Solution of ECE 504 Final Examination S08

1. Two independent ergodic stochastic processes X and Y with expected values of 2 and -5 respectively, have autocorrelations

$$\mathrm{R}_X(\tau) = 12e^{-100|\tau|} + 4 \text{ and } \mathrm{R}_Y(\tau) = 8\,\mathrm{sinc}(60\tau) + 25 \ .$$

X and Y are added to a sinusoidal signal $\mathrm{W}(t) = 5\cos(120\pi t)$ to form $\mathrm{V}(t)$.

(a) Find the autocorrelation function of $\mathrm{V}(t)$, $\mathrm{R}_V(\tau)$.

$$\mathrm{R}_V(\tau) = \mathrm{E}(\mathrm{V}(t)\mathrm{V}(t+\tau)) = \mathrm{E}((\mathrm{X}(t)+\mathrm{Y}(t)+\mathrm{W}(t))(\mathrm{X}(t+\tau)+\mathrm{Y}(t+\tau)+\mathrm{W}(t+\tau)))$$

$$\mathrm{R}_V(\tau) = \mathrm{E}\begin{pmatrix} \mathrm{X}(t)\mathrm{X}(t+\tau) + \mathrm{X}(t)\mathrm{Y}(t+\tau) + \mathrm{X}(t)\mathrm{W}(t+\tau) \\ +\mathrm{Y}(t)\mathrm{X}(t+\tau) + \mathrm{Y}(t)\mathrm{Y}(t+\tau) + \mathrm{W}(t)\mathrm{W}(t+\tau) \\ +\mathrm{W}(t)\mathrm{X}(t+\tau) + \mathrm{W}(t)\mathrm{Y}(t+\tau) + \mathrm{W}(t)\mathrm{W}(t+\tau) \end{pmatrix}$$

$$\mathrm{R}_V(\tau) = \begin{bmatrix} \mathrm{E}(\mathrm{X}(t)\mathrm{X}(t+\tau)) + \mathrm{E}(\mathrm{X}(t)\mathrm{Y}(t+\tau)) + \mathrm{E}(\mathrm{X}(t)\mathrm{W}(t+\tau)) \\ +\mathrm{E}(\mathrm{Y}(t)\mathrm{X}(t+\tau)) + \mathrm{E}(\mathrm{Y}(t)\mathrm{Y}(t+\tau)) + \mathrm{E}(\mathrm{Y}(t)\mathrm{W}(t+\tau)) \\ +\mathrm{E}(\mathrm{W}(t)\mathrm{X}(t+\tau)) + \mathrm{E}(\mathrm{W}(t)\mathrm{Y}(t+\tau)) + \mathrm{E}(\mathrm{W}(t)\mathrm{W}(t+\tau)) \end{bmatrix}$$

$$\mathrm{R}_V(\tau) = \begin{bmatrix} \mathrm{R_X}(\tau) + \mathrm{E}(\mathrm{X}(t))\mathrm{E}(\mathrm{Y}(t+\tau)) + \mathrm{E}(\mathrm{X}(t))\mathrm{E}(\mathrm{W}(t+\tau)) \\ +\mathrm{E}(\mathrm{Y}(t))\mathrm{E}(\mathrm{X}(t+\tau)) + \mathrm{R_Y}(\tau) + \mathrm{E}(\mathrm{Y}(t))\mathrm{E}(\mathrm{W}(t+\tau)) \\ +\mathrm{E}(\mathrm{W}(t))\mathrm{E}(\mathrm{X}(t+\tau)) + \mathrm{E}(\mathrm{W}(t))\mathrm{E}(\mathrm{Y}(t+\tau)) + \mathscr{R}_W(\tau) \end{bmatrix}$$

$$\mathrm{R}_V(\tau) = \begin{bmatrix} \mathrm{R_X}(\tau) - 10 + 0 \\ -10 + \mathrm{R_Y}(\tau) + 0 \\ +0 + 0 + \mathscr{R}_W(\tau) \end{bmatrix} = \mathrm{R_X}(\tau) + \mathrm{R_Y}(\tau) + \mathscr{R}_W(\tau) - 20$$

$$\mathscr{R}_W(\tau) = 25\,\mathrm{E}(\cos(120\pi t)\cos(120\pi(t+\tau)))$$

$$\mathscr{R}_W(\tau) = \frac{25}{2}\mathrm{E}(\cos(120\pi\tau) + \cos(240\pi t + 120\pi\tau))$$

$$\mathscr{R}_W(\tau) = \frac{25}{2}\cos(120\pi\tau)$$

$$\mathrm{R}_V(\tau) = 12e^{-100|\tau|} + 4 + 8\,\mathrm{sinc}(60\tau) + 25 + \frac{25}{2}\cos(120\pi\tau) - 20$$

$$\mathrm{R}_V(\tau) = 12e^{-100|\tau|} + 8\,\mathrm{sinc}(60\tau) + \frac{25}{2}\cos(120\pi\tau) + 9$$

(b) Find the numerical mean-squared value of $\mathrm{V}(t)$.

$$\mathrm{E}\left(\mathrm{V}^2(t)\right) = \mathrm{R}_V(0) = 12 + 8 + 12.5 + 9 = 41.5$$

(c) What function does the autocovariance of *V* approach as $\tau \to \infty$?

$$\mathrm{C}_V(\tau) = 12e^{-100|\tau|} + 8\,\mathrm{sinc}(60\tau) + \frac{25}{2}\cos(120\pi\tau)$$

$$\lim_{\tau \to \infty} \mathrm{C}_V(\tau) = \frac{25}{2}\cos(120\pi\tau)$$

(d) If you wanted to make a good estimate of the signal power of *W* (knowing it is a sinusoid) but with access only to *V*, how could you do it using the autocorrelation function of *V*?

