Design of Digital Filters

Paley-Wiener Theorem

If $\mathrm{h}[n]$ is a causal energy signal, then $\int_{-\infty}^{\infty} \left| \ln \left| \mathrm{H}\left(e^{j\Omega}\right) \right| \right| d\Omega < B$ where B is a finite upper bound.

One implication of the Paley-Wiener theorem is that a transfer function can be zero at isolated points but it cannot be zero over a frequency range of finite width because, if it were, the integral would be infinite ($\ln\left(\varepsilon\right) \to -\infty$ as $\varepsilon \to 0$).

Implications of Causality

If a system's impulse response $\mathrm{h}[n]$ is causal, then it is possible to determine $\mathrm{h}[n]$ from its even part $\mathrm{h}_e[n]$ where

$$\mathrm{h}_e[n] = \frac{\mathrm{h}[n] + \mathrm{h}[-n]}{2} \text{ and } \mathrm{h}_o[n] = \frac{\mathrm{h}[n] - \mathrm{h}[-n]}{2},$$

$\mathrm{h}_o[n]$ is the odd part and $\mathrm{h}[\mathrm{n}] = \mathrm{h}_e[n] + \mathrm{h}_o[n]$.

If $\mathrm{h}[n]$ is causal then

$$\mathrm{h}[n] = \mathrm{h}[n]\mathrm{u}[n] = \left(\mathrm{h}_e[n] + \mathrm{h}_o[n]\right)\mathrm{u}[n] = \mathrm{h}_e[n]\mathrm{u}[n] + \mathrm{h}_o[n]\mathrm{u}[n]$$

$$\mathrm{h}_e[n]\mathrm{u}[n] = \frac{\mathrm{h}[n]\mathrm{u}[n] + \mathrm{h}[-n]\mathrm{u}[n]}{2} = \frac{\mathrm{h}[n] + \mathrm{h}[0]\delta[n]}{2}$$

and

$$\mathrm{h}[n] = 2\,\mathrm{h}_e[n]\mathrm{u}[n] - \mathrm{h}[0]\delta[n]$$

Implications of Causality

Since $\mathrm{h}_o[0]=0$, $\mathrm{h}_e[0]=\mathrm{h}[0]$ and $\mathrm{h}[n]=2\,\mathrm{h}_e[n]\mathrm{u}[n]-\mathrm{h}_e[0]\delta[n]$.

Therefore a causal $\mathrm{h}[n]$ can be found from $\mathrm{h}_e[n]$ alone and

$$\mathrm{h}_o[n]=\frac{2\,\mathrm{h}_e[n]\mathrm{u}[n]-\mathrm{h}_e[0]\delta[n]-\left(2\,\mathrm{h}_e[-n]\mathrm{u}[-n]-\mathrm{h}_e[0]\delta[n]\right)}{2}$$

$$=\mathrm{h}_e[n]\mathrm{u}[n]-\mathrm{h}_e[-n]\mathrm{u}[-n]$$

proving that both $\mathrm{h}[n]$ and $\mathrm{h}_o[n]$ can be found from $\mathrm{h}_e[n]$.

Implications of Causality

The Fourier transform of any even function is purely real and the Fourier transform of any odd function is purely imaginary. Therefore the real part $\mathrm{H}_R\left(e^{j\Omega}\right)$ of $\mathrm{H}\left(e^{j\Omega}\right)$ for a causal system is completely determined by $\mathrm{h}_e[n]$, so is the imaginary part $\mathrm{H}_I\left(e^{j\Omega}\right)$ and therefore the real and imaginary parts are interrelated and cannot be specified separately. It also follows that the magnitude and phase cannot be specified separately either.

Implications of Causality

$$\mathrm{h}[n] \xleftrightarrow{\mathcal{F}} \mathrm{H}\left(e^{j\Omega}\right)$$

$$2\,\mathrm{h}_e[n]\,\mathrm{u}[n] - \mathrm{h}_e[0]\,\delta[n] \xleftrightarrow{\mathcal{F}} 2\times(1/2\pi)\,\mathrm{H}_R\left(e^{j\Omega}\right) \circledast \mathrm{U}\left(e^{j\Omega}\right) - \mathrm{h}_e[0]$$

$$\mathrm{H}\left(e^{j\Omega}\right) = -\mathrm{h}_e[0] + (1/\pi)\int_{-\pi}^{\pi} \mathrm{H}_R\left(e^{j\lambda}\right)\mathrm{U}\left(e^{j(\Omega-\lambda)}\right)d\lambda$$

where $\mathrm{u}[n] \xleftrightarrow{\mathcal{F}} \mathrm{U}\left(e^{j\Omega}\right) = \dfrac{1}{1-e^{-j\Omega}} + \pi\sum_{k=-\infty}^{\infty}\delta(\Omega - 2\pi k)$

Implications of Causality

$$\cot\left(\frac{x}{2}\right)=\frac{\sin(x)}{1-\cos(x)}\Rightarrow\cot\left(\frac{\Omega}{2}\right)=\frac{\sin(\Omega)}{1-\cos(\Omega)}$$

$$\cot\left(\frac{\Omega}{2}\right)=\frac{\dfrac{e^{j\Omega}-e^{-j\Omega}}{j2}}{1-\dfrac{e^{j\Omega}+e^{-j\Omega}}{2}}=-j\frac{e^{j\Omega}-e^{-j\Omega}}{2-e^{j\Omega}-e^{-j\Omega}}$$

$$\frac{1}{2}-\frac{j}{2}\cot\left(\frac{\Omega}{2}\right)=\frac{1}{2}-\frac{1}{2}\times\frac{e^{j\Omega}-e^{-j\Omega}}{2-e^{j\Omega}-e^{-j\Omega}}$$

$$=\frac{2-e^{j\Omega}-e^{-j\Omega}-e^{j\Omega}+e^{-j\Omega}}{2\left(2-e^{j\Omega}-e^{-j\Omega}\right)}=\frac{1-e^{j\Omega}}{2-e^{j\Omega}-e^{-j\Omega}}$$

$$\frac{1}{2}-\frac{j}{2}\cot\left(\frac{\Omega}{2}\right)=\frac{1-e^{j\Omega}}{\left(1-e^{j\Omega}\right)\left(1-e^{-j\Omega}\right)}=\frac{1}{1-e^{-j\Omega}}$$

Implications of Causality

$$\mathrm{H}\left(e^{j\Omega}\right) = -\mathrm{h}_e[0] + (1/\pi)\int_{-\pi}^{\pi}\left\{\mathrm{H}_R\left(e^{j\lambda}\right)\left[\frac{1}{2} - \frac{j}{2}\cot\left(\frac{\Omega-\lambda}{2}\right) + \pi\sum_{k=-\infty}^{\infty}\delta(\Omega-\lambda-2\pi k)\right]\right\}d\lambda$$

$$\mathrm{H}\left(e^{j\Omega}\right) = -\mathrm{h}_e[0] + (1/\pi)\int_{-\pi}^{\pi}\mathrm{H}_R\left(e^{j\lambda}\right)\left[\frac{1}{2} - \frac{j}{2}\cot\left(\frac{\Omega-\lambda}{2}\right) + \pi\delta(\Omega-\lambda)\right]d\lambda \quad , \quad -\pi < \Omega < \pi$$

$$\mathrm{H}\left(e^{j\Omega}\right) = -\mathrm{h}_e[0] + (1/2\pi)\int_{-\pi}^{\pi}\mathrm{H}_R\left(e^{j\lambda}\right)d\lambda - (j/2\pi)\int_{-\pi}^{\pi}\mathrm{H}_R\left(e^{j\lambda}\right)\cot\left(\frac{\Omega-\lambda}{2}\right)d\lambda + \mathrm{H}_R\left(e^{j\Omega}\right) \quad , \quad -\pi < \Omega < \pi$$

$$\mathrm{h}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{H}\left(e^{j\Omega}\right)d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\mathrm{H}_R\left(e^{j\Omega}\right) + j\,\mathrm{H}_I\left(e^{j\Omega}\right)\right]d\Omega = \frac{1}{2\pi}\left\{\int_{-\pi}^{\pi}\underbrace{\mathrm{H}_R\left(e^{j\Omega}\right)}_{\text{even}}d\Omega + \int_{-\pi}^{\pi}j\,\underbrace{\mathrm{H}_I\left(e^{j\Omega}\right)}_{\text{odd}}d\Omega\right\}$$

$$\therefore \mathrm{h}[0] = \mathrm{h}_e[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{H}_R\left(e^{j\Omega}\right)d\Omega$$

$$\mathrm{H}\left(e^{j\Omega}\right) = -(j/2\pi)\int_{-\pi}^{\pi}\mathrm{H}_R\left(e^{j\lambda}\right)\cot\left(\frac{\Omega-\lambda}{2}\right)d\lambda + \mathrm{H}_R\left(e^{j\Omega}\right) \quad , \quad -\pi < \Omega < \pi$$

