Solution of ECE 504 Test 2 S08

1. Two random variables X and Y have a joint pdf $f_{XY}(x,y) = K\begin{cases} xy & , \quad 0 < x < y < 1 \\ 0 & , \quad \text{otherwise} \end{cases}$ and $Z = XY$.

   (a) Find the numerical value of K.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dxdy = K\int_0^1\int_0^y xydxdy = K\int_0^1 ydy\int_0^y xdx = K\int_0^1 \left(y^3/2\right)dy = K(1/8) = 1$$

$$\therefore K = 8$$

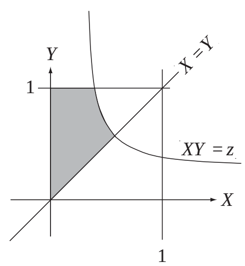
   (b) Z lies in a range $A \le Z \le B$. What are the numerical values of A and B?

   Z is never negative. Therefore $f_Z(z) = 0 \ , \ z < 0$.

   Z cannot be greater than 1. Therefore $f_Z(z) = 0 \ , \ z > 1$

   $A = 0 \ , \ B = 1$

   (c) On the graph below, shade the region in the XY plane over which $0 < x < y < 1$ and $Z \le z$.



   (d) If $F_Z(z) = 1 - K\int_c^d\int_a^b xydxdy = 1 - K\int_c^d ydy\int_a^b xdx$ what are a, b, c and d?

   a = ________ , b = ________ , c = ________ , d = ________

   The point of interesection of $X = Y$ and $XY = z$ is the point $X = Y = \sqrt{z}$.

   $a = z/y \ , \ b = y \ , \ c = \sqrt{z} \ , \ d = 1$

2. The weights of adult men and women in the United States can be assumed to be Gaussian distributed with these means and standard deviations (in kg)

	Men	Women
Mean	88	75
Standard Deviation	11	9

(a) If a man is chosen at random what is the numerical probability that he weighs more than 100 kg?

$$P\left[\text{man} > 100\text{ kg}\right] = 1 - G\left(\frac{100-88}{11}\right) = 0.1377$$

(b) If a man and woman are chosen at random what is the numerical probability that both of them weigh less than 70 kg?

$$P\left[\text{man and woman both} < 70\text{ kg}\right] = G\left(\frac{70-88}{11}\right) \times G\left(\frac{70-75}{9}\right) = 0.0147$$

(c) If we choose a woman at random from the population of all women who weigh less than 60 kg, what is the numerical probability that she weighs between 50 and 60 kg?

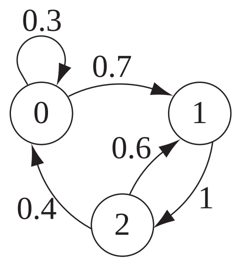
$$P\left[50 < \text{woman} < 60 \mid \text{woman} < 60\right] = \frac{P\left[50 < \text{woman} < 60 \cap \text{woman} < 60\right]}{P\left[\text{woman} < 60\right]}$$

$$P\left[50 < \text{woman} < 60 \mid \text{woman} < 60\right] = \frac{P\left[50 < \text{woman} < 60\right]}{P\left[\text{woman} < 60\right]} = \frac{G\left(\frac{60-75}{9}\right) - G\left(\frac{50-75}{9}\right)}{G\left(\frac{60-75}{9}\right)} = 0.9427$$

(d) If we randomly choose 20 men and average their weights, what is the numerical probability that the average is greater than 90 kg?

$$P\left[\text{avg} > 90\text{ kg}\right] = 1 - G\left(\frac{90-88}{11/\sqrt{20}}\right) = 0.2081$$

3. A Markov process has the chain illustrated below. Assume that it has been in operation for a long time and then we look at its state at some random time. What is the numerical probability of finding it in state 1?



$$\pi_0 = 0.3\pi_0 + 0.4\pi_2$$
$$\pi_1 = 0.7\pi_0 + 0.6\pi_2$$
$$\pi_2 = \pi_1$$
$$\pi_0 + \pi_1 + \pi_2 = 1$$

We can use any two of the first three equations along with the last equation to solve for the limiting state probabilities.

$$\begin{bmatrix} -0.7 & 1 & -0.6 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \pi_0 \\ \pi_1 \\ \pi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\Delta = -0.7(-2) + 0.4 = 1.8$$

$$\pi_1 = \frac{\begin{vmatrix} -0.7 & 0 & -0.6 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}}{1.8} = \frac{0.7}{1.8} = 0.389$$

Alternate Solution:

$$\boldsymbol{\pi}^T = \boldsymbol{\pi}^T \mathbf{P} \;\;,\;\; \boldsymbol{\pi}^T \mathbf{1} = 1$$

$$\mathbf{P} = \begin{bmatrix} 0.3 & 0.7 & 0 \\ 0 & 0 & 1 \\ 0.4 & 0.6 & 0 \end{bmatrix}$$

$$\begin{bmatrix}\pi_0 & \pi_1 & \pi_2\end{bmatrix} = \begin{bmatrix}\pi_0 & \pi_1 & \pi_2\end{bmatrix}\begin{bmatrix}0.3 & 0.7 & 0 \\ 0 & 0 & 1 \\ 0.4 & 0.6 & 0\end{bmatrix}$$

$$\begin{aligned}\pi_0 &= 0.3\pi_0 + 0.4\pi_2 \\ \pi_1 &= 0.7\pi_0 + 0.6\pi_2 \quad , \quad \pi_0 + \pi_1 + \pi_2 = 1 \\ \pi_2 &= \pi_1\end{aligned}$$

These are exactly the same as the equations in the first solution form and the solution is therefore also the same from this point on.