Lattice Structures

Lattices

Consider an *m*th-order FIR filter with

$$\mathrm{H}_m(z) = \frac{\mathrm{Y}(z)}{\mathrm{X}(z)} = \sum_{k=0}^{m} \alpha_m[k] z^{-k}$$

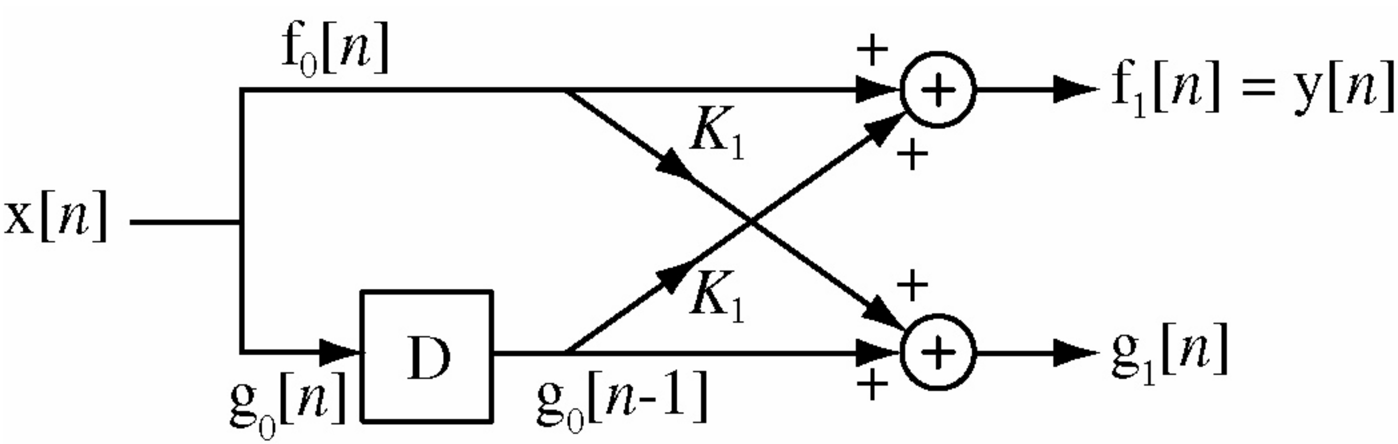
where the α's are the filter coefficients and $\alpha_m[0] = 1$ for any *m*.

Then $\mathrm{h}_m[n] = \sum_{k=0}^{m} \alpha_m[k]\delta[n-k]$ and $\mathrm{y}[n] = \mathrm{x}[n] * \alpha_m[n]$

For $m = 1$, $\mathrm{y}[n] = \alpha_1[0]\mathrm{x}[n] + \alpha_1[1]\mathrm{x}[n-1]$.

Lattices

The recursion relation $y[n] = \alpha_1[0]x[n] + \alpha_1[1]x[n-1]$ can be realized by this **lattice** structure if $K_1 = \alpha_1[1]$ and $y[n]. = f_1[n]$.



Lattices

The filter can also be described by

$$\begin{bmatrix} f_1[n] \\ g_1[n] \end{bmatrix} = \begin{bmatrix} 1 & K_1 \\ K_1 & 1 \end{bmatrix} \begin{bmatrix} f_0[n] \\ g_0[n-1] \end{bmatrix} \Rightarrow \begin{bmatrix} F_1(z) \\ G_1(z) \end{bmatrix} = \begin{bmatrix} 1 & K_1 \\ K_1 & 1 \end{bmatrix} \begin{bmatrix} F_0(z) \\ z^{-1} G_0(z) \end{bmatrix}$$

or

$$\begin{bmatrix} y[n] \\ g_1[n] \end{bmatrix} = \begin{bmatrix} 1 & K_1 \\ K_1 & 1 \end{bmatrix} \begin{bmatrix} x[n] \\ x[n-1] \end{bmatrix} \Rightarrow \begin{bmatrix} Y(z) \\ G_1(z) \end{bmatrix} = \begin{bmatrix} 1 & K_1 \\ K_1 & 1 \end{bmatrix} \begin{bmatrix} X(z) \\ z^{-1} X(z) \end{bmatrix}$$