Implications of Causality

From the previous slide,

$$\mathrm{H}\left(e^{j\Omega}\right) = -\left(j/2\pi\right)\int_{-\pi}^{\pi} \mathrm{H}_R\left(e^{j\lambda}\right)\cot\left(\frac{\Omega-\lambda}{2}\right)d\lambda + \mathrm{H}_R\left(e^{j\Omega}\right) \quad , \quad -\pi < \Omega < \pi$$

Subtract the real part from both sides.

$$j\,\mathrm{H}_I\left(e^{j\Omega}\right) = -\left(j/2\pi\right)\int_{-\pi}^{\pi} \mathrm{H}_R\left(e^{j\lambda}\right)\cot\left(\left(\Omega-\lambda\right)/2\right)d\lambda \quad , \quad -\pi < \Omega < \pi$$

$$\mathrm{H}_I\left(e^{j\Omega}\right) = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{H}_R\left(e^{j\lambda}\right)\cot\left(\frac{\Omega-\lambda}{2}\right)d\lambda \quad , \quad -\pi < \Omega < \pi$$

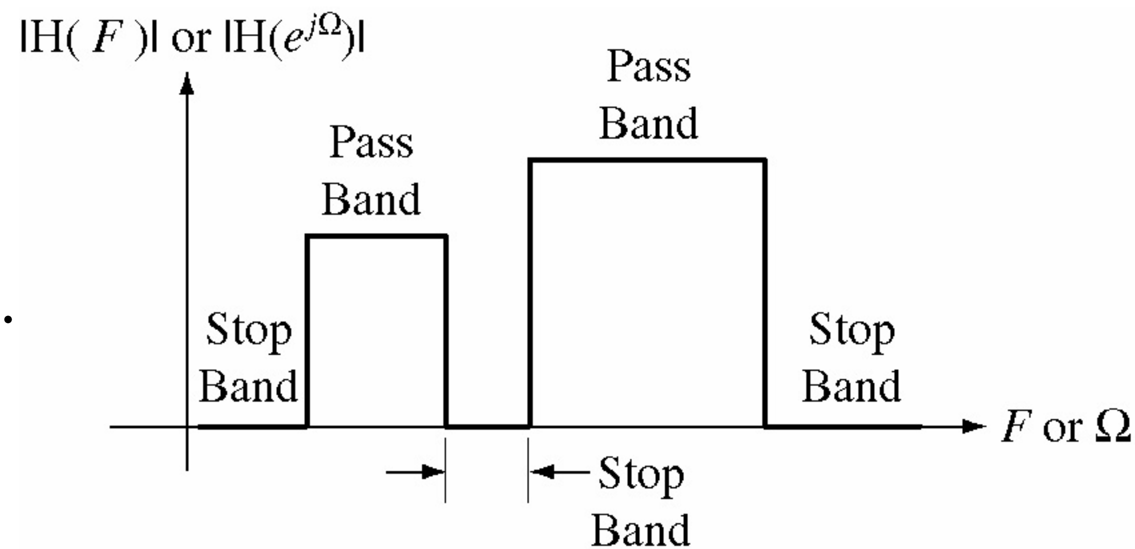
This integral is called the discrete Hilbert transform. It shows how the real part and imaginary part of the transfer function of a causal system are interrelated.

Filter Specifications

For discrete-time systems described by difference equations the transfer function is a ratio of polynomials in z.

$$\mathrm{H}(z)=\frac{\sum_{k=0}^{N} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

Filters of this type are causal and cannot be ideal. Ideal filters are specifed by their passbands and stopbands.

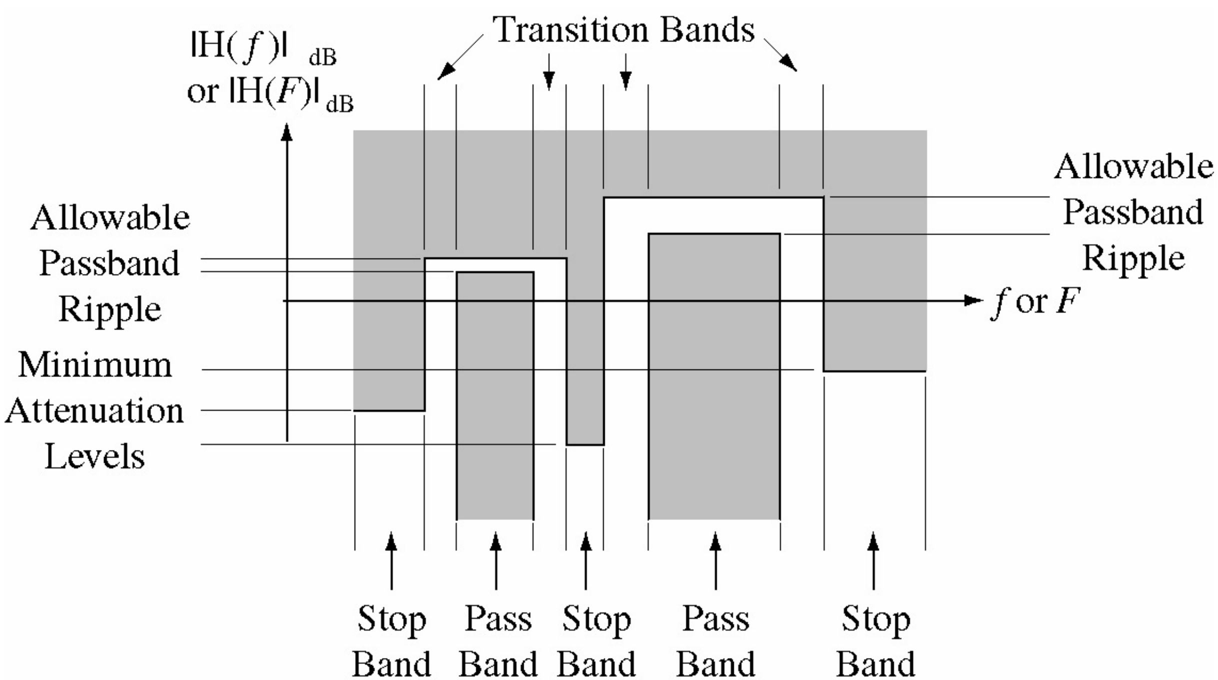


Filter Specifications

A typical practical-filter specification usually includes several elements,

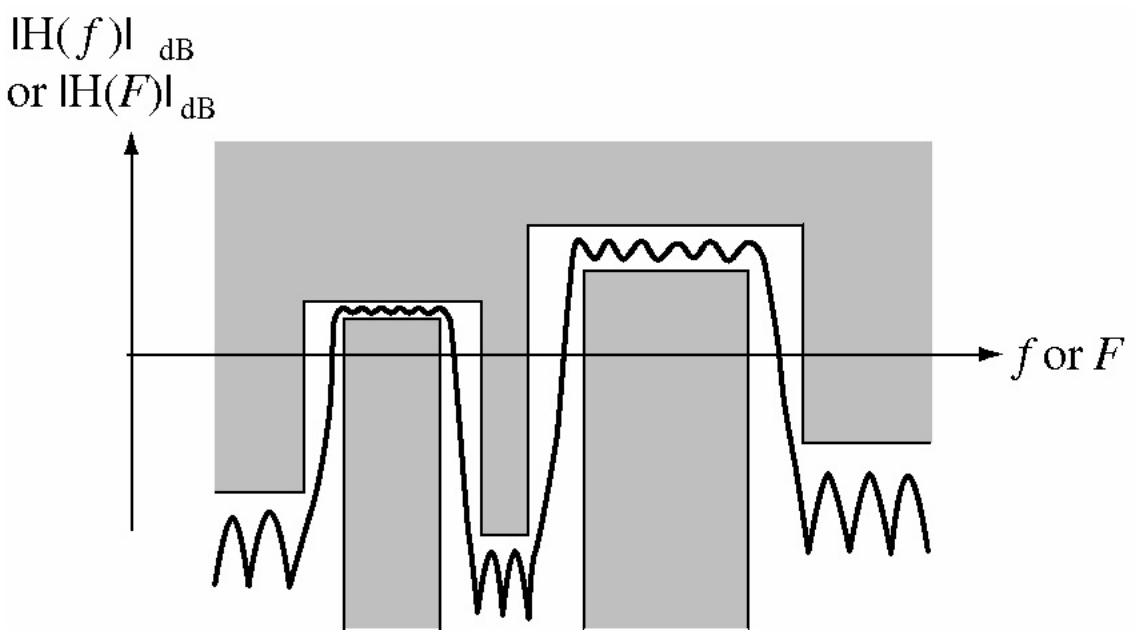
1. one or more pass bands,
2. one or more stop bands,
3. transition bands between pass and stop bands,
4. allowable ripple in the pass band(s),

and 5. minimum required attenuation in the stop band(s).



