Example of Decimation and Interpolation

Sample the signal

$$\mathrm{x}(t) = 5\sin(2000\pi t)\cos(20{,}000\pi t)$$

at 80 kHz to form a discrete-time signal $\mathrm{x}[n]$, take every fourth sample of $\mathrm{x}[n]$ to form $\mathrm{x}_s[n]$ and decimate $\mathrm{x}_s[n]$ to form $\mathrm{x}_d[n]$. Then upsample $\mathrm{x}_d[n]$ by a factor of four to form $\mathrm{x}_i[n]$ and compare it to $\mathrm{x}[n]$.

$$\mathrm{X}(f) = (j5/2)\left[\delta(f+1000) - \delta(f-1000)\right] * (1/2)\left[\delta(f-10000) + \delta(f+10000)\right]$$

$$\mathrm{X}(f) = (j5/4)\left[\delta(f-9000) + \delta(f+11000) - \delta(f-11000) - \delta(f+9000)\right]$$

or in the ω form

$$\mathrm{X}(j\omega) = (j5\pi/2)\left[\delta(\omega - 18000\pi) + \delta(\omega + 22000\pi) - \delta(\omega - 22000\pi) - \delta(\omega + 18000\pi)\right].$$

The signal power of $\mathrm{x}(t)$ is the average of its squared magnitude.

$$P_{\mathrm{x}} = \frac{1}{0.001}\int_{-0.0005}^{0.0005} \left|5\sin(2000\pi t)\cos(20{,}000\pi t)\right|^2 dt$$
$$= 2000 \times 25 \int_0^{0.0005} \sin^2(2000\pi t)\cos^2(20{,}000\pi t)\,dt$$

$$P_{\mathrm{x}} = 50000\int_0^{0.0005}\left[\frac{1}{2} - \frac{1}{2}\cos(4000\pi t)\right]\left[\frac{1}{2} + \frac{1}{2}\cos(40000\pi t)\right]dt$$

$$P_{\mathrm{x}} = 50000\left[\begin{array}{l} \displaystyle\int_0^{0.0005}\frac{1}{4}dt - \underbrace{\int_0^{0.0005}\frac{1}{4}\cos(4000\pi t)\,dt}_{=0} \\ \displaystyle + \underbrace{\int_0^{0.0005}\frac{1}{4}\cos(40000\pi t)\,dt}_{=0} - \underbrace{\int_0^{0.0005}\frac{1}{4}\cos(4000\pi t)\cos(4000\pi t)\,dt}_{=0} \end{array}\right]$$

$$P_{\mathrm{x}} = 50000 \times 0.0005/4 = 6.25$$

The signal power is also the integral of its power spectral density. The power spectral density is

$$\mathrm{G_x}(f) = (25/16)\left[\delta(f-9000)+\delta(f+11000)-\delta(f-11000)-\delta(f+9000)\right] .$$

This integral over all frequency is just the sum of the impulse strengths which is 25/4 or 6.25.

The sampled signal is

$$\mathrm{x}[n] = 5\sin(2000\pi n T_s)\cos(20{,}000\pi n T_s) = 5\sin(\pi n/40)\cos(\pi n/4)$$

and its Fourier transform is

$$\mathrm{X}(F) = j\frac{5}{2}\left[\delta_1\left(F+\frac{1}{80}\right)-\delta_1\left(F-\frac{1}{80}\right)\right] \circledast \frac{1}{2}\left[\delta_1\left(F-\frac{1}{8}\right)+\delta_1\left(F+\frac{1}{8}\right)\right]$$

where $\delta_T(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT)$ and the $\circledast$ (periodic convolution) operation is defined by

$$\mathrm{x}(t) \circledast \mathrm{y}(t) = \int_{\tau=t_0}^{t_0+T}\mathrm{x}(\tau)\mathrm{y}(t-\tau)\,d\tau = \int_T \mathrm{x}(\tau)\mathrm{y}(t-\tau)\,d\tau$$

where t_0 is arbitrary and T is any period that is common to both x and y. Also

$$\mathrm{x}(t) \circledast \mathrm{y}(t) = \mathrm{x}_{ap}(t) * \mathrm{y}(t) = \mathrm{x}(t) * \mathrm{y}_{ap}(t)$$

where $\mathrm{x}_{ap}(t) = \begin{cases}\mathrm{x}(t) & , \ t_0 \le t < t_0+T \\ 0 & , \ \text{otherwise}\end{cases}$ and $\mathrm{y}_{ap}(t) = \begin{cases}\mathrm{y}(t) & , \ t_0 \le t < t_0+T \\ 0 & , \ \text{otherwise}\end{cases}$