Multiplying out the matrices,

$$Y(z) = X(z) + K_1 z^{-1} X(z) \Rightarrow y[n] = x[n] + K_1\, x[n-1]$$

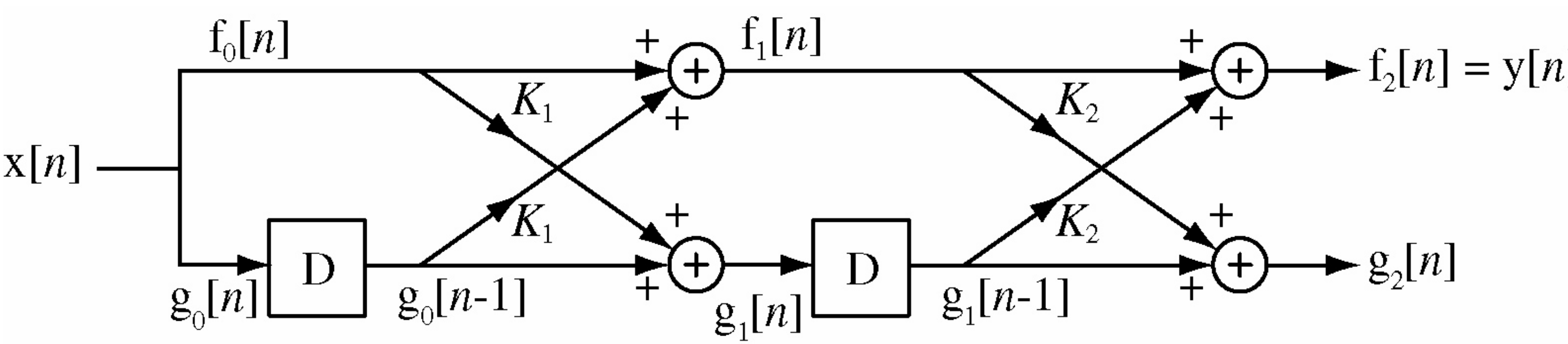
$$G_1(z) = K_1\, X(z) + z^{-1} X(z) \Rightarrow g_1[n] = K_1\, x[n] + x[n-1]$$

Notice that in these two recursion relations the coefficients occur in reverse order.

Lattices

Next consider a second-order FIR filter

$$y[n] = \underbrace{\alpha_2[0]}_{=1} x[n] + \alpha_2[1] x[n-1] + \alpha_2[2] x[n-2].$$

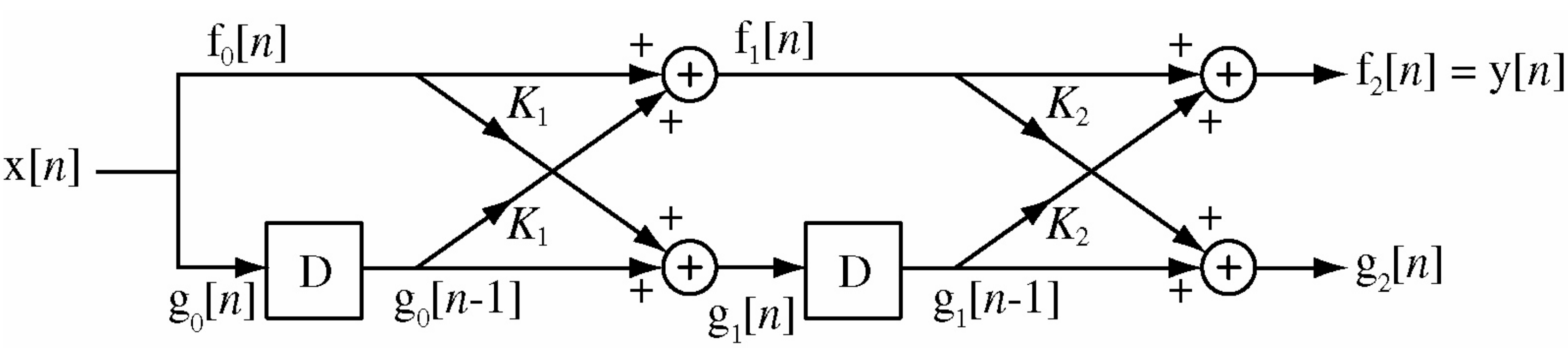


Lattices

$$\begin{bmatrix} F_1(z) \\ G_1(z) \end{bmatrix} = \begin{bmatrix} 1 & K_1 \\ K_1 & 1 \end{bmatrix} \begin{bmatrix} F_0(z) \\ z^{-1} G_0(z) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} F_1(z) \\ G_1(z) \end{bmatrix} = \begin{bmatrix} F_1(z) \\ z^{-1} G_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & K_1 \\ K_1 & 1 \end{bmatrix} \begin{bmatrix} F_0(z) \\ z^{-1} G_0(z) \end{bmatrix}$$

$$\begin{bmatrix} F_1(z) \\ z^{-1} G_1(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & K_1 \\ K_1 & 1 \end{bmatrix} \begin{bmatrix} X(z) \\ z^{-1} X(z) \end{bmatrix}$$