Filter Specifications

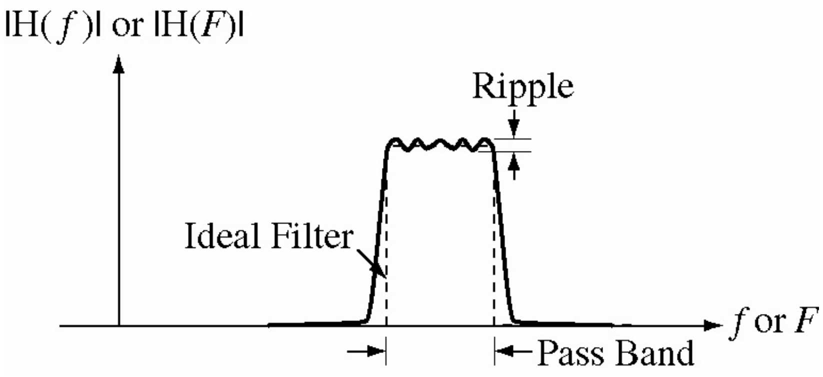
The actual filter must have a magnitude response that lies totally within the white area of the specification.

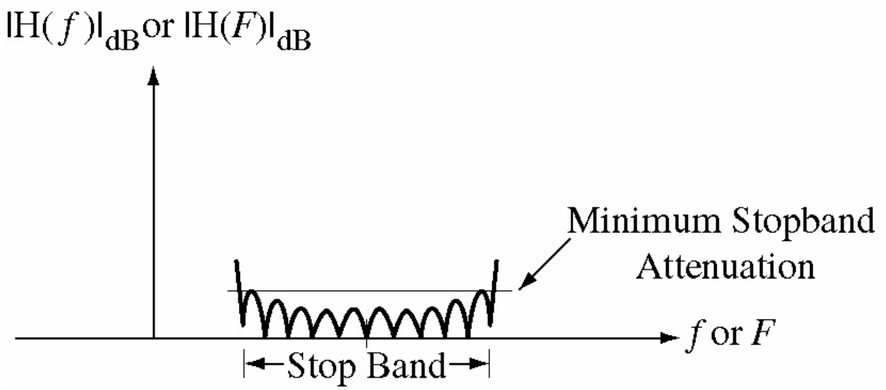


Filter Specifications

"Ripple" refers to the variation of a filter's transfer function magnitude in its passband.

"Stopband Attenuation" refers to the number of dB by which the filter's transfer function magnitude in the stopband is reduced compared with the passband.





FIR Filter Design

If an FIR filter has a symmetric impulse response

$$\mathrm{h}[n] = \mathrm{h}[M-1-n]$$

or an antisymmetric impulse response

$$\mathrm{h}[n] = -\mathrm{h}[M-1-n]$$

it has a linear phase.

FIR Filter Design

If a filter has an antisymmetric impulse response

$$\mathrm{H}\left(e^{j\Omega}\right)_{\Omega=0} = \sum_{n=0}^{M-1} \mathrm{h}[n] = \underbrace{\mathrm{h}\left[(M-1)/2\right]}_{=0} + \sum_{n=0}^{(M-3)/3} \mathrm{h}[n] + \sum_{n=(M+1)/2}^{M-1} \mathrm{h}[n]$$

Let $m = M-1-n$. Then

$$\mathrm{H}\left(e^{j\Omega}\right)_{\Omega=0} = \sum_{n=0}^{(M-3)/2} \mathrm{h}[n] + \sum_{m=(M-3)/2}^{0} \mathrm{h}[M-1-m]$$

$$\mathrm{H}\left(e^{j\Omega}\right)_{\Omega=0} = \sum_{n=0}^{(M-3)/2} \left\{\mathrm{h}[n] + \mathrm{h}[M-1-n]\right\} = \sum_{n=0}^{(M-3)/2} \left\{\mathrm{h}[n] - \mathrm{h}[n]\right\} = 0$$

Therefore, its frequency response at zero frequency is zero and it could not be used as a lowpass filter. It can also be shown that if M is odd, it cannot be a highpass filter either.

FIR Filter Design

A straightforward design process is to first find the impulse response of an ideal filter (which is non-causal), then truncate the impulse response so that all its values for times $n < 0$ and $n > M - 1$ are zero. Then modify the shape of the impulse response by multiplying it by a window chosen to reduce passband ripple and increase stopband attenuation. Consider a lowpass filter

$$\mathrm{H}\left(e^{j2\pi F}\right) = A\begin{Bmatrix} e^{-j2\pi F n_0} & , & 0 < |F| < F_c \\ 0 & , & F_c < |F| < 1/2 \end{Bmatrix}$$

or

$$\mathrm{H}\left(e^{j\Omega}\right) = A\begin{Bmatrix} e^{-j\Omega n_0} & , & 0 < |\Omega| < \Omega_c \\ 0 & , & \Omega_c < |\Omega| < \pi \end{Bmatrix}$$

FIR Filter Design

Its impulse response is

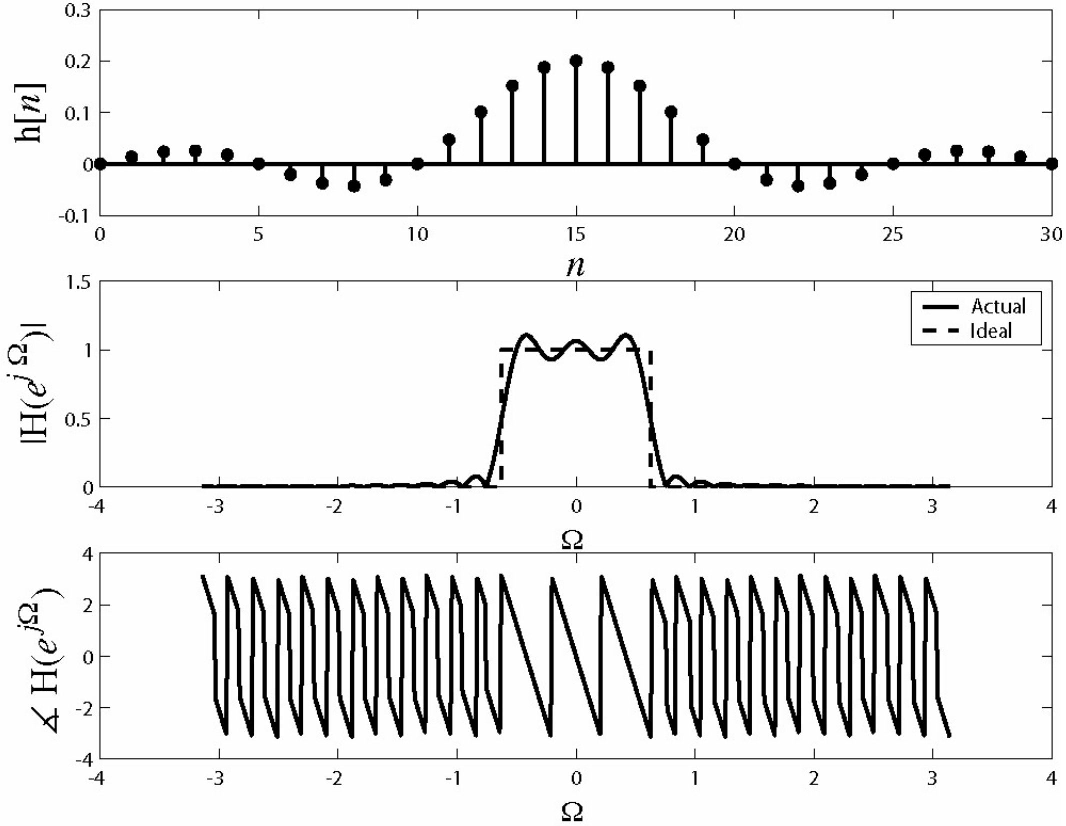
$$\mathrm{h}[n] = 2AF_c \operatorname{sinc}\left(2F_c\left(n-n_0\right)\right)$$

$$= \left(\frac{A\Omega_c}{\pi}\right)\operatorname{sinc}\left(\frac{\Omega_c}{\pi}\left(n-n_0\right)\right) = \frac{A\Omega_c}{\pi}\frac{\sin\left(\Omega_c\left(n-n_0\right)\right)}{\Omega_c\left(n-n_0\right)}$$

The center point of the sinc function is at $n = n_0$. If we truncate the impulse response below $n = 0$ and above $2n_0$ we get an FIR filter with $2n_0 + 1$ coefficients.

FIR Filter Design

For example, let $F_c = 0.1 \Rightarrow \Omega_c = 0.2\pi$, $A = 1$ and $n_0 = 15$.

