Solution of ECE 504 Test 1 S08

1. In a city suburban area, there are 20,000 residences. 17,500 of them have cable TV, 18,000 have wired telephones (the rest have only cellular telephones). 16,000 have both cable TV and a wired telephone.

   (a) If you choose a residence at random in this area, what is the numerical probability that you choose a residence which has no cable TV and no wired telephone? Probability = ____________________________

   Let *A* be the event "residence has cable TV".
   Let *B* be the event "residence has a wired telephone".

$$\mathrm{P}\left[\overline{A+B}\right] = 1 - \mathrm{P}\left[A+B\right]$$

$$\mathrm{P}\left[A+B\right] = \mathrm{P}\left[A\right] + \mathrm{P}\left[B\right] - \mathrm{P}\left[AB\right] = 0.875 + 0.90 - 0.8 = 0.975$$

$$\mathrm{P}\left[\overline{A+B}\right] = 1 - 0.975 = 0.025$$

   (b) If you choose a residence at random in this area, what is the numerical probability that you choose a residence which has either

   1. cable TV and no wired telephone

   or 2. no cable TV and a wired telephone?

   Probability = ____________________________

$$\mathrm{P}\left[A\bar{B} + B\bar{A}\right] = \mathrm{P}\left[A\bar{B}\right] + \mathrm{P}\left[B\bar{A}\right] - \mathrm{P}\left[\underbrace{A\bar{B} \cap B\bar{A}}_{=\varnothing}\right]$$

$$\mathrm{P}\left[A\bar{B}\right] = \mathrm{P}\left[A\right] - \mathrm{P}\left[AB\right] = 0.875 - 0.8 = 0.075$$

$$\mathrm{P}\left[B\bar{A}\right] = \mathrm{P}\left[B\right] - \mathrm{P}\left[AB\right] = 0.9 - 0.8 = 0.1$$

$$\mathrm{P}\left[A\bar{B} + B\bar{A}\right] = 0.075 + 0.1 - \underbrace{\mathrm{P}\left[\varnothing\right]}_{=0} = 0.175$$

2. In the city of Basel, Switzerland, 90% of the people speak fluent French, 70% speak fluent Swiss-German and 10% speak fluent Italian. An experiment is conducted in which random people on the street are stopped and tested for their language skills.

(a) What is the numerical probability that in the first 8 people tested none of them speaks fluent French?

Probability = ____________________________

$$\text{P}\left[\text{no fluent French speakers in first 8}\right] = 0.1^8 = 10^{-8}$$

(b) What is the numerical probability that in the first 5 people tested exactly 3 of them speak fluent Italian?

Probability = ____________________________

$$\text{P}\left[\text{3 of 5 speak fluent Italian}\right] = \binom{5}{3}(0.1)^3(0.9)^2 = 0.0081$$

(c) What is the numerical probability that in the first 5 people tested at least 2 of them speak fluent Swiss-German?

Probability = ____________________________

$$\text{P}\left[\geq 2 \text{ speak fluent Swiss-German}\right] = \left\{\begin{matrix} 1 - \text{P}\left[0 \text{ speak fluent Swiss-German}\right] \\ - \text{P}\left[1 \text{ speaks fluent Swiss-German}\right] \end{matrix}\right\}$$

$$\text{P}\left[\geq 2 \text{ speak fluent Swiss-German}\right] = \left[1 - 0.3^5 - \binom{5}{1}(0.7)^1(0.3)^4\right]$$

$$\text{P}\left[\geq 2 \text{ speak fluent Swiss-German}\right] = 1 - 0.00243 - 0.02835 = 0.9692$$

3. A discrete-value random variable X has a PMF

$$\mathrm{P}(x) = \begin{cases} Kx(6-x) & , \ x = 0,1,2,3,4,5,6 \\ 0 & , \ \text{otherwise} \end{cases}.$$

(a) What is the numerical value of K? $K =$ ________________

$$\sum_{x=0}^{6} Kx(6-x) = 1 \Rightarrow K(0+5+8+9+8+5+0) = 1 \Rightarrow K = 1/35$$

(b) What is the numerical expected value of X?
$\mathrm{E}(X) =$ ______________

$$\mathrm{E}(X) = \sum_{x=0}^{6} x\left[(1/35)x(6-x)\right] = (1/35)(0+5+16+27+32+25+0) = 3$$

(c) What is the numerical variance of X? $\sigma_X^2 =$ ___________________

$$\mathrm{E}\left(X^2\right) = \sum_{x=0}^{6} x^2\left[(1/35)x(6-x)\right] = (1/35)(0+5+32+81+128+125+0) = 10.6$$

$$\sigma_X^2 = \mathrm{E}\left(X^2\right) - \mathrm{E}^2(X) = 10.6 - 3^2 = 1.6$$

(d) What is the numerical probability that $X = 3$, given the condition $X < 4$?
$\mathrm{P}[X = 3 \mid X < 4] =$ _______________________

$$\mathrm{P}[X = 3 \mid X < 4] = \frac{\mathrm{P}[X = 3 \cap X < 4]}{\mathrm{P}[X < 4]}$$

$$\mathrm{P}[X = 3 \mid X < 4] = \frac{\mathrm{P}[X = 3]}{\mathrm{P}[X = 0] + \mathrm{P}[X = 1] + \mathrm{P}[X = 2] + \mathrm{P}[X = 3]}$$

$$\mathrm{P}[X = 3 \mid X < 4] = \frac{9/35}{(0+5+8+9)/35} = \frac{9}{22}$$

4. A random variable X has a PDF

$$\mathrm{f}_X(x) = 0.3\delta(x-2) + (0.7/3)\mathrm{rect}((x-1)/3).$$

Another random variable Y is related to X through $Y = 3X - 2$.

