Spectral Estimation

Power Spectral Density

Estimation of PSD of a stochastic process X is most commonly done by sampling it for a finite time and analyzing the samples with the discrete Fourier transform (DFT).

$$\mathrm{X}[k] = \sum_{n=0}^{N-1} \mathrm{x}[n] e^{-j2\pi nk/N}$$

The estimate of the PSD at N points in frequency is

$$\mathrm{G}_X[k] = \frac{\left|\mathrm{X}[k]/N\right|^2}{\Delta f}$$

where k is the harmonic number, an integer multiple of the fundamental frequency which is $f_s / N = \Delta f$.

Power Spectral Density

If the samples come from a bandlimited white noise process sampled at the Nyquist rate the expected value of the PSD estimates is

$$\mathrm{E}\left(\mathrm{G}_X\left[k\right]\right)=T_s\sigma_x^2=\sigma_x^2/f_s$$

The variance of the PSD estimates is

$$\mathrm{Var}\left(\mathrm{G}_X\left[k\right]\right)=\frac{T_s^2}{N}\left[\mathrm{E}\left(x^4\right)+\left(2N-3\right)\left(\sigma_x^2\right)^2\right]\ ,\ \ k=0 \text{ or } k=N/2$$

$$\mathrm{Var}\left(\mathrm{G}_X\left[k\right]\right)=\frac{T_s^2}{N}\left[\mathrm{E}\left(x^4\right)+\left(N-3\right)\left(\sigma_x^2\right)^2\right]\ ,\ \ k\neq 0 \text{ and } k\neq N/2$$

Power Spectral Density

It is important to point out that the variance of the PSD estimates **does not approach zero as N approaches infinity**.

$$\lim_{N\to\infty} \mathrm{Var}\left(\mathrm{G}_X\left[k\right]\right) = 2\left(\sigma_x^2 T_s\right)^2 = 2\,\mathrm{E}^2\left(\mathrm{G}\left[k\right]\right) \quad , \quad k = 0 \text{ or } k = N/2$$

$$\lim_{N\to\infty} \mathrm{Var}\left(\mathrm{G}_X\left[k\right]\right) = \left(\sigma_x^2 T_s\right)^2 = \mathrm{E}^2\left(\mathrm{G}\left[k\right]\right) \quad , \quad k \neq 0 \text{ and } k \neq N/2$$

This fact makes this kind of estimate of the PSD **inconsistent**.

Power Spectral Density

It can be shown (and is in the text) that the distributions of the PSD estimates are chi-squared distributed with either one or two degrees of freedom

$$\mathrm{G}\left[k\right]/\,\sigma_x^2 T_s \text{ is } \chi_1^2,\text{ for } k = 0 \text{ or } k = N/2$$

$$2\mathrm{G}\left[k\right]/\,\sigma_x^2 T_s \text{ is } \chi_2^2\;,\; k \neq 0 \text{ and } k \neq N/2$$

where σ_x^2 is the variance of the stochastic process sampled and T_s is the time between samples.

Power Spectral Density

The problem of inconsistent PSD estimates can be solved by analyzing the data in a different way. If N samples are available, divide the total of N samples into K data blocks of M points each. Then estimate the PSD for each block separately and average the PSD estimates from all the blocks. Since each block is now shorter than the full N points, the spectral resolution will not be as great as if all N points were used but the variance of the PSD estimates will be reduced by averaging the results from multiple blocks.

Power Spectral Density

Now the final PSD estimates are given by

$$\mathrm{G}\left[k\right] = \frac{\mathrm{G}_1\left[k\right] + \mathrm{G}_2\left[k\right] + \cdots + \mathrm{G}_K\left[k\right]}{K} = \frac{1}{K}\sum_{i=1}^{K}\mathrm{G}_i\left[k\right]$$

