Frequency Sampling Structures

Specifying Frequency Response

Set the frequency response of an FIR filter at M equally-spaced points $\Omega_k = 2\pi k / M \quad , \; k = 0,1,2,\cdots,M-1$ and find the impulse response $\mathrm{h}[n]$.

Frequency response and impulse response are related by

$$\mathrm{H}\left(e^{j\Omega}\right) = \sum_{n=0}^{M-1} \mathrm{h}[n] e^{-j\Omega n}$$

and it follows that

$$\mathrm{H}\left(e^{j\Omega_k}\right) = \mathrm{H}\left(e^{j2\pi k/M}\right) = \sum_{n=0}^{M-1} \mathrm{h}[n] e^{-j2\pi kn/M} \quad , \; k = 0,1,2,\cdots,M-1$$

This is the DFT of $\mathrm{h}[n]$.

Specifying Frequency Response

Since $\mathrm{H}\left(e^{j2\pi k/M}\right)$ is the DFT of $\mathrm{h}[n]$, $\mathrm{h}[n]$ must be the inverse DFT of $\mathrm{H}\left(e^{j2\pi k/M}\right)$.

$$\mathrm{h}[n]=\frac{1}{M}\sum_{k=0}^{M-1}\mathrm{H}\left(e^{j2\pi k/M}\right)e^{j2\pi kn/M} \quad , \; n=0,1,2,\cdots,M-1$$

The transfer function and impulse response are related by

$$\mathrm{H}(z)=\sum_{n=0}^{M-1}\mathrm{h}[n]z^{-n}=\frac{1}{M}\sum_{n=0}^{M-1}\sum_{k=0}^{M-1}\mathrm{H}\left(e^{j2\pi k/M}\right)e^{j2\pi kn/M}z^{-n} \; .$$

Exchanging the order of summation,

$$\mathrm{H}(z)=\frac{1}{M}\sum_{k=0}^{M-1}\mathrm{H}\left(e^{j2\pi k/M}\right)\sum_{n=0}^{M-1}e^{j2\pi kn/M}z^{-n}$$

Specifying Frequency Response

Using the formula for summing a geometric series,

$$\mathrm{H}(z) = \frac{1}{M}\sum_{k=0}^{M-1}\mathrm{H}\left(e^{j2\pi k/M}\right)\frac{1-z^{-M}e^{j2\pi}}{1-z^{-1}e^{j2\pi k/M}} = \frac{1-z^{-M}}{M}\sum_{k=0}^{M-1}\frac{\mathrm{H}\left(e^{j2\pi k/M}\right)}{1-z^{-1}e^{j2\pi k/M}}$$

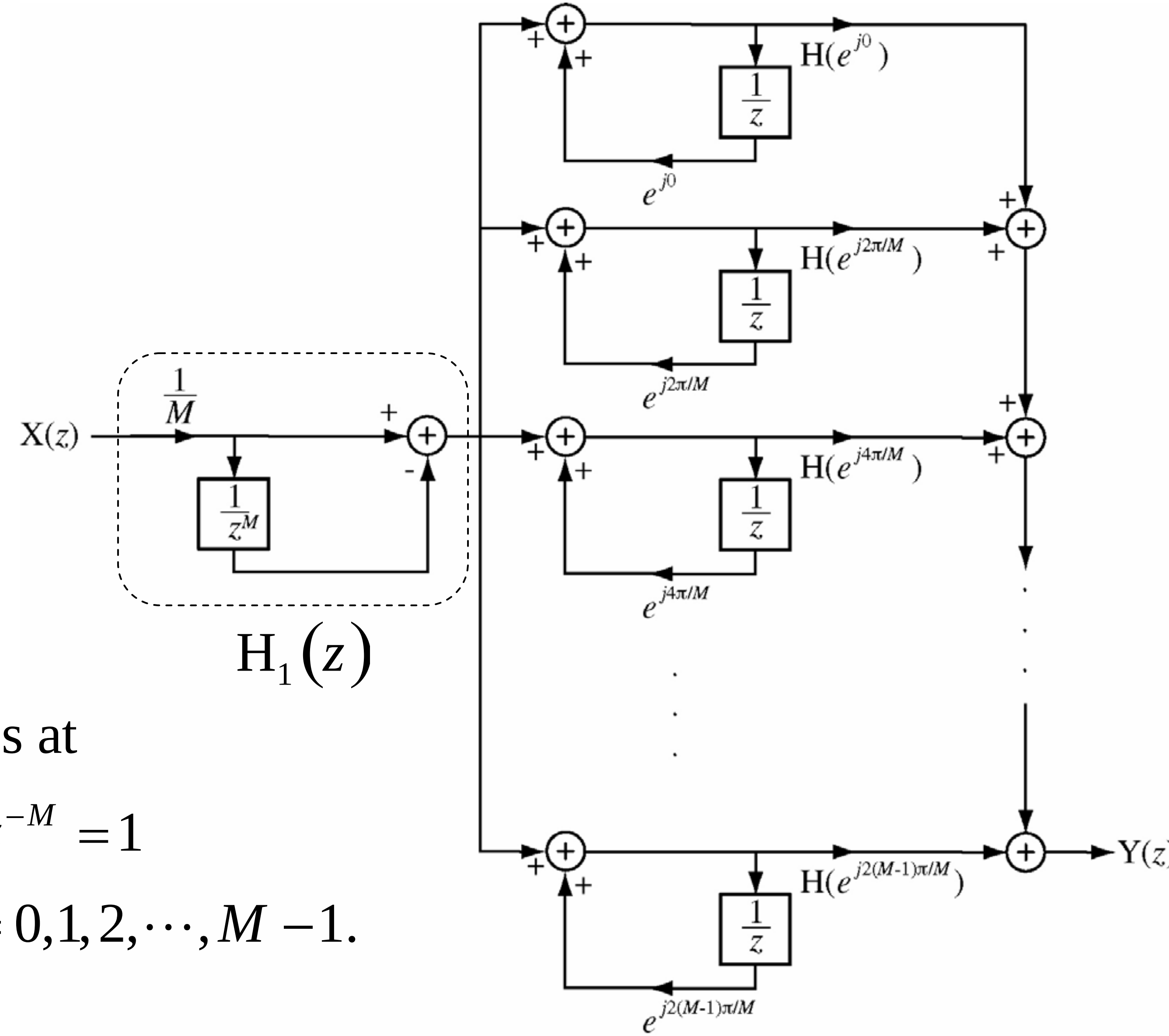
We can think of this as the cascade of two filters, one with transfer function