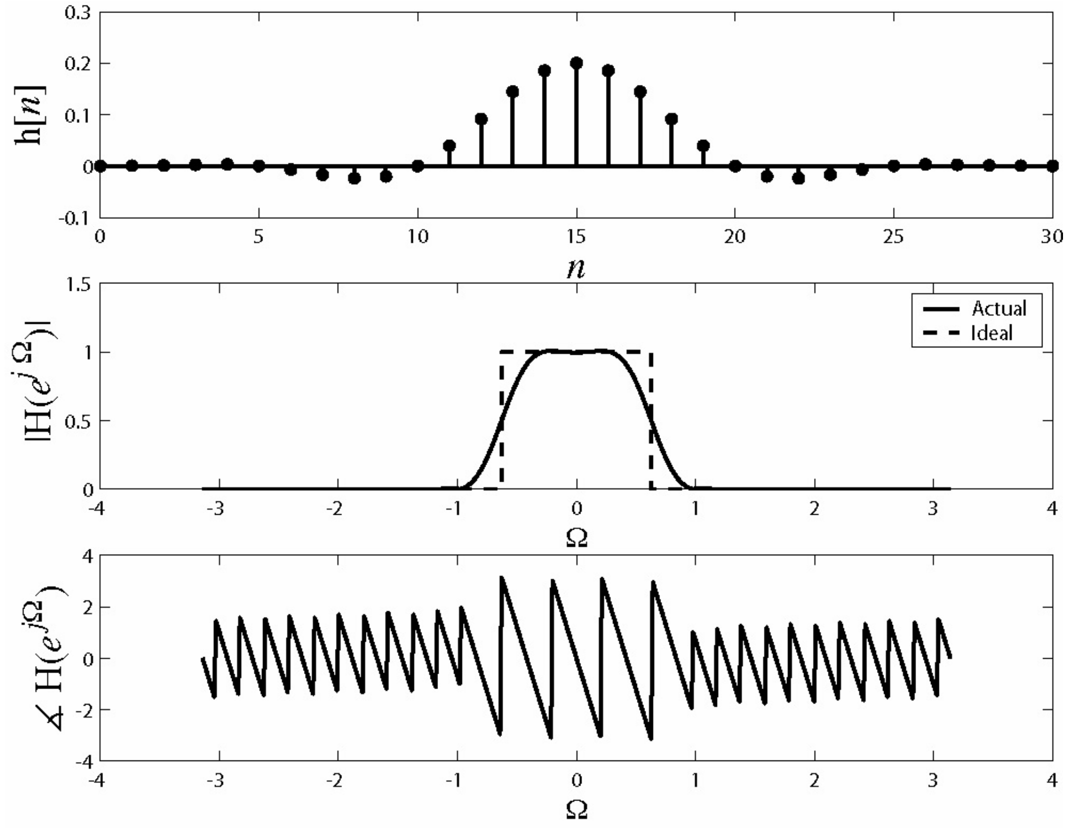


FIR Filter Design

We can reduce passband ripple and increase stopband attenuation by multiplying the truncated impulse response by a von Hann window

$$w[n] = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{N-1}\right)\right]$$

$$0 \le n < N$$



FIR Filter Design

Some common windows are

von Hann $$\mathrm{w}[n] = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{N-1}\right)\right] \;,\; 0 \le n < N$$

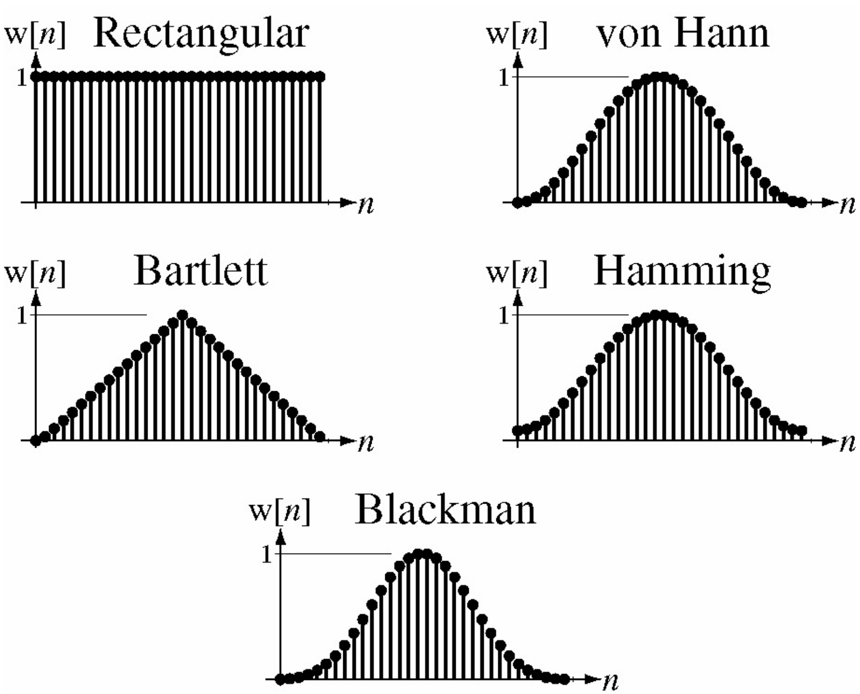
Hamming $$\mathrm{w}[n] = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right) \;,\; 0 \le n < N$$

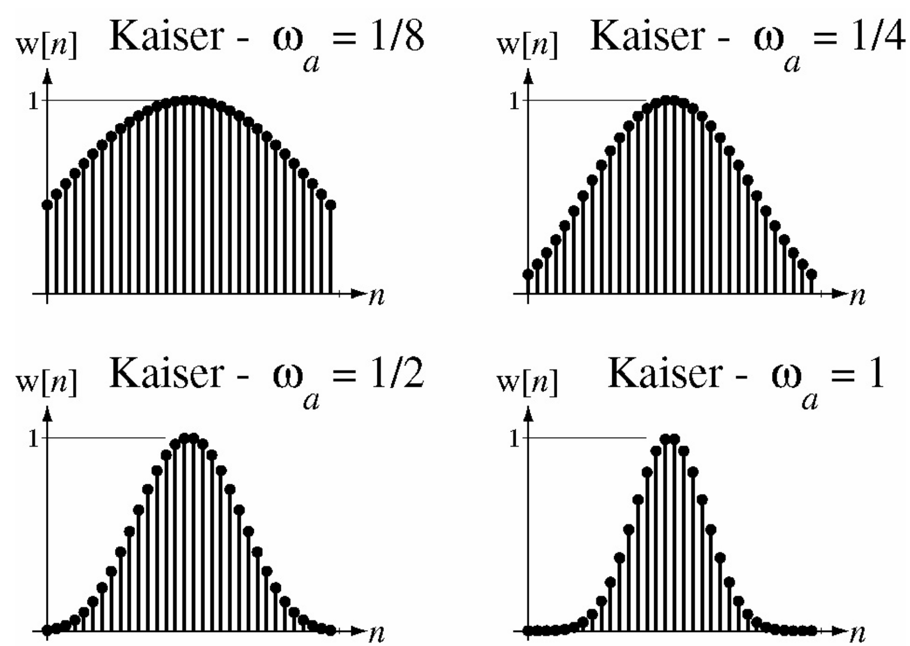
Blackman $$\mathrm{w}[n] = 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right) \;,\; 0 \le n < N$$

Kaiser $$\mathrm{w}[n] = \frac{I_0\left(\omega_a\sqrt{\left(\frac{N-1}{2}\right)^2 - \left(n - \frac{N-1}{2}\right)^2}\right)}{I_0\left(\omega_a \frac{N-1}{2}\right)}$$

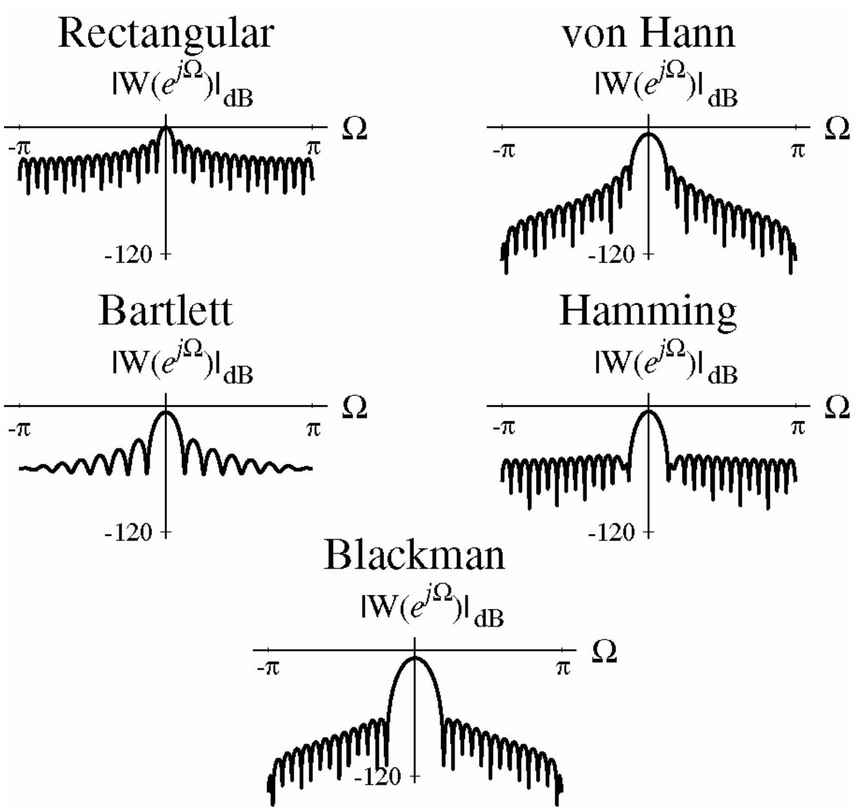
where I_0 is the modified zeroth order Bessel function of the first kind and ω_a is a parameter adjusted to trade off between transition bandwidth and side-lobe amplitude.