and the variance of the PSD estimates is

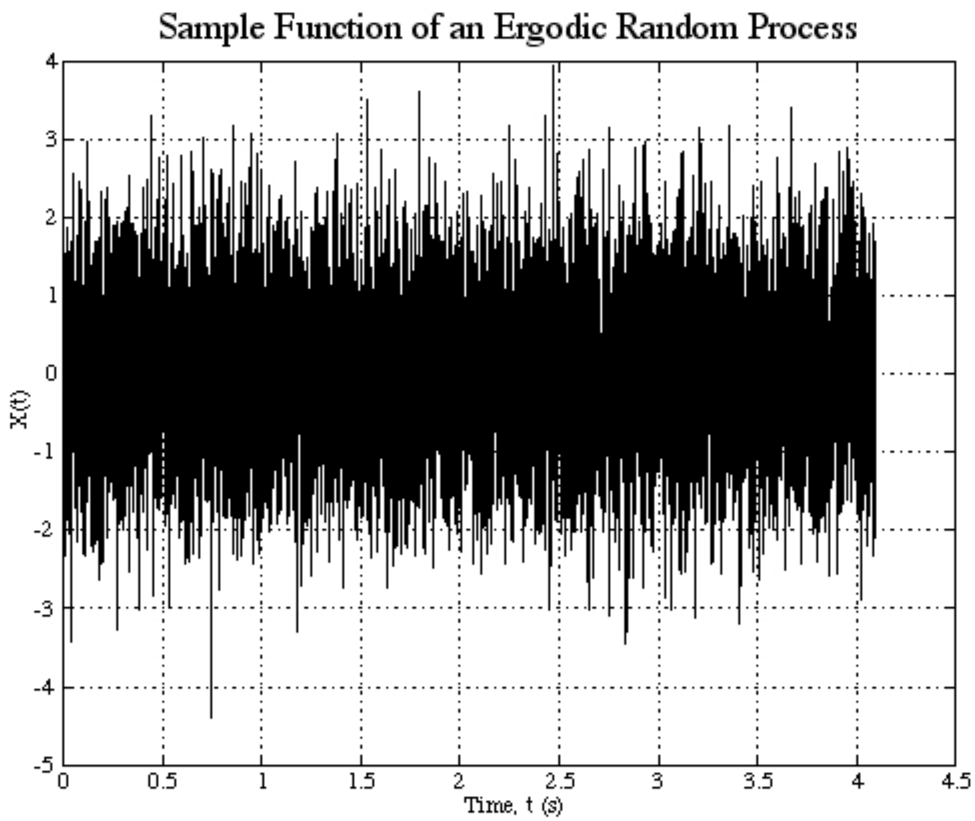
$$\sigma^2_{\mathrm{G}\left[k\right]} = \sum_{i=1}^{K}\left(\frac{1}{K}\right)^2 \sigma^2_{\mathrm{G}_i\left[k\right]} = \sigma^2_{\mathrm{G}_i\left[k\right]} / K$$

$$\sigma^2_{\mathrm{G}\left[0\right]} = \sigma^2_{\mathrm{G}\left[N/2\right]} = 2\left(\sigma_x^2 T_s\right)^2 / K$$

$$\sigma^2_{\mathrm{G}\left[k\right]} = \left(\sigma_x^2 T_s\right)^2 / K \quad , \quad k \neq 0 \ \text{ and } \ k \neq N/2$$