$$\mathrm{H}_1(z) = \frac{1-z^{-M}}{M}$$

and one with transfer function

$$\mathrm{H}_2(z) = \sum_{k=0}^{M-1}\frac{\mathrm{H}\left(e^{j2\pi k/M}\right)}{1-z^{-1}e^{j2\pi k/M}} \ .$$

Specifying Frequency Response



$\mathrm{H}_1(z)$ has zeros at

$$1 - z^{-M} = 0 \Rightarrow z^{-M} = 1$$

$$z = e^{j2\pi q/M},\ q = 0, 1, 2, \cdots, M-1.$$

Specifying Frequency Response

$\mathrm{H}_2(z)$ is the sum of the transfer functions of M subsystems, each of which has a pole where

$$1 - z^{-1}e^{j2\pi k/M} = 0 \Rightarrow z^{-1}e^{j2\pi k/M} = 1 \Rightarrow z = e^{j2\pi k/M} \quad , \ k = 0,1,2,\cdots,M-1$$

So the transfer function can be written as

$$\mathrm{H}(z) = \underbrace{\frac{1}{M}\left[\prod_{q=0}^{M-1} 1 - z^{-1}e^{j2\pi q/M}\right]}_{\mathrm{H}_1(z)} \underbrace{\sum_{k=0}^{M-1} \frac{\mathrm{H}\left(e^{j2\pi k/M}\right)}{1 - z^{-1}e^{j2\pi k/M}}}_{\mathrm{H}_2(z)}$$

or

$$\mathrm{H}\left(e^{j2\pi p/M}\right) = \frac{1}{M}\sum_{k=0}^{M-1} \frac{\prod_{q=0}^{M-1} 1 - e^{-j2\pi(p-q)/M}}{1 - e^{-j2\pi(p-k)/M}} \mathrm{H}\left(e^{j2\pi k/M}\right) \quad , \ 0 \le p < M$$

Specifying Frequency Response

The poles of $\mathrm{H}_2(z)$ are at the same locations as the zeros of $\mathrm{H}_1(z)$. So for any particular zero of $\mathrm{H}_1(z)$, all the terms in the summation are zero except the one for which the pole and zero cancel. Therefore

$$\mathrm{H}\left(e^{j2\pi p/M}\right) = \frac{\mathrm{H}\left(e^{j2\pi p/M}\right)}{M} \prod_{\substack{q=0 \\ q \neq p}}^{M-1} 1 - e^{-j2\pi(p-q)/M} \quad , \ 0 \le p < M$$

This result implies that

$$\prod_{\substack{q=0 \\ q \neq p}}^{M-1} 1 - e^{-j2\pi(p-q)/M} = M \quad , \ 0 \le p < M$$

Specifying Frequency Response

The poles and zeros could theoretically be anwhere on the unit circle, but for practical designs they must occur in complex conjugate pairs and

$$\mathrm{H}\left(e^{j2\pi k/M}\right) = \mathrm{H}^*\left(e^{-j2\pi k/M}\right) = \mathrm{H}^*\left(e^{j2\pi(M-k)/M}\right)$$