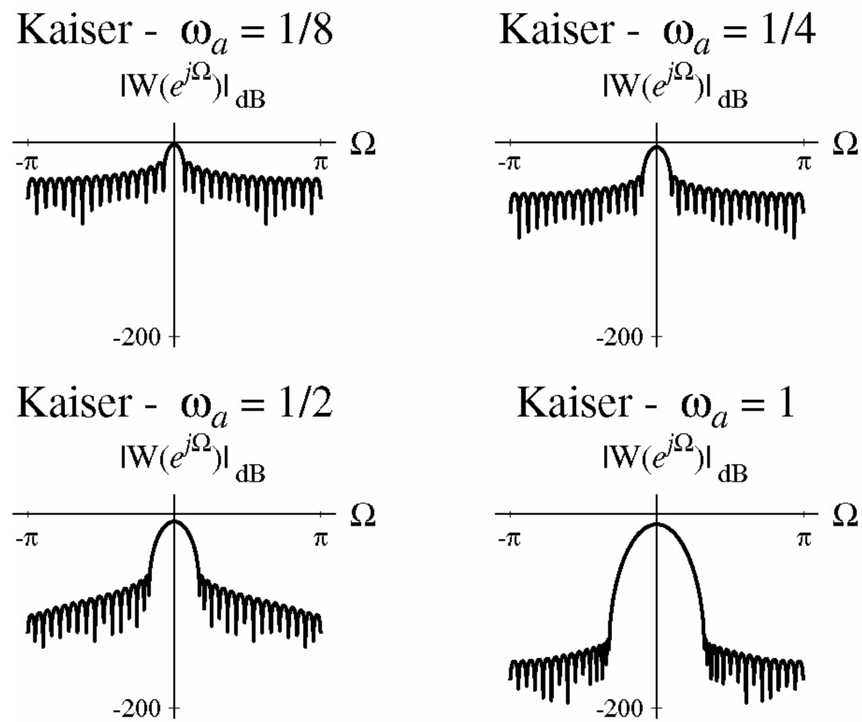
FIR Filter Design





FIR Filter Design





Approximating Standard Analog Filters

There are multiple methods for approximating analog filters with digital filters. We will consider six, approximation of derivatives, impulse invariance, step invariance, direct substitution, the matched z-transform and the bilinear z transform.

Approximation of Derivatives

A derivative is defined by $\frac{d}{dt}\mathrm{x}(t) = \lim_{\Delta t \to 0} \frac{\mathrm{x}(t+\Delta t) - \mathrm{x}(t)}{\Delta t}$. If we just make Δt small, but not zero, we approximate the derivative with a finite difference

$$\frac{d}{dt}\mathrm{x}(t) \cong \frac{\mathrm{x}(t+\Delta t) - \mathrm{x}(t)}{\Delta t} = \frac{\mathrm{x}[n+1] - \mathrm{x}[n]}{T_s}$$

This is called a **forward** difference. We can also approximate a derivative with a **backward** difference.

$$\frac{d}{dt}\mathrm{x}(t) \cong \frac{\mathrm{x}(t) - \mathrm{x}(t-\Delta t)}{\Delta t} = \frac{\mathrm{x}[n] - \mathrm{x}[n-1]}{T_s}$$

Because a backward difference is causal and is the inverse of the accumulation operation $\sum_{m=-\infty}^{n} \mathrm{x}[m]$, it is usually used.

Approximation of Derivatives

In the Laplace domain, multiplication by s is equivalent to a time derivative. The first backward difference in discrete time is equivalent to a multiplication by $\dfrac{1-z^{-1}}{T_s} = \dfrac{z-1}{zT_s}$ in the z domain.

So the substitution $s \rightarrow \dfrac{1-z^{-1}}{T_s}$ converts s-domain expressions into z-domain expressions. The relationship $s = \dfrac{1-z^{-1}}{T_s}$ or $z = \dfrac{1}{1-sT_s}$ is a mapping between the s and z planes.

Approximation of Derivatives

Letting $s = \sigma + j\omega$,

$$z = \frac{1}{1-(\sigma + j\omega)T_s} = \frac{1-\sigma T_s}{(1-\sigma T_s)^2 + (\omega T_s)^2} + \frac{j\omega T_s}{(1-\sigma T_s)^2 + (\omega T_s)^2}$$

For the special case of $\sigma = 0$, $z = \dfrac{1}{1+(\omega T_s)^2} + \dfrac{j\omega T_s}{1+(\omega T_s)^2}$ and the ω axis maps into a circle of radius $1/2$, centered at $z = 1/2$. For the special case $\omega = 0$, $z = \dfrac{1}{1-\sigma T_s}$ and for any negative value of σ, z lies on the real axis between $z = 0$ and $z = 1$. Overall, the left half of the s plane maps into the interior of the circle described above. Therefore stable s-domain filters map into stable z-domain filters. But a highpass analog filter does not become a highpass digital filter.

Approximation of Derivatives Example

Simulate $\mathrm{H}_s(s) = \dfrac{s}{s^2 + 400s + 2\times10^5}$ using a sampling rate of 1 kHz.

$$\mathrm{H}_z(z) = \left(\frac{s}{s^2 + 400s + 2\times10^5}\right)_{s\to\frac{1-z^{-1}}{T_s}} = \frac{\frac{1-z^{-1}}{T_s}}{\left(\frac{1-z^{-1}}{T_s}\right)^2 + 400\frac{1-z^{-1}}{T_s} + 2\times10^5}$$

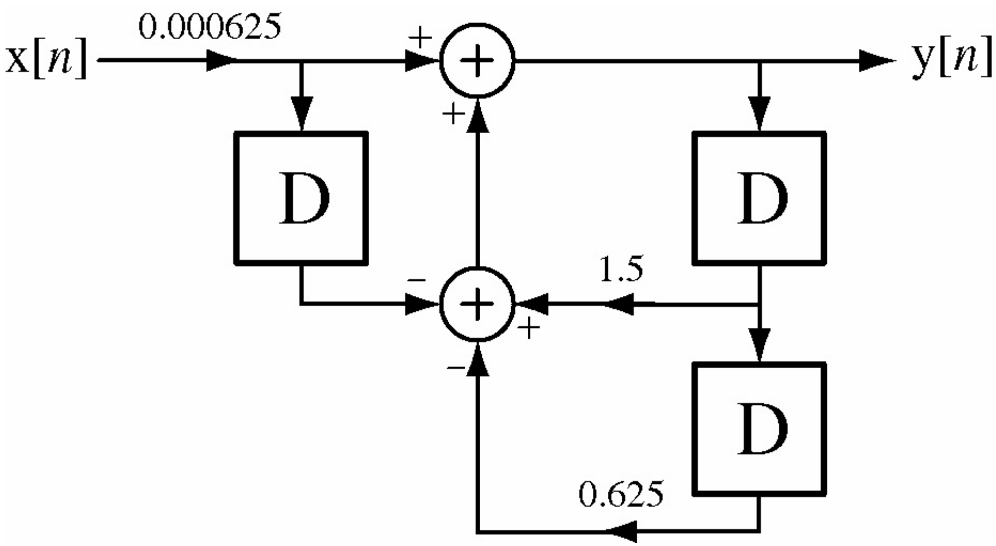
$$\mathrm{H}_z(z) = T_s \frac{z(z-1)}{\left(1+400T_s + 2\times10^5 T_s^2\right)z^2 - \left(2+400T_s\right)z + 1}$$