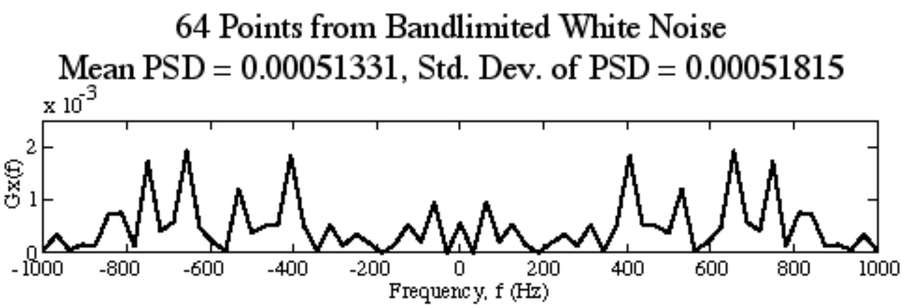
PSD Estimation Example

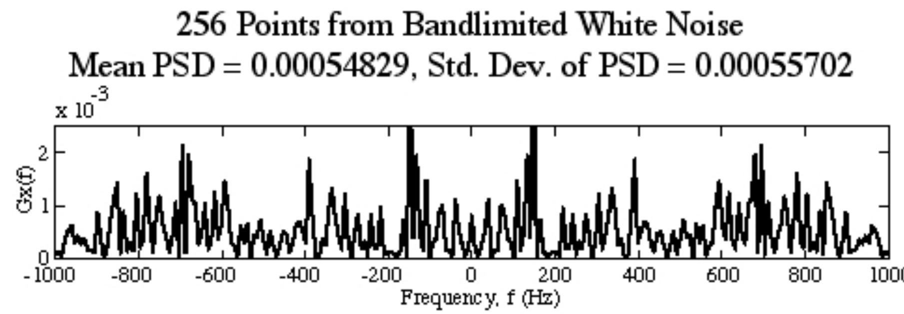
Acquire a block of 8192 points from a bandlimited white noise process.

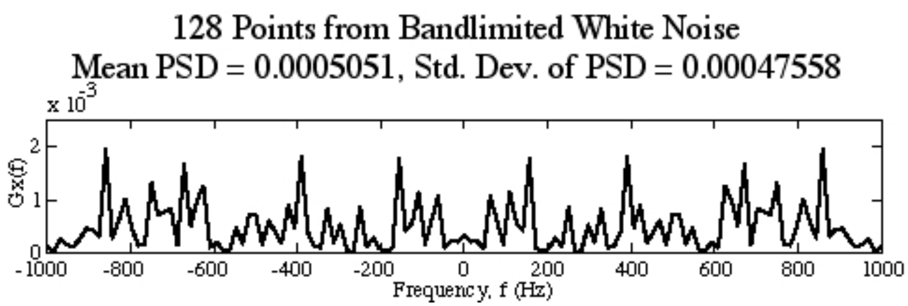


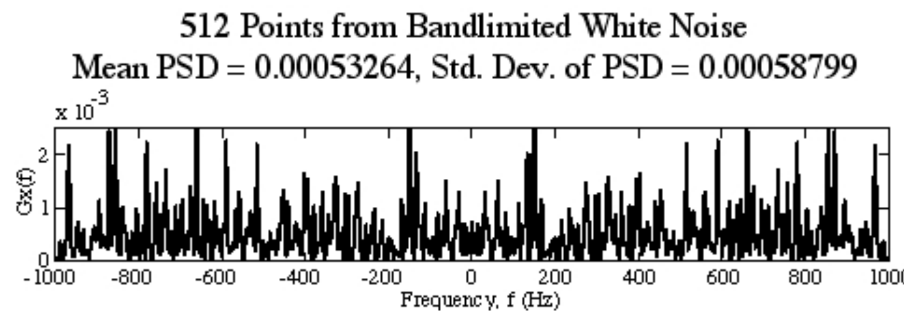
PSD Estimation Example

Estimate the PSD from a single block of 64 points, 128 points, 256 points and 512 points.