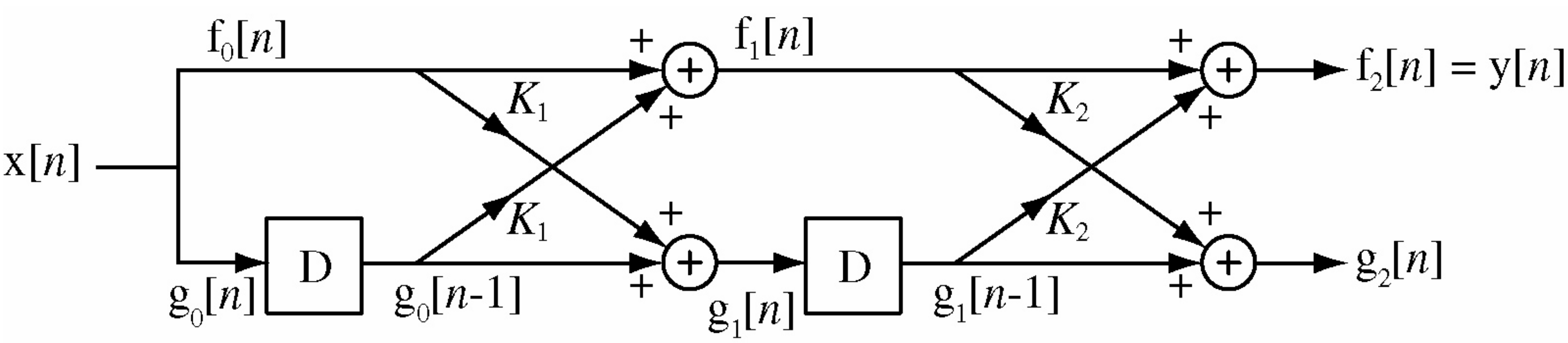
Lattices

$$\begin{bmatrix} \mathrm{Y}(z) \\ \mathrm{G}_2(z) \end{bmatrix} = \begin{bmatrix} \mathrm{F}_2(z) \\ \mathrm{G}_2(z) \end{bmatrix} = \begin{bmatrix} 1 & K_2 \\ K_2 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{F}_1(z) \\ z^{-1}\mathrm{G}_1(z) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & K_2 \\ K_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix} \begin{bmatrix} 1 & K_1 \\ K_1 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{X}(z) \\ z^{-1}\mathrm{X}(z) \end{bmatrix}$$

Multiplying the matrices,

$$\begin{bmatrix} \mathrm{Y}(z) \\ \mathrm{G}_2(z) \end{bmatrix} = \begin{bmatrix} 1 + z^{-1}K_1K_2 & K_1 + z^{-1}K_2 \\ K_2 + z^{-1}K_1 & K_1K_2 + z^{-1} \end{bmatrix} \begin{bmatrix} \mathrm{X}(z) \\ z^{-1}\mathrm{X}(z) \end{bmatrix}$$



Lattices

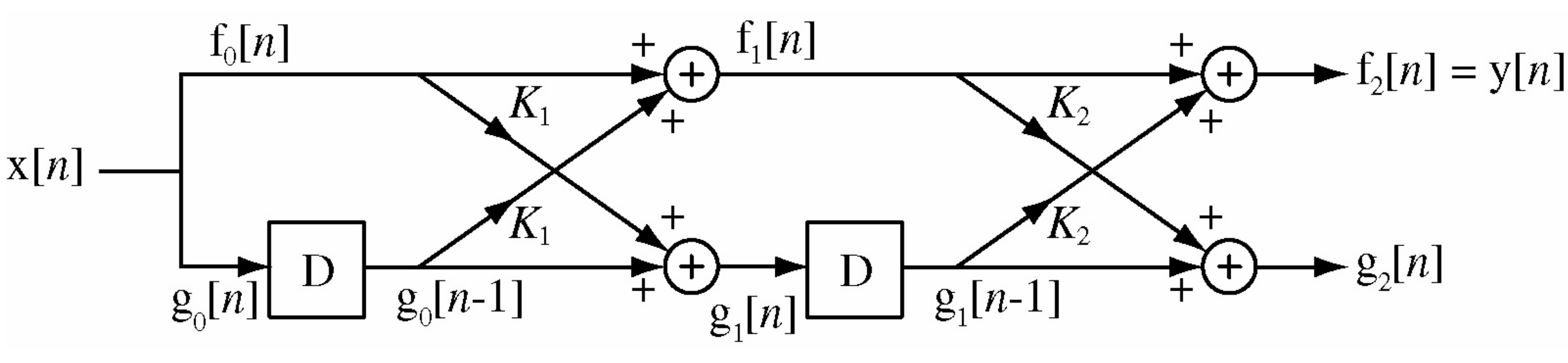
In the time domain

$$y[n] = x[n] + K_1(K_2+1)x[n-1] + K_2\,x[n-2]$$

$$g_2[n] = K_2\,x[n] + K_1(K_2+1)x[n-1] + x[n-2]$$

Again the coefficients occur in reverse order in the two recursion relations. This occurs for any value of *m*. If $K_1(K_2+1) = \alpha_2[1]$ and $K_2 = \alpha_2[2]$ the upper lattice signal is the desired response and

$$K_1 = \frac{\alpha_2[1]}{\alpha_2[2]+1} \quad \text{and} \quad K_2 = \alpha_2[2].$$



