Lattice and Lattice-Ladder Structures for IIR Systems

Lattice IIR Structure

Consider a system with finite poles and all its zeros at $z = 0$ whose transfer function is of the form

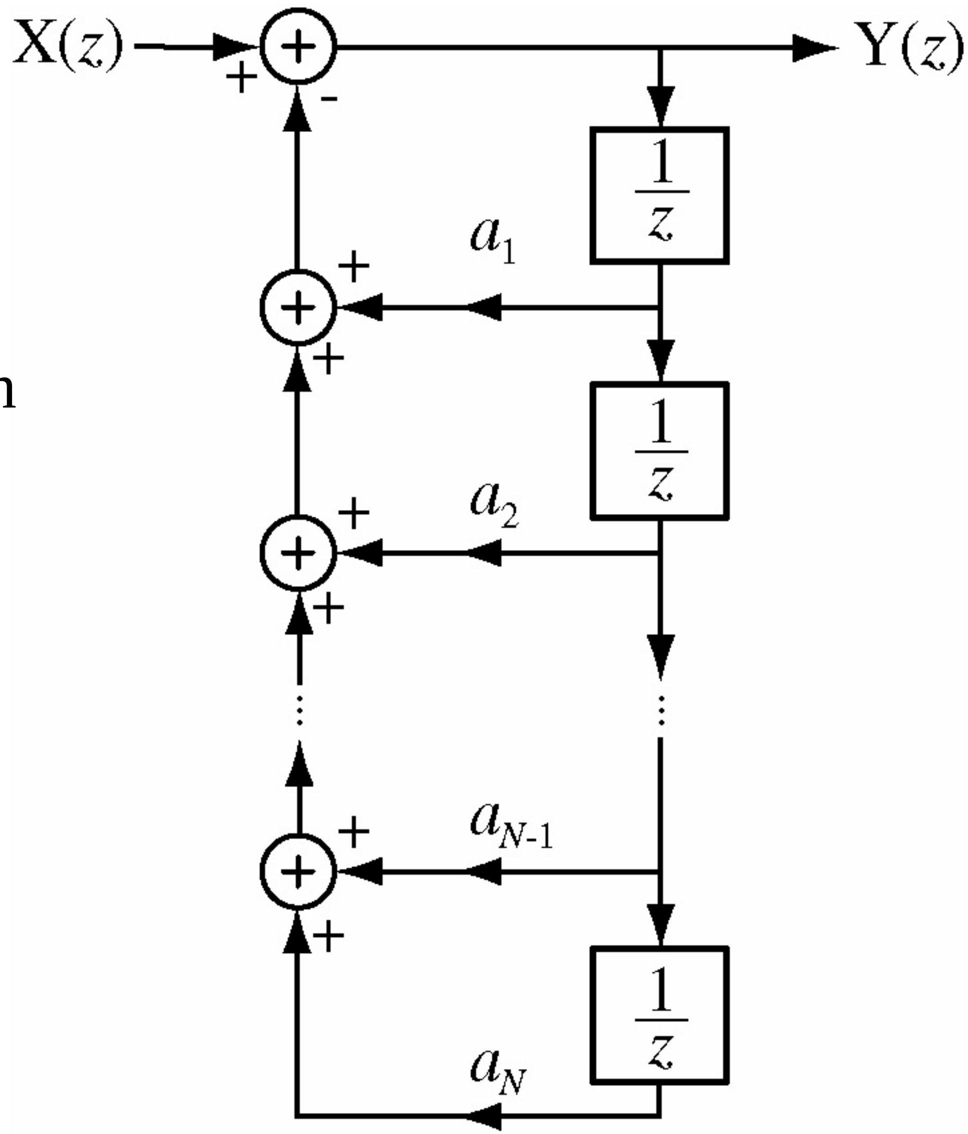
$$\mathrm{H}(z) = \frac{1}{\mathrm{A}_N(z)}$$

where $\mathrm{A}_N(z) = 1 + \sum_{k=1}^{N} a_N[k] z^{-k}$ is an Nth degree polynomial in z.

This system has N finite poles and N zeros at $z = 0$.