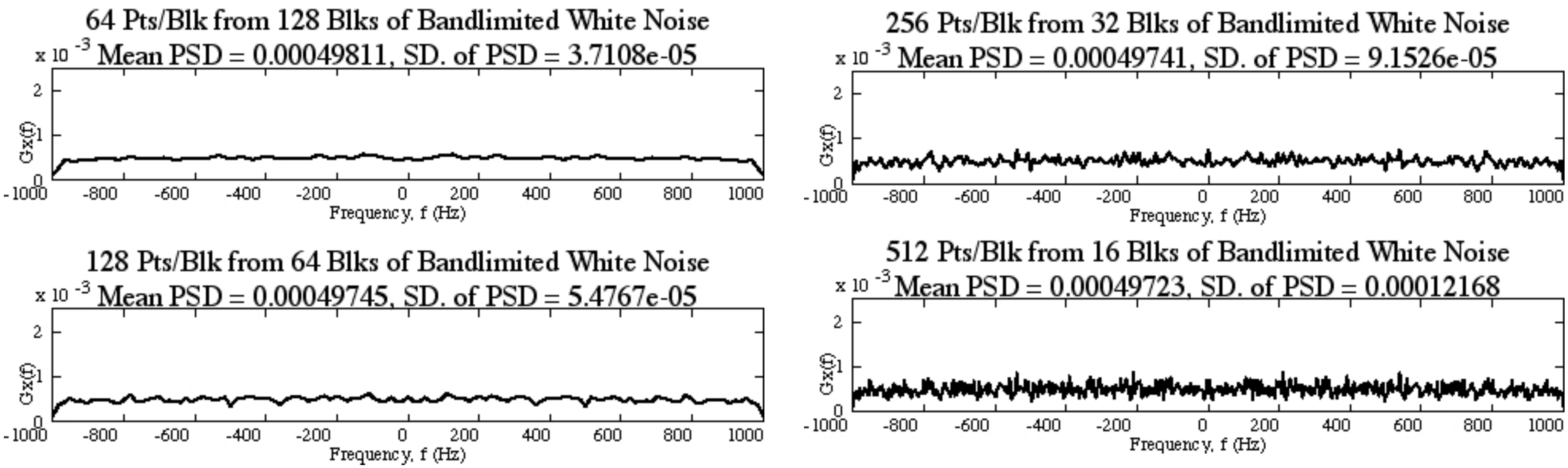




Resolution is improved with a larger block of data but the standard deviation of the PSD is not improved.

PSD Estimation Example

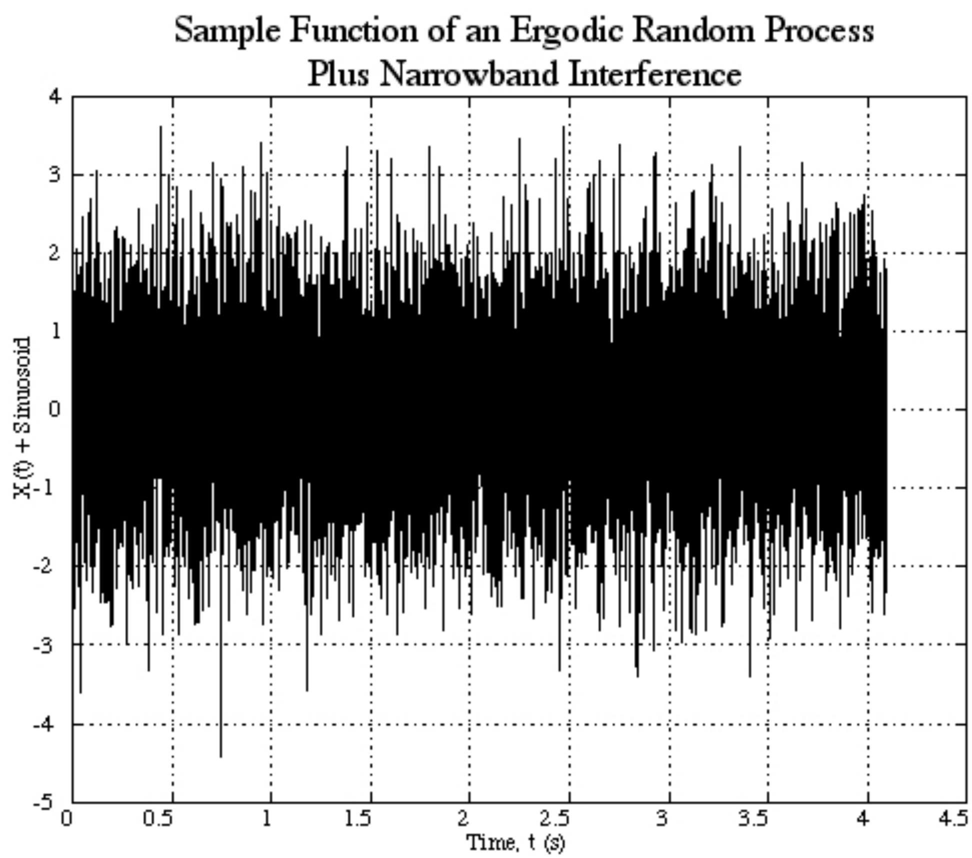
Now use the same block sizes but use all of the data and average block PSD estimates.



The resolution effect of block size is still apparent but the effect of averaging multiple blocks is also apparent. The standard deviation of the PSD estimates is smaller when more blocks are averaged.