(a) Find the numerical expected value and variance of X.

$\mathrm{E}(X) =$ ________________ , $\sigma_X^2 =$ ________________

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x\,\mathrm{f}_X(x)dx = \int_{-\infty}^{\infty} x\left[0.3\delta(x-2) + (0.7/3)\mathrm{rect}((x-1)/3)\right]dx$$

$$\mathrm{E}(X) = 0.3\int_{-\infty}^{\infty} x\delta(x-2)dx + (0.7/3)\int_{-0.5}^{2.5} x dx = 0.3\times 2 + (0.7/3)\left[x^2/2\right]_{-0.5}^{2.5}$$

$$\mathrm{E}(X) = 0.6 + (0.7/3)\frac{6.25 - 0.25}{2} = 0.6 + 0.7 = 1.3$$

$$\mathrm{E}(X^2) = \int_{-\infty}^{\infty} x^2\,\mathrm{f}_X(x)dx = \int_{-\infty}^{\infty} x^2\left[0.3\delta(x-2) + (0.7/3)\mathrm{rect}((x-1)/3)\right]dx$$

$$\mathrm{E}(X^2) = 0.3\int_{-\infty}^{\infty} x^2\delta(x-2)dx + (0.7/3)\int_{-0.5}^{2.5} x^2 dx = 0.3\times 4 + (0.7/3)\left[x^3/3\right]_{-0.5}^{2.5}$$

$$\mathrm{E}(X^2) = 1.2 + (0.7/3)\frac{15.625 + 0.125}{3} = 1.2 + 1.225 = 2.425$$

$$\sigma_X^2 = \mathrm{E}(X^2) - \mathrm{E}^2(X) = 2.425 - 1.3^2 = 0.735$$

(b) Find the PDF of Y. (You may want to use the scaling property of the impulse, $\delta\left(a\left(x-x_0\right)\right)=\frac{1}{|a|}\delta\left(x-x_0\right)$).

$f_Y(y)=$ ____

The function $Y=3X-2$ is invertible. Therefore $X=\left(Y+2\right)/3$, $dY/dX=3$ and

$$f_Y(y)=\frac{f_X(x)}{\left|\frac{dY}{dX}\right|}=0.1\delta\left(\left(y+2\right)/3-2\right)+\left(0.7/9\right)\text{rect}\left(\left(\left(y+2\right)/3-1\right)/3\right)$$

$$f_Y(y)=0.1\delta\left(\left(y-4\right)/3\right)+\left(0.7/9\right)\text{rect}\left(\left(y-1\right)/9\right)$$

$$f_Y(y)=0.3\delta\left(y-4\right)+\left(0.7/9\right)\text{rect}\left(\left(y-1\right)/9\right)$$

(c) Find the numerical expected value and variance of Y.

$E(Y)=$ __________________ , $\sigma_Y^2=$ __________________

$$E(Y)=\int_{-\infty}^{\infty}y\,f_Y(y)\,dy=\int_{-\infty}^{\infty}y\left[0.3\delta\left(y-4\right)+\left(0.7/9\right)\text{rect}\left(\left(y-1\right)/9\right)\right]dy$$

$$E(Y)=0.3\int_{-\infty}^{\infty}y\delta\left(y-4\right)dy+\left(0.7/9\right)\int_{-3.5}^{5.5}y\,dy$$

$$E(Y)=1.2+\left(0.7/9\right)\left[y^2/2\right]_{-3.5}^{5.5}=1.2+\left(0.7/9\right)\left[y^2/2\right]_{-3.5}^{5.5}=1.2+0.7=1.9$$

$$E\left(Y^2\right)=\int_{-\infty}^{\infty}y^2 f_Y(y)\,dy=\int_{-\infty}^{\infty}y^2\left[0.3\delta\left(y-4\right)+\left(0.7/9\right)\text{rect}\left(\left(y-1\right)/9\right)\right]dy$$

$$E\left(Y^2\right)=0.3\int_{-\infty}^{\infty}y^2\delta\left(y-4\right)dy+\left(0.7/9\right)\int_{-3.5}^{5.5}y^2\,dy$$

$$E\left(Y^2\right)=4.8+\left(0.7/9\right)\left[y^3/3\right]_{-3.5}^{5.5}=4.8+5.425=10.225$$

$$\sigma_Y^2=E\left(Y^2\right)-E^2\left(Y\right)=10.225-1.9^2=6.616$$