Specifying Frequency Response

Taking advantage of the complex-conjugate symmetry, for M even,

$$\mathrm{H}_2(z) = \sum_{k=0}^{M-1} \frac{\mathrm{H}\left(e^{j2\pi k/M}\right)}{1 - z^{-1} e^{j2\pi k/M}} = \frac{\mathrm{H}\left(e^{j0}\right)}{1 - z^{-1}} + \sum_{k=1}^{M/2-1} \frac{\mathrm{H}\left(e^{j2\pi k/M}\right)}{1 - z^{-1} e^{j2\pi k/M}} + \frac{\mathrm{H}\left(e^{j2\pi(M-k)/M}\right)}{1 - z^{-1} e^{j2\pi(M-k)/M}}$$

$$\mathrm{H}_2(z) = \frac{\mathrm{H}\left(e^{j0}\right)}{1 - z^{-1}} + \sum_{k=1}^{M/2-1} \frac{\mathrm{A}(k) - \mathrm{B}(k) z^{-1}}{1 - 2\cos(2\pi k / M) z^{-1} + z^{-2}}$$

where $\mathrm{A}(k) = \mathrm{H}\left(e^{j2\pi k/M}\right) + \mathrm{H}\left(e^{j2\pi(M-k)/M}\right)$

and $\mathrm{B}(k) = e^{-j2\pi k/M}\,\mathrm{H}\left(e^{j2\pi k/M}\right) + e^{j2\pi k/M}\,\mathrm{H}\left(e^{j2\pi(M-k)/M}\right)$

Similarly, for M odd,

$$\mathrm{H}_2(z) = \frac{\mathrm{H}\left(e^{j0}\right)}{1 - z^{-1}} + \frac{\mathrm{H}\left(e^{j\pi}\right)}{1 + z^{-1}} + \sum_{k=1}^{(M-1)/2} \frac{\mathrm{A}(k) - \mathrm{B}(k) z^{-1}}{1 - 2\cos(2\pi k / M) z^{-1} + z^{-2}}$$

Example

Design a filter whose frequency response goes through these points.

k	0	1	2	3	4	5	6	7
$\mathrm{H}\left(e^{j2\pi k/M}\right)$	0	1	j	0	0	0	$-j$	1

$$\mathrm{H}(z) = \frac{1 - z^{-M}}{M} \sum_{k=0}^{M-1} \frac{\mathrm{H}\left(e^{j2\pi k/M}\right)}{1 - z^{-1} e^{j2\pi k/M}}$$

$$\mathrm{H}(z) = \frac{1 - z^{-8}}{8} \left[\frac{1}{1 - z^{-1} e^{j2\pi/8}} + \frac{j}{1 - z^{-1} e^{j4\pi/8}} + \frac{-j}{1 - z^{-1} e^{j12\pi/8}} + \frac{1}{1 - z^{-1} e^{j14\pi/8}} \right]$$

Example

Taking advantage of the periodicity of $e^{j2\pi k/8}$

$$H(z) = \frac{1 - z^{-8}}{8}\left[\frac{1}{1 - z^{-1}e^{j2\pi/8}} + \frac{1}{1 - z^{-1}e^{-j2\pi/8}} + \frac{j}{1 - z^{-1}e^{j4\pi/8}} + \frac{-j}{1 - z^{-1}e^{-j4\pi/8}}\right]$$

or

$$H(z) = \frac{1 - z^{-8}}{8}\left[\frac{2 - 2z^{-1}\cos(\pi/4)}{1 - 2z^{-1}\cos(\pi/4) + z^{-2}} - \frac{2z^{-1}\sin(\pi/2)}{1 - 2z^{-1}\cos(\pi/2) + z^{-2}}\right]$$

and the frequency response is

$$H(e^{j\Omega}) = \frac{1 - e^{-j8\Omega}}{8}\left[\frac{2 - 2e^{-j\Omega}\cos(\pi/4)}{1 - 2e^{-j\Omega}\cos(\pi/4) + e^{-j2\Omega}} - \frac{2e^{-j\Omega}\sin(\pi/2)}{1 - 2e^{-j\Omega}\cos(\pi/2) + e^{-j2\Omega}}\right]$$

The impulse response is

$$h[n] = \left\{\begin{matrix} \underset{\uparrow}{0.25}, -0.0732, 0, 0.0732, \\ -0.25, -0.4268, 0, 0.4268 \end{matrix}\right\} \xleftrightarrow{\mathcal{DFT}} H(e^{j2\pi k/M}) = \{0, 1, j, 0, 0, 0, -j, 1\}$$