where t_0 is arbitrary. Therefore

$$\mathrm{X}(F) = j\frac{5}{4}\left[\delta_1\left(F+\frac{1}{80}\right)-\delta_1\left(F-\frac{1}{80}\right)\right] * \left[\delta\left(F-\frac{1}{8}\right)+\delta\left(F+\frac{1}{8}\right)\right]$$

$$\mathrm{X}(F) = j\frac{5}{4}\left[\delta_1\left(F-\frac{9}{80}\right)+\delta_1\left(F+\frac{11}{80}\right)-\delta_1\left(F-\frac{11}{80}\right)-\delta_1\left(F+\frac{9}{80}\right)\right]$$

or, in the Ω form

$$\mathrm{X}(e^{j\Omega}) = j\frac{5\pi}{2}\left[\delta_{2\pi}\left(\Omega-\frac{18\pi}{80}\right)+\delta_{2\pi}\left(\Omega+\frac{22\pi}{80}\right)-\delta_{2\pi}\left(\Omega-\frac{22\pi}{80}\right)-\delta_{2\pi}\left(\Omega+\frac{18\pi}{80}\right)\right] .$$

The signal power is

$$P_x = (1/80)\sum_{n=0}^{79}\left|5\sin(\pi n/40)\cos(\pi n/4)\right|^2 = (25/80)\sum_{n=0}^{79}\sin^2(\pi n/40)\cos^2(\pi n/4)$$

$$P_x = (25/80)\sum_{n=0}^{79}\left[\frac{1}{2}-\frac{1}{2}\cos\left(\frac{\pi n}{20}\right)\right]\left[\frac{1}{2}-\frac{1}{2}\cos\left(\frac{\pi n}{2}\right)\right]$$

$$= (25/80)\times(1/4)\left[\sum_{n=0}^{79}1+\underbrace{\sum_{n=0}^{79}\cos\left(\frac{\pi n}{20}\right)}_{=0}+\underbrace{\sum_{n=0}^{79}\cos\left(\frac{\pi n}{2}\right)}_{=0}+\underbrace{\sum_{n=0}^{79}\cos\left(\frac{\pi n}{20}\right)\cos\left(\frac{\pi n}{20}\right)}_{=0}\right]$$

$$P_x = (25/80)\times(1/4)\times 80 = 25/4$$

The signal power is also the summation of the power spectral density over one period.

$$\mathrm{G}_x(F) = \frac{25}{16}\left[\delta_1\left(F-\frac{9}{80}\right)+\delta_1\left(F+\frac{11}{80}\right)-\delta_1\left(F-\frac{11}{80}\right)-\delta_1\left(F+\frac{9}{80}\right)\right].$$

The summation over one period (1) is 25/4.

Proakis would write these in his notation as

$$\mathrm{X}(f) = j\frac{5}{4}\left[\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{9}{80}k\right)+\sum_{k=-\infty}^{\infty}\delta\left(f+\frac{11}{80}k\right)+\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{11}{80}k\right)+\sum_{k=-\infty}^{\infty}\delta\left(f+\frac{9}{80}k\right)\right]$$

or

$$\mathrm{X}(f) = j\frac{5\pi}{2}\left[\begin{array}{l}\sum_{k=-\infty}^{\infty}\delta\left(\Omega-\frac{18\pi}{80}k\right)+\sum_{k=-\infty}^{\infty}\delta\left(\Omega+\frac{22\pi}{80}k\right)\\+\sum_{k=-\infty}^{\infty}\delta\left(\Omega-\frac{22\pi}{80}k\right)+\sum_{k=-\infty}^{\infty}\delta\left(\Omega+\frac{18\pi}{80}k\right)\end{array}\right]$$

$$\mathrm{x}_s[n] = \mathrm{x}[n]\delta_4[n] = 5\sin(\pi n/40)\cos(\pi n/4)\sum_{m=-\infty}^{\infty}\delta[n-mN]$$

$$\mathrm{X}_s(F) = j\frac{5}{4}\left[\delta_1\left(F-\frac{9}{80}\right)+\delta_1\left(F+\frac{11}{80}\right)-\delta_1\left(F-\frac{11}{80}\right)-\delta_1\left(F+\frac{9}{80}\right)\right]\circledast\frac{1}{4}\delta_{1/4}(F)$$

$$\mathrm{X}_s(F) = j\frac{5}{16}\begin{bmatrix} \delta_1\left(F-\frac{9}{80}\right)+\delta_1\left(F+\frac{11}{80}\right) \\ -\delta_1\left(F-\frac{11}{80}\right)-\delta_1\left(F+\frac{9}{80}\right) \end{bmatrix} * \begin{bmatrix} \delta(F)+\delta\left(F-\frac{1}{4}\right) \\ +\delta\left(F-\frac{1}{2}\right)+\delta\left(F-\frac{3}{4}\right) \end{bmatrix}$$

where the period T common to both signals is 1.