Lattices

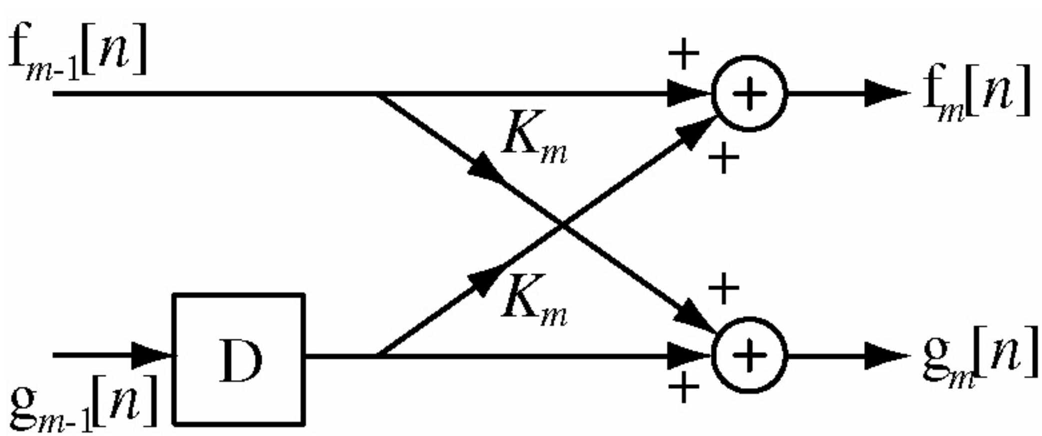
In the general $(M-1)$th order case,

$$f_0[n] = g_0[n] = x[n]$$

$$f_m[n] = f_{m-1}[n] + K_m\, g_{m-1}[n-1] \;\;,\;\; m = 0,1,2,\cdots,M-1$$

$$g_m[n] = K_m\, f_{m-1}[n] + g_{m-1}[n-1] \;\;,\;\; m = 0,1,2,\cdots,M-1$$

$$y[n] = f_{M-1}[n]$$



Lattices

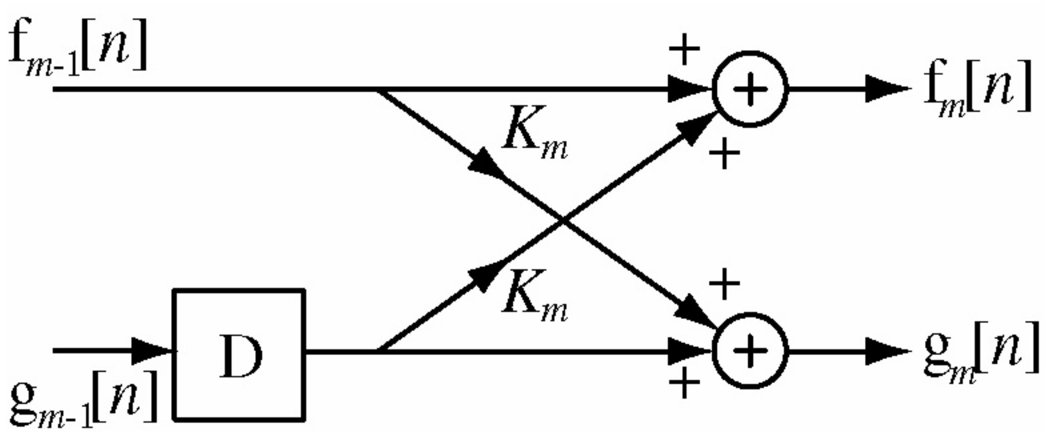
The output signal from the *m*th stage is

$$\mathrm{f}_m[n] = \mathrm{x}[n] * \alpha_m[n].$$

Therefore

$$\mathrm{F}_m(z) = \mathrm{X}(z)\mathrm{A}_m(z) = \mathrm{F}_0(z)\mathrm{A}_m(z)$$

where $\alpha_m[n] \xleftrightarrow{\mathcal{Z}} \mathrm{A}_m(z) = \sum_{k=0}^{m} \alpha_m[k] z^{-k}.$



