac{(n-k_1)!}{k_2!(n-k_1-k_2)!} = \frac{(n-k_1)!}{k_2!k_3!}$ permutations on event 2 which occurs k_2 times. This accounts for all possible permutations because when we permute event 2 we are also permuting event 3 since it is the only event left. Multiplying the two numbers of permutations yields the overall number of permutations of three events with three numbers of occurrences

$$\frac{n!}{k_1!(n-k_1)!}\frac{(n-k_1)!}{k_2!k_3!} = \frac{n!}{k_1!k_2!k_3!}$$

Generalization of Bernoulli Trials

By induction, the number of ways events 1 through m can occur with their numbers of occurrences being $k_1, \cdots, k_m$ is

$$\binom{n}{k_1, k_2, \cdots, k_m} = \frac{n!}{k_1!k_2!\cdots k_m!}$$

which is called the **multinomial coefficient**. The probability in n trials that event 1 will occur k_1 times, event 2 will occur k_2 times, ... event m will occur k_m times is

$$\mathrm{P}\left[1-k_1, 2-k_2, \cdots, m-k_m\right] = \binom{n}{k_1, k_2, \cdots, k_m} p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$$