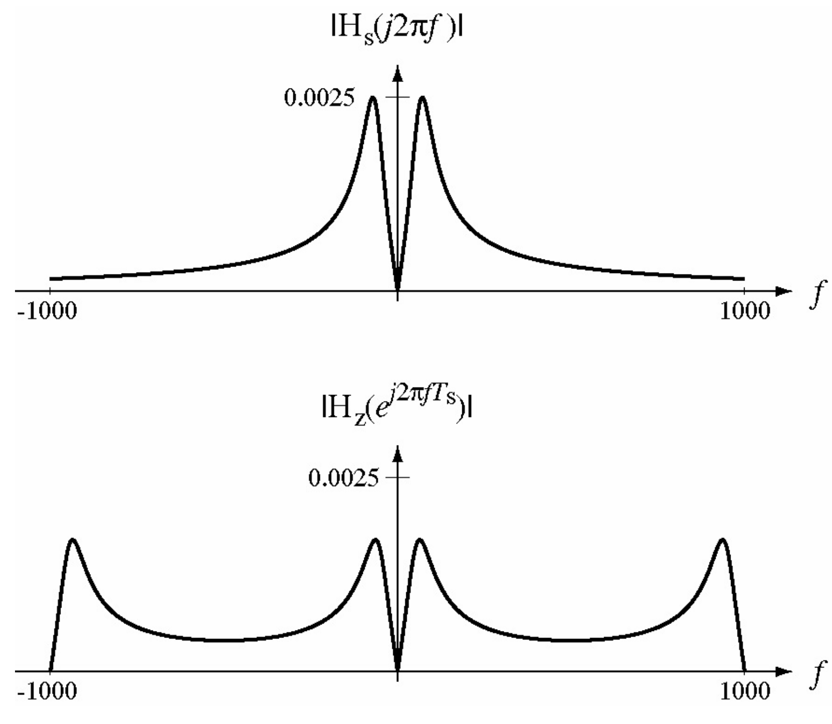
For $f_s = 1000$ ($T_s = 1/1000$),

$$\mathrm{H}_z(z) = 6.25\times10^{-4}\frac{z(z-1)}{z^2 - 1.5z + 0.625}$$

Approximation of Derivatives Example

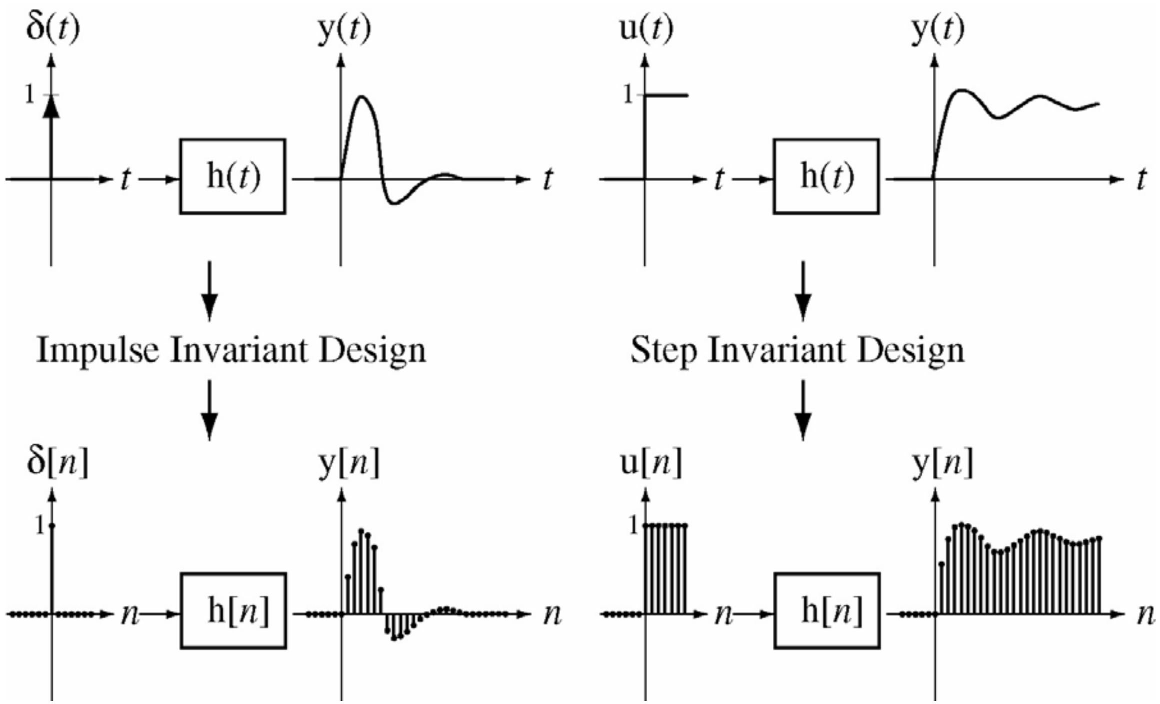
$$y[n] = 6.25\times10^{-4}\left(x[n]-x[n-1]\right)+1.5\,y[n-1]-0.625\,y[n-2]$$





Impulse and Step Invariant Design

Impulse invariant design makes the impulse response of the digital filter be a sampled version of the impulse response of the analog filter. Step invariant design makes the step response of the digital filter be a sampled version of the step response of the analog filter.



Impulse and Step Invariant Design

Impulse Invariant Design:

$$\underbrace{\mathrm{H}_s(s)}_{\substack{s\text{-Domain}\\\text{transfer}\\\text{function}}} \xrightarrow{\mathscr{L}^{-1}} \underbrace{\mathrm{h}(t)}_{\substack{\text{Continuous-}\\\text{time impulse}\\\text{response}}} \xrightarrow{\text{Sampling}} \underbrace{\mathrm{h}[n]}_{\substack{\text{Discrete-}\\\text{time impulse}\\\text{response}}} \xrightarrow{\mathscr{Z}} \underbrace{\mathrm{H}_z(z)}_{\substack{z\text{-Domain}\\\text{transfer}\\\text{function}}}$$

Step Invariant Design:

$$\underbrace{\mathrm{H}_s(s)}_{\substack{s\text{-Domain}\\\text{transfer}\\\text{function}}} \xrightarrow{\div s} \underbrace{\mathrm{H}_{-1,s}(s)}_{\substack{\text{Laplace transform}\\\text{of step response}}} \xrightarrow{\mathscr{L}^{-1}} \underbrace{\mathrm{h}_{-1}(t)}_{\substack{\text{Continuous-}\\\text{time step}\\\text{response}}} \xrightarrow{\text{Sampling}} \cdots$$

$$\cdots \xrightarrow{\text{Sampling}} \underbrace{\mathrm{h}_{-1}[n]}_{\substack{\text{Discrete-}\\\text{time step}\\\text{response}}} \xrightarrow{\mathscr{Z}} \underbrace{\mathrm{H}_{-1,z}(z)}_{\substack{z\text{ Transform}\\\text{of step response}}} \xrightarrow[z]{\frac{z-1}{}} \underbrace{\mathrm{H}_z(z)}_{\substack{z\text{-Domain}\\\text{transfer}\\\text{function}}}$$

Impulse Invariant Example

Simulate $\mathrm{H}_s(s) = \dfrac{s}{s^2 + 400s + 2\times 10^5}$ using a sampling rate of 1 kHz.

The continuous-time impulse response is

$$\mathrm{h}(t) = e^{-200t}\left[\cos(400t) - \frac{1}{2}\sin(400t)\right]\mathrm{u}(t).$$

The discrete-time impulse response is

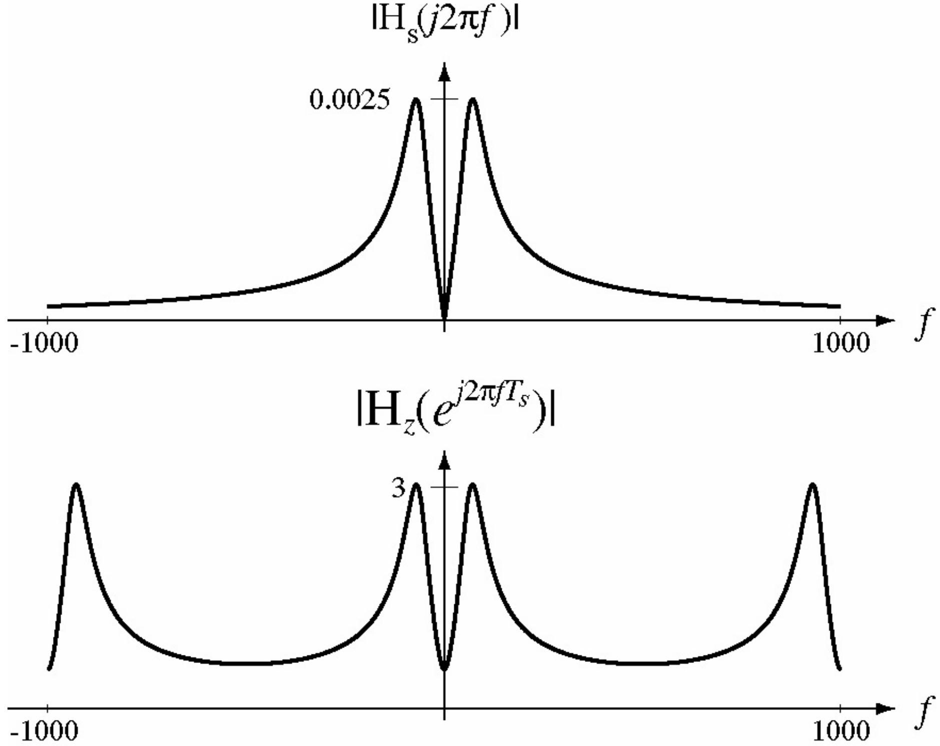
$$\mathrm{h}[n] = (0.8187)^n\left[\cos(0.4n) - \frac{1}{2}\sin(0.4n)\right]\mathrm{u}[n].$$