Lattices

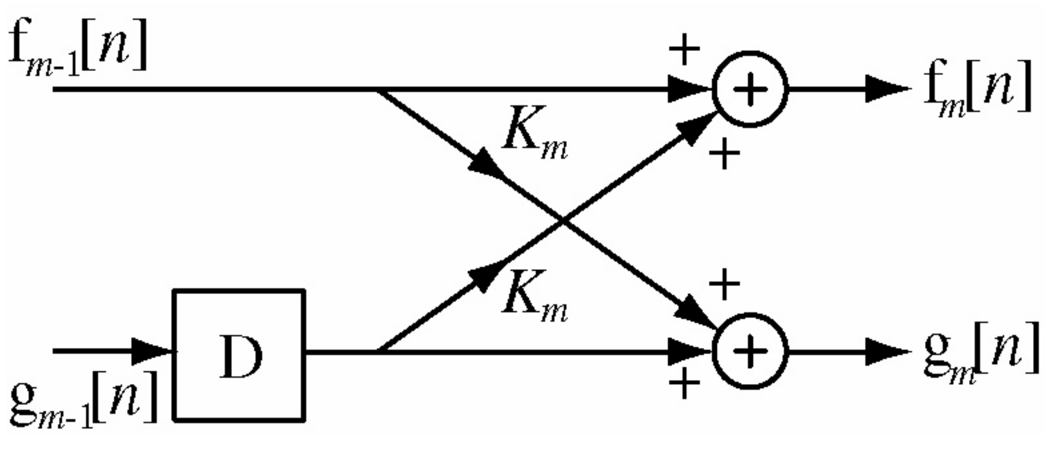
Given that the coefficients for g are always the reverse of the coefficients for f, the other output signal of the *m*th stage is

$$\mathrm{g}_m[n] = \sum_{k=0}^{m} \alpha_m[m-k]\,\mathrm{x}[n-k]$$

Let $\beta_m[k] = \alpha_m[m-k]$. Then

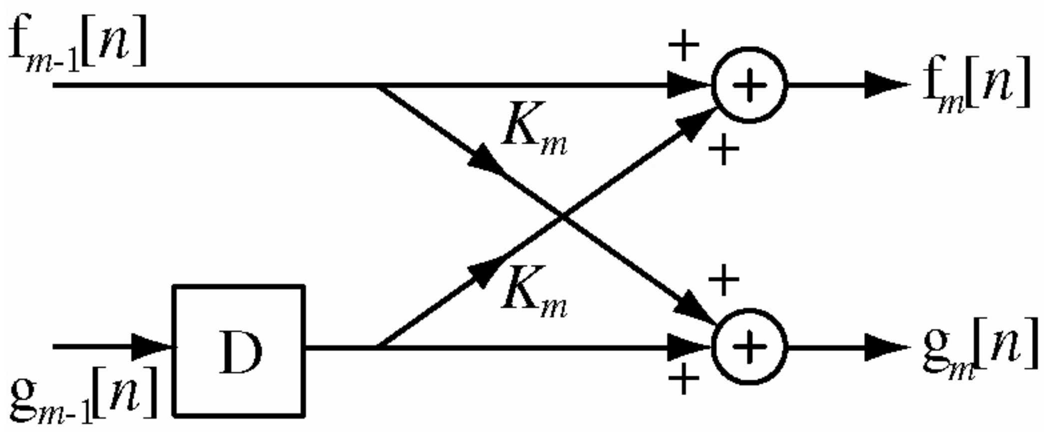
$$\mathrm{g}_m[n] = \sum_{k=0}^{m} \beta_m[k]\,\mathrm{x}[n-k] = \beta_m[n] * \mathrm{x}[n]$$

and $\mathrm{G}_m(z) = \mathrm{X}(z)\mathrm{B}_m(z)$ where $\beta_m[n] \xleftrightarrow{\mathcal{Z}} \mathrm{B}_m(z)$.



Lattices

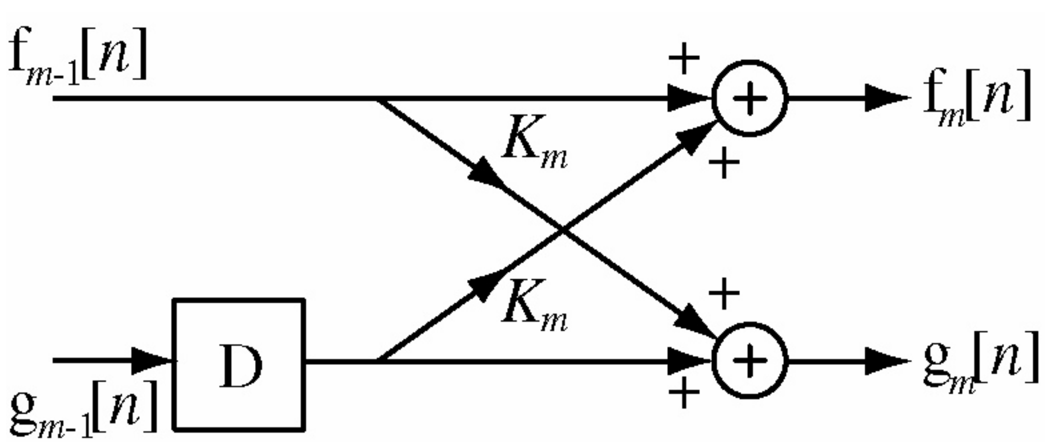
From the definition $\beta_m[k] = \alpha_m[m-k] \Rightarrow \mathrm{B}_m(z) = \sum_{k=0}^{m} \alpha_m[m-k] z^{-k}$. Let $q = m-k$. Then $\mathrm{B}_m(z) = \sum_{q=m}^{0} \alpha_m[q] z^{q-m} = z^{-m} \sum_{q=0}^{m} \alpha_m[q] z^{q}$. It was shown above that $\mathrm{A}_m(z) = \sum_{k=0}^{m} \alpha_m[k] z^{-k}$ therefore $\mathrm{B}_m(z) = z^{-m} \mathrm{A}_m(1/z)$ which implies that the zeros of $\mathrm{B}_m(z)$ are the reciprocals of the zeros of $\mathrm{A}_m(z)$.