$$\mathrm{X}_s(F) = j\frac{5}{16}\begin{bmatrix} \delta_1\left(F-\frac{9}{80}\right)+\delta_1\left(F-\frac{1}{4}-\frac{9}{80}\right)+\delta_1\left(F-\frac{1}{2}-\frac{9}{80}\right)+\delta_1\left(F-\frac{3}{4}-\frac{9}{80}\right) \\ +\delta_1\left(F+\frac{11}{80}\right)+\delta_1\left(F-\frac{1}{4}+\frac{11}{80}\right)+\delta_1\left(F-\frac{1}{2}+\frac{11}{80}\right)+\delta_1\left(F-\frac{3}{4}+\frac{11}{80}\right) \\ -\delta_1\left(F-\frac{11}{80}\right)-\delta_1\left(F-\frac{1}{4}-\frac{11}{80}\right)-\delta_1\left(F-\frac{1}{2}-\frac{11}{80}\right)-\delta_1\left(F-\frac{3}{4}-\frac{11}{80}\right) \\ -\delta_1\left(F+\frac{9}{80}\right)-\delta_1\left(F-\frac{1}{4}+\frac{9}{80}\right)-\delta_1\left(F-\frac{1}{2}+\frac{9}{80}\right)-\delta_1\left(F-\frac{3}{4}+\frac{9}{80}\right) \end{bmatrix}$$

$$\mathrm{X}_s(F) = j\frac{5}{16}\begin{bmatrix} \delta_1\left(F-\frac{9}{80}\right)+\delta_1\left(F-\frac{29}{80}\right)+\delta_1\left(F-\frac{49}{80}\right)+\delta_1\left(F-\frac{69}{80}\right) \\ +\delta_1\left(F+\frac{11}{80}\right)+\delta_1\left(F-\frac{9}{80}\right)+\delta_1\left(F-\frac{29}{80}\right)+\delta_1\left(F-\frac{49}{80}\right) \\ -\delta_1\left(F-\frac{11}{80}\right)-\delta_1\left(F-\frac{31}{80}\right)-\delta_1\left(F-\frac{51}{80}\right)-\delta_1\left(F-\frac{71}{80}\right) \\ -\delta_1\left(F+\frac{9}{80}\right)-\delta_1\left(F-\frac{11}{80}\right)-\delta_1\left(F-\frac{31}{80}\right)-\delta_1\left(F-\frac{51}{80}\right) \end{bmatrix}$$

Combining equivalent periodic impulses (using the periodicity of the periodic impulse),

$$\mathrm{X}_s(F) = j\frac{5}{8}\begin{bmatrix} \delta_1\left(F-\frac{9}{80}\right)+\delta_1\left(F-\frac{29}{80}\right)+\delta_1\left(F-\frac{49}{80}\right)+\delta_1\left(F-\frac{69}{80}\right) \\ -\delta_1\left(F-\frac{11}{80}\right)-\delta_1\left(F-\frac{31}{80}\right)-\delta_1\left(F-\frac{51}{80}\right)-\delta_1\left(F-\frac{71}{80}\right) \end{bmatrix}$$

The signal power is $P_{\mathrm{x}_s} = (25/64)\times 8 = 25/8$.

$$X_s\left(e^{j\Omega}\right) = j\frac{5\pi}{4}\begin{bmatrix}\delta_{2\pi}\left(\Omega - \frac{18\pi}{80}\right) + \delta_{2\pi}\left(\Omega - \frac{58\pi}{80}\right) + \delta_{2\pi}\left(\Omega - \frac{98\pi}{80}\right) + \delta_{2\pi}\left(\Omega - \frac{138\pi}{80}\right) \\ -\delta_{2\pi}\left(\Omega - \frac{22\pi}{80}\right) - \delta_{2\pi}\left(\Omega - \frac{62\pi}{80}\right) - \delta_{2\pi}\left(\Omega - \frac{102\pi}{80}\right) - \delta_{2\pi}\left(\Omega - \frac{142\pi}{80}\right)\end{bmatrix}$$