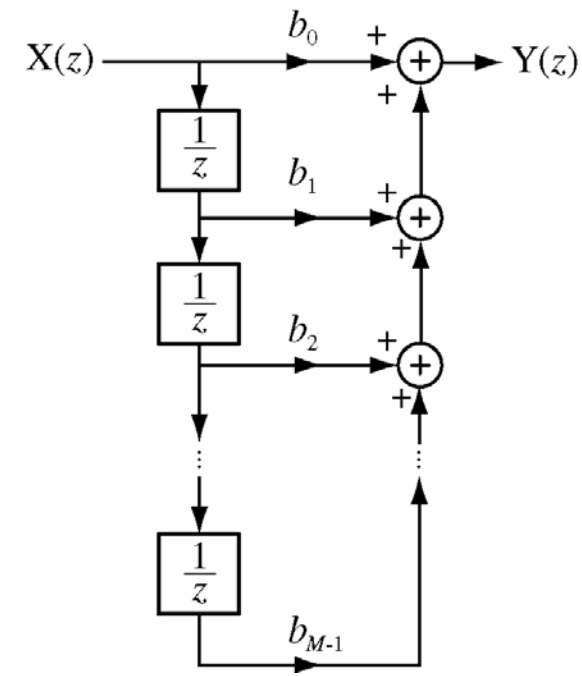
Lattice IIR Structure

A Direct Form II system with *N* finite poles and *N* zeros at *z* = 0.

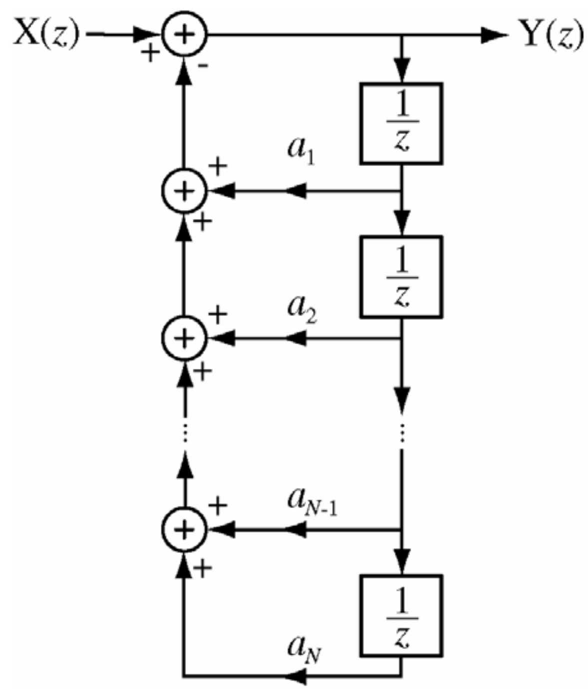


Lattice IIR Structure

Compare the Direct Form II structures for FIR and IIR systems. If, in the FIR system, we exchange the roles of $X(z)$ and $Y(z)$, change all b's to $-a$'s (with $b_0 = 1$) and let $N = M - 1$, we get the IIR system.



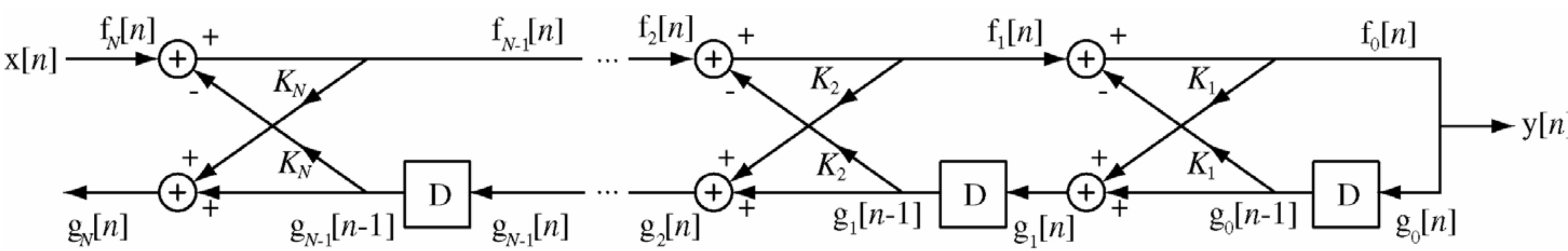
FIR



IIR

Lattice-Ladder IIR Structure

Modify the FIR lattice structure as illustrated below. Reverse the arrows on all the "f" signals. Reverse the lattice and apply $x[n]$ to the previous output and take $y[n]$ from the previous input. Also reverse the signs of the signals arriving from the bottom. This is now a recursive or feedback structure which can implement an IIR filter.



Lattice-Ladder IIR Structure

Take the case $N = 1$.