Lattices

Now the transfer function of the mth stage can be expressed in terms of the transfer function of the $(m-1)$th stage as

$$\begin{bmatrix} A_m(z) \\ B_m(z) \end{bmatrix} = \begin{bmatrix} 1 & K_m \\ K_m & 1 \end{bmatrix} \begin{bmatrix} A_{m-1}(z) \\ B_{m-1}(z) \end{bmatrix}$$



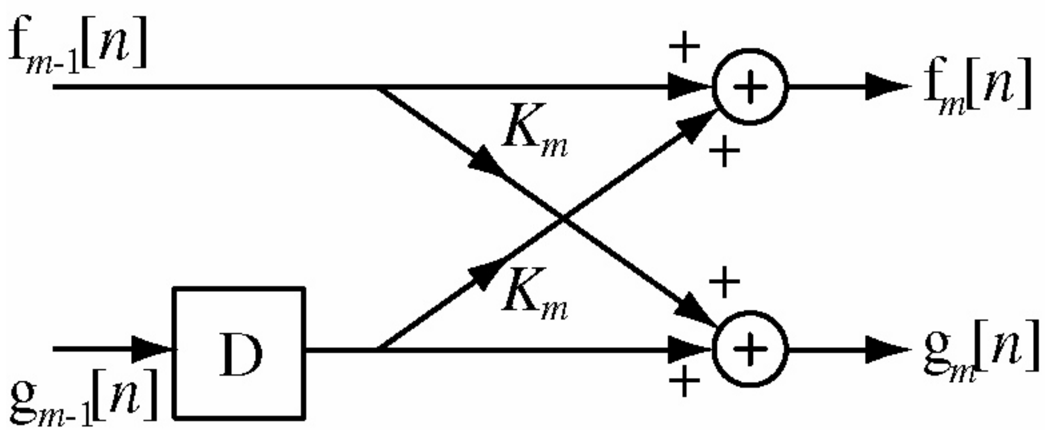
Lattices

The coefficients α of the Direct Form II filter and the reflection coefficients of the lattice structure are related and the α's can be found from the K's recursively by using

$$A_0(z) = B_0(z) = 1$$

$$A_m(z) = A_{m-1}(z) + K_m z^{-1} B_{m-1}(z)$$

$$B_m(z) = z^{-m} A_m(1/z)$$



Lattices

Example

Let the reflection coefficients be $K_1 = 1/2$, $K_2 = 1/5$ and $K_3 = 3/4$.

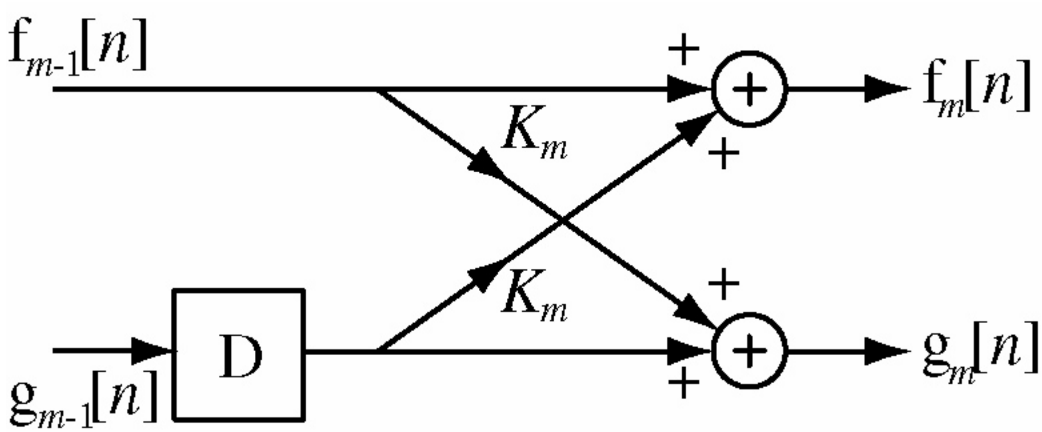
Find the Direct Form II filter coefficients α.