$$x_d[n] = x_s[4n]$$

$$X_d(F) = X_s(F/4) = j\frac{5}{8}\begin{bmatrix}\delta_1\left(\frac{F}{4} - \frac{9}{80}\right) + \delta_1\left(\frac{F}{4} - \frac{29}{80}\right) + \delta_1\left(\frac{F}{4} - \frac{49}{80}\right) + \delta_1\left(\frac{F}{4} - \frac{69}{80}\right) \\ -\delta_1\left(\frac{F}{4} - \frac{11}{80}\right) - \delta_1\left(\frac{F}{4} - \frac{31}{80}\right) - \delta_1\left(\frac{F}{4} - \frac{51}{80}\right) - \delta_1\left(\frac{F}{4} - \frac{71}{80}\right)\end{bmatrix}$$

$$X_d(F) = j\frac{5}{2}\begin{bmatrix}\delta_4\left(F - \frac{36}{80}\right) + \delta_4\left(F - \frac{116}{80}\right) + \delta_4\left(F - \frac{196}{80}\right) + \delta_4\left(F - \frac{276}{80}\right) \\ -\delta_4\left(F - \frac{44}{80}\right) - \delta_4\left(F - \frac{124}{80}\right) - \delta_4\left(F - \frac{204}{80}\right) - \delta_4\left(F - \frac{284}{80}\right)\end{bmatrix}$$

$$X_d(F) = j\frac{5}{2}\left[\delta_1\left(F - \frac{36}{80}\right) - \delta_1\left(F - \frac{44}{80}\right)\right]$$

The signal power is $P_{x_d} = (25/4) \times 2 = 25/2$.
or

$$X_d\left(e^{j\Omega}\right) = j5\pi\left[\delta_{2\pi}\left(\Omega - \frac{72\pi}{80}\right) - \delta_{2\pi}\left(F - \frac{88\pi}{80}\right)\right]$$

To recover the original $x[n]$ from $x_d[n]$, replace the zeros to form $x_s[n]$ and then lowpass filter $x_s[n]$.

$$X_s(F) = j\frac{5}{8}\begin{bmatrix}\delta_1\left(F - \frac{9}{80}\right) + \delta_1\left(F - \frac{29}{80}\right) + \delta_1\left(F - \frac{49}{80}\right) + \delta_1\left(F - \frac{69}{80}\right) \\ -\delta_1\left(F - \frac{11}{80}\right) - \delta_1\left(F - \frac{31}{80}\right) - \delta_1\left(F - \frac{51}{80}\right) - \delta_1\left(F - \frac{71}{80}\right)\end{bmatrix}$$

If we lowpass filter this with lowpass filter with a transfer function

$$\mathrm{H}(F) = \begin{cases} 1 \ , \ 0 < |F| < F_c \\ 0 \ , \ \text{otherwise} \end{cases}$$

where $11/80 < F_c < 69/80$, we get

$$\mathrm{X}_i(F) = \mathrm{X}_s(F)\mathrm{H}(F) = j\frac{5}{8}\left[\delta_1\left(F - \frac{9}{80}\right) + \delta_1\left(F - \frac{69}{80}\right) - \delta_1\left(F - \frac{11}{80}\right) - \delta_1\left(F - \frac{71}{80}\right)\right]$$

or

$$\mathrm{X}_i(F) = j\frac{5}{8}\left[\delta_1\left(F - \frac{9}{80}\right) - \delta_1\left(F + \frac{9}{80}\right) + \delta_1\left(F + \frac{11}{80}\right) - \delta_1\left(F - \frac{11}{80}\right)\right]$$

The inverse Fourier transform of this is

$$\mathrm{x}_i[n] = (5/2)\left[\sin(11\pi n/40) - \sin(9\pi n/40)\right] .$$

The original discrete-time signal is $\mathrm{x}[n] = 5\sin(\pi n/40)\cos(\pi n/4)$. Using a trigonometric identityt this can be written as

$$\mathrm{x}[n] = 5\left[\sin(11\pi n/40) - \sin(9\pi n/40)\right].$$

The two signals are the same except for a factor of 2. The signal power of $\mathrm{x}_i[n]$ is 25/16 which is 1/4 of the signal power of $\mathrm{x}[n]$. This loss of signal power can be compensated for by using a lowpass filter with a gain of 2 instead of a gain of one.