The z-domain transfer function is

$$\mathrm{H}_z(z) = \frac{\mathrm{Y}(z)}{\mathrm{X}(z)} = \frac{z(z - 0.9135)}{z^2 - 1.508z + 0.6703}.$$

Impulse Invariant Example

A comparison of frequency responses reveals that the impulse invariant digital filter has a peak gain that is a factor of about 1200 too large. Also, the gain at zero frequency is not zero as it is in the analog filter. This is undesirable for a filter that is intended to be bandpass. Also, the gain falls from the peak but does not go to zero at high frequencies as in the analog case. The peak gain can be easily compensated for but the non-zero, zero-frequency gain cannot. The gain error near half the sampling rate will always be wrong due to aliasing, therefore an increase in sampling rate will improve the design.



Step Invariant Example

Simulate $\mathrm{H}_s(s) = \dfrac{s}{s^2 + 400s + 2\times 10^5}$ using a sampling rate of 1 kHz.

The continuous-time step response is

$$\mathrm{h}_{-1}(t) = \mathscr{L}^{-1}\left(\frac{\mathrm{H}_s(s)}{s}\right) = \frac{e^{-200t}\sin(400t)}{400}\mathrm{u}(t)$$

The discrete-time step response is

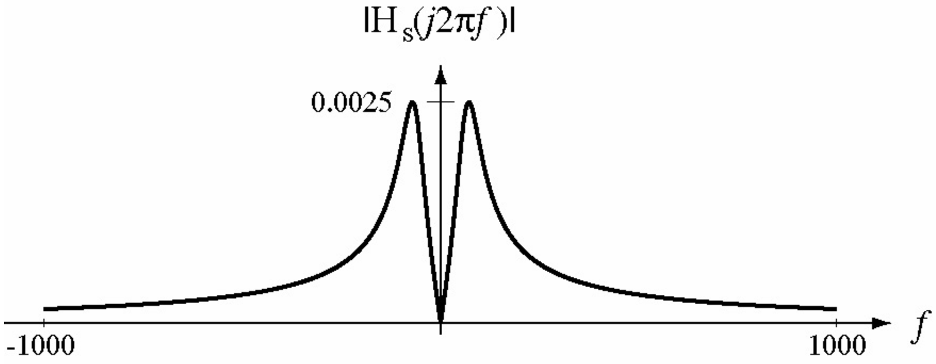
$$\mathrm{h}_{-1}[n] = \frac{e^{-200nT_s}\sin(400nT_s)}{400}\mathrm{u}[n]$$

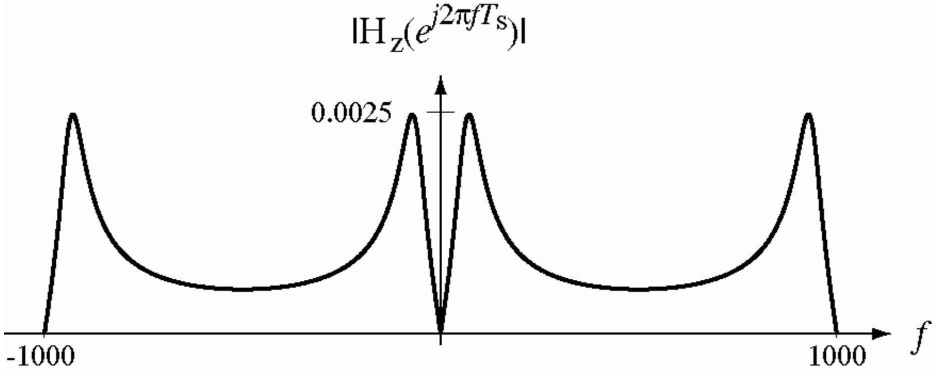
The z-domain transfer function is

$$\mathrm{H}_z(z) = \frac{z-1}{z}\mathscr{Z}\left(\mathrm{h}_{-1}[n]\right) = 7.97\times 10^{-4}\,\frac{(z-1)}{z^2 - 1.509z + 0.6708}$$

Step Invariant Example

The peak gain is correct and the gain at zero frequency is zero as it should be. The aliasing effect near half the sampling rate is there as in the impulse-invariant case and can only be minimized by increasing the sampling rate.





Direct Substitution and Matched z Transform

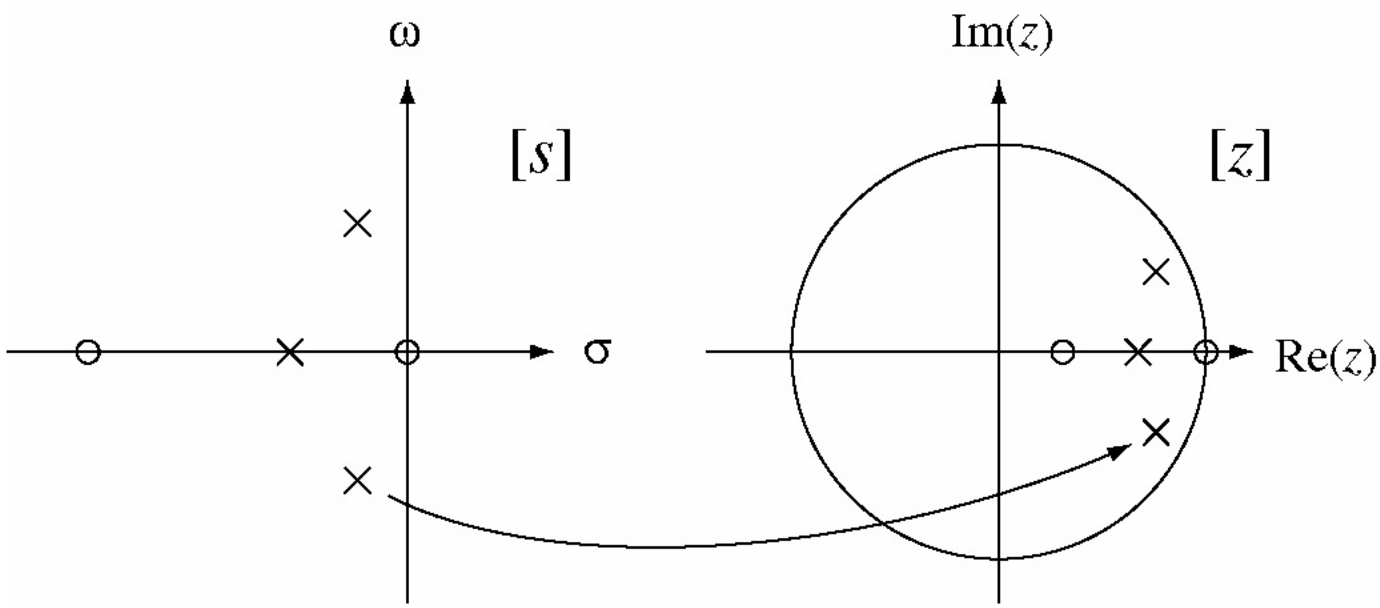
A popular approach to the design of digital filters is to find a transformation from s to z which maps the s plane into the z plane, converts the poles and zeros of the s-domain transfer function into appropriate corresponding locations in the z plane and converts stable s-domain systems into stable z-domain systems. The most common techniques that use this idea are the "matched z transform", "direct substitution" and the "bilinear transformation".

Direct Substitution and Matched z Transform

To directly transform s-domain transfer functions into z-domain transfer functions we need a relationship between s and z. The relation between discrete- and continuous-time frequencies is $\Omega = \omega \mathrm{T}_s$. If we map ω to Ω this way it is equivalent to mapping ω values in the s plane to to z values on the unit circle through $z = e^{j\omega T_s}$. If we generalize to s values anywhere in the complex s plane, we get $z = e^{sT_s}$. This mapping can be used to map poles and zeros in the s plane into poles and zeros in the z plane.