Solution

$$\mathrm{A}_0(z) = \mathrm{B}_0(z) = 1$$

$$\mathrm{A}_1(z) = \mathrm{A}_0(z) + K_1 z^{-1}\mathrm{B}_0(z) = 1 + z^{-1}K_1 = 1 + 0.5z^{-1}$$

$$\mathrm{B}_1(z) = z^{-1}\mathrm{A}_1(1/z) = z^{-1}(1 + 0.5z) = 0.5 + z^{-1}$$



Lattices

Example (cont.)
Solution

$$A_2(z) = A_1(z) + K_2 z^{-1} B_1(z) = 1 + 0.5z^{-1} + 0.2z^{-1}\left(0.5 + z^{-1}\right)$$

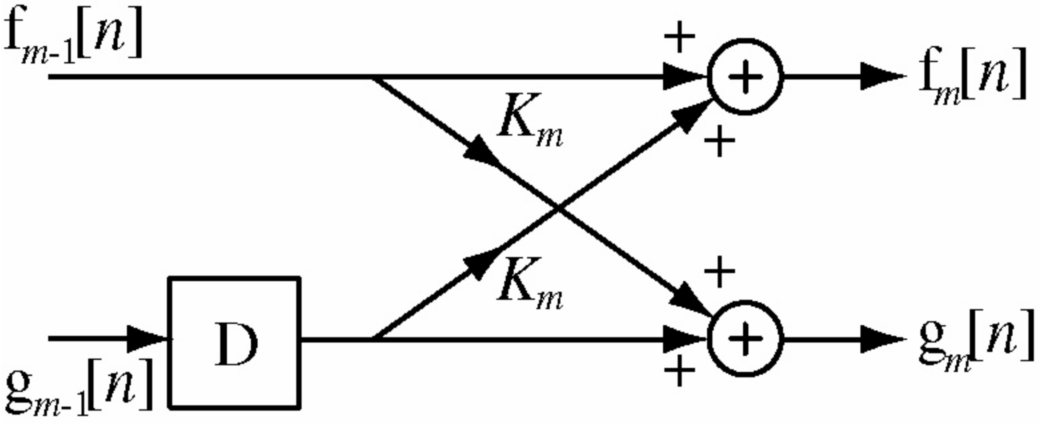
$$= 1 + 0.6z^{-1} + 0.2z^{-2}$$

$$B_2(z) = z^{-2} A_2(1/z) = z^{-2}\left(1 + 0.6z + 0.2z^2\right) = 0.2 + 0.6z^{-1} + z^{-2}$$

$$A_3(z) = A_2(z) + K_3 z^{-1} B_2(z) = 1 + 0.6z^{-1} + 0.2z^{-2}$$

$$+ 0.75z^{-1}\left(0.2 + 0.6z^{-1} + z^{-2}\right)$$

$$= 1 + 0.75z^{-1} + 0.65z^{-2} + 0.75z^{-3}$$



Lattices

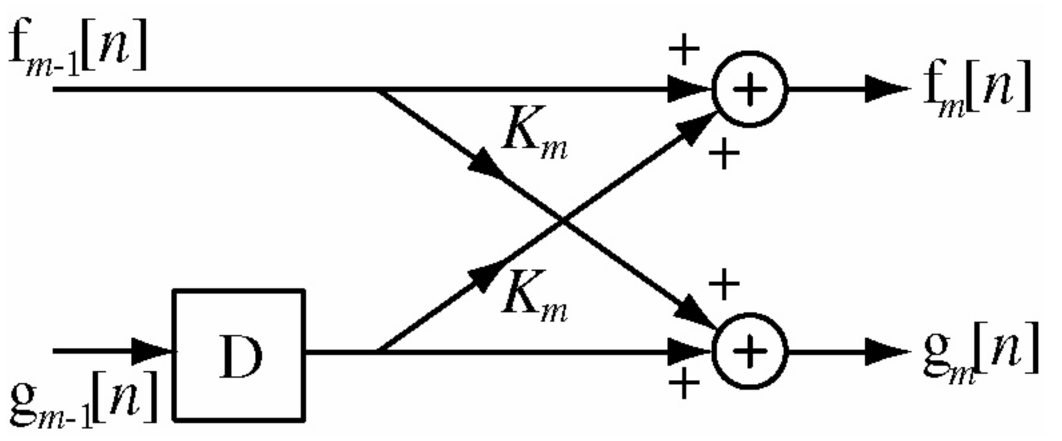
Example (cont.)