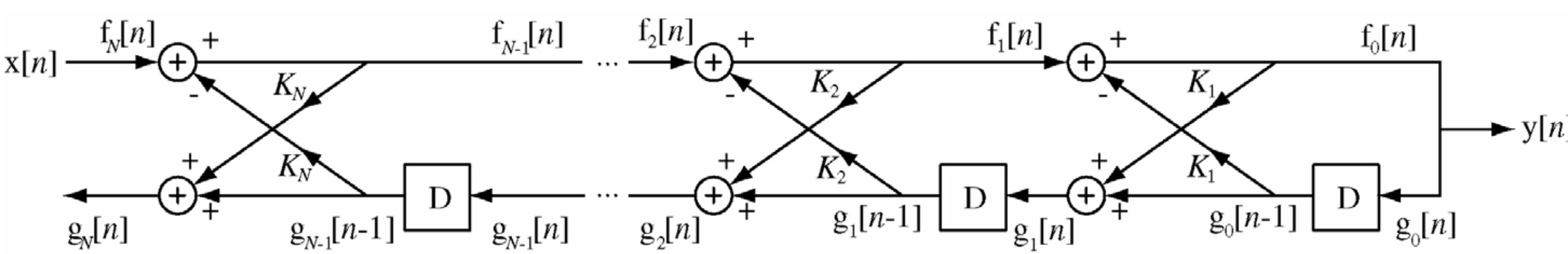
$\mathrm{x}[n] = \mathrm{f}_1[n]$, $\mathrm{f}_0[n] = \mathrm{f}_1[n] - K_1 \mathrm{g}_0[n-1]$, $\mathrm{g}_1[n] = K_1 \mathrm{f}_0[n] + \mathrm{g}_0[n-1]$

and $\mathrm{y}[n] = \mathrm{f}_0[n] = \mathrm{g}_0[n] = \underbrace{\mathrm{f}_1[n]}_{=\mathrm{x}[n]} - K_1 \underbrace{\mathrm{g}_0[n-1]}_{=\mathrm{y}[n-1]}$

z transforming

$$\mathrm{Y}(z) + K_1 z^{-1} \mathrm{Y}(z) = \mathrm{X}(z) \Rightarrow \mathrm{H}_1(z) = \frac{1}{1 + z^{-1} K_1} = \frac{z}{z + K_1}$$

Single pole at $z = -K_1$ and a zero at $z = 0$.



Lattice-Ladder IIR Structure

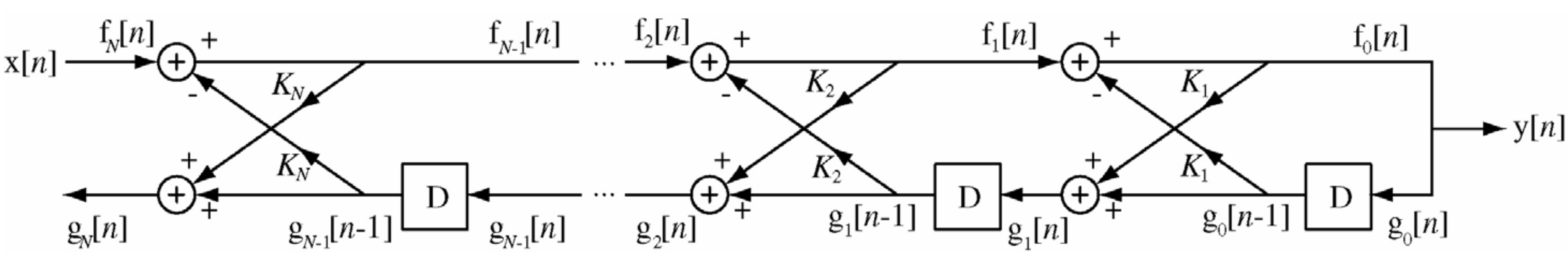
Also, for the $N = 1$ case

$$g_1[n] = K_1\, y[n] + y[n-1]$$

$$G_1(z) = K_1\, Y(z) + z^{-1}\, Y(z) \Rightarrow \frac{G_1(z)}{Y(z)} = K_1 + z^{-1} = K_1 \frac{z + 1/K_1}{z}$$

$\dfrac{G_1(z)}{Y(z)}$ is the transfer function of a system with a single zero at

$z = -1/K_1$ and a pole at zero.