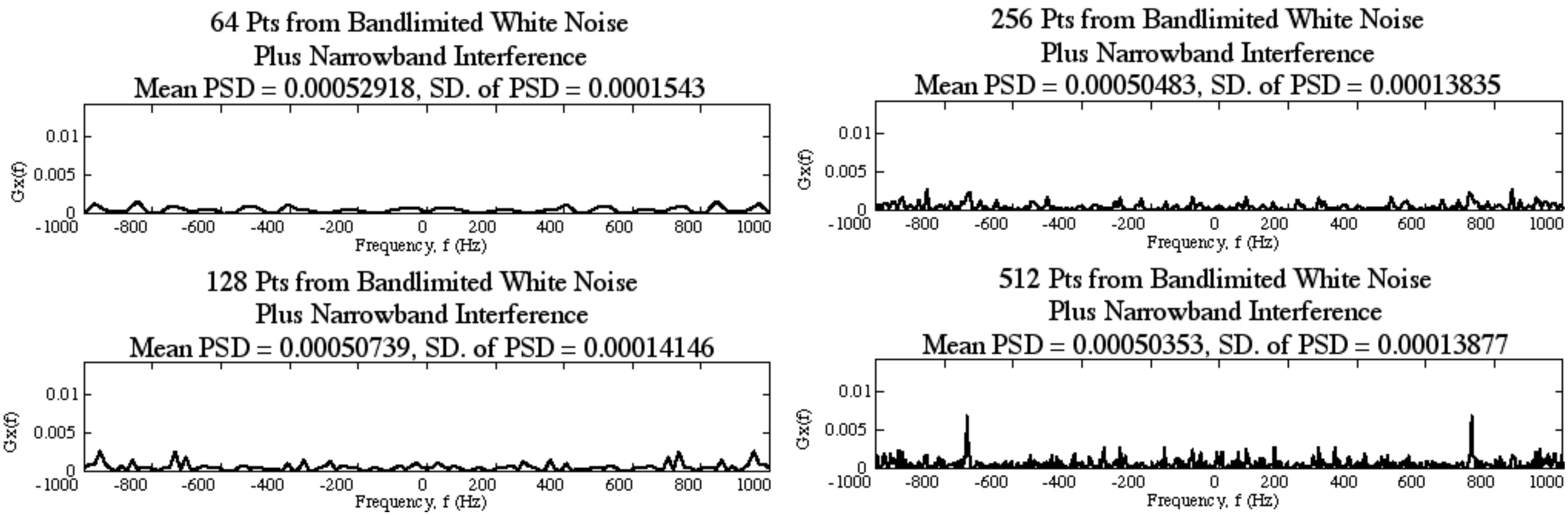
PSD Estimation Example

Now acquire a block of 8192 points from a bandlimited white noise process plus narrowband interference.



PSD Estimation Example

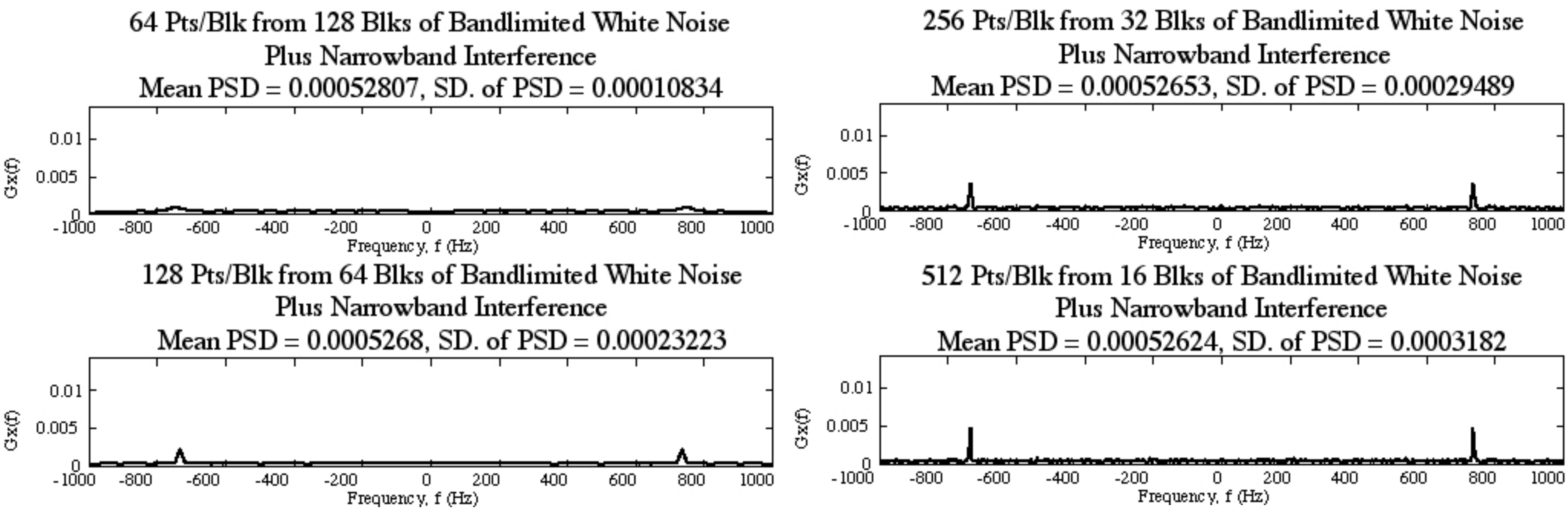
Estimate the PSD from a single block of 64 points, 128 points, 256 points and 512 points.