Solution

$$B_3(z) = z^{-3}A_3(1/z) = z^{-3}\left(1 + 0.75z + 0.65z^2 + 0.75z^3\right)$$

$$= 0.75 + 0.65z^{-1} + 0.75z^{-2} + z^{-3}$$

So $\alpha_3[0] = 1$, $\alpha_3[1] = 0.75$, $\alpha_3[2] = 0.65$ and $\alpha_3[3] = 0.75$



Lattices

We can also find the reflection coefficients from the Direct Form II coefficients using

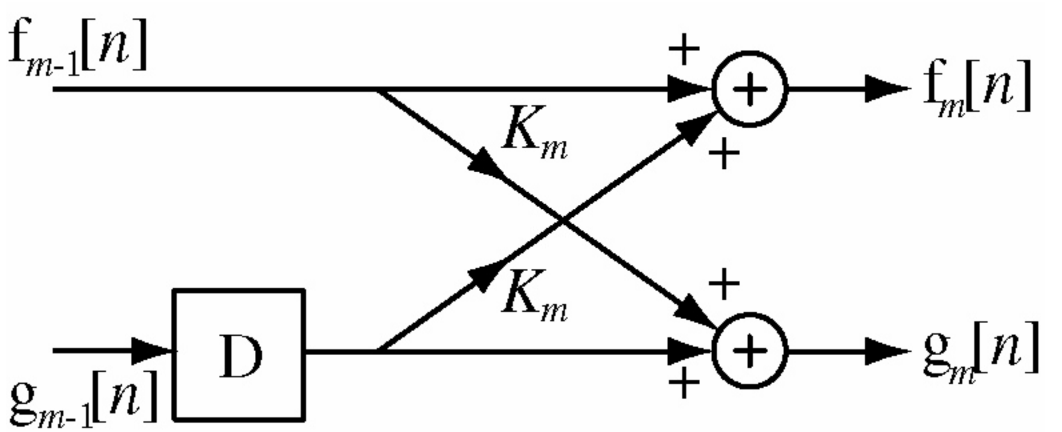
$$\mathrm{A}_m(z) = \mathrm{A}_{m-1}(z) + K_m z^{-1}\mathrm{B}_{m-1}(z)$$

and

$$\mathrm{g}_m[n] = K_m\,\mathrm{f}_{m-1}[n] + \mathrm{g}_{m-1}[n-1] \ \ , \ \ m = 0,1,2,\cdots,M-1$$

z transforming we get

$$\mathrm{G}_m(z) = K_m\,\mathrm{F}_{m-1}(z) + z^{-1}\mathrm{G}_{m-1}(z)$$



Lattices

Dividing through by $\mathrm{X}(z)$ we get

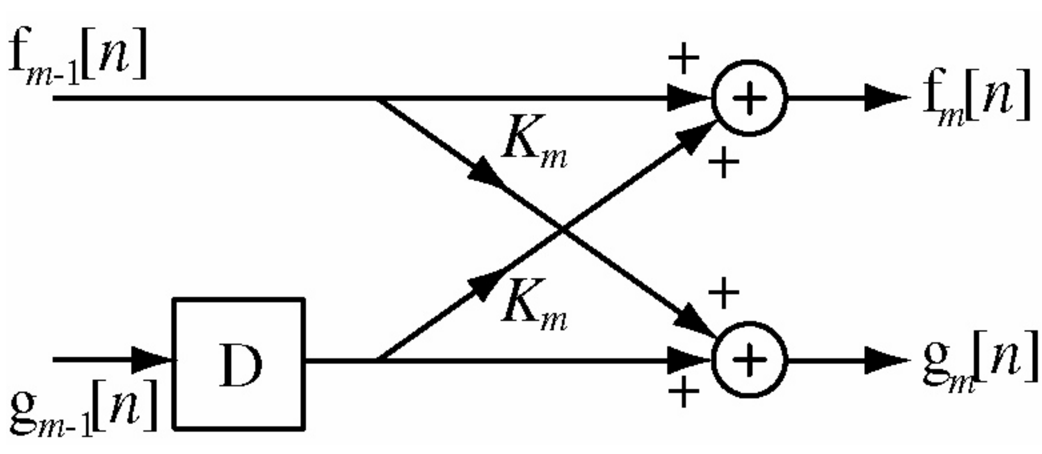
$$\mathrm{B}_m(z) = K_m \mathrm{A}_{m-1}(z) + z^{-1}\mathrm{B}_{m-1}(z) \Rightarrow z^{-1}\mathrm{B}_{m-1}(z) = \mathrm{B}_m(z) - K_m \mathrm{A}_{m-1}(z)$$

Combining this with

$$\mathrm{A}_m(z) = \mathrm{A}_{m-1}(z) + K_m z^{-1}\mathrm{B}_{m-1}(z)$$

we get

$$\mathrm{A}_m(z) = \mathrm{A}_{m-1}(z) + K_m\left[\mathrm{B}_m(z) - K_m \mathrm{A}_{m-1}(z)\right]$$



Lattices

Finally, solving for $\mathrm{A}_{m-1}(z)$ we get

$$\mathrm{A}_{m-1}(z)=\frac{\mathrm{A}_m(z)-K_m\mathrm{B}_m(z)}{1-K_m^2}$$

where $K_m=\alpha_m[m]$. Iterating on this equation we can find all the coefficients α.