Direct Substitution and Matched z Transform

Suppose we have a pole or zero at $s = s_0$. Its corresponding location in the z plane is $z = e^{s_0 T_s}$.



Direct Substitution and Matched z Transform

There are two closely-related methods for mapping poles and zeros from the s to the z plane. If there is a factor in the numerator or denominator of the transfer function of the form $s - a$, we can replace it with

$$z - e^{aT_s} \quad \rightarrow \quad \text{Direct Substitution}$$

or with

$$1 - e^{aT_s} z^{-1} \quad \rightarrow \quad \text{Matched } z \text{ Transform}$$

The only difference is an extra delay in the matched z transform.

Matched z Transform Example

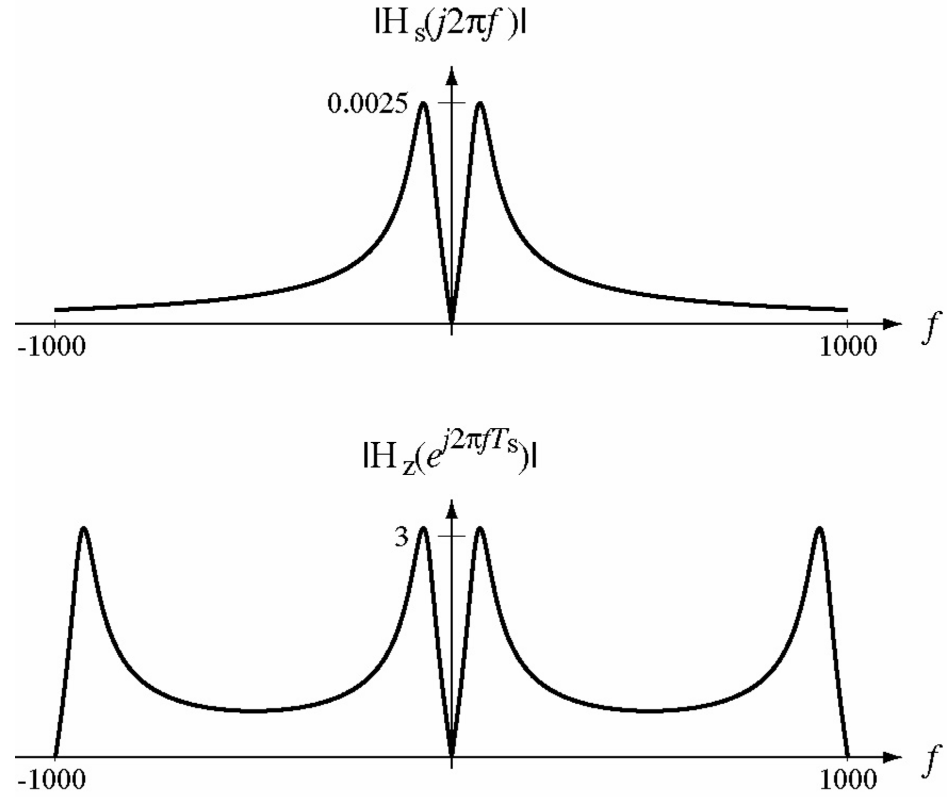
Simulate $\mathrm{H}_s(s) = \dfrac{s}{s^2 + 400s + 2\times 10^5}$ using the matched z transform and a sampling rate of 1 kHz. $\mathrm{H}_s(s)$ has a zero at $s = 0$ and poles at $s = -200 \pm j400$.

$$z = e^{(-200 \pm j400)T_s} = e^{-0.2 \pm j0.4}$$

$$= 0.7541 \pm j0.3188$$

$$\mathrm{H}_z(z) = \frac{1 - z^{-1}}{1 - 1.509z^{-1} + 0.6708z^{-2}}$$

$$= \frac{z(z-1)}{z^2 - 1.509z + 0.6708}$$



Bilinear Transformation

A natural idea is to use the mapping $z = e^{sT_s}$ or $s = (1/T_s)\ln(z)$ to form $\mathrm{H}_z(z) = \mathrm{H}_s(s)\big|_{s \to \frac{1}{T_s}\ln(z)}$. But this transforms a rational function of s into a rational function of $(1/T_s)\ln(z)$. This function is transcendental and has infinitely many poles and/or zeros. One way to avoid this problem is to use an approximation to the exponential function. Start with the infinite-series expression

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and truncate the series after two terms.

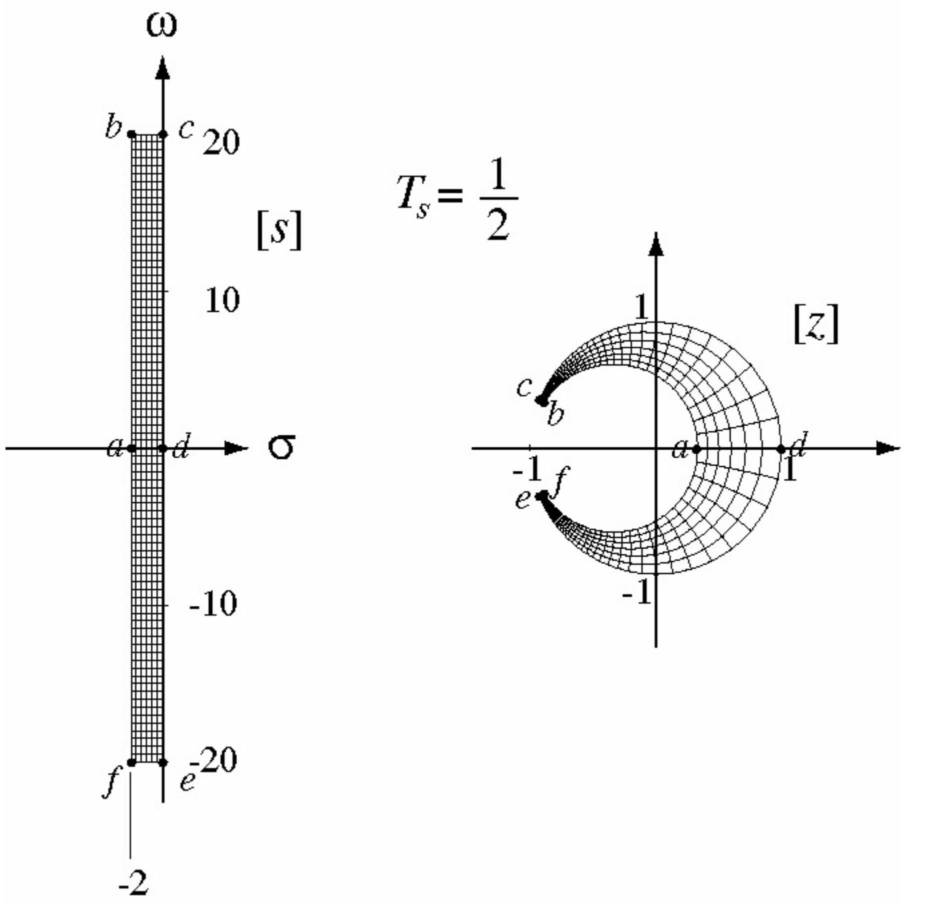
Bilinear Transformation

Truncating the infinite-series after two terms yields $1+sT_s \to z$ or $s \to (z-1)/T_s$. For small T_s (therefore high sampling rates) this is a good approximation. This is identical to approximating a derivative with a forward difference. This mapping maps the entire left half of the s plane into the interior of the z plane infinitely many time. Also, for some sampling rates, this mapping can convert a stable system in s to an unstable system in z. A very clever modification solves both problems. Write the transformation as $e^{sT_s} = \dfrac{e^{sT_s/2}}{e^{-sT_s/2}} \to z$ and then approximate the numerator and denominator separately by truncating the series to two terms.

Bilinear Transformation

This last approximation yields $\dfrac{1+sT_s/2}{1-sT_s/2} \rightarrow z$ or $s \rightarrow \dfrac{2}{T_s}\dfrac{z-1}{z+1}$.

This is called the bilinear (not bilateral) transformation. This mapping has the favorable quality that the left half of the s plane maps into the interior of the unit circle in the z plane only one time and stable s-domain filters are transformed into stable z-domain filters.



Bilinear Transformation

The bilinear transformation has a "warping" effect because of the way the $s = j\omega$ axis is mapped into the unit circle in the z plane. Letting $z = e^{j\Omega}$ with Ω real determines the unit circle in the z plane. The corresponding contour in the s plane is

$$s = \frac{2}{T_s}\frac{e^{j\Omega}-1}{e^{j\Omega}+1} = j\frac{2}{T_s}\tan\left(\frac{\Omega}{2}\right)$$

or $\Omega = 2\tan^{-1}\left(\omega T_s / 2\right)$.