Lattice-Ladder IIR Structure

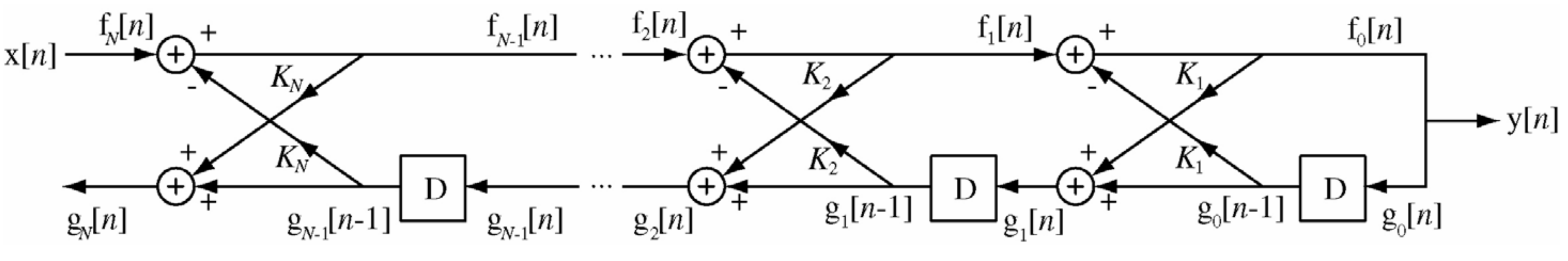
For the $N = 2$ case, it can be shown that

$$H_2(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 + K_1(K_2+1)z^{-1} + K_2 z^{-2}} = \frac{z^2}{z^2 + K_1(K_2+1)z + K_2}$$

and

$$\frac{G_2(z)}{Y(z)} = K_2 + K_1(K_2+1)z^{-1} + z^{-2} = K_2 \frac{z^2 + K_1(1+1/K_2)z + 1/K_2}{z^2}$$

Notice that the coefficients for the FIR and IIR systems occur in reverse order as before.



Lattice-Ladder IIR Structure

For any *m*,

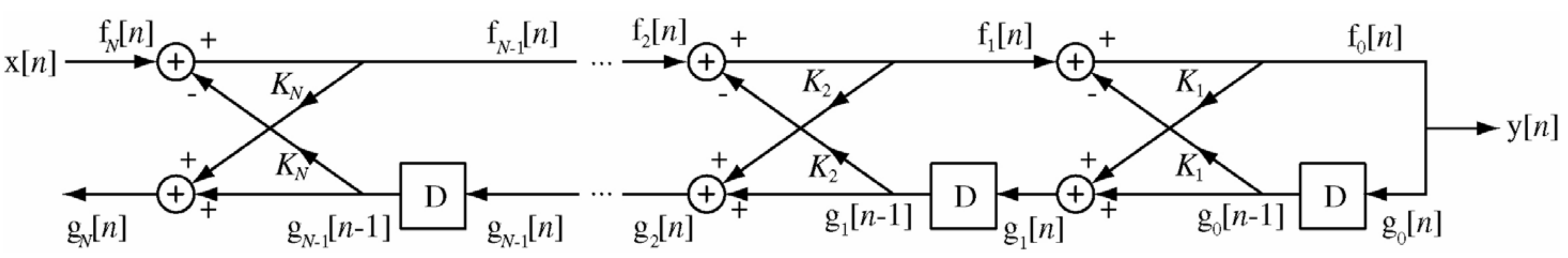
$$\mathrm{H}_m(z) = \frac{\mathrm{Y}(z)}{\mathrm{X}(z)} = \frac{1}{\mathrm{A}_m(z)} \quad \text{and} \quad \frac{\mathrm{G}_m(z)}{\mathrm{Y}(z)} = \mathrm{B}_m(z) = z^{-m}\mathrm{A}_m(1/z)$$

and the previous relations for FIR lattices still hold.

$$\mathrm{A}_0(z) = \mathrm{B}_0(z) = 1 \ , \ \mathrm{A}_m(z) = \mathrm{A}_{m-1}(z) + K_m z^{-1}\mathrm{B}_{m-1}(z)$$