The presence of narrowband interference obvious from a block of 512 points but not at all obvious from a block of 64 points.

PSD Estimation Example

Now use the same block sizes but use all of the data and average block PSD estimates.



Again the resolution effect of block size is apparent and the effect of averaging multiple blocks is also apparent.

Cross Power Spectral Density

The analysis of CPSD is similar to the analysis of PSD but a little more complicated. The CPSD estimates for two stochastic processes X and Y are

$$\mathrm{G}_{XY}\left[k\right]=\frac{\left(\mathrm{X}\left[k\right]/N\right)\left(\mathrm{Y}^{*}\left[k\right]/N\right)}{\Delta f}$$

If X and Y are independent the expected value of these CPSD estimates is zero.

Cross Power Spectral Density

Let $\mathrm{x}(t) = K_X\,\mathrm{z}(t) + \mathrm{x}_N(t)$ and $\mathrm{y}(t) = K_Y\,\mathrm{z}(t) + \mathrm{y}_N(t)$

and let z, x_N and y_N all be bandlimited white noise processes sampled at the Nyqist rate. Then

$$\mathrm{E}\left(\mathrm{G}_{XY}\left[k\right]\right) = T_s K_X K_Y \sigma_{\mathrm{z}}^2 = K_X K_Y \sigma_{\mathrm{z}}^2 / f_s$$

Cross Power Spectral Density

The variance of the CPSD estimates is

$$\mathrm{Var}\left(\mathrm{G}_{XY}\left[k\right]\right)=\frac{T_s^2}{N}\left\{\begin{array}{l}\left(K_x K_y\right)^2\left[\mathrm{E}\left(\mathrm{z}^4\right)+\left(2N-3\right)\left(\sigma_{\mathrm{z}}^2\right)^2\right]\\+N\sigma_{\mathrm{z}}^2\left(K_X^2\sigma_{Y_N}^2+K_Y^2\sigma_{X_N}^2\right)+N\sigma_{X_N}^2\sigma_{Y_N}^2\end{array}\right\}\;,\;k=0\text{ or }k=N/2$$

$$\mathrm{Var}\left(\mathrm{G}_{XY}\left[k\right]\right)=\frac{T_s^2}{N}\left\{\begin{array}{l}\left(K_x K_y\right)^2\left[\mathrm{E}\left(\mathrm{z}^4\right)+\left(N-3\right)\left(\sigma_{\mathrm{z}}^2\right)^2\right]\\+N\sigma_{\mathrm{z}}^2\left(K_X^2\sigma_{Y_N}^2+K_Y^2\sigma_{X_N}^2\right)+N\sigma_{X_N}^2\sigma_{Y_N}^2\end{array}\right\}\;,\;k\neq 0\;,\;k\neq N/2$$

Just like the PSD estimates, these estimates are inconsistent but the variance can be reduced by using a block averaging technique.