Look at the autocovariance for large values of τ where it will be sinusoidal. That sinusoid must be related to the sinusoid *W* and, from it, one can determine the signal power of *W*.

2. Two independent stochastic processes X and Y are both bandlimited white noise with PSD's

$$\mathrm{G}_X(f) = \mathrm{G}_Y(f) = 10\,\mathrm{rect}(f/50) = \begin{cases} 10 & , \ |f| < 25 \\ 0 & , \ \text{otherwise} \end{cases} .$$

If $\mathrm{W}(t) = 2\,\mathrm{X}(t) - 5\,\mathrm{Y}(t-0.2)$ and W is sampled at 30 Hz, what is the numerical correlation between successive samples, $\mathrm{R}_W(1/30)$?

$$\mathrm{R}_X(\tau) = \mathrm{R}_Y(\tau) = 500\,\mathrm{sinc}(50\tau)$$

$$\mathrm{R}_W(\tau) = \mathrm{E}\big(\mathrm{W}(t)\mathrm{W}(t+\tau)\big) = \mathrm{E}\big(\big[2\,\mathrm{X}(t) - 5\,\mathrm{Y}(t-0.2)\big]\big[2\,\mathrm{X}(t+\tau) - 5\,\mathrm{Y}(t+\tau-0.2)\big]\big)$$

$$\mathrm{R}_W(\tau) = 4\,\mathrm{R}_X(\tau) + 25\,\mathrm{R}_Y(\tau) - 10\underbrace{\mathrm{R}_{YX}(\tau+0.2)}_{=0} - 10\underbrace{\mathrm{R}_{XY}(\tau-0.2)}_{=0}$$

$$\mathrm{R}_W(\tau) = 2000\,\mathrm{sinc}(50\tau) + 12500\,\mathrm{sinc}(50\tau) = 14500\,\mathrm{sinc}(50\tau)$$

$$\mathrm{R}_W(1/30) = 14500\,\mathrm{sinc}(50/30) = -2398.3$$

3. A lowpass filter with frequency response $\mathrm{H}(f) = \dfrac{-j5}{f - j15}$ is excited by bandlimited white noise $\mathrm{X}(t)$ with PSD

$$\mathrm{G}_X(f) = 5\,\mathrm{rect}(f/30) = \begin{cases} 5 & , \ |f| < 15 \\ 0 & , \ \text{otherwise} \end{cases}.$$

Find the numerical mean squared value of the filter response $\mathrm{Y}(t)$.

$$\mathrm{E}(Y^2) = \int_{-\infty}^{\infty} \mathrm{G}_\mathrm{X}(f)\left|\mathrm{H}(f)\right|^2 df$$

$$\left|\mathrm{H}(f)\right|^2 = \frac{25}{f^2 + 225}$$

$$\mathrm{E}(Y^2) = \int_{-\infty}^{\infty} 5\,\mathrm{rect}(f/30)\frac{25}{f^2+225}\,df = 125\int_{-15}^{15}\frac{df}{f^2+225}$$

Using

$$\int \frac{dx}{a^2 + (bx)^2} = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) \quad \text{from the formula sheet}$$

$$\mathrm{E}(Y^2) = 125\int_{-15}^{15}\frac{df}{f^2+225} = \left[\frac{125}{15}\tan^{-1}\left(\frac{f}{15}\right)\right]_{-15}^{15} = \frac{125}{15}\big(0.7854 - (-0.7854)\big) = 13.09$$

4. A deterministic stochastic process has sample functions

$$\mathrm{X}(t) = \cos(\omega t + \theta)$$

where θ is a random variable uniformly distributed between $-a$ and a. For what numerical values of a is the expected value of X a constant (independent of time)?

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} \cos(\omega t + \theta) \mathrm{f}_\Theta(\theta) d\theta = \frac{1}{2a} \int_{-a}^{a} \cos(\omega t + \theta) d\theta$$

$$\mathrm{E}(X) = \frac{1}{2a} \Big[\sin(\omega t + \theta) \Big]_{-a}^{a} = \frac{\sin(\omega t + a) - \sin(\omega t - a)}{2a}$$

To be independent of time the difference between the two sine functions must be a constant. The only way that can be true is if the difference between a and $-a$ is a non-zero integer multiple of 2π.

$$2a = 2n\pi \ , \ n \text{ an integer} \ , \ n \neq 0$$

or

$$a = n\pi \ , \ n \text{ an integer} \ , \ n \neq 0 \ .$$