$$\mathrm{B}_m(z) = z^{-m}\mathrm{A}_m(1/z) \ , \ K_m = \alpha_m[m]$$

$$\mathrm{A}_{m-1}(z) = \frac{\mathrm{A}_m(z) - K_m\mathrm{B}_m(z)}{1 - K_m^2}$$

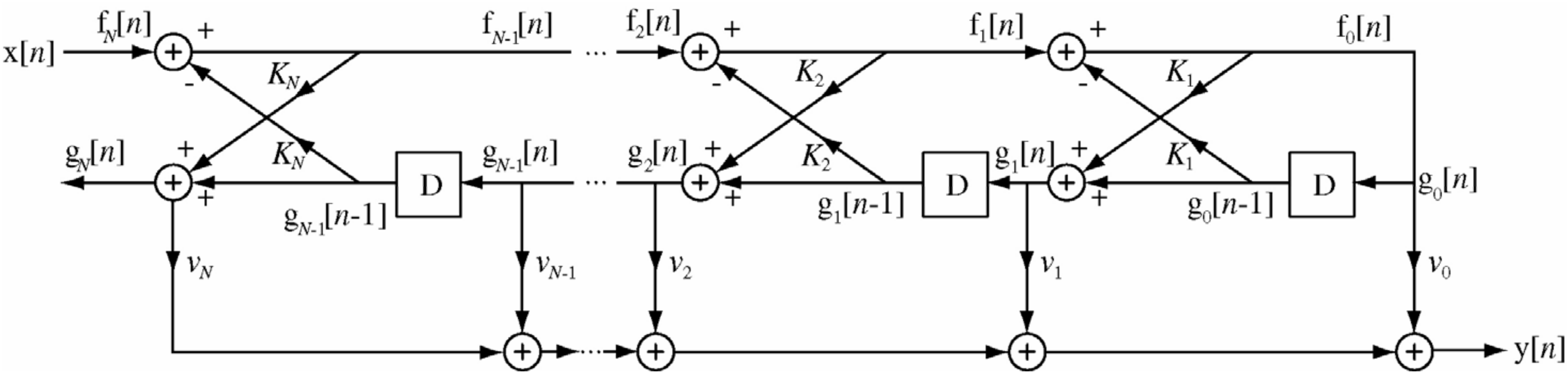


Lattice-Ladder IIR Structure

If we want to add finite zeros to $\mathrm{H}_m(z)$ we can add a ladder network to the lattice. Then the transfer function will be of the form

$$\mathrm{H}(z) = \frac{\gamma_N(0) + \gamma_N(1)z^{-1} + \cdots + \gamma_N(N)z^{-N}}{1 + \alpha_N(1)z^{-1} + \alpha_N(2)z^{-2} + \cdots + \alpha_N(N)z^{-N}} = \frac{\Gamma_N(z)}{\mathrm{A}_N(z)}$$

$$\mathrm{y}[n] = \sum_{m=0}^{N} v_m \, \mathrm{g}_m[n] \Rightarrow \mathrm{Y}(z) = \sum_{m=0}^{N} v_m \, \mathrm{G}_m(z)$$



Lattice-Ladder Example

Synthesize the transfer function

$$\mathrm{H}(z) = \frac{1 - z^{-1} + 0.5z^{-2}}{1 + 0.2z^{-1} - 0.15z^{-2}} = \frac{z^2 - z + 0.5}{z^2 + 0.2z - 0.15}$$

using a lattice-ladder network.

$$\mathrm{A}_2(z) = 1 + 0.2z^{-1} - 0.15z^{-2}$$

$$K_2 = -0.15 \text{ and } \mathrm{B}_2(z) = -0.15 + 0.2z^{-1} + z^{-2}$$

$$\Gamma_2(z) = \sum_{m=0}^{2} v_m \mathrm{B}_m(z) = 1 - z^{-1} + 0.5z^{-2} \Rightarrow v_2 = 0.5$$

$$\Gamma_1(z) = \Gamma_2(z) - v_2\mathrm{B}_2(z) = 1 - z^{-1} + 0.5z^{-2} - 0.5\left(-0.15 + 0.2z^{-1} + z^{-2}\right)$$

$$\Gamma_1(z) = 1.075 - 1.1z^{-1} \Rightarrow v_1 = -1.1$$

$$\Gamma_0(z) = \Gamma_1(z) - v_1\mathrm{B}_1(z)$$

Lattice-Ladder Example

Using $A_{m-1}(z) = \dfrac{A_m(z) - K_m B_m(z)}{1 - K_m^2}$

$$A_1(z) = \frac{1 + 0.2z^{-1} - 0.15z^{-2} - (-0.15)\left(-0.15 + 0.2z^{-1} + z^{-2}\right)}{1 - (-0.15)^2}$$

$$A_1(z) = \frac{0.9775 + 0.23z^{-1}}{0.9775} = 1 + 0.23529z^{-1}$$

$K_1 = 0.23529$ and $B_1(z) = 0.23529 + z^{-1}$

$$\Gamma_0(z) = 1.075 - 1.1z^{-1} - (-1.1)\left(0.23529 + z^{-1}\right) = 1.3382 \Rightarrow v_0 = 1.3382$$