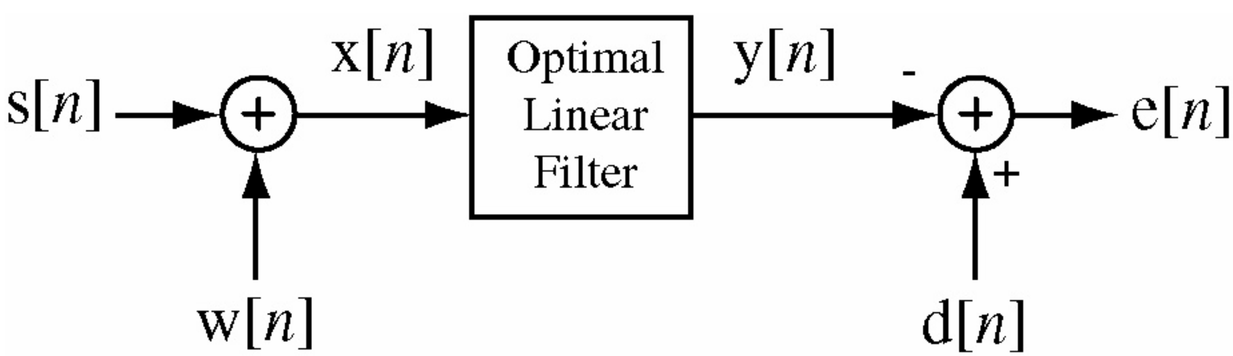
Adaptive Filters

The LMS Algorithm

In adaptive filtering the object is to constantly revise the impulse response of a filter to adapt it to changing signal conditions. The optimal filter to estimate a signal $\mathrm{d}[n] = \mathrm{s}[n - n_0]$ is a Wiener filter whose impulse response is determined by the Wiener-Hopf equations

$$\sum_{m=0}^{M-1} \mathrm{h}[m]\mathrm{R}_{xx}[l-m] = \mathrm{R}_{dx}[-l] \ , \ l = 0,1,\cdots,M-1$$



The LMS Algorithm

The autocorrelation of x is measured and can be constantly updated. Any convenient method can be used to solve the Wiener-Hopf equations. In adaptive filtering a simple direct method is the LMS (least-mean-square) algorithm. If the autocorrelation matrix $\mathbf{R}_M$ and the cross correlation vector $\mathbf{r}_d$ are known, the filter coefficients can be found by iteration. The coefficients at the $\left[q+1\right]$th iteration can be found from the coefficients at the qth iteration by

$$\mathbf{h}_M\left[q+1\right]=\mathbf{h}_M\left[q\right]+\left(1/2\right)\Delta\left[q\right]\mathbf{S}\left[q\right]$$

where $\Delta\left[q\right]$ is the step size at the qth iteration and $\mathbf{S}\left[q\right]$ is the direction vector for the qth iteration.

The LMS Algorithm

There are multiple ways of determining a direction vector. The simplest one is to use the method of steepest descent in which $\mathbf{S}[q] = -\mathbf{g}[q]$ where

$$\mathbf{g}[q] = \frac{d\varepsilon_M[q]}{d\mathbf{h}_M[q]} = 2\left(\mathbf{R}_M \mathbf{h}_M[q] - \mathbf{r}_d\right)$$

and $\varepsilon_M[q]$ is the mean-squared error at the qth iteration. In adaptive filtering applications of this theory we don't know the autocorrelation matrix or the cross correlation vector but we can still estimate the gradient as

$$\hat{\mathbf{g}}[q] = -2\,\mathrm{e}[q]\mathbf{X}^*_M[q]$$

where $\mathrm{e}[q]$ is the error signal at the qth iteration and $\mathbf{X}^*_M[q]$ is the vector of the last M values of the signal x at the qth iteration.

The LMS Algorithm

Then the iteration algorithm becomes

$$\mathbf{h}_M\left[q+1\right]=\mathbf{h}_M\left[q\right]+\Delta\left[q\right]\mathrm{e}\left[q\right]\mathbf{X}_M^*\left[q\right]$$

Usually the step size is fixed and the iteration simplifies to

$$\mathbf{h}_M\left[q+1\right]=\mathbf{h}_M\left[q\right]+\Delta\,\mathrm{e}\left[q\right]\mathbf{X}_M^*\left[q\right]$$

The adaptation algorithm can be modeled as a feedback system with the step size as a parameter that determines the dynamic response of the system. It has been shown that the step size should be less than $2/M\,\mathrm{R}_{xx}\left[0\right]$. However in practice the autocorrelation matrix is not known. various studies have shown that in the real case in which only estimates of the autocorrelation matrix and cross correlation vector are known the upper bound on the step size should instead be $1/M\,\hat{\mathrm{R}}_{xx}\